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Transforming linear functional systems into fully integrable systems☆

Ziming Li [a], Min Wu [b]

[a] Key Lab of Mathematics-Mechanization, AMSS, Chinese Academy of Sciences, Zhong Guan Cun, Beijing 100190, China
[b] Shanghai Key Lab of Trustworthy Computing, East China Normal University, Shanghai 200062, China

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ABSTRACT

A linear (partial) functional system consists of linear partial differential, difference equations or any mixture thereof. We present an algorithm that determines whether linear functional systems are ∂-finite, and transforms ∂-finite systems to fully integrable ones. The algorithm avoids using Gröbner bases in Laurent–Ore modules when ∂-finite systems correspond to finite-dimensional Ore modules.

1. Introduction

A system of linear ordinary differential equations with coefficients in $\mathbb{C}(t)$ can be transformed into a first-order linear differential system

$$\frac{d}{dt}\,\mathbf{z}(t) = A\,\mathbf{z}(t)$$

where A is a square matrix over $\mathbb{C}(t)$ of size, say n, and $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^\tau$, provided that the original system has a finite-dimensional solution space. There is a one-to-one correspondence between the solutions of these two systems. Consequently, the solution spaces of both systems have dimension n over $\mathbb{C}$. From Proposition 1.20 in van der Put and Singer (2003), this conclusion remains true for systems over a differential field (F, δ), provided that F is of characteristic zero and has an algebraically closed field of constants.

Assume that F is a field endowed with an automorphism σ, and that Σ stands for a system of linear homogeneous ordinary difference equations over F whose solution space is finite-dimensional over

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E-mail addresses: zmli@mmrc.iss.ac.cn (Z. Li), mwu@sei.ecnu.edu.cn (M. Wu).

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the constants. Like in the differential case, Σ can be converted into a first-order system $\sigma(\mathbf{z}) = B\mathbf{z}$ where B is a square matrix over F of size, say n, and $\mathbf{z}$ is a column vector $(z_1, \ldots, z_n)^\tau$ of unknowns. Unlike the differential case, if the coefficient matrix B is singular, the linear relations among its rows allow us to transform the system further into a new one whose coefficient matrix has smaller size, since σ is an automorphism. Doing this recursively yields a partition

$$\{z_1, \ldots, z_n\} = \{y_1, \ldots, y_d\} \cup \{y_{d+1}, \ldots, y_n\},$$

a $d \times d$ invertible matrix P, and an $(n-d) \times d$ matrix Q such that the new system consists of a first-order difference system

$$\sigma(y_1, \ldots, y_d)^\tau = P(y_1, \ldots, y_d)^\tau, \tag{1}$$

and $n - d$ linear relations

$$(y_{d+1}, \ldots, y_n)^\tau = Q(y_1, \ldots, y_d)^\tau.$$

There is a one-to-one correspondence between the solutions of Σ and (1). By the construction of Picard–Vessiot rings in van der Put and Singer (1997), the solution spaces of Σ and (1) have dimension d, provided that F is of characteristic zero and has an algebraically closed field of constants.

A linear functional system consists of linear partial differential, difference equations or any mixture thereof. Such a system is said to be ∂-finite if its module of formal solutions is a finite-dimensional vector space over the ground field (Bronstein et al., 2005). The dimension of this module is called the linear dimension of the system, and corresponds to the dimension of its solution space. It is shown in Bronstein et al. (2005) and Wu (2005) that a ∂-finite system is equivalent to a fully integrable system, whose linear dimension equals the number of its unknowns. For fully integrable systems, a factorization algorithm is developed in Li et al. (2006) and Wu and Li (2007), and a Galois theory is presented in Hardouin and Singer (2008). These results motivate us to transform ∂-finite systems into fully integrable ones.

A naive way to transform a ∂-finite system is to compute a Gröbner basis of its corresponding submodule over a Laurent–Ore algebra (see Wu, 2005 and Zhou and Winkler, 2008), and construct the desired fully integrable system from the basis. As Laurent–Ore algebras are localizations of Ore algebras, it is easier to compute Gröbner bases in free modules over Ore algebras (see Cox et al., 2004, Ch. 5, Chyzak and Salvy, 1998, Chyzak et al., 2004). This observation motivates us to transform ∂-finite systems by the latter Gröbner bases whenever possible. Moreover, we avoid computing Gröbner bases of any kind when transforming an integrable system, which is a common special case of linear functional systems.

The contributions of this paper include: an algorithm for determining the reflexive closure of the zero submodule of a finite-dimensional module over a noncommutative domain (see Section 3.2), and an algorithm for transforming a ∂-finite system into a fully integrable one (see Section 5). The former algorithm evolves from discussions with Manuel Bronstein and the algorithm LinearReduction in Wu (2005, Section 2.5.2). It enables us to use linear algebras to transform an integrable system. The latter algorithm uses the method in Chyzak et al. (2004) to compute a Gröbner basis of the Ore submodule defined by the input system. This Gröbner basis tells us whether to use the former algorithm or to compute a Gröbner basis over Laurent–Ore algebras. Indeed, we have only one artificial example (see Example 33), for which a Gröbner basis over some Laurent–Ore algebra has to be computed.

The rest of this paper is organized as follows. In Section 2, we recall how to localize modules over a noncommutative domain, and introduce the notion of reflexive closures of submodules. An algorithm is presented in Section 3 for computing a linear basis of the reflexive closure of the zero submodule in a finite-dimensional module. We extend an equivalence relation among linear ordinary differential (difference) equations to linear functional systems in Section 4, and describe in Section 5 a method for transforming a linear functional system to its integrable connection, which is fully integrable and equivalent to the given system. Our results are summarized in Section 6.

Throughout this paper, rings are not necessarily commutative, while fields are always commutative. An (integral) domain is a ring without zero-divisors. All modules, vector spaces and ideals are left ones, unless mentioned otherwise. Vectors are denoted by the boldfaced letters $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, etc., and vectors of unknowns by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. The notation $(\cdot)^\tau$ stands for the transpose of a vector or matrix.

2. Localizations and reflexive submodules

In this section, we recall a standard way to localize a module over a noncommutative domain by a left Ore set described in Cohn (1985, Section 0.9) or Rowen (1988, Section 3.1). The localizations help us to describe the transformation algorithm concisely. We define the notion of reflexive closures, which enables us to get information about localizations.

Let R be a (noncommutative) domain, and denote $R \setminus \{0\}$ by $R^\times$, which is a monoid. A submonoid T of $R^\times$ is called a left Ore set of R if $Rt \cap Tr \neq \emptyset$ for all $t \in T$ and $r \in R^\times$. If t_1 and t_2 are in a left Ore set T, then $Rt_1 \cap Tt_2$ contains an element t such that $t = r_1' t_1 = t_2' t_2$ for some $r_1' \in R$ and $t_2' \in T$. So t is in T. We say that t is a common left multiple of t_1 and t_2 in T. An easy induction implies that a finite number of elements in T have a common left multiple in T.

Let M be a module over R. The (left) localization of M at T is defined to be

$$T^{-1}M = \{t^{-1}\mathbf{v} \mid t \in T,\ \mathbf{v} \in M\}.$$

Two elements $t_1^{-1}\mathbf{v}_1$ and $t_2^{-1}\mathbf{v}_2$ in the localization are equal if there exist $r_1, r_2 \in R^\times$ such that $r_1t_1 = r_2t_2 \in T$ and $r_1\mathbf{v}_1 = r_2\mathbf{v}_2$ in M.

For any two elements $t_1{}^{-1}\mathbf{v}_1, t_2{}^{-1}\mathbf{v}_2 \in T^{-1}M$ with $t_1, t_2 \in T$ and $\mathbf{v}_1, \mathbf{v}_2 \in M$, their sum is defined as:

$$t_1{}^{-1}\mathbf{v}_1 + t_2{}^{-1}\mathbf{v}_2 = t^{-1}\left(r_1\mathbf{v}_1 + r_2\mathbf{v}_2\right),$$

where t is a common left multiple of t_1 and t_2 in T, and $t = r_it_i$ with $r_i \in R$ for $i = 1, 2$.

Observe that, for $t \in T$ and $r \in R$, there exist $r' \in R$ and $t' \in T$ such that $r't = t'r$. So we define the left-hand scalar multiplication as:

$$r\left(t^{-1}\mathbf{v}\right) = \left(t'\right)^{-1} r'\mathbf{v} \quad \text{for all } \mathbf{v} \in M.$$

Equipped with these two operations, $T^{-1}M$ becomes a left module over R.

Take M to be the ring R itself. For $t_1^{-1}r_1, t_2^{-1}r_2 \in T^{-1}R$ with $t_1, t_2 \in T$ and $r_1, r_2 \in R$, we define the product of $t_1^{-1}r_1$ and $t_2^{-1}r_2$ as

$$\left(t_1^{-1}r_1\right)\left(t_2^{-1}r_2\right) = (t_3t_1)^{-1}r_3r_2 \quad \text{where } t_3r_1 = r_3t_2 \text{ for some } t_3 \in T \text{ and } r_3 \in R.$$

Then the left R-module $T^{-1}R$ becomes a domain.

The reader is referred to Fu et al. (2009) for a detailed account that verifies the above three operations are well-defined. Elementary constructions of $T^{-1}R$ and $T^{-1}M$ are also presented and verified in Rowen (1988, Section 3.1).

The following examples will be frequently used in the sequel.

Example 1. Let R be a commutative domain and T a submonoid of $R^\times$. Then T is an Ore set, and the (left) localization $T^{-1}R$ of R coincides with the usual localization defined in commutative algebra.

Example 2. Let F be a field and σ an automorphism of F. The ring of shift operators with respect to σ is denoted by $R = F[\partial; \sigma]$, whose commutation rule is $\partial f = \sigma(f)\partial$ for all $f \in F$. Let T be the monoid generated by ∂, which is a left Ore set of R. The localization $T^{-1}\left(F[\partial; \sigma]\right)$ is the ring $F[\partial, \partial^{-1}]$ defined in van der Put and Singer (1997).

Example 3. Let F be a field, $\delta_1, \ldots, \delta_\ell$ be derivations on F, and $\sigma_{\ell+1}, \ldots, \sigma_m$ be automorphisms of F. Assume that all these maps commute pairwise. According to Chyzak and Salvy (1998), the ring of Ore polynomials over F is $F[\partial_1, \ldots, \partial_\ell, \partial_{\ell+1}, \ldots, \partial_m]$ endowed with the following commutation rules:

(i) $\partial_i\partial_j = \partial_j\partial_i$ for all i, j with $1 \le i < j \le m$;
(ii) $\partial_i f = f\partial_i + \delta_i(f)$ for all i with $1 \le i \le \ell$ and $f \in F$; and
(iii) $\partial_j f = \sigma_j(f)\partial_j$ for all j with $\ell + 1 \le j \le m$ and $f \in F$.

Let T be the monoid generated by $\partial_{\ell+1}, \ldots, \partial_m$. Then T is a left Ore set by rules (i) and (iii). The localization $T^{-1}R$ is the Laurent–Ore algebra

$$F[\partial_1, \ldots, \partial_\ell, \partial_{\ell+1}, \partial_{\ell+1}^{-1}, \ldots, \partial_m, \partial_m^{-1}]$$

defined in Bronstein et al. (2005).

The canonical R-homomorphism ϕ from M to $T^{-1}M$ maps $\mathbf{v}$ to $1^{-1}\mathbf{v}$. It is straightforward to see that $\ker(\phi) = \{\mathbf{v} \in M \mid t\mathbf{v} = 0 \text{ for some } t \in T\}$. This observation motivates us to borrow a terminology from Cohn (1965).

Definition 4. Let R be a domain, T a left Ore set of R and M a module over R. A submodule N of M is said to be *reflexive* (with respect to T) if $t\mathbf{v} \in N$ implies $\mathbf{v} \in N$ for every $t \in T$ and $\mathbf{v} \in M$. The *reflexive closure* of a submodule N, denoted $\widehat{N}$, is the intersection of all reflexive submodules containing N.

Since the intersection of reflexive submodules is again reflexive, the reflexive closure of a submodule N is the smallest reflexive submodule containing N.

Example 5. Let R and T be given in Example 2. The submodule $R(\partial^2 + \partial)$ is not reflexive, because it does not contain $\partial + 1$. Its reflexive closure is the submodule $R(\partial + 1)$.

We call the submodule $\{0\}$ of an R-module M the *zero submodule of* M and denote it by 0_M. A set-theoretic characterization of reflexive closures is given in

Proposition 6. *Let R be a domain and T a left Ore set of R. If N is a submodule of an R-module M, then $\widehat{N} = \{\mathbf{v} \in M \mid \exists\, t \in T,\ t\mathbf{v} \in N\}$. In particular, $\widehat{0_M}$ is the kernel of the canonical homomorphism from M to $T^{-1}M$.*

Proof. Let $N' = \{\mathbf{v} \in M \mid \exists\, t \in T,\ t\mathbf{v} \in N\}$. It is a submodule because it equals the kernel of the composition of the canonical homomorphisms $M \to M/N \to T^{-1}(M/N)$ sending $\mathbf{v}$ to $1^{-1}(\mathbf{v} + N)$. Clearly, N' is reflexive by Definition 4. Thus, $\widehat{N} = N'$ because N' is a subset of every reflexive submodule containing N. □

It follows from Proposition 6 that

$$\widehat{0_{M/N}} = \{\mathbf{v}+N \mid \exists t \in T,\ t(\mathbf{v} + N) = 0\} = \{\mathbf{v} + N \mid \exists t \in T,\ t\mathbf{v} \in N\} = \widehat{N}/N,$$

which leads to

Corollary 7. *With the notation introduced in Proposition 6, we have $\widehat{0_{M/N}} = \widehat{N}/N$. In particular, N is reflexive if and only if $0_{M/N}$ is reflexive.*

The above corollary enables us to characterize reflexive submodules by their quotients.

Corollary 8. *With the notation introduced in Proposition 6, we have that N is reflexive if and only if N contains a submodule L such that N/L is reflexive in M/L.*

Proof. Let $M' = M/L$ and $N' = N/L$. Then M/N and M'/N' are isomorphic. Therefore, $0_{M/N}$ is reflexive if and only if $0_{M'/N'}$ is reflexive. Consequently, N is reflexive if and only if N' is reflexive by Corollary 7. □

Remark 9. Since the ring $T^{-1}R$ is a bi-module over R, $T^{-1}R \otimes_R M$ is well-defined, and isomorphic to $T^{-1}M$ canonically by the discussion on page 47 of Cohn (1985) (see also Fu et al. (2009, Section 5)). Therefore, $T^{-1}M$ is a module over $T^{-1}R$. Let N be a submodule of M. It is clear that $\psi : T^{-1}N \to T^{-1}M$ defined by $t^{-1}\mathbf{v} \mapsto t^{-1}\mathbf{v}$ for all $t \in T$ and $\mathbf{v} \in N$ is a monomorphism, which, together with the right exactness of $\otimes_R$, implies that $T^{-1}(M/N)$ and $(T^{-1}M)/(T^{-1}N)$ are isomorphic.

3. Finite-dimensional modules and their localizations

In this section, we assume that R is a domain containing a field F. Then a module over R is also a vector space over F. An R-module M is said to be *finite-dimensional* if $\dim_F M$ is finite.

Let T be a left Ore set of R. Assume further that, for every $t \in T$, there exists an automorphism σ_t of F such that

$$tf = \sigma_t(f)t \quad \text{for all } f \in F. \tag{2}$$

Typical examples for such domains are Ore algebras over F (see Examples 2 and 3).

Under these assumptions, we describe a relation between the dimension of M and that of $T^{-1}M$ in Section 3.1, and present an algorithm for computing an F-basis of $T^{-1}M$ by solving linear systems over F in Section 3.2.

3.1. Dimensions and bases

For two elements $a, b \in \mathbb{Z} \cup \{+\infty\}$, by $a = b$ we mean either $a, b \in \mathbb{Z}$ and $a = b$, or both $a = +\infty$ and $b = +\infty$.

Lemma 10. *For an R-module L,* $\dim_F L/\widehat{0_L} = \dim_F T^{-1}L$*, and* $L/\widehat{0_L}$ *is R-isomorphic to* $T^{-1}L$ *whenever either one is finite-dimensional.*

Proof. We claim that $\dim_F L \geq \dim_F T^{-1}L$. Suppose that $t_1{}^{-1}\mathbf{v}_1, \ldots, t_n{}^{-1}\mathbf{v}_n$ are linearly independent over F, where $\mathbf{v}_i \in L$ and $t_i \in T$ for all i with $1 \leq i \leq n$. Set t to be a common left multiple of $t_1, \ldots, t_n$ in T. Then $t = r_i t_i$ for some $r_i \in R$. Suppose that $\sum_{i=1}^n f_i r_i \mathbf{v}_i = 0$ in L with $f_i \in F$. Then $0 = t^{-1}(\sum_{i=1}^n f_i r_i \mathbf{v}_i) = \sum_{i=1}^n t^{-1}(f_i r_i \mathbf{v}_i)$ in $T^{-1}L$. By (2), $t\sigma_t^{-1}(f_i) = f_i t$. So the scalar multiplication of $T^{-1}L$ by R implies that

$$t^{-1}(f_i r_i \mathbf{v}_i) = \sigma_t^{-1}(f_i)\left(t^{-1}(r_i\mathbf{v}_i)\right) = \sigma_t^{-1}(f_i)\left(t_i^{-1}\mathbf{v}_i\right)$$

for each i. Hence $\sum_{i=1}^n \sigma_t^{-1}(f_i)\left(t_i^{-1}\mathbf{v}_i\right) = 0$, which implies $f_i = 0$ for all i with $1 \leq i \leq n$. Hence, $r_1\mathbf{v}_1, \ldots, r_n\mathbf{v}_n$ are linearly independent over F. The claim is proved.

Let ϕ be the canonical homomorphism from L to $T^{-1}L$. By Proposition 6, $L/\widehat{0_L}$ and $\phi(L)$ are isomorphic. Thus $\dim_F L/\widehat{0_L}$ cannot exceed $\dim_F T^{-1}L$. On the other hand, $T^{-1}L = T^{-1}(\phi(L)) \cong T^{-1}(L/\widehat{0_L})$. Note that $T^{-1}(\phi(L))$ is the localization of the R-submodule $\phi(L)$ of $T^{-1}L$, in which the action is defined as $t^{-1}(1^{-1}\mathbf{v}) = t^{-1}\mathbf{v}$ for any $t \in T$ and $\mathbf{v} \in L$. By the claim, $T^{-1}(L/\widehat{0_L})$ has dimension no more than $\dim_F L/\widehat{0_L}$. So $\dim_F T^{-1}L$ cannot exceed $\dim_F L/\widehat{0_L}$ either. Consequently, $\dim_F L/\widehat{0_L} = \dim_F T^{-1}L$.

The rest follows from the above equality and the observation that the canonical injection from $L/\widehat{0_L}$ to $T^{-1}(L/\widehat{0_L})$ is an R-isomorphism if either $\dim_F L/\widehat{0_L}$ or $\dim_F T^{-1}L$ is finite. □

By Lemma 10, we have that $\dim_F L \geq \dim_F T^{-1}L$. It is possible that $\dim_F L$ is infinite, while $\dim_F T^{-1}L$ is finite.

Example 11. Let $R = F[\partial_1, \partial_2]$ be given in Example 3 with $\ell = 0$ and $m = 2$. Then

$$T = \left\{\partial_1^{d_1}\partial_2^{d_2} \mid d_1, d_2 \in \mathbb{N}\right\}.$$

Let I be the (left) ideal of R generated by $L_1 = \partial_1\partial_2(\partial_1 + 1)$ and $L_2 = \partial_1\partial_2(\partial_2 + 1)$. Then

$$T^{-1}I = \left(T^{-1}R\right)L_1 + \left(T^{-1}R\right)L_2 = \left(T^{-1}R\right)(\partial_1 + 1) + \left(T^{-1}R\right)(\partial_2 + 1).$$

By Remark 9, $T^{-1}(R/I) \cong T^{-1}R/T^{-1}I$ is a one-dimensional vector space over F. However, computing a Gröbner basis of I yields that R/I is infinite-dimensional over F.

R-modules are usually infinite-dimensional, while their quotient modules may be finite-dimensional.

Corollary 12. *Let M be an R-module and N a submodule of M. If* $M/\widehat{N}$ *is finite-dimensional, then the map*

$$\begin{array}{rcl}\bar{\phi}: M/\widehat{N} & \rightarrow & T^{-1}M/T^{-1}N \\ \mathbf{v} + \widehat{N} & \mapsto & \mathbf{v} + T^{-1}N\end{array}$$

is an R-isomorphism.

Proof. By Corollary 7, $(M/N)/\widehat{0}_{M/N} = (M/N)/(\widehat{N}/N) \cong M/\widehat{N}$. Putting $L = M/N$, we see that $\bar{\phi}$ is the R-isomorphism induced by ϕ in the proof of Lemma 10. □

We are going to present some special properties of reflexive submodules in finite-dimensional modules over R in order to develop an algorithm for computing F-bases of their localizations with respect to T.

Let M be an R-module with a finite F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. For every $t \in T$, there exists an $n \times n$ matrix A_t over F such that

$$t(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = A_t(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau.$$

We call A_t the *matrix associated with* t and the F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. When the basis is clear from context, A_t is simply called the matrix associated with t.

Lemma 13. *Let M be a finite-dimensional R-module. Then 0_M is reflexive if and only if all the matrices associated with $t \in T$ and an F-basis are invertible.*

Proof. Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be an F-basis of M, and A_t the matrix associated with $t \in T$. Let $\mathbf{v} = \sum_{i=1}^n f_i \mathbf{b}_i \in M$ with $f_i \in F$. By (2),

$$t\mathbf{v} = (\sigma_t(f_1), \ldots, \sigma_t(f_n))\, A_t\, (\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau \quad \text{for all } t \in T. \tag{3}$$

If A_t is invertible for all $t \in T$, then $t\mathbf{v} = 0$ implies that $\sigma_t(f_i) = 0$, and, hence, $f_i = 0$ for all i with $1 \le i \le n$. Consequently, $\mathbf{v} = 0$ and 0_M is reflexive. Conversely, suppose that A_t is singular for some $t \in T$. Since σ_t is an automorphism of F, there exist $f_1, \ldots, f_n \in F$, not all zero, such that the nonzero vector $(\sigma_t(f_1), \ldots, \sigma_t(f_n))$ is in the left kernel of A_t. By (3), the vector $t\left(\sum_{i=1}^n f_i \mathbf{b}_i\right)$ equals zero. So 0_M is not reflexive. □

For a finite-dimensional R-module M, determining $\widehat{0_M}$ plays a key role in determining the reflexive closure of any submodule of M, as described in the next proposition.

Proposition 14. *Let M be a finite-dimensional R-module. Then*

(i) *all submodules of M are reflexive if and only if 0_M is reflexive;*
(ii) *for every submodule N of M, $\widehat{N} = N + \widehat{0_M}$.*

Proof. Let N be a submodule of M and $\mathbf{b}_1, \ldots, \mathbf{b}_d$ an F-basis of N. Extend this basis to an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_d, \mathbf{b}_{d+1}, \ldots, \mathbf{b}_n$ of M. Then the matrix associated with $t \in T$ and the extended basis is of the form

$$A_t = \begin{pmatrix} B_t & 0 \\ C_t & D_t \end{pmatrix},$$

where B_t and D_t are $d \times d$ and $(n-d) \times (n-d)$ matrices over F, respectively. Assume that 0_M is reflexive. Then Lemma 13 implies that A_t is invertible, and so is D_t, which is the matrix associated with t and the F-basis $\mathbf{b}_{d+1} + N, \ldots, \mathbf{b}_n + N$ in M/N. By Lemma 13, $0_{M/N}$ is reflexive, and so is N by Corollary 7. The first assertion holds.

For the second assertion, since $N + \widehat{0_M}$ is a subset of $\widehat{N}$, it suffices to show that $N + \widehat{0_M}$ is reflexive. By Corollary 8, it suffices to prove that the quotient $(N + \widehat{0_M})/\widehat{0_M}$ is a reflexive submodule in $M/\widehat{0_M}$. By the first assertion, it is sufficient to show that the zero submodule of $M/\widehat{0_M}$ is reflexive, which is, however, immediate from Corollary 7. □

Note that the assumption on finite dimensionality in Proposition 14 cannot be dropped. For instance, the domain R in Example 2 is an R-module such that 0_R is reflexive, but it contains non-reflexive submodules.

Proposition 14 (ii) indicates that, once we have an F-basis of $\widehat{0_M}$, an F-basis of the reflexive closure of any submodule in M can be obtained easily.

3.2. Computing an F-basis of $\widehat{0_M}$

Let R_0 be the F-linear subspace spanned by T. By (2), R_0 is closed under multiplication. So R_0 is a subring of R. The following lemma allows us to construct reflexive closures of R-submodules by R_0-submodules.

Lemma 15. *Assume that T is a left Ore set of both R and R_0. If M is an R-module and N is an R-submodule of M, then $\widehat{N}$ equals the intersection of all reflexive R_0-submodules (with respect to T) containing N, that is, $\widehat{N}$ is also the reflexive closure of N regarded as an R_0-submodule.*

Proof. Let N' be the intersection of all reflexive R_0-submodules containing N. Then both N and N' are equal to $\{\mathbf{v} \in M \mid \exists t \in T \text{ such that } t\mathbf{v} \in N\}$ by Proposition 6 and the assumption that T is a left Ore set of both R and R_0. □

In the rest of this section, we assume that T is a left Ore set of both R and R_0, and is generated by $t_1, \ldots, t_p$. Let M be an R-module with an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Denote by A_i the matrix associated with t_i for all i with $1 \le i \le p$. Note that all the A_t with $t \in T$ are invertible if and only if $A_1, \ldots, A_p$ are invertible. For brevity, the automorphism σ_{t_i} in (2) is denoted by σ_i.

Let U be a finite subset of M whose elements are given as linear combinations of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ over F. We can find an F-basis G of FU using Gaussian elimination. By (2), FU is an R_0-module if and only if $t_i G$ is a subset of FU for all i with $1 \le i \le p$.

Assume that FU is not an R_0-submodule. We form

$$U' = G \cup \{t_i\mathbf{g} \mid \mathbf{g} \in G,\ t_i\mathbf{g} \notin FU \text{ for some } i \text{ with } 1 \le i \le p\}.$$

Then $FU \subsetneq FU' \subset R_0U \subset M$. Replacing U by U' and repeating the above computation finitely many times yields an F-basis of R_0U, because M is finite-dimensional. This basis, together with $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and $A_1, \ldots, A_p$, allows us to construct an F-basis of M/R_0U and the associated matrices. These considerations lead to

Algorithm LinearBasis. *Given an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of an R-module M, the associated matrices $A_1, \ldots, A_p$, and a finite set U of nonzero elements of M, compute an F-basis of the R_0-module R_0U, an F-basis of the R_0-module $M/(R_0U)$ and the associated matrices with the latter basis.*

The details of this algorithm are given in the Appendix.

An idea for computing an F-basis of $\widehat{0_M}$ was outlined in terms of first-order matrix equations by Dr. Manuel Bronstein during email discussions with us in May, 2005. Its correctness is proved in Wu (2005, Section 2.5.2). We translate the idea into a module-theoretic language. If $A_1, \ldots, A_p$ are all invertible, then $\widehat{0_M} = 0_M$ by Lemma 13, and we are done. Otherwise, the nontrivial left kernel of A_i for some i with $1 \le i \le p$ leads to some nonzero elements in $\widehat{0_M}$. Let U be the set of all the nonzero elements in $\widehat{0_M}$ obtained from left-kernel computations. Then FU is contained in $\widehat{0_M}$. Applying Algorithm LinearBasis to U yields an F-basis of R_0U, which is contained in $\widehat{0_M}$, and an F-basis of $M/(R_0U)$ together with the associated matrices. We then apply the same idea to $M/(R_0U)$ recursively.

We would like to attribute the following algorithm to M. Bronstein. Our proof of its correctness is less involved than the one in Wu (2005, Section 2.5.2).

Bronstein's Algorithm. *Given an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of an R-module M with the associated matrices $A_1, \ldots, A_p$, compute: (a) an F-basis of $\widehat{0_M}$; (b) an F-basis of $M/\widehat{0_M}$; (c) the matrices associated with the basis in (b).*

(1) [*Recursive base*] When $n = 1$,
  (1.1) if none of the A_i is zero, then **return**
    (a) $\emptyset$; (b) $\mathbf{b}_1$; (c) $A_1, \ldots, A_p$; [*In this case,* $\widehat{0_M} = 0_M$.]
  (1.2) otherwise, **return**
    (a) $\mathbf{b}_1$; (b) $\emptyset$; (c) $\emptyset$. [*In this case,* $\widehat{0_M} = M$.]

(2) [*Compute left kernels*] For $i = 1, \ldots, p$, compute an F-basis W_i of the left kernel of A_i. If $W_i = \emptyset$ for all i with $1 \le i \le p$, then **return**
  (a) $\emptyset$; (b) $\mathbf{b}_1, \ldots, \mathbf{b}_n$; (c) $A_1, \ldots, A_p$. [*In this case* $\widehat{0_M} = 0_M$.]

(3) [*Construct a nontrivial F-subspace in* $\widehat{0_M}$] Suppose that $W_{i_1}, \ldots, W_{i_s}$ are nonempty sets among all the W_i's, where $\{i_1, \ldots, i_s\} \subset \{1, \ldots, p\}$ and $1 \le s \le p$.
  (3.1) Represent W_j as a $|W_j| \times n$ matrix P_j for $j = i_1, \ldots, i_s$.
  (3.2) Set U to be the set of nonzero elements in the column vectors
    $$\sigma_j^{-1}(P_j)(\mathbf{b}_1, \ldots, \mathbf{b}_n)^{\tau} \quad \text{for } j = i_1, \ldots, i_s.$$

(4) [*Construct a nontrivial R_0-submodule in* $\widehat{0_M}$] Call Algorithm LinearBasis to compute an F-basis $\mathbf{u}_1, \ldots, \mathbf{u}_q$ of R_0U, and an F-basis

$$\mathbf{u}_{q+1} + R_0U, \ldots, \mathbf{u}_n + R_0U$$

  of $M/(R_0U)$ with associated matrices $B_1, \ldots, B_p$. If $q = n$, then **return**
  (a) $\mathbf{b}_1, \ldots, \mathbf{b}_n$; (b) $\emptyset$; (c) $\emptyset$. [*In this case* $\widehat{0_M} = M$.]

(5) [*Recursion*] Apply Bronstein's algorithm to the quotient module $M/(R_0U)$ recursively to find:
  (5.1) F-linearly independent elements $\mathbf{v}_1, \ldots, \mathbf{v}_r$ in M such that
    $$\mathbf{v}_1 + R_0U, \ldots, \mathbf{v}_r + R_0U$$
  form an F-basis of the reflexive closure H of the zero submodule of $M/(R_0U)$;

(5.2) F-linearly independent elements $\mathbf{w}_1, \ldots, \mathbf{w}_d$ in M such that
$$(\mathbf{w}_1 + R_0U) + H, \ldots, (\mathbf{w}_d + R_0U) + H$$
form an F-basis of $(M/(R_0U))/H$; and

(5.3) the matrices $B_1, \ldots, B_p$ associated with the latter basis.
[*Note that* $r + d = \dim_F M/(R_0U) = n - q$.]

(6) **Return**

(a) $\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_r$; [*an F-basis of* $\widehat{0_M}$]

(b) $\mathbf{w}_1 + \widehat{0_M}, \ldots, \mathbf{w}_d + \widehat{0_M}$; [*an F-basis of* $M/\widehat{0_M}$]

(c) $B_1, \ldots, B_p$. [*matrices associated with the basis in (b)*]

The above algorithm terminates evidently. To prove its correctness, we remark that $\widehat{0_M}$ is also the reflexive closure of the zero submodule over R_0 by Lemma 15.

Step 1.1 is correct by Lemma 13. Step 1.2 is correct, because there exists some i with $1 \le i \le p$ such that $t_iM = 0$. If all the left kernels of $A_1, \ldots, A_p$ are trivial, so is $\widehat{0_M}$ by Lemma 13. Hence, the algorithm is correct if it stops in Step 2.

Suppose now that $(w_1, \ldots, w_n)$ is a nonzero vector in the left kernel of A_i for some i with $1 \le i \le p$. Then $\mathbf{w} = \sum_{j=1}^n \sigma_i^{-1}(w_j)\mathbf{b}_j \in \widehat{0_M}$ by a direct verification. Hence, U obtained in Step 3.2 is a nonempty subset of $\widehat{0_M}$. If $\dim_F R_0U = n$, then $\widehat{0_M} = M$. The algorithm is correct if it stops in Step 4.

Inductively, we assume that Step 5 is correct. Since $0_M \subset R_0U \subset \widehat{0_M}$, $\widehat{0_M}$ is equal to $\widehat{R_0U}$. By Corollary 7,
$$H = \widehat{0}_{M/(R_0U)} = \widehat{R_0U}/R_0U = \widehat{0_M}/(R_0U). \tag{4}$$
Hence, $\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_r$ form an F-basis of $\widehat{0_M}$. Moreover, (4) implies
$$(M/(R_0U))/H = (M/(R_0U))/(\widehat{0_M}/(R_0U)) \cong M/\widehat{0_M}.$$
So $\mathbf{w}_1 + \widehat{0_M}, \ldots, \mathbf{w}_d + \widehat{0_M}$ form an F-basis of $M/\widehat{0_M}$. The correctness is proved.

Some byproducts of the above algorithm are summarized in

Corollary 16. *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_r$ *and* $\mathbf{w}_1, \ldots, \mathbf{w}_d$ *be in the outputs of Bronstein's algorithm. Then*

(i) $\dim_F T^{-1}M = d$;

(ii) $1^{-1}\mathbf{w}_1, \ldots, 1^{-1}\mathbf{w}_d$ *form an F-basis of* $T^{-1}M$;

(iii) *for every submodule N of M*, $\widehat{N} = N + \left(\oplus_{i=1}^q F\mathbf{u}_i\right) \oplus \left(\oplus_{j=1}^r F\mathbf{v}_j\right)$.

Proof. The first conclusion is direct from Lemma 10. The second follows from Lemma 10 and the fact that $\mathbf{w}_1+\widehat{0_M}, \ldots, \mathbf{w}_d+\widehat{0_M}$ form an F-basis of $M/\widehat{0_M}$. The last is immediate from Proposition 14 (ii). □

Example 17. Let $F = \mathbb{C}(x, n, k)$, $\delta_x = \frac{d}{dx}$ be the derivation with respect to x, and σ_n, σ_k be the shift operators with respect to n and k, respectively. Let $R = F[\partial_x, \partial_n, \partial_k]$ and $R_0 = F[\partial_n, \partial_k]$. Suppose that M is an R-module of dimension five, and that $\mathbf{b}_1, \ldots, \mathbf{b}_5$ is an F-basis of M with the associated matrices

$$A_x = \begin{pmatrix} \frac{k+1}{2xk} & -\frac{1}{2xk} & -\frac{nk}{x(k+1)} & \frac{n}{x(k+1)} & -\frac{1}{2xk} \\ -\frac{-k+1+2x}{2xk} & \frac{1+2x}{2kx} & -\frac{k^2n}{x(k+1)} & \frac{nk}{x(k+1)} & -\frac{-1+2kx-2x+2k}{2kx} \\ \frac{k^2+k+2xn+2n}{2kx} & -\frac{k+2xn+2n}{2xk} & \frac{k}{x(k+1)} & -\frac{1}{x(k+1)} & \frac{-k+2kxn-2xn+2nk-2n}{2xk} \\ \frac{k^2+k+2xn+2n}{2x} & -\frac{k+2xn+2n}{2x} & -\frac{k}{x(k+1)} & \frac{1}{x(k+1)} & \frac{-k+2kxn-2xn+2nk-2n}{2x} \\ \frac{x+1}{xk} & -\frac{x+1}{xk} & -\frac{nk}{x(k+1)} & \frac{n}{x(k+1)} & \frac{(x+1)(k-1)}{xk} \end{pmatrix},$$

$$A_n = \begin{pmatrix} \frac{(n+1)(k+1)}{nk} & -\frac{n+1}{nk} & -\frac{(n+1)nk}{k+1} & \frac{(n+1)n}{k+1} & -\frac{n+1}{nk} \\ \frac{1+nk+k}{nk} & -\frac{1}{nk} & -\frac{k^2(n+1)n}{k+1} & \frac{(n+1)nk}{k+1} & -\frac{1+nk}{nk} \\ \frac{(n+1)(k^2+k+n)}{nk} & -\frac{(n+1)(n+k)}{nk} & \frac{nk}{k+1} & -\frac{n}{k+1} & \frac{(n+1)(-k+nk-n)}{nk} \\ \frac{k^2n+nk+k^2+k+n^2+n}{n} & -\frac{nk+k+n^2+n}{n} & -\frac{nk}{k+1} & \frac{n}{k+1} & \frac{(n+1)(-k+nk-n)}{n} \\ \frac{1}{k} & -\frac{1}{k} & -\frac{(n+1)nk}{k+1} & \frac{(n+1)n}{k+1} & \frac{k-1}{k} \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} -\frac{k+1}{k} & \frac{1}{k} & -n & \frac{n}{k} & \frac{1}{k} \\ -\frac{2k+1}{k} & \frac{k+1}{k} & -n(k+1) & \frac{n(k+1)}{k} & -\frac{k^2-k-1}{k} \\ -\frac{2k+1+k^2-nk}{k} & \frac{k+1-nk}{k} & 1 & -\frac{1}{k} & \frac{k+1+k^2n-nk}{k} \\ -\frac{(k+1)(2k+1+k^2-nk)}{k} & \frac{(k+1)(k+1-nk)}{k} & -1 & \frac{1}{k} & \frac{(k+1)(k+1+k^2n-nk)}{k} \\ 1 & -1 & -n & \frac{n}{k} & k-1 \end{pmatrix}.$$

We now compute $\widehat{0_M}$ via the $F[\partial_n, \partial_k]$-module structure of M, that is, the action of the differential operators ∂_x is ignored.

One verifies easily that both A_n and A_k are singular. We get that

$$P_1 = \begin{pmatrix} -\frac{k^2+nk+3k+2+2n}{nk-n^2k+n+1} & \frac{2n+nk}{nk-n^2k+n+1} & -\frac{2nk+n+nk^2-1-n^2k-n^2k^2}{nk-n^2k+n+1} & 1 & 0 \\ \frac{nk^2+3nk+1+n^2+2n}{nk-n^2k+n+1} & -\frac{n^2+nk+n+1}{nk-n^2k+n+1} & \frac{nk+n}{nk-n^2k+n+1} & 0 & 1 \end{pmatrix}$$

is an F-basis of the left kernel of A_n, and

$$P_2 = \begin{pmatrix} \frac{k^2+1+2k+nk+n}{n^2k+nk-n^2-2-2n} & -\frac{nk+n+k+1}{n^2k+nk-n^2-2-2n} & -\frac{n^2k^2+nk^2-2nk-k-n^2k+1}{n^2k+nk-n^2-2-2n} & 1 & 0 \\ \frac{2+nk^2+k^2+nk+k+n^2+2n}{n^2k+nk-n^2-2-2n} & -\frac{nk+k+n^2+2+2n}{n^2k+nk-n^2-2-2n} & -\frac{nk+k}{n^2k+nk-n^2-2-2n} & 0 & 1 \end{pmatrix}$$

is that of A_k.

Set U to be the set consisting of the non-zero elements of $\sigma_n^{-1}(P_1)(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5)^\tau$ and $\sigma_k^{-1}(P_2)(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5)^\tau$. Applying Algorithm LinearBasis to U, we find that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an F-basis of R_0U where $\mathbf{w}_1 = \mathbf{b}_1 + \frac{nk^2+n^2k-1}{(k+1)k}\mathbf{b}_3 - \frac{n^2+nk+1}{(k+1)k}\mathbf{b}_4 + \frac{n}{k}\mathbf{b}_5$ and

$$\mathbf{w}_2 = \mathbf{b}_2 + \frac{nk^3+nk^2+n^2k-1}{(k+1)k}\mathbf{b}_3 - \frac{nk^2+nk+1+n^2}{(k+1)k}\mathbf{b}_4 + \frac{n+k}{k}\mathbf{b}_5,$$

and that $\{\mathbf{b}_3 + R_0U, \mathbf{b}_4 + R_0U, \mathbf{b}_5 + R_0U\}$ is an F-basis of $M/(R_0U)$ with the associated matrices

$$B_n = \begin{pmatrix} \frac{n+n^2k+1}{n(k+1)} & \frac{n-n^2+1}{n(k+1)} & 0 \\ \frac{k(n-n^2+1)}{n(k+1)} & \frac{nk+n^2+k}{n(k+1)} & 0 \\ -\frac{n^2k}{k+1} & \frac{n^2}{k+1} & 1 \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} \frac{n^2k-1+n^2k^3+n^2k^2+k^2}{k(k+1)} & -\frac{n^2+2+2k+n^2k^2+n^2k}{k(k+1)} & \frac{n(1+k+k^2)}{k} \\ \frac{n^2k-1-2k+n^2k^3+n^2k^2}{k} & -\frac{n^2+k+n^2k^2+n^2k}{k} & \frac{(k+1)n}{k(1+k+k^2)} \\ \frac{n(-1+k^2-k)}{k+1} & \frac{n(1-k^2+k)}{(k+1)k} & k \end{pmatrix}.$$

Since both B_n and B_k are invertible, the algorithm stops. So $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an F-basis of $\widehat{0_M}$ and $\{\mathbf{b}_3 + \widehat{0_M}, \mathbf{b}_4 + \widehat{0_M}, \mathbf{b}_5 + \widehat{0_M}\}$ is an F-basis of $M/\widehat{0_M}$ with the associated matrices B_n and B_k.

Let us look at the case where R is a commutative domain. Assume that the elements $t_1, \ldots, t_p$ of $R^\times$ generate a multiplicative monoid T, which is clearly an Ore set. Note that each σ_{t_i} in (2) is an identity map on F for all i with $1 \leq i \leq p$. Bronstein's algorithm enables us to compute an F-basis of $\widehat{0_M}$ and an F-basis of $T^{-1}M$, provided that a finite F-basis of M is given.

Below is an example from Kehrein et al. (2005, Ex. 4.2.8).

Example 18. Let $R = \mathbb{Q}[X, Y]$ be a commutative domain. Then

$$T := \{X^{k_1} Y^{k_2} \mid \text{ for any } k_1, k_2 \in \mathbb{N}\}$$

is a left (and right) Ore set of R and $R_0 = R$. Let $M = \mathbb{Q}^3$ with the standard basis

$$\mathbf{e}_1 = (1, 0, 0)^\tau, \qquad \mathbf{e}_2 = (0, 1, 0)^\tau, \qquad \mathbf{e}_3 = (0, 0, 1)^\tau.$$

Define an R-module structure on M by two actions $X(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\tau = A_X(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\tau$ and $Y(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\tau = A_Y(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\tau$ where

$$A_X = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We now apply Bronstein's algorithm to compute $\widehat{0}_M$. Note that both A_X and A_Y are singular. We compute that $P = (1, 0, -1)$ is both an F-basis of the left kernel of A_X and that of A_Y. Set $\mathbf{v} = P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^\tau = \mathbf{e}_1 - \mathbf{e}_3$. Applying Algorithm LinearBasis, we find that $\mathbf{v}$ is an F-basis of $R\mathbf{v}$, and that $\{\mathbf{e}_2 + R\mathbf{v}, \mathbf{e}_3 + R\mathbf{v}\}$ is an F-basis of $M/R\mathbf{v}$ with the associated matrices B_X and B_Y, which are

$$B_X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B_Y = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since both B_X and B_Y are invertible, the algorithm stops. So $\mathbf{e}_1 - \mathbf{e}_3$ is an F-basis of $\widehat{0}_M$ and $\{\mathbf{e}_2 + \widehat{0}_M, \mathbf{e}_3 + \widehat{0}_M\}$ is an F-basis of $M/\widehat{0}_M$ with the associated matrices B_X and B_Y. By Corollary 16, $\{1^{-1}\mathbf{e}_2, 1^{-1}\mathbf{e}_3\}$ is an F-basis of $T^{-1}M$.

The above example was used in Kehrein et al. (2005) to illustrate the Buchberger–Möller algorithm that computes a $\mathbb{Q}$-basis of $\text{ann}(M)$ and a $\mathbb{Q}$-basis of $R/\text{ann}(M)$, where $\text{ann}(M)$ stands for the set of polynomials in R annihilating all elements of M. Although the Buchberger–Möller algorithm for matrices and Bronstein's in the usual commutative case have different goals, they share certain similarity. For instance, both take a finite set of commutative matrices as part of the inputs, and both compute linear bases without forming S-polynomials of any sort.

In summary, we have proved in this section that $M/\widehat{0}_M$ and $T^{-1}M$ are isomorphic as R-modules if M is finite-dimensional. An algorithm is described in this case for computing an F-basis of $\widehat{0}_M$ and an F-basis of $T^{-1}M$, provided that T is a finitely generated submonoid and a left Ore set of both R and R_0. The algorithm enables us to determine reflexive closures of submodules in M.

4. Equivalence

In this section, we define an equivalence relation among linear functional systems, which allows us to describe the notion of integrable connections more concisely than in Bronstein et al. (2005).

In the rest of this paper, F stands for a field. Assume that $\delta_1, \ldots, \delta_\ell$ are derivations on F, $\sigma_{\ell+1}, \ldots, \sigma_m$ are automorphisms of F, and all these maps commute pairwise. An element c of F is called a *constant* if $\delta_i(c) = 0$ for all i with $1 \le i \le \ell$ and $\sigma_j(c) = c$ for all j with $\ell + 1 \le j \le m$. The set of all constants in F form a subfield, which is denoted by C_F.

Let $\mathcal{S}$ be the Ore algebra $F[\partial_1, \ldots, \partial_\ell, \partial_{\ell+1}, \ldots, \partial_m]$, whose commutation rules are given in Example 3. Let T be the submonoid generated by $\partial_{\ell+1}, \ldots, \partial_m$, which is a left Ore set as shown in the same example. In terms of the notation introduced in previous sections, we have that $R = \mathcal{S}$ and $R_0 = F[\partial_{\ell+1}, \ldots, \partial_m]$. Moreover, the ring $T^{-1}\mathcal{S}$ is denoted by $\mathcal{L}$, which is the Laurent–Ore algebra defined by the δ_i and σ_j over F. The modules of $p \times n$ matrices over $\mathcal{S}$ and $\mathcal{L}$ are denoted by $\mathcal{S}^{p\times n}$ and $\mathcal{L}^{p\times n}$, respectively.

Remark 19. By identifying ∂_i with $1^{-1}\partial_i$ for all i with $1 \le i \le m$, and ∂_j^{-1} with $\partial_j^{-1} 1$ for all j with $\ell + 1 \le j \le m$, we can write $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ and view it as an extension of $\mathcal{S}$.

A linear (homogeneous) functional system over F is of the form

$$A(\mathbf{y}) = 0 \tag{5}$$

where $A \in \mathcal{S}^{p\times n}$ and $\mathbf{y}$ is a column vector of n unknowns. Let V be an $\mathcal{L}$-module. By a solution of (5) in V, we mean a vector $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^\tau$ with $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ such that $A(\mathbf{v}) = 0$. The set of all solutions of (5) in V is denoted by $\mathrm{sol}_V(A(\mathbf{y}) = 0)$, which is a linear space over C_F.

The next example illustrates why we consider the solutions of (5) in $\mathcal{L}$-modules rather than in $\mathcal{S}$-modules.

Example 20. Let $m = 1$ and $\ell = 0$ in Example 3. Then $\mathcal{S} = F[\partial]$ and $\mathcal{L} = F[\partial, \partial^{-1}]$. The difference equation $\sigma^2(y) = 0$ has only trivial solutions in any difference ring extension of F. The equation is expressed as $\partial^2 y = 0$ in terms of module-theoretic notation. It annihilates a nonzero element $1 + \mathcal{S}\partial^2$ in the $\mathcal{S}$-module $\mathcal{S}/\left(\mathcal{S}\partial^2\right)$, but has no nonzero solution in any $\mathcal{L}$-module due to the presence of ∂^{-1}.

We recall the notion of modules of formal solutions, which connects linear functional systems with $\mathcal{L}$-modules. Let $A \in \mathcal{S}^{p\times n}$ and N be the $\mathcal{S}$-submodule generated by the row vectors of A in $\mathcal{S}^{1\times n}$. For convenience, we call N the *Ore submodule* associated with the system (5). The $\mathcal{L}$-module $\mathcal{L}^{1\times n}/\mathcal{L}N$, where $\mathcal{L}N$ stands for the $\mathcal{L}$-submodule generated by the row vectors of A in $\mathcal{L}^{1\times n}$, is called *the module of formal solutions* of (5).

Since $\mathcal{L} = T^{-1}\mathcal{S}$, $\mathcal{L}^{1\times n}/\mathcal{L}N = T^{-1}\left(\mathcal{S}^{1\times n}\right)/T^{-1}N$, which is $\mathcal{L}$-isomorphic to the localization $T^{-1}(\mathcal{S}^{1\times n}/N)$ by Remark 9. In $\mathcal{S}^{1\times n}$, for $k = 1, \ldots, n$, set

$$\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$$

with 1 appearing in the kth coordinate. We call $\mathbf{e}_1 + \mathcal{L}N, \ldots, \mathbf{e}_n + \mathcal{L}N$ the canonical generators of $\mathcal{L}^{1\times n}/\mathcal{L}N$, the module of formal solutions of (5). It is clear that

$$(\mathbf{e}_1 + \mathcal{L}N, \ldots, \mathbf{e}_n + \mathcal{L}N)^\tau$$

is a solution of (5) in $\mathcal{L}^{1\times n}/\mathcal{L}N$. By Theorem 4 in Bronstein et al. (2005) or Theorem 2.4.1 in Wu (2005), for every solution $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^\tau$ of (5) in an $\mathcal{L}$-module V, there exists a (unique) $\mathcal{L}$-homomorphism $\phi_{\mathbf{v}}$ such that $\phi_{\mathbf{v}}(\mathbf{e}_k + \mathcal{L}N) = \mathbf{v}_k$ for all k with $1 \le k \le n$. In other words, the canonical generators give rise to a generic solution of the system (5).

Definition 21. Two linear functional systems are said to be *equivalent* if their modules of formal solutions are isomorphic as $\mathcal{L}$-modules.

The next proposition provides a one-to-one correspondence between the solutions of two equivalent linear functional systems.

Proposition 22. *Assume that A and A' are two matrices in $\mathcal{S}^{p\times n}$ and $\mathcal{S}^{p'\times n'}$, respectively. If $A(\mathbf{y}) = 0$ and $A'(\mathbf{y}') = 0$ are equivalent, then there exist $P \in \mathcal{L}^{n\times n'}$ and $Q \in \mathcal{L}^{n'\times n}$ such that, for every $\mathcal{L}$-module V, both*

$$\begin{array}{rccc} \phi: & \mathrm{sol}_V(A(\mathbf{y}) = 0) & \to & \mathrm{sol}_V\left(A'(\mathbf{y}') = 0\right) \\ & \mathbf{v} & \mapsto & Q\mathbf{v} \end{array}$$

and

$$\begin{array}{rccc} \phi': & \mathrm{sol}_V\left(A'(\mathbf{y}') = 0\right) & \to & \mathrm{sol}_V(A(\mathbf{y}) = 0) \\ & \mathbf{v}' & \mapsto & P\mathbf{v}' \end{array}$$

are well-defined C_F-linear isomorphisms with $\phi^{-1} = \phi'$.

Proof. Let M and M' be the modules of formal solutions of the given two systems, respectively. Set $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ and $\mathbf{b}' = \left(\mathbf{b}'_1, \ldots, \mathbf{b}'_{n'}\right)^\tau$, where $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and $\mathbf{b}'_1, \ldots, \mathbf{b}'_{n'}$ are the canonical generators of M and M', respectively.

Assume that θ is an $\mathcal{L}$-isomorphism from M to M'. Then there exist $P \in \mathcal{L}^{n\times n'}$ and $Q \in \mathcal{L}^{n'\times n}$ such that $\theta(\mathbf{b}) = P\mathbf{b}'$ and $\theta^{-1}(\mathbf{b}') = Q\mathbf{b}$. In particular, we have

$$\mathbf{b} = \theta^{-1} \circ \theta(\mathbf{b}) = \theta^{-1}(P\mathbf{b}') = P\theta^{-1}(\mathbf{b}') = PQ\mathbf{b} \tag{6}$$

and, similarly, $\mathbf{b}' = QP\mathbf{b}'$.

For every $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^\tau$ in $\mathrm{sol}_V(A(\mathbf{y}) = 0)$, the $\mathcal{L}$-homomorphism from $\mathcal{L}^{1\times n}$ to V sending $\mathbf{e}_k$ to $\mathbf{v}_k$, $k = 1, \ldots, n$, induces an $\mathcal{L}$-homomorphism f from M to V sending $\mathbf{b}_k$ to $\mathbf{v}_k$, $k = 1, \ldots, n$. Therefore, $f \circ \theta^{-1}$ belongs to $\mathrm{Hom}_{\mathcal{L}}(M', V)$. Consequently, $f \circ \theta^{-1}(\mathbf{b}')$ belongs to $\mathrm{sol}_V\left(A'(\mathbf{y}') = 0\right)$, because $\mathbf{b}'$ is in $\mathrm{sol}_{M'}\left(A'(\mathbf{y}') = 0\right)$. On the other hand,

$$f \circ \theta^{-1}(\mathbf{b}') = f(Q\mathbf{b}) = Qf(\mathbf{b}) = Q\mathbf{v}. \tag{7}$$

So ϕ is well-defined. In the same vein, ϕ' is well-defined. For every $\mathbf{v} \in \mathrm{sol}_V(A(\mathbf{y}) = 0)$, we compute:

$$\begin{aligned}\phi' \circ \phi(\mathbf{v}) &= P(Q\mathbf{v}) = Pf \circ \theta^{-1}(\mathbf{b}') \quad (\text{by } (7))\\ &= f\left(P\theta^{-1}(\mathbf{b})\right) \quad (\text{since } f \in \mathrm{Hom}_{\mathcal{L}}(M, V))\\ &= f(PQ\mathbf{b}) = f(\mathbf{b}) \quad (\text{by } (6))\\ &= \mathbf{v}.\end{aligned}$$

Similarly, $\phi \circ \phi'(\mathbf{v}') = \mathbf{v}'$ for all $\mathbf{v}' \in \mathrm{sol}_V\left(A'(\mathbf{y}') = 0\right)$. Therefore, $\phi^{-1} = \phi'$. □

Given a linear functional system Σ, the dimension of its module of formal solutions as a vector space over F is called its *linear dimension*. We say that Σ is *∂-finite* if its linear dimension is finite. We are going to show that a ∂-finite system is equivalent to a fully integrable system defined below.

Consider a first-order system of the form

$$\partial_i(\mathbf{z}) = B_i\mathbf{z}, \quad \text{where } B_i \in F^{n\times n} \text{ for } i = 1, \ldots, m. \tag{8}$$

The system (8) is said to be *integrable* if

$$\begin{aligned}&B_sB_i + \delta_i(B_s) = B_iB_s + \delta_s(B_i) \quad (1 \le i < s \le \ell),\\ &\sigma_j(B_s)B_j = \sigma_s(B_j)B_s \quad (\ell + 1 \le j < s \le m),\\ &B_jB_i + \delta_i(B_j) = \sigma_j(B_i)B_i \quad (1 \le i \le \ell,\ \ell + 1 \le j \le m).\end{aligned} \tag{9}$$

These integrability conditions are derived from $\partial_i\partial_j(\mathbf{z}) = \partial_j\partial_i(\mathbf{z})$ with $\mathbf{z}$ viewed as a vector of indeterminates. Moreover, (8) is said to be *fully integrable* if it is integrable, and $B_{\ell+1}, \ldots, B_m$ are all invertible.

Note that the system (8) can be rewritten as a linear functional system $B(\mathbf{z}) = 0$, where $B \in \mathcal{S}^{n^2\times n}$ is the stacking of $n \times n$ blocks $\partial_1 \cdot I_n - B_1, \ldots, \partial_m \cdot I_n - B_m$ with I_n the identity matrix of size n.

The next lemma will help us construct an F-basis of the module of formal solutions of an integrable system using merely Ore algebras.

Lemma 23. *Let N be the Ore submodule associated with the first-order matrix system (8). Then we have the following.*

(i) *If (8) is integrable, then $\mathbf{e}_1 + N, \ldots, \mathbf{e}_n + N$ form an F-basis of the $\mathcal{S}$-module $\mathcal{S}^{1\times n}/N$, and B_i is the matrix associated with ∂_i for all i with $1 \le i \le m$.*
(ii) *If (8) is integrable, then its module of formal solutions is $\mathcal{S}$-isomorphic to $\mathcal{S}^{1\times n}/\widehat{N}$.*
(iii) *If (8) is fully integrable, then $\mathbf{e}_1 + \mathcal{L}N, \ldots, \mathbf{e}_n + \mathcal{L}N$ form an F-basis of its module of formal solutions, and $B_1, \ldots, B_m$ are the respective associated matrices.*

Proof. For every i with $1 \le i \le m$, the row vectors in the block $\partial_i \cdot I_n - B_i$ are

$$\partial_i\mathbf{e}_1 - \sum_{h=1}^{n} b_{1h}^{(i)}\mathbf{e}_h, \ldots, \partial_i\mathbf{e}_n - \sum_{h=1}^{n} b_{nh}^{(i)}\mathbf{e}_h,$$

where $b_{jh}^{(i)}$ stands for the element at the jth row and hth column of B_i. Denote by G the set consisting of these row vectors. Then N is generated by G over $\mathcal{S}$. Remark that the integrability conditions (9) imply that G is a Gröbner basis of N in $\mathcal{S}^{1\times n}$ with respect to a monomial order, in which $\partial_i\mathbf{e}_j$ is higher than $\mathbf{e}_k$ for all $i \in \{1, \ldots, m\}$ and $j, k \in \{1, \ldots, n\}$. Thus, every element of $\mathcal{S}^{1\times n}$ is congruent to a unique F-linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$ modulo N. It follows that $\mathbf{e}_1 + N, \ldots, \mathbf{e}_n + N$ form an F-basis of $\mathcal{S}^{1\times n}/N$. Expressing (8) in terms of the elements of $\mathcal{S}^{1\times n}$ yields

$$\partial_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau \equiv B_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau \quad \mod N,$$

for $1 \le i \le m$. So B_i is the matrix associated with ∂_i and the basis $\mathbf{e}_1+N, \ldots, \mathbf{e}_n+N$. The first assertion holds.

From the first assertion, $\mathcal{S}^{1\times n}/N$ is finite-dimensional, so is $\mathcal{S}^{1\times n}/\widehat{N}$. Then the second assertion follows from Corollary 12.

Assume that (8) is fully integrable. Then $B_{\ell+1}, \ldots, B_m$ are all invertible. The zero submodule of $\mathcal{S}^{1\times n}/N$ is reflexive by the first assertion and Lemma 13. Hence by Lemma 10 and Remark 9, $\mathcal{S}^{1\times n}/N$ is $\mathcal{S}$-isomorphic to $\mathcal{L}^{1\times n}/\mathcal{L}N$. This implies the last assertion. □

Corollary 24. *If Σ and Σ' are two equivalent fully-integrable systems, then there exists an invertible matrix P over F such that the map $\mathbf{v} \mapsto P\mathbf{v}$ is a C_F-linear isomorphism from $\mathrm{sol}_V(\Sigma)$ to $\mathrm{sol}_V(\Sigma')$ for any $\mathcal{L}$-module V.*

Proof. Since Σ and Σ' are equivalent, by Lemma 23(iii) they both have the same size, say n. Then the matrices P and Q given in Proposition 22 are $n \times n$ matrices. The canonical generators of the module of formal solutions of Σ (resp. Σ') form an F-basis by Lemma 23(iii). So both P and Q can be chosen as invertible matrices over F. □

By a Δ-extension of F, we mean a commutative ring E containing F such that the maps $\delta_1, \ldots, \delta_\ell$ and $\sigma_{\ell+1}, \ldots, \sigma_m$ can be extended to the derivations on E and automorphisms of E, respectively. A Δ-extension E of F can be viewed as an $\mathcal{L}$-module, in which $\partial_i a = \delta_i(a)$, $\partial_j a = \sigma_j(a)$ and $\partial_j^{-1} a = \sigma_j^{-1}(a)$ for all $a \in E$, $i \in \{1, \ldots, \ell\}$ and $j \in \{\ell+1, \ldots, m\}$. In practice, we are more interested in solutions contained in a Δ-extension than solutions in an $\mathcal{L}$-module.

For a fully integrable system of size n, there exists a Δ-extension E of F and an $n \times n$ invertible matrix W over E such that each column vector of W is a solution of (8) (see Theorem 1 in Bronstein et al. (2005)), We call W a *fundamental matrix* for the given system. The next proposition characterizes two equivalent fully-integrable systems in terms of their fundamental matrices.

Proposition 25. *Let Σ and Σ' be two fully integrable systems over F. Assume that W is a fundamental matrix of Σ in a Δ-extension E of F.*

(i) *If Σ and Σ' are equivalent, then there exists an invertible (square) matrix Q over F such that QW is a fundamental matrix of Σ' in E.*
(ii) *If there exist a fundamental matrix W' of Σ' in E and an invertible matrix Q over F such that $W' = QW$, then Σ and Σ' are equivalent.*

Proof. Assume that Σ is of the form (8) and $\Sigma' = \{\partial_1(\mathbf{z}) = B_1'\mathbf{z}, \ldots, \partial_m(\mathbf{z}) = B_m'\mathbf{z}\}$.

First, we assume that Σ and Σ' are equivalent. Then both Σ and Σ' are of the same size n by Lemma 23 (iii). By Corollary 24, there exists an $n \times n$ invertible matrix Q over F such that $\mathbf{v} \mapsto Q\mathbf{v}$ is a C_F-linear map from $\mathrm{sol}_E(\Sigma)$ to $\mathrm{sol}_E(\Sigma')$. So QW is a fundamental matrix of Σ'. The first assertion holds.

Assume now that W' and Q are given as in the second assertion. Since Q is a square matrix, both W' and Q have size n, and so does Σ'. Denote by M and M' the modules of formal solutions of Σ and Σ', respectively. Assume further that $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ (resp. $\mathbf{b}' = (\mathbf{b}_1', \ldots, \mathbf{b}_n')^\tau$) is the column vector consisting of the canonical generators of M (resp. M'). It follows from Lemma 23 (iii) that, for all i with $1 \le i \le m$,

$$\partial_i \mathbf{b} = B_i \mathbf{b} \quad \text{in } M \quad \text{and} \quad \partial_i \mathbf{b}' = B_i' \mathbf{b}' \quad \text{in } M'.$$

Define θ to be the F-linear isomorphism from M' to M given by $\theta(\mathbf{b}') = Q\mathbf{b}$. We claim that

$$\partial_i \theta(\mathbf{b}') = \theta(\partial_i \mathbf{b}') \quad \text{for all } i \text{ with } 1 \le i \le m. \tag{10}$$

The F-linearity of θ implies that

$$\partial_i \theta(\mathbf{b}') = \partial_i (Q\mathbf{b}) \quad \text{and} \quad \theta(\partial_i \mathbf{b}') = \theta(B_i'\mathbf{b}') = B_i'\theta(\mathbf{b}') = B_i'Q\mathbf{b}.$$

Therefore, the claim holds if $\partial_i (Q\mathbf{b}) = B_i'Q\mathbf{b}$ for all i with $1 \le i \le m$, which is equivalent to that

$$\delta_i(Q) + QB_i = B_i'Q \quad (1 \le i \le \ell) \quad \text{and} \quad \sigma_i(Q)B_i = B_i'Q \quad (\ell+1 \le i \le m), \tag{11}$$

because $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an F-basis. By the discussion after Corollary 24, $\partial_i W = B_i W$ (resp. $\partial_i W' = B_i' W'$) means $\delta_i(W) = B_i W$ (resp. $\delta_i(W') = B_i' W'$) for any $i \in \{1, \ldots, \ell\}$, because the coefficients of W and W' are in Δ-extensions. Applying δ_i to $W' = QW$ yields

$$B_i' W' = B_i' QW = (\delta_i(Q) + QB_i)W.$$

It follows from the invertibility of W that the first equality in (11) holds. The second follows from a similar calculation. This proves claim (10).

By Lemma 23 (iii), every element $\mathbf{v}'$ in M' can be written as $\mathbf{f}'\mathbf{b}'$, where $\mathbf{f}' \in F^{1\times n}$. Thus, $\partial_i \theta(\mathbf{v}') = \theta(\partial_i \mathbf{v}')$ follows from the commutation rules (ii) and (iii) in Example 3, claim (10) and the F-linearity of θ. □

The next corollary is immediate from the above proof. It shows that the notion of equivalence is a generalization of that on page 7 of van der Put and Singer (2003).

Corollary 26. *Let*

$$\{\partial_1(\mathbf{z}) = B_1\mathbf{z}, \ldots, \partial_m(\mathbf{z}) = B_m\mathbf{z}\} \quad \textit{and} \quad \{\partial_1(\mathbf{z}) = B_1'\mathbf{z}, \ldots, \partial_m(\mathbf{z}) = B_m'\mathbf{z}\}$$

be two fully integrable systems of the same size. Then they are equivalent if and only if the equalities in (11) hold.

A fully integrable system is called an *integrable connection* of a ∂-finite system if it is equivalent to the ∂-finite system. Clearly, all the integrable connections of a ∂-finite system are equivalent to each other. One way to construct integrable connections is given in Bronstein et al. (2005) and Wu (2005, §2.4.4). Another way to compute them will be described in the next section.

5. Computing integrable connections

In this section, we present an algorithm for computing the integrable connection of a ∂-finite system. The algorithm is based on the following lemma.

Lemma 27. *Let Σ be a ∂-finite system with n unknowns, and N the Ore module associated with Σ.*

(i) *If $\mathcal{L}^{1\times n}/\mathcal{L}N$ has an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$ with the associated matrices $B_1, \ldots, B_m$, then $\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}$ is an integrable connection of Σ.*

(ii) *If $\widehat{\mathcal{S}^{1\times n}/N}$ has an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$ with the associated matrices $B_1, \ldots, B_m$, then $\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}$ is an integrable connection of Σ.*

Proof. For brevity, we denote $\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}$ by Σ'. The matrices $B_1, \ldots, B_m$ satisfy (9) by the linear independence of $\mathbf{b}_1, \ldots, \mathbf{b}_d$ and the commutativity of ∂_i and ∂_j for all i, j with $1 \le i < j \le m$. For every j with $\ell + 1 \le j \le m$, we compute

$$\begin{aligned}(\mathbf{b}_1, \ldots, \mathbf{b}_d)^\tau &= \partial_j^{-1}\partial_j(\mathbf{b}_1, \ldots, \mathbf{b}_d)^\tau = \partial_j^{-1}\left(B_j(\mathbf{b}_1, \ldots, \mathbf{b}_d)^\tau\right)\\ &= \sigma_j^{-1}(B_j)\partial_j^{-1}(\mathbf{b}_1, \ldots, \mathbf{b}_d)^\tau.\end{aligned}$$

The linear independence of $\mathbf{b}_1, \ldots, \mathbf{b}_d$ then implies that B_j is invertible. Hence, Σ' is fully integrable. Denote by N' the Ore module associated with Σ'. For $k = 1, \ldots, d$, write

$$\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 appears in the kth position. By Lemma 23 (iii), $\mathbf{e}_1 + \mathcal{L}N', \ldots, \mathbf{e}_d + \mathcal{L}N'$ form an F-basis of $\mathcal{L}^{1\times d}/\mathcal{L}N'$ and

$$\partial_i(\mathbf{e}_1 + \mathcal{L}N', \ldots, \mathbf{e}_d + \mathcal{L}N')^\tau = B_i(\mathbf{e}_1 + \mathcal{L}N', \ldots, \mathbf{e}_d + \mathcal{L}N')^\tau$$

for all i with $1 \le i \le m$. Since both $\mathbf{b}_1, \ldots, \mathbf{b}_d$ and $\mathbf{e}_1 + \mathcal{L}N', \ldots, \mathbf{e}_d + \mathcal{L}N'$ have the same associated matrices, the F-linear map defined by $\mathbf{b}_k \mapsto \mathbf{e}_k + \mathcal{L}N'$ for all k with $1 \le k \le d$ is an $\mathcal{L}$-isomorphism from $\mathcal{L}^{1\times n}/\mathcal{L}N$ to $\mathcal{L}^{1\times d}/\mathcal{L}N'$. The module of formal solutions of Σ is $\mathcal{L}$-isomorphic to that of Σ'. The first assertion is proved.

Recall that T is the submonoid generated by $\partial_{\ell+1}, \ldots, \partial_m$ and that $\mathcal{L} = T^{-1}\mathcal{S}$. Set $M = \mathcal{S}^{1\times n}$ and $\bar{\phi}$ to be the $\mathcal{S}$-isomorphism from $\widehat{M/N}$ to $T^{-1}M/T^{-1}N$ given in Corollary 12. Note that $T^{-1}M/T^{-1}N = \mathcal{L}^{1\times n}/\mathcal{L}N$. Then $\bar{\phi}(\mathbf{b}_1), \ldots, \bar{\phi}(\mathbf{b}_d)$ form an F-basis of $\mathcal{L}^{1\times n}/\mathcal{L}N$ with the associated matrices $B_1, \ldots, B_m$. It follows from the first assertion that Σ' is an integrable connection of Σ. □

With the notation introduced in Lemma 27, we proceed as follows to find an integrable connection of Σ. First, compute a Gröbner basis G of the submodule N in the free Ore module $\mathcal{S}^{1\times n}$. The basis G allows us to determine if $\mathcal{S}^{1\times n}/N$ is finite-dimensional over F. If it is, we construct an F-basis of $\mathcal{S}^{1\times n}/\widehat{N}$ using Bronstein's algorithm. The F-basis yields an integrable connection by Lemma 27(ii). Otherwise, we compute a Gröbner basis of $\mathcal{L}N$ in the free Laurent–Ore module $\mathcal{L}^{1\times n}$, and apply Lemma 27(i).

Remark that when Σ is a first-order system of the form (8) then $\mathcal{S}^{1\times n}/N$ is clearly finite-dimensional. Moreover, if Σ is an integrable (first-order) system then an F-basis of $\mathcal{S}^{1\times n}/N$ and the associated matrices are already known from Lemma 23(i). In this case there is no need to compute any Gröbner bases, and we can directly apply Bronstein's algorithm to obtain an integrable connection.

These considerations lead to the following algorithm.

Algorithm IntegrableConnection. Given a $p \times n$ matrix A over $\mathcal{S}$, determine whether the system $A(\mathbf{y}) = 0$ is ∂-finite. When it is ∂-finite, compute matrices $B_1, \ldots, B_m \in F^{d\times d}$ and $P \in F^{n\times d}$ such that

(i) d is the linear dimension of $A(\mathbf{y}) = 0$;
(ii) $\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}$ is an integrable connection of $A(\mathbf{y}) = 0$;
(iii) $\xi \mapsto P\xi$ is a C_F-isomorphism from $\mathrm{sol}_V\left(\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}\right)$ to $\mathrm{sol}_V\,(A(\mathbf{y}) = 0)$ for any $\mathcal{L}$-module V.

In the following description, we assume that N is the $\mathcal{S}$-submodule generated by the row vectors of A in $\mathcal{S}^{1\times n}$. Recall that, for all k with $1 \le k \le n$, $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 appearing in the kth position.

(1) If $A(\mathbf{y}) = 0$ is of the form of a first-order system

$$\{\partial_i(\mathbf{y}) = A_i\mathbf{y}\}_{1\le i\le m} \quad \text{where } A_i \in F^{n\times n}, \tag{12}$$

then do the following.
   (1.1) Determine if (12) is fully integrable.
   (1.2) If (12) is fully integrable, then **return** $A_1, \ldots, A_m$ and I_n. [*$A(\mathbf{y}) = 0$ is itself an integrable connection.*]
   (1.3) If (12) is integrable, then set $q := n$, $\mathbf{b}_i := \mathbf{e}_i + N$ for all i with $1 \le i \le n$, and $V_j := A_j$ for all j with $1 \le j \le m$, and go to Step (4.2). [*There is no need to do any Gröbner basis computation.*]
(2) Compute a Gröbner basis G of N in $\mathcal{S}^{1\times n}$.
(3) Set $q := \dim_F \mathcal{S}^{1\times n}/N$. If $q = 0$, then **return** $\emptyset$. [*$A(\mathbf{y}) = 0$ is inconsistent.*]
(4) If q is finite, then do the following.
   (4.1) Use G to compute an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_q$ of $\mathcal{S}^{1\times n}/N$ and the matrix V_i associated with ∂_i for all i with $1 \le i \le m$.
   (4.2) Call Bronstein's algorithm to the $F[\partial_{\ell+1}, \ldots, \partial_m]$-module $\mathcal{S}^{1\times n}/N$ to compute an F-basis $\mathbf{w}_{d+1} + N, \ldots, \mathbf{w}_q + N$ of $\widehat{N}/N$ with $\mathbf{w}_s \in \widehat{N}$ for $d+1 \le s \le q$, an F-basis $\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N}$ of $\mathcal{S}^{1\times n}/\widehat{N}$ with $\mathbf{v}_t \in \mathcal{S}^{1\times n}$ for $1 \le t \le d$, and the $d \times d$ matrices $B_{\ell+1}, \ldots, B_m$ associated with $\partial_{\ell+1}, \ldots, \partial_m$ and the latter basis, respectively.
   (4.3) If $d = 0$, then **return** $\emptyset$. [*$A(\mathbf{y}) = 0$ is inconsistent*].
   (4.4) Use the two F-bases $\mathbf{v}_1 + N, \ldots, \mathbf{v}_d + N, \mathbf{w}_{d+1} + N, \ldots, \mathbf{w}_q + N$ and $\mathbf{b}_1, \ldots, \mathbf{b}_q$ of $\mathcal{S}^{1\times n}/N$ to construct an invertible matrix $Q \in F^{q\times q}$ such that
   $$(\mathbf{v}_1 + N, \ldots, \mathbf{v}_d + N, \mathbf{w}_{d+1} + N, \ldots, \mathbf{w}_q + N)^\tau = Q(\mathbf{b}_1, \ldots, \mathbf{b}_q)^\tau. \tag{13}$$
   Set $U_j = \delta_j(Q)Q^{-1} + QV_jQ^{-1}$ for $j = 1, \ldots, \ell$. Take the first d rows and the first d columns of U_j to form a $d \times d$ matrix B_j for $1 \le j \le \ell$.
   (4.5) Compute a matrix $P \in F^{n\times d}$ such that
   $$(\mathbf{e}_1 + \widehat{N}, \ldots, \mathbf{e}_n + \widehat{N})^\tau = P(\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N})^\tau,$$
   [*which yields* $(\mathbf{e}_1 \ldots, \mathbf{e}_n)^\tau \equiv P(\mathbf{v}_1, \ldots, \mathbf{v}_d)^\tau \mod \widehat{N}$.]

   Return $B_1, \ldots, B_m$ and P.

(5) If q is infinite, then compute a Gröbner basis H of $\mathcal{L}N$ in $\mathcal{L}^{1\times n}$ and set

$$d := \dim_F \mathcal{L}^{1\times n}/\mathcal{L}N.$$

If $d = \infty$, then **return** ∞; [$A(\mathbf{y}) = 0$ *is not* ∂*-finite*.] If $d = 0$, then **return** $\emptyset$; [$A(\mathbf{y}) = 0$ *is inconsistent*.]

Otherwise, use H to compute an F-basis $\mathbf{v}_1 + \mathcal{L}N, \ldots, \mathbf{v}_d + \mathcal{L}N$ of $\mathcal{L}^{1\times n}/\mathcal{L}N$, and the matrix B_i associated with ∂_i and the basis for all i with $1 \le i \le m$. Find a matrix $P \in F^{n\times d}$ such that

$$(\mathbf{e}_1 + \mathcal{L}N, \ldots, \mathbf{e}_n + \mathcal{L}N)^\tau = P(\mathbf{v}_1 + \mathcal{L}N, \ldots, \mathbf{v}_d + \mathcal{L}N)^\tau,$$

[*which yields* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau \equiv P(\mathbf{v}_1, \ldots, \mathbf{v}_d)^\tau \mod \mathcal{L}N$.]

Return $B_1, \ldots, B_m$ and P.

The above algorithm terminates obviously. To prove its correctness, let us first consider the case in which $\dim_F \mathcal{S}^{1\times n}/N$ is finite. Steps (1.1) and (1.2) are clear. Step (1.3) yields desired results for Step (4.2) by Lemma 23(i).

Steps (2), (3) and (4.1) are evident. By Corollary 12, d is the linear dimension of $A(\mathbf{y}) = 0$. Assume further that d is positive. Note that, for all k with $\ell + 1 \le k \le m$,

$$\partial_k(\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N})^\tau = B_k(\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N})^\tau \tag{14}$$

by the definition of the matrices B_k's.

It follows from (13) and the definition of U_j for $j = 1, \ldots, \ell$ that

$$\begin{aligned}&\partial_j(\mathbf{v}_1 + N, \ldots, \mathbf{v}_d + N, \mathbf{w}_{d+1} + N, \ldots, \mathbf{w}_q + N)^\tau\\ &\quad = U_j(\mathbf{v}_1 + N, \ldots, \mathbf{v}_d + N, \mathbf{w}_{d+1} + N, \ldots, \mathbf{w}_q + N)^\tau.\end{aligned}$$

Since $\mathbf{w}_{d+1}, \ldots, \mathbf{w}_q$ belong to $\widehat{N}$, we have that, for all j with $1 \le j \le \ell$,

$$\partial_j(\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N})^\tau = B_j(\mathbf{v}_1 + \widehat{N}, \ldots, \mathbf{v}_d + \widehat{N})^\tau. \tag{15}$$

Lemma 27(ii), together with (14) and (15), implies that $\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}$ is an integrable connection of $A(\mathbf{y}) = 0$.

The matrix P obtained from Step (4.5) is the same as the matrix defining θ given in the proof of Proposition 22. Thus, the same proposition implies that P gives rise to a C_F-isomorphism from $\mathrm{sol}_V\left(\{\partial_i(\mathbf{z}) = B_i\mathbf{z}\}_{1\le i\le m}\right)$ to $\mathrm{sol}_V\,(A(\mathbf{y}) = 0)$ for every $\mathcal{L}$-module V. We can choose $P \in F^{n\times d}$ because $\mathbf{b}_1 + \widehat{N}, \ldots, \mathbf{b}_d + \widehat{N}$ form an F-basis of $\mathcal{S}^{1\times n}/\widehat{N}$. This proved the correctness of Step (4).

Lemma 27(i) and the same argument used for the correctness of Step (4.5) assert that Step (5) is correct.

[Convention] For a matrix A, its submatrix consisting of entries in the $i_1, \ldots, i_m$ rows and $j_1, \ldots, j_n$ columns is denoted

$$A\begin{pmatrix} i_1, & \ldots, & i_m \\ j_1, & \ldots, & j_n\end{pmatrix}.$$

Example 28. Set $F = \mathbb{C}(x, n, k)$. Let $\delta_x = \frac{d}{dx}$ be the derivation with respect to x, σ_n and σ_k be the shift operators with respect to n and k, respectively, and $\mathcal{S} = F[\partial_x, \partial_n, \partial_k]$. Consider the first-order differential–difference system of size five

$$\{\partial_x(\mathbf{y}) = A_x\mathbf{y},\ \partial_n(\mathbf{y}) = A_n\mathbf{y},\ \partial_k(\mathbf{y}) = A_k\mathbf{y}\}$$

where A_x, A_n and A_k are the same as those in Example 17.

One verifies easily that A_x, A_n, A_k satisfy the integrability conditions but both A_n and A_k are singular, so the given system is integrable but not fully integrable. Let N be its associated Ore submodule. Then $\mathcal{S}^{1\times 5}/N$ is the module M in Example 17 with $\mathbf{b}_i := \mathbf{e}_i + N$ for $i = 1, \ldots, 5$. According to the computation in Example 17, we have that

(i) $\widehat{N}/N = \widehat{0}_{\mathcal{S}^{1\times 5}/N} = F\mathbf{w}_1 \oplus F\mathbf{w}_2$ with $\mathbf{w}_1 = \mathbf{b}_1 + \frac{k^2n+n^2k-1}{(k+1)k}\mathbf{b}_3 - \frac{n^2+nk+1}{(k+1)k}\mathbf{b}_4 + \frac{n}{k}\mathbf{b}_5$ and $\mathbf{w}_2 = \mathbf{b}_2 + \frac{k^3n+k^2n+n^2k-1}{(k+1)k}\mathbf{b}_3 - \frac{k^2n+nk+1+n^2}{(k+1)k}\mathbf{b}_4 + \frac{n+k}{k}\mathbf{b}_5$;

(ii) $\mathbf{e}_3 + \widehat{N}, \mathbf{e}_4 + \widehat{N}, \mathbf{e}_5 + \widehat{N}$ form an F-basis of $\mathcal{S}^{1\times 5}/\widehat{N}$ with the matrices

$$B_n = \begin{pmatrix} \frac{n+n^2k+1}{(k+1)n} & -\frac{n^2-n-1}{(k+1)n} & 0 \\ -\frac{k(n^2-n-1)}{(k+1)n} & \frac{nk+n^2+k}{(k+1)n} & 0 \\ -\frac{n^2k}{k+1} & \frac{n^2}{k+1} & 1 \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} \frac{n^2k-1+k^3n^2+k^2n^2+k^2}{k(k+1)} & -\frac{n^2+2+2k+k^2n^2+n^2k}{k(k+1)} & \frac{n(k+1+k^2)}{k} \\ \frac{n^2k-1-2k+k^3n^2+k^2n^2}{k} & -\frac{n^2+k+k^2n^2+n^2k}{k} & \frac{(k+1)n(k+1+k^2)}{k} \\ \frac{n(-1+k^2-k)}{k+1} & -\frac{n(-1+k^2-k)}{k(k+1)} & k \end{pmatrix}$$

associated with ∂_n and ∂_k, respectively.

Clearly, the transforming matrix from the F-basis $\{\mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{w}_1, \mathbf{w}_2\}$ to the F-basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_5\}$ of $\mathcal{S}^{1\times 5}/N$ is

$$Q = \begin{pmatrix} \mathbf{0}_{3\times 2} & I_3 \\ I_2 & B \end{pmatrix}$$

where $\mathbf{0}_{3\times 2}$ denotes a (3×2) zero matrix and B is a (2×3) matrix of the form

$$B = \begin{pmatrix} \frac{k^2n+n^2k-1}{(k+1)k} & -\frac{n^2+nk+1}{(k+1)k} & \frac{n}{k} \\ \frac{k^3n+k^2n+n^2k-1}{(k+1)k} & -\frac{k^2n+nk+1+n^2}{(k+1)k} & \frac{n+k}{k} \end{pmatrix}.$$

Note that $Q^{-1} = \begin{pmatrix} -B & I_2 \\ I_3 & \mathbf{0}_{3\times 2} \end{pmatrix}$ and partition A_x as $\begin{pmatrix} A_{x11} & A_{x12} \\ A_{x21} & A_{x22} \end{pmatrix}$ in which

$$A_{x11} = A_x\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad A_{x12} = A_x\begin{pmatrix} 1 & & 2 \\ 3 & 4 & 5 \end{pmatrix},$$

$$A_{x21} = A_x\begin{pmatrix} 3 & 4 & 5 \\ 1 & & 2 \end{pmatrix}, \qquad A_{x22} = A_x\begin{pmatrix} 3 & 4 & 5 \\ 3 & 4 & 5 \end{pmatrix}.$$

It follows that

$$\begin{aligned} U_x &= \delta_x(Q)Q^{-1} + QA_xQ^{-1} \\ &= \begin{pmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} \\ \delta_x(B) & \mathbf{0}_{2\times 2} \end{pmatrix} + \begin{pmatrix} A_{x21} & A_{x22} \\ A_{x11} + BA_{x21} & A_{x12} + BA_{x22} \end{pmatrix}\begin{pmatrix} -B & I_2 \\ I_3 & \mathbf{0}_{3\times 2} \end{pmatrix}. \end{aligned}$$

Taking the first 3 rows and the first 3 columns of U_x yields the matrix

$$B_x = -A_{x21}B + A_{x22} = \begin{pmatrix} \frac{n^2k+1+2n^2kx+2k}{2(k+1)x} & -\frac{n^2+1+2xn^2}{(2k+2)x} & \frac{n(1+2x)}{2x} \\ \frac{k(n^2k-1+2n^2kx)}{(2k+2)x} & -\frac{n^2k-k-2+2n^2kx}{(2k+2)x} & \frac{kn(1+2x)}{2x} \\ \frac{kn}{k+1} & -\frac{n}{k+1} & \frac{x+1}{x} \end{pmatrix}$$

associated with ∂_x and the F-basis $\{\mathbf{e}_3 + \widehat{N}, \mathbf{e}_4 + \widehat{N}, \mathbf{e}_5 + \widehat{N}\}$ of $\mathcal{S}^{1\times 5}/\widehat{N}$. So

$$\{\partial_x(\mathbf{z}) = B_x\mathbf{z},\ \partial_n(\mathbf{z}) = B_n\mathbf{z},\ \partial_k(\mathbf{z}) = B_k\mathbf{z}\}$$

is an integrable connection of the original system.

In addition, the matrix defining a $\mathbb{C}$-linear isomorphism from the solution space of the integrable connection to that of the given system, can be read off from the above F-linear expressions of $\mathbf{w}_1$ and $\mathbf{w}_2$ as:

$$P = \begin{pmatrix} -\frac{k^2n+n^2k-1}{(k+1)k} & \frac{n^2+nk+1}{(k+1)k} & -\frac{n}{k} \\ -\frac{k^3n+k^2n+n^2k-1}{(k+1)k} & \frac{k^2n+nk+1+n^2}{(k+1)k} & -\frac{n+k}{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the linear dimension of the given system is three.

Example 29. Let $F = \mathbb{C}(n, k)$, and σ_n and σ_k be two shift operators with respect to n and k respectively. Let $\mathcal{S} = F[\partial_n, \partial_k]$ be the corresponding Ore algebra. We now compute linear dimension of the partial difference system $A(\mathbf{y}) = 0$ where

$$A = \begin{pmatrix} \partial_n + \partial_k & -1 & -1 + k\partial_n & -k \\ 0 & \partial_n + \frac{k^2-n^2+3k+1-2n}{k^2+k-1-2n-n^2} & 0 & \partial_n + \frac{k^2-4n-3-n^2+k}{k^2+k-1-2n-n^2} \\ 0 & 0 & \partial_n & -1 \\ \partial_n & \frac{k-n}{k^2+k-1-2n-n^2} & 0 & \partial_n - \frac{-k+n+1}{k^2+k-1-2n-n^2} \\ \partial_n + \partial_k & \frac{k-n}{k^2+k-1-2n-n^2} & -1 & \partial_n - \frac{-k+n+1}{k^2+k-1-2n-n^2} \\ \partial_n & -1 + \partial_k & 0 & -1 \\ -\frac{k^2+3k+2-n^2}{-n^2+k^2+k} & 0 & \partial_k + \frac{2n}{-n^2+k^2+k} & 0 \\ 0 & -\frac{k^2-n^2+3k+1-2n}{k^2+k-1-2n-n^2} & 0 & \partial_k - \frac{-2n-2}{k^2+k-1-2n-n^2} \end{pmatrix}$$

is an 8×4 matrix over $\mathcal{S}$. Let N be the associated Ore submodule. Computing a Gröbner basis of N yields that

$$\mathbf{b}_1 := \mathbf{e}_1 + N, \qquad \mathbf{b}_2 := \mathbf{e}_2 + N, \qquad \mathbf{b}_3 := \mathbf{e}_3 + N, \qquad \mathbf{b}_4 := \mathbf{e}_4 + N$$

form an F-basis of $\mathcal{S}^{1\times 4}/N$ with the associated matrices

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-k-n-2}{k^2+k-1-2n-n^2} & 0 & \frac{-k^2+3n+2+n^2}{k^2+k-1-2n-n^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-k^2+n^2-2k+1+3n}{k^2+k-1-2n-n^2} & 0 & \frac{-k+n+1}{k^2+k-1-2n-n^2} \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k^2+3k+2-n^2}{-n^2+k^2+k} & 0 & \frac{-2n}{-n^2+k^2+k} & 0 \\ 0 & \frac{k^2-n^2+3k+1-2n}{k^2+k-1-2n-n^2} & 0 & \frac{-2n-2}{k^2+k-1-2n-n^2} \end{pmatrix}.$$

Applying Bronstein's algorithm to the $\mathcal{S}$-module

$$\mathcal{S}^{1\times 4}/N = F\mathbf{b}_1 \oplus F\mathbf{b}_2 \oplus F\mathbf{b}_3 \oplus F\mathbf{b}_4,$$

we find that

(i) $\widehat{N}/N = F\mathbf{w}_1 \oplus F\mathbf{w}_2$ where $\mathbf{w}_1 = \mathbf{b}_1 + \frac{k-n}{2k-n+1+k^2-n^2}\mathbf{b}_3 + \frac{k^2-n^2+k}{2k-n+1+k^2-n^2}\mathbf{b}_4$ and $\mathbf{w}_2 = \mathbf{b}_2 + \frac{k^2+k-1-2n-n^2}{2k-n+1+k^2-n^2}\mathbf{b}_3 - \frac{k+n+1}{2k-n+1+k^2-n^2}\mathbf{b}_4$;

(ii) $\{\mathbf{e}_3 + \widehat{N}, \mathbf{e}_4 + \widehat{N}\}$ is an F-basis of $S^{1\times 4}/\widehat{N}$ with the associated matrices

$$B_n = \begin{pmatrix} 0 & 1 \\ \frac{k^2-n^2+2k-1-3n}{2k-n+1+k^2-n^2} & -\frac{2k+2}{2k-n+1+k^2-n^2} \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} -\frac{k+n+2}{2k-n+1+k^2-n^2} & -\frac{k^2+3k+2-n^2}{2k-n+1+k^2-n^2} \\ -\frac{k^2-n^2+3k+1-2n}{2k-n+1+k^2-n^2} & \frac{k+1-n}{2k-n+1+k^2-n^2} \end{pmatrix}.$$

Therefore, an integrable connection of the given system is $\{\partial_n(\mathbf{z}) = B_n\mathbf{z}, \partial_k(\mathbf{z}) = B_k\mathbf{z}\}$. The matrix defining a $\mathbb{C}$-linear isomorphism from the solution space of the integrable connection to that of the given system is

$$P = \begin{pmatrix} \frac{n-k}{2k-n+1+k^2-n^2} & \frac{n^2-k^2-k}{2k-n+1+k^2-n^2} \\ \frac{n^2+2n+1-k^2-k}{2k-n+1+k^2-n^2} & \frac{k+n+1}{2k-n+1+k^2-n^2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the given system has linear dimension two.

To illustrate Step 5 in Algorithm IntegrableConnection, we recall the method in Wu (2005, §2.4.4) for computing Gröbner bases in finitely-generated free modules over $\mathcal{L}$. Another method is given in Zhou and Winkler (2008).

Recall that $\mathcal{S} = F[\partial_1, \ldots, \partial_\ell, \partial_{\ell+1}, \ldots, \partial_m]$. To construct an extended Ore algebra of $\mathcal{S}$, note that σ_i is an automorphism for all i with $\ell + 1 \le i \le m$ so is σ_i^{-1}. Let $\theta_{\ell+1}, \ldots, \theta_m$ be indeterminates independent of $\partial_{\ell+1}, \ldots, \partial_m$. Then $\bar{\mathcal{S}} = \mathcal{S}[\theta_{\ell+1}; \sigma_{\ell+1}^{-1}, \mathbf{0}] \cdots [\theta_m; \sigma_m^{-1}, \mathbf{0}]$ is also an Ore algebra over F. Recall that, for $k = 1, \ldots, n$, $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 appearing in the kth position. Consider the $\mathcal{S}$-module homomorphism $\Phi : \bar{\mathcal{S}}^{1\times n} \to \mathcal{L}^{1\times n}$ given by $\theta_j^{d_j}\partial_i^{d_i}\mathbf{e}_k \mapsto \partial_j^{-d_j}\partial_i^{d_i}\mathbf{e}_k$ for $1 \le i \le m$, $\ell + 1 \le j \le m$ and $1 \le k \le n$. It follows that $\ker(\Phi)$ equals the (left and right) $\bar{\mathcal{S}}$-module generated by $\theta_j\partial_j\mathbf{e}_k - \mathbf{e}_k$ for $j = \ell + 1, \ldots, m$ and $k = 1, \ldots, n$.

Let P be a subset of $\mathcal{L}$ consisting of power products of the form $\partial_1^{k_1} \cdots \partial_\ell^{k_\ell}\partial_{\ell+1}^{k_{\ell+1}} \cdots \partial_m^{k_m}$ for all $k_1, \ldots, k_\ell \in \mathbb{N}$ and $k_{\ell+1}, \ldots, k_m \in \mathbb{Z}$, and let $\bar{P}$ denote a subset of $\bar{\mathcal{S}}$ consisting of power products of the form $\partial_1^{k_1} \cdots \partial_\ell^{k_\ell}\partial_{\ell+1}^{k_{\ell+1}} \cdots \partial_m^{k_m}$ for all $k_1, \ldots, k_m \in \mathbb{N}$. We define

Definition 30. Let $p, q \in P$. We say that *p divides q in the sense of Laurent* if the following conditions are both satisfied:

(i) $\deg_{\partial_i} p \le \deg_{\partial_i} q$ for all i with $1 \le i \le \ell$;
(ii) either $0 \le \deg_{\partial_j} p \le \deg_{\partial_j} q$ or $\deg_{\partial_j} q \le \deg_{\partial_j} p \le 0$ for all j with $\ell + 1 \le j \le m$.

Remark that, unlike in the usual sense, ∂_j^{-s} does not divide ∂_j^t in the sense of Laurent for any $s, t \in \mathbb{Z}^+$ and j with $\ell + 1 \le j \le m$.

A *monomial* of $\mathcal{L}^{1\times n}$ is an element of the form $p\,\mathbf{e}_i$ where $p \in P$ and $i \in \{1, \ldots, n\}$. Set $P_{\mathcal{L}}$ to be the set of all monomials in $\mathcal{L}^{1\times n}$. For two monomials $p\,\mathbf{e}_i$ and $q\,\mathbf{e}_j$ in $\bar{\mathcal{S}}^{1\times n}$ with $p, q \in \bar{P}$, we say that $p\,\mathbf{e}_i$ *divides* $q\,\mathbf{e}_j$ if i equals j and p divides q in $\bar{P}$. Denote by $P_{\bar{\mathcal{S}}}$ the set of all monomials in $\bar{\mathcal{S}}^{1\times n}$ that are not divisible by any $\partial_j\theta_j\mathbf{e}_k$ for all j with $\ell + 1 \le j \le m$ and k with $1 \le k \le n$. Then the map $\rho : P_{\bar{\mathcal{S}}} \to P_{\mathcal{L}}$ given by $\partial_i\mathbf{e}_k \mapsto \partial_i\mathbf{e}_k$ and $\theta_j\mathbf{e}_k \mapsto \partial_j^{-1}\mathbf{e}_k$, for $1 \le i \le m$, $\ell + 1 \le j \le m$ and $1 \le k \le n$, is a restriction of Φ and gives a well-defined correspondence between monomials of $\bar{\mathcal{S}}^{1\times n}$ and those of $\mathcal{L}^{1\times n}$. Clearly, ρ is bijective.

Let $\prec$ be a monomial order in $\bar{\mathcal{S}}^{1\times n}$. For two monomials $p\,\mathbf{e}_i, q\,\mathbf{e}_j \in P_{\mathcal{L}}$ with $p, q \in P$, we define $p\,\mathbf{e}_i \prec q\,\mathbf{e}_j$ if $\rho^{-1}(p\,\mathbf{e}_i) \prec \rho^{-1}(q\mathbf{e}_j)$ in $P_{\bar{\mathcal{S}}}$. Such an ordering is called an *induced* order on $P_{\mathcal{L}}$ with respect to $\prec$. Leading monomials (coefficients) and a division algorithm can be defined for elements of $\mathcal{L}^{1\times n}$ likewise. Then the following definition is quite natural.

Definition 31. Let M be a submodule in $\mathcal{L}^{1\times n}$. Given a monomial order $\prec$ in $\bar{\mathcal{S}}^{1\times n}$, a finite subset $G \subset M$ is called a *Gröbner basis* with respect to an induced order on $P_{\mathcal{L}}$, if the leading monomial of every element of M is divisible in the sense of Laurent by the leading monomial of some element of G.

The next proposition yields an algorithm for computing Gröbner bases in $\mathcal{L}^{1\times n}$.

Proposition 32. *Let M be a submodule of $\mathcal{L}^{1\times n}$ and Φ be defined as above. If G is a Gröbner basis of $\Phi^{-1}(M)$ with respect to a monomial order in $\bar{\mathcal{S}}^{1\times n}$, then $\Phi(G)$ is a Gröbner basis of M with respect to the induced order on $P_{\mathcal{L}}$.*

Example 33. Let $F = \mathbb{C}(n_1, n_2)$. For $i = 1, 2$, let σ_i be the shift operator with respect to n_i respectively, $\mathcal{S} = F[\partial_1, \partial_2]$ and $\bar{\mathcal{S}} = F[\partial_1, \partial_2, \theta_1, \theta_2]$. We now compute linear dimension of the ideal I generated by two partial difference operators

$$L_1 = \partial_1\partial_2(\partial_1 + 1) \quad \text{and} \quad L_2 = \partial_1\partial_2(\partial_2 + 1)$$

in $\mathcal{S}$. An easy Gröbner basis computation shows that $\mathcal{S}/I$ is infinite dimensional over F. Now view L_1, L_2 as elements of $\bar{\mathcal{S}}$ and compute a Gröbner basis of the ideal $\bar{I}$ generated by

$$L_1,\ L_2,\ \partial_1\theta_1 - 1,\ \partial_2\theta_2 - 1,$$

in $\bar{\mathcal{S}}$ with respect to an elimination order on $\bar{\mathcal{S}}$ in which any monomial in θ_i's is greater than those in ∂_j's. We get that $\{\partial_1 + 1, \partial_2 + 1, \theta_1 + 1, \theta_2 + 1\}$ is a Gröbner basis of $\bar{I}$. By Proposition 32, $\{\partial_1 + 1, \partial_2 + 1, \partial_1^{-1} + 1, \partial_2^{-1} + 1\}$ is a Gröbner basis of the ideal $\mathcal{L}I$ of $\mathcal{L}$. So the linear dimension of I is one.

6. Summary

In this paper, we studied how to construct a linear basis of an Ore localization of a finite-dimensional module M, and proved that $\widehat{N} = N + \widehat{0}_M$ for all submodules N of M. Using module-theoretic language, we described Bronstein's algorithm for determining $\widehat{0}_M$ and $M/\widehat{0}_M$. An equivalence relation among linear differential (difference) equations was extended to linear functional systems. An algorithm was presented for transforming a ∂-finite system Σ to its integrable connection, which is fully integrable and equivalent to Σ.

Appendix. A detailed description of Algorithm LinearBasis

Let R be a noncommutative domain containing a field F, T a left Ore set of R, and R_0 the F-linear subspace spanned by T. Assume that T is also a left Ore set of R_0, and is generated by $t_1, \ldots, t_p$. Let M be an R-module with an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$, and denote by A_i the matrix associated with t_i for all i with $1 \leq i \leq p$.

Let V be the subspace generated by a given finite set of nonzero elements of M. From the generators, one can obtain an F-basis of V using Gaussian elimination. Without loss of generality, we assume that an F-basis $\mathbf{b}'_1, \ldots, \mathbf{b}'_m$ of V, with $0 < m < n$, is given by

$$\left(\mathbf{b}'_1, \ldots, \mathbf{b}'_m\right)^\tau = (I_m,\ B)(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau, \tag{16}$$

where I_m is the identity matrix of size m and B is an $m \times (n - m)$ matrix over F. Then $\mathbf{b}'_1, \ldots, \mathbf{b}'_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_n$ form a new F-basis of M with

$$\left(\mathbf{b}'_1, \ldots, \mathbf{b}'_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_n\right)^\tau = \begin{pmatrix} I_m & B \\ 0 & I_{n-m} \end{pmatrix} (\mathbf{b}_1, \ldots, \mathbf{b}_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_n)^\tau .$$

Since $t_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = A_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ for $i = 1, \ldots, p$, the matrix associated with t_i and $\mathbf{b}'_1, \ldots, \mathbf{b}'_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_n$ is

$$C_i = \begin{pmatrix} I_m & \sigma_i(B) \\ 0 & I_{n-m} \end{pmatrix} A_i \begin{pmatrix} I_m & -B \\ 0 & I_{n-m} \end{pmatrix} \quad \text{for } i = 1, \ldots, p, \tag{17}$$

where $\sigma_i(B)$ means applying the action of σ_i on each entry of B. Partition the matrices

$$A_i = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} \\ A_i^{(21)} & A_i^{(22)} \end{pmatrix} \quad \text{and} \quad C_i = \begin{pmatrix} C_i^{(11)} & C_i^{(12)} \\ C_i^{(21)} & C_i^{(22)} \end{pmatrix},$$

where $A_i^{(11)}, C_i^{(11)} \in F^{m\times m}$, $A_i^{(12)}, C_i^{(12)} \in F^{m\times(n-m)}$, $A_i^{(21)}, C_i^{(21)} \in F^{(n-m)\times m}$, and $A_i^{(22)}, C_i^{(22)} \in F^{(n-m)\times(n-m)}$ for $i = 1, \ldots, p$. Then

$$C_i^{(12)} = A_i^{(12)} - A_i^{(11)}B + \sigma_i(B)\left(A_i^{(22)} - A_i^{(21)}B\right)$$

and $C_i^{(22)} = A_i^{(22)} - A_i^{(21)}B$ for $i = 1, \ldots, p$. From (17), V is an R_0-module if and only if $C_i^{(12)} = 0$ for all i with $1 \le i \le p$. when this is the case, $C_i^{(22)}$ is the matrix associated with t_i and the F-basis $\mathbf{b}_{m+1} + V, \ldots, \mathbf{b}_n + V$ for the R_0-module M/V.

The next proposition is a summary for the above discussion.

Proposition 34. *Let M be a finite-dimensional module over R. Assume that $\mathbf{b}_1, \ldots, \mathbf{b}_n$ form an F-basis of M, and $A_1, \ldots, A_p$ are the associated matrices. Let V be the F-subspace generated by an F-basis $\mathbf{b}'_1, \ldots \mathbf{b}'_m$ given in (16) where $0 < m < n$. Then*

(i) *the elements $\mathbf{b}'_1, \ldots, \mathbf{b}'_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_n$ form an F-basis of M with associated matrices $C_1, \ldots, C_p$ given in (17);*
(ii) *the F-subspace V is an R_0-submodule if and only if $C_i^{(12)}$ is equal to zero for all i with $1 \le i \le p$, When this is the case, the R_0-module M/V has an F-basis $\mathbf{b}_{m+1} + V, \ldots, \mathbf{b}_n + V$ with associated matrices $C_1^{(22)}, \ldots, C_p^{(22)}$;*
(iii) *for $i = 1, \ldots, p$, every nonzero entry in the column vector $C_i^{(12)}(\mathbf{b}_{m+1}, \ldots, \mathbf{b}_n)^\tau$ belongs to $R_0V \setminus V$.*

Proof. The first two assertions hold due to the above discussion. From

$$\partial_i\left(\mathbf{b}'_1, \ldots, \mathbf{b}'_m\right)^\tau = C_i^{(11)}\left(\mathbf{b}'_1, \ldots, \mathbf{b}'_m\right)^\tau + C_i^{(12)}(\mathbf{b}_{m+1}, \ldots, \mathbf{b}_n)^\tau,$$

it follows that $C_i^{(12)}(\mathbf{b}_{m+1}, \ldots, \mathbf{b}_n)^\tau \equiv 0 \mod R_0V$, which, together with the decomposition $M = V \oplus (F\mathbf{b}_{m+1}) \oplus \cdots \oplus (F\mathbf{b}_n)$, implies the last assertion. □

Proposition 34 leads to the following algorithm for constructing an F-basis of a given R_0-submodule of M.

Algorithm LinearBasis. Given an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of an R-module M, the associated matrices $A_1, \ldots, A_p$, and a finite set U of nonzero elements of M, compute

(a) an F-basis of the R_0-module R_0U;
(b) an F-basis of the R_0-module $M/(R_0U)$;
(c) the associated matrices with the latter basis.

(1) [*Initialize*] Set $U_0 := U$.
(2) [*Compute an F-basis of FU_0*] Construct a subsequence: $i_1, i_2, \ldots, i_m$ of $1, 2, \ldots, n$ such that $\mathbf{b}'_{i_1}, \ldots, \mathbf{b}'_{i_m}$ form an F-basis of FU_0, and an $m \times (n-m)$ matrix B over F such that

$$\left(\mathbf{b}'_{i_1}, \ldots, \mathbf{b}'_{i_m}\right)^\tau = (I_m,\ B)\left(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_m}, \mathbf{b}_{i_{m+1}}, \ldots, \mathbf{b}_{i_n}\right)^\tau.$$

[*Note that $i_{m+1}, \ldots, i_n$ is the complementary subsequence of $i_1, \ldots, i_m$.*]
Set $U_0 := \left\{\mathbf{b}'_{i_1}, \ldots, \mathbf{b}'_{i_m}\right\}$. If $m = n$, then **return**
(a) [*an F-basis of R_0U*] $\mathbf{b}_1, \ldots, \mathbf{b}_n$;
(b) [*an F-basis of $M/(R_0U)$*] $\emptyset$;
(c) [*associated matrices with the latter basis*] $\emptyset$.

(3) [*Determine if FU_0 is an R_0-submodule*] For $j = 1, \ldots, p$, set

$$A_j^{(11)} := A_j \begin{pmatrix} i_1, \ldots, i_m \\ i_1, \ldots, i_m \end{pmatrix}, \qquad A_j^{(12)} := A_j \begin{pmatrix} i_1, \ldots, i_m \\ i_{m+1}, \ldots, i_n \end{pmatrix},$$

$$A_j^{(21)} := A_j \begin{pmatrix} i_{m+1}, \ldots, i_n \\ i_1, \ldots, i_m \end{pmatrix}, \qquad A_j^{(22)} := A_j \begin{pmatrix} i_{m+1}, \ldots, i_n \\ i_{m+1}, \ldots, i_n \end{pmatrix},$$

where the submatrices are defined under the notational convention in Section 5.
For $j = 1, \ldots, p$, set $C_j^{(22)} := A_j^{(22)} - A_j^{(21)} B$ and

$$C_j^{(12)} := A_j^{(12)} - A_j^{(11)} B + \sigma_j(B) C_j^{(22)}.$$

If $C_j^{(12)} = 0$ for all j with $1 \le j \le p$, then **return** three sequences:
(a) [*an F-basis of R_0U*] $\mathbf{b}'_{i_1}, \ldots, \mathbf{b}'_{i_m}$;
(b) [*an F-basis of $M/(R_0U)$*] $\mathbf{b}_{i_{m+1}} + R_0U, \ldots, \mathbf{b}_{i_n} + R_0U$;
(c) [*associated matrices with the latter basis*] $C_1^{(22)}, \ldots, C_p^{(22)}$.

(4) [Update U_0] Set

$$U_0 := U_0 \cup \{\text{nonzero elements in } C_j^{(12)} (\mathbf{b}_{i_{m+1}}, \ldots, \mathbf{b}_{i_n})^\tau \mid j = 1, \ldots, p\}.$$

Go to Step 2.

Algorithm LinearBasis terminates, because by Proposition 34 (iii), the dimension of FU_0 increases whenever U_0 gets updated in Step 4. It is correct by Proposition 34 (i) and (ii).

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