Submersive Rational Difference Systems and Their Accessibility

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ABSTRACT

The paper describes an algebraic construction of the inversive difference field associated with a discrete-time rational nonlinear control system under the assumption that the system is submersive. We prove that a system is submersive iff its associated difference ideal is proper, prime and reflexive. Next, we show that Kähler differentials of the above inversive field define a module over the corresponding ring of Ore operators, and relate its torsion submodule to the vector space of autonomous one-forms, introduced elsewhere. The above results allow us to check accessibility property and simplify transfer functions with computer algebra techniques.

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1. INTRODUCTION

Many algebraic concepts such as Ore (skew) polynomial rings, modules over skew polynomial rings, difference fields and ideals play a prominent role in (nonlinear) control theory. Their applications, in general, require the help of computer algebra software. In Maple there exists a built-in package, OreTools [1], that addresses the computations with Ore polynomials. Unfortunately, this package, though being a good starting point for control applications, cannot handle the Ore polynomials associated to nonlinear control systems without further extension. The reason is that one cannot describe the relations between control system variables explicitly

in this package, as well as take these relationships into account in calculations with Ore polynomials.

The extension of the package so that it can address the control problems, is not immediate and trivial, especially for discrete-time systems. The main goal of this paper is to build a solid mathematical ground that will help to implement the required changes.

An Ore (called alternatively pseudo-linear) algebra approach has been applied in [15] for a unification of the study of nonlinear control systems, both discrete- and continuous-time. Mathematically speaking, the approach consists of three steps. Namely, one first associates a differential or inversive difference field to a continuous- or discrete-time nonlinear control system, respectively; second, one applies the differential operator to "linearize" the given system, yielding the system description in terms of differential one-forms; and finally, one introduces pseudo-linear operators acting on differential one-forms. This approach allows us to handle different control problems, including accessibility, system equivalence and reduction [19], realization [20], feedback linearization [2]. Furthermore, this approach enables us to extend the concept of the transfer functions for nonlinear systems [14, 15, 16, 26].

Although the last two steps can be carried out in a similar manner both for continuous- and discrete-time systems, it is much easier to associate a field to a continuous-time system than to a discrete-time system. Note that for a continuous-time system, such a field is proved to exist and its construction is given in [9] using differential algebra. For a discrete-time system of the form

$$\Sigma : \begin{cases} x_1(t+1) &=& f_1(u_1(t),\ldots,u_m(t),x_1(t),\ldots,x_n(t)) \\ &\cdots& \\ x_n(t+1) &=& f_n(u_1(t),\ldots,u_m(t),x_1(t),\ldots,x_n(t)), \end{cases}$$

such a field exists, if the system is generically submersive, that is, if the Jacobian matrix $\frac{\partial(f_1,\ldots,f_n)}{\partial(u_1,\ldots,u_m,x_1,\ldots,x_n)}$ has rank n almost everywhere, except on a set of measure zero [13, 2, 15]. In this case, the field has been constructed in [2], using the tools both from algebra and classical analysis.

The main focus of this paper is to provide an algebraic construction of the inversive difference field associated with Σ, when all $f_1, f_2, \ldots, f_n$ are rational functions. This paves a way to apply computer algebra techniques (see for example [1, 5]) for the last two steps. We show that Σ corresponds to a proper, prime and reflexive ideal iff it is submersive. From a technical point of view, completing the second step, we apply the notion of Kähler differentials which has been widely used in control theory to study the structural properties of the nonlinear systems; see for instance [10]

and references therein. We show that Kähler differentials of the elements of the inversive field, associated with Σ, define a module over the corresponding ring of Ore (pseudo-linear) operators, and relate its torsion submodule to the vector space of autonomous one-forms, introduced in [2, 19]. Finally, we apply these constructions to check accessibility (controllability) and simplify transfer functions. Similar results are obtained for an input-output system description: $y(t+n)=f(u(t),\ldots,u(t+n-1),y(t),\ldots,y(t+n-1))$, where f is a rational function.

The rest of this paper is organized as follows. In Section 2, we define the submersivity property of rational difference systems, and connect submersivity with algebraic independence. In Section 3, we prove that a rational difference system is submersive iff its associated ideal is proper, prime and reflexive, and define its associated field whose inversive closure is constructed explicitly in Section 4. Kähler differentials are described in module-theoretic language in Section 5. Similar results for rational difference equations of any order are given in Section 6. We discuss how to define and check accessibility property, and simplify transfer functions in Section 7.

2. A DIFFERENCE FIELD EXTENSION

In this section, we connect submersivity property with algebraic independence, and construct a difference field containing a solution of a given submersive rational system.

Let us recall some terminologies from difference algebra. The reader is referred to [8] for more details. Let R be a commutative ring, and $\sigma : R \to R$ be a monomorphism, *i.e.* an injective homomorphism. The pair (R, σ) is called an (ordinary) *difference ring*. It is called a *difference domain* (resp. *a difference field*) if it is an integral domain (resp. a field). If σ is bijective, then R is said to be *inversive*. Let (R_1, σ_1) and (R_2, σ_2) be two difference rings. We say that R_2 is a *difference ring extension* of R_1 if $R_1 \subset R_2$ and $\sigma_2|_{R_1} = \sigma_1$.

Throughout the paper, (k, σ) stands for an inversive difference field of characteristic zero. Let $\mathbf{u} = \{u_1, \ldots, u_m\}$ be a set of m indeterminates, and $\mathbf{U} = \{\sigma^\ell(u)\,|\,u \in \mathbf{u}, \ell \in \mathbb{N}\}$ an infinite set of indeterminates containing $\mathbf{u}$. We extend σ from k to $k(\mathbf{U})$ by mapping u to $\sigma(u)$ for all $u \in \mathbf{U}$. Then $(k(\mathbf{U}), \sigma)$ becomes a difference field. Note that σ is not surjective, because none of the u_i has preimages.

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a set of new indeterminates. By a rational difference system over k, we mean a system of the form

$$\Sigma : \{\sigma(x_1) = f_1, \ldots, \sigma(x_n) = f_n\} \tag{1}$$

where $f_1, \ldots, f_n$ are in $k(\mathbf{U}, \mathbf{x})$. Following [13, 2], we say that (1) is *submersive* if the $n \times (m+n)$ matrix

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(u_1, \ldots, u_m, x_1, \ldots, x_n)}, \quad \text{abbreviated as: } \frac{\partial(f_1, \ldots, f_n)}{\partial(\mathbf{u}, \mathbf{x})},$$

has rank n.

System (1) has a solution in some difference field extension of k if the following question has an affirmative answer:

Given $f_1, \ldots, f_n \in k(\mathbf{U}, \mathbf{x})$, can σ be extended to a monomorphism θ from $k(\mathbf{U}, \mathbf{x})$ to $k(\mathbf{U}, \mathbf{x})$ with $\theta(x_j) = f_j$ for all j with $1 \le j \le n$?

First, we extend $\sigma : k(\mathbf{U}) \to k(\mathbf{U})$ to $\theta : k(\mathbf{U})[\mathbf{x}] \to k(\mathbf{U}, \mathbf{x})$ by mapping x_j to f_j for all j with $1 \le j \le n$. The homomorphism θ can be extended further to the fraction field $k(\mathbf{U}, \mathbf{x})$, provided that it is injective. Observe that θ is injective iff any nonzero polynomial in $k(\mathbf{U})[\mathbf{x}]$ is not in its kernel, which is equivalent to that

$$\sigma(\mathbf{U}) \cup \{\theta(x_1), \ldots, \theta(x_n)\}, \quad \text{i.e.} \quad \sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$$

is algebraically independent over $\sigma(k)$. Note that $\sigma(k) = k$, since σ is an automorphism. So we seek a criterion for the algebraic independence of $\sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$ over k.

To do this, we need the notion of Kähler differentials [18]. A tensor-free definition of Kähler differentials is given in [28], and basic results about Kähler differentials are contained in [3, §9.1]. Applications of Kähler differentials in nonlinear control theory are given in [10] and references therein.

For two fields $F \subset E$, there exist a linear space $\Omega_{E/F}$ over E, and a derivation $d : E \to \Omega_{E/F}$ such that (i) $dc = 0$ for all $c \in F$; and (ii) $\Omega_{E/F} = \mathrm{span}_E\{da\,|\,a \in E\}$, where da denotes the image of a under d. An element of $\Omega_{E/F}$ is called an *F-differential*, and the pair $\left(\Omega_{E/F}, d\right)$ is called the *E-linear space of F-differentials*. The next lemma is important for this paper. Its proof can be found in [18, 28] and [3, Corollary 9.1.1].

Lemma 2.1 *Let $F \subset E$ be fields of characteristic zero. Then $P \subset E$ is algebraically independent over F iff $\{dp\}_{p \in P} \subset \Omega_{E/F}$ is linearly independent over E.*

Corollary 2.2 *In $\Omega_{k(\mathbf{U}, \mathbf{x})/k}$, the subset $\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$ is linearly independent over $k(\mathbf{U}, \mathbf{x})$.*

Proof. Observe that $\mathbf{U} \cup \mathbf{x}$ is algebraically independent over k, and then apply Lemma 2.1. □

The algebraic independence of $\sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$ is equivalent to submersivity of (1), as stated in the next lemma.

Lemma 2.3 *Let $f_1, \ldots, f_n$ be in $k(\mathbf{U}, \mathbf{x})$. Then $\sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$ is algebraically independent over k iff $\frac{\partial(f_1, \ldots, f_n)}{\partial(\mathbf{u}, \mathbf{x})}$ has rank n.*

Proof. For brevity, we denote $\frac{\partial(f_1, \ldots, f_n)}{\partial(\mathbf{u}, \mathbf{x})}$ by J. Let (Ω, d) be the linear space of k-differentials over $k(\mathbf{U}, \mathbf{x})$,

$$L = \{dv\,|\,v \in \sigma(\mathbf{U})\} \cup \{df_1, \ldots, df_n\},$$

and V be the $k(\mathbf{U}, \mathbf{x})$-linear subspace spanned by $\{dv\,|\,v \in \sigma(\mathbf{U})\}$. By Lemma 2.1, it suffices to show that L is linearly dependent over $k(\mathbf{U}, \mathbf{x})$ iff $\mathrm{rank}(J) < n$. A straightforward calculation yields

$$(df_1, \ldots, df_n)^T \equiv J(du_1, \ldots, du_m, dx_1, \ldots, dx_n)^T \mod V, \quad (2)$$

where $(\cdot)^T$ stands for the transpose of a matrix (vector).

If the rank of J is less than n, then there exists a nonzero vector $(r_1, \ldots, r_n)$ in the left kernel of J. Thus, $\sum_{j=1}^n r_j df_j$ belongs to V. So L is linearly dependent over $k(\mathbf{U}, \mathbf{x})$. Conversely, suppose that L is linearly dependent over $k(\mathbf{U}, \mathbf{x})$. It follows from Corollary 2.2 that $\{dv\,|\,v \in \sigma(\mathbf{U})\}$ is linearly independent over $k(\mathbf{U}, \mathbf{x})$. So a nontrivial linear relation among elements in L implies that there exist $r_1, \ldots, r_n$ in $k(\mathbf{U}, \mathbf{x})$, not all zero, such that $\sum_{j=1}^n r_j df_j$ belongs to V. Multiplying $(r_1, \ldots, r_n)$ to (2) yields

$$(r_1, \ldots, r_n)J(du_1, \ldots, du_m, dx_1, \ldots, dx_n)^T \in V.$$

Thus, $(r_1, \ldots, r_n)$ is in the left kernel of J, for, otherwise, the subset $\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$ would be linearly dependent, a contradiction to Corollary 2.2. The rank of J is less than n. □

The next proposition is immediate from Lemma 2.3.

Proposition 2.4 *Let $f_1, \ldots, f_n$ be in $k(\mathbf{U}, \mathbf{x})$, and θ be the homomorphism from $k(\mathbf{U})[\mathbf{x}]$ to $k(\mathbf{U}, \mathbf{x})$ such that $\theta|_k = \sigma$, $\theta(u) = \sigma(u)$ for all $u \in \mathbf{U}$ and $\theta(x_j) = f_j$ for all j with $1 \le j \le n$. Then θ is injective iff the matrix $\frac{\partial(f_1, \ldots, f_n)}{\partial(\mathbf{u}, \mathbf{x})}$ has rank n. When this is the case, θ can be extended to a monomorphism from $k(\mathbf{U}, \mathbf{x})$ to itself, that is, $(k(\mathbf{U}, \mathbf{x}), \theta)$ is a difference field extension of (k, σ).*

System (1) has a solution $(\mathbf{u}, \mathbf{x})$ in the difference field defined in Proposition 2.4, provided that (1) is submersive.

3. ASSOCIATED DIFFERENCE IDEALS

In this section, we characterize a submersive system by a proper, prime and reflexive ideal, and show that the field associated with the system is isomorphic to $(k(\mathbf{U},\mathbf{x}),\theta)$ defined in Proposition 2.4.

Let (R,σ) be a difference ring and I an ideal of R. We say that I is *proper* if $I \neq R$, and that it is a *difference* ideal if $\sigma(I) \subset I$.

Assume now that I is a difference ideal. Then I is said to be *reflexive* if, for every $r \in R$, $\sigma(r) \in I$ implies that $r \in I$. The induced homomorphism $\bar{\sigma}: R/I \to R/I$ with $r+I \mapsto \sigma(r)+I$ for all $r \in R$, is injective iff I is reflexive. In other words, $(R/I,\bar{\sigma})$ is a difference ring iff I is reflexive. A homomorphism ϕ from (R_1,σ_1) to (R_2,σ_2) is called a *difference homomorphism* if $\phi \circ \sigma_1 = \sigma_2 \circ \phi$. The kernel of a difference homomorphism is proper and reflexive by Theorem V in [8, page 71].

Lemma 3.1 *Let ϕ be a surjective difference homomorphism from (R',σ') to (R,σ), and I a difference ideal of R. Then $\phi^{-1}(I)$ is proper (resp. prime, reflexive) iff I is.*

Proof. Denote $\phi^{-1}(I)$ by I'. Note that I' is a difference ideal. Clearly, I' is proper iff I is. Assume that I' is proper. Then the induced homomorphism $\bar{\phi}$ from R'/I' to R/I is a ring isomorphism, since ϕ is surjective. So I' is prime iff I is. Let $\bar{\sigma}': R'/I' \to R'/I'$ and $\bar{\sigma}: R/I \to R/I$ be the homomorphisms induced by σ' and σ, respectively. Then $\bar{\phi} \circ \bar{\sigma}' = \bar{\sigma} \circ \bar{\phi}$, because $\phi \circ \sigma' = \sigma \circ \phi$. Thus, $\bar{\sigma}'$ is injective iff $\bar{\sigma}$ is, and I' is reflexive iff I is. $\square$

For a subset B of R, the algebraic and difference ideals generated by B are denoted by (B) and $[B]$, respectively. Note that $[B]$ is the algebraic ideal generated by $\cup_{\ell \in \mathbb{N}} \sigma^\ell(B)$. Let I be a difference ideal and H a subset of R closed under σ and multiplication. Then the saturation ideal

$$I:H = \{r \in R \mid \exists\, h \in H,\ hr \in I\}$$

is a difference ideal.

We retain the indeterminates in $\mathbf{U}$ and $\mathbf{x}$ introduced in Section 2, and let $\mathbf{X} = \mathbf{x} \cup \{\sigma^\ell(x_j) \mid j \in \{1,\dots,n\}, \ell \in \mathbb{Z}^+\}$ be an infinite set of indeterminates disjoint with $\mathbf{U}$. Extend σ from k to $k[\mathbf{U},\mathbf{X}]$ by mapping z to $\sigma(z)$ for all $z \in \mathbf{U} \cup \mathbf{X}$ so that $k[\mathbf{U},\mathbf{X}]$ becomes a difference ring, which is usually denoted by $k\{\mathbf{u},\mathbf{x}\}$. We opt for the former notation, as $k[\mathbf{U},\mathbf{x}]$ plays a significant role in the sequel.

Definition 3.1 *Given a system Σ of the form (1), let $f_j = \frac{b_j}{a_j}$ with a_j and b_j in $k[\mathbf{U},\mathbf{x}]$ and $\gcd(a_j,b_j)=1$, and let $p_j = a_j\sigma(x_j)-b_j$ for all j with $1 \leq j \leq n$. Denote by H_Σ the monoid generated by $\sigma^\ell(a_j)$ for all $j \in \{1,\dots,n\}$ and $\ell \in \mathbb{N}$. We call $[p_1,\dots,p_n]:H_\Sigma$ the difference ideal associated with Σ, and denote it by I_Σ.*

The reader is referred to [8, page 74] for the definition of solutions of a difference equation. Roughly speaking, Σ and I_Σ are related as follows. A solution of Σ is a solution of I_Σ. Conversely, a solution of I_Σ that annihilates none of the a_j, is a solution of Σ.

We recall a reduction process for difference polynomials. The order of an element $r \in k[\mathbf{U},\mathbf{X}]$ (with respect to $\mathbf{x}$) is defined to be ℓ if $\sigma^\ell(x)$ appears in r for some $x \in \mathbf{x}$, and $\sigma^e(y)$ does not appear in r for all $e > \ell$ and $y \in \mathbf{x}$. By convention, the order of any elements in $k[\mathbf{U}]$ is understood to be $-\infty$. A ranking $\prec$ is a total order on $\mathbf{X}$ such that $x \prec \sigma(x)$ for all $x \in \mathbf{X}$. A ranking is *orderly* if, for all x,y in $\mathbf{x}$, $\sigma^d(x) \prec \sigma^\ell(y)$ whenever $d < \ell$.

Assume that r is an element of order ℓ in $k[\mathbf{U},\mathbf{X}]$. If $\ell > 0$, then, using the successive difference pseudo-division by $p_1,\dots,p_n$ with respect to an orderly ranking, we have that

$$hr = q + r_0 \tag{3}$$

where h is in H_Σ with order less than ℓ, q is in the algebraic ideal generated in the subring $k[\mathbf{U},\mathbf{x},\sigma(\mathbf{x}),\dots,\sigma^\ell(\mathbf{x})]$ by $\sigma^e(p_j)$ for all e in $\{0,1,\dots,\ell-1\}$ and j in $\{1,\dots,n\}$, and r_0 is in $k[\mathbf{U},\mathbf{x}]$. If $\ell \leq 0$, we set $h=1$, $q=0$ and $r_0 = r$ in (3). The reader is referred to [12] for the definition of a successive difference pseudo-division with respect to any ranking.

Example 3.2 *Compute the successive difference pseudo-remainder of $r = \sigma^2(x_1)$ by $p_1 = u\sigma(x_1)-x_2$ and $p_2 = u\sigma(x_2)-x_1$.*

Since r is of order two, we first compute the pseudo-remainder of r by $\sigma(p_1)$. This yields $\sigma(u)r = \sigma(p_1)+\sigma(x_2)$. Next, compute the pseudo-remainder of $\sigma(x_2)$ by p_2 to get $u\sigma(x_2) = p_2+x_1$. As x_1 has order zero, (3) becomes $u\sigma(u)r = u\sigma(p_1)+p_2+x_1$.

Most of results in this section are based on the next key lemma.

Lemma 3.2 *If I_Σ is proper, then $k[\mathbf{U},\mathbf{x}] \cap I_\Sigma = \{0\}$.*

Proof. For every $\ell \in \mathbb{N}$, we let $R^{(\ell)} = k[\mathbf{U},\mathbf{x},\sigma(\mathbf{x}),\dots,\sigma^\ell(\mathbf{x})]$, and $H^{(\ell)}$ the monoid generated by $\sigma^d(a_j)$ for all $d \in \{0,\dots,\ell\}$ and $j \in \{1,\dots,n\}$. Then $H^{(\ell)}$ is contained in $R^{(\ell)}$. Put $J^{(0)} = \{0\}$ and, for all $\ell > 0$, let $J^{(\ell)}$ be the algebraic ideal generated in $R^{(\ell)}$ by $\sigma^e(p_j)$ for all $e \in \{0,\dots,\ell-1\}$ and $j \in \{1,\dots,n\}$. Moreover, we put $I^{(0)} = \{0\}$ and, for every $\ell > 0$, $I^{(\ell)} = J^{(\ell)} : H^{(\ell-1)}$, which is an algebraic ideal in $R^{(\ell)}$. Observe that $I_\Sigma = \cup_{\ell \in \mathbb{N}} I^{(\ell)}$. So it suffices to show that $R^{(0)} \cap I^{(\ell)} = \{0\}$ for all $\ell \in \mathbb{N}$.

This clearly holds for $\ell = 0$. Assume now that $\ell > 0$. Let $D_{\ell-1}$ be the determinant of the $n\ell \times n\ell$ matrix

$$\frac{\partial\left(p_1,\dots,p_n,\sigma(p_1),\dots,\sigma(p_n),\dots,\sigma^{\ell-1}(p_1),\dots,\sigma^{\ell-1}(p_n)\right)}{\partial\left(\sigma(x_1),\dots,\sigma(x_n),\sigma^2(x_1),\dots,\sigma^2(x_n),\dots,\sigma^\ell(x_1),\dots,\sigma^\ell(x_n)\right)},$$

which is in triangular form. Thus, $D_{\ell-1}$ equals the product of all the $\sigma^e(a_j)$'s with $0 \leq e \leq \ell-1$ and $1 \leq j \leq n$. Consequently, $I^{(\ell)}$ is the ideal of $J^{(\ell)}$ saturated at $D_{\ell-1}$. It follows from Proposition 3.2 in [24] that $R^{(0)} \cap I^{(\ell)} = \{0\}$. $\square$

A longer but more elementary proof of Lemma 3.2 is given in the appendix of [17, page 56].

Corollary 3.3 *I_Σ is a prime ideal.*

Proof. It is sufficient to consider the case in which I_Σ is proper. Let r_1 and r_2 be in $K[\mathbf{U},\mathbf{X}]$ such that their product belongs to I_Σ. By (3), $h_i r_i \equiv r_i' \bmod I_\Sigma$ for some h_i in H_Σ and r_i' in $k[\mathbf{U},\mathbf{x}]$, $i=1,2$. Then $r_1' r_2'$ is in I_Σ. By Lemma 3.2, $r_1' r_2' = 0$, which implies that either r_1 or r_2 is in I_Σ. $\square$

The ideal I_Σ is not always proper, that is, Σ may be inconsistent; and I_Σ is not necessarily reflexive, that is, there may exist a nonzero element in $k[\mathbf{U},\mathbf{x}] \setminus I_\Sigma$ vanishing at any solution of Σ.

Example 3.3 *(i) Let $\Sigma = \left\{\sigma(x_1) = x_1,\ \sigma(x_2)=x_1,\ \sigma(x_3)=\frac{u_1}{x_2-x_1}\right\}$. Then $p_1 = \sigma(x_1)-x_1$, $p_2 = \sigma(x_2)-x_1$ and $p_3 = a_3\sigma(x_3)-u_1$. with $a_3 = x_2-x_1$. Since $\sigma(a_3) = p_2-p_1$, $\sigma(a_3) \in [p_1,p_2,p_3]\cap H_\Sigma$. So I_Σ is improper, that is, Σ has no solution in any difference field.*
(ii) Let $\Sigma = \{\sigma(x_1) = x_2,\ \sigma(x_2) = x_2\}$. Then $\sigma(x_1) = \sigma(x_2)$. Hence, $x_1 - x_2$ vanishes at every solution of Σ, but it is not in I_Σ as will be shown in Lemma 3.2. So I_Σ is not reflexive.

Note that neither of the systems in Example 3.3 is submersive.

Proposition 3.4 *Let Σ be a rational difference system given by (1). Then I_Σ is proper, prime and reflexive iff Σ is submersive. When*

this is the case, $(k(\mathbf{U}, \mathbf{x}), \theta)$ is k-isomorphic to $(K_\Sigma, \bar{\sigma})$, where θ is defined in Proposition 2.4, K_Σ is the quotient field of $k[\mathbf{U}, \mathbf{X}]/I_\Sigma$, and $\bar{\sigma}$ is induced by σ.

Proof. Assume that Σ is submersive. Then $(k(\mathbf{U}, \mathbf{x}), \theta)$ is a well-defined difference field by Proposition 2.4. Define ϕ to be a difference homomorphism from $(k[\mathbf{U}, \mathbf{X}], \sigma)$ to $(k(\mathbf{U}, \mathbf{x}), \theta)$ that fixes k and maps z to z for all $z \in \mathbf{U} \cup \mathbf{x}$. Then

$$\phi(\sigma(x_j)) = \theta(\phi(x_j)) = \theta(x_j) = f_j \quad \text{for all } j \text{ with } 1 \le j \le n.$$

Thus, $[p_1, \ldots, p_n]$ is a subset of $\ker(\phi)$. Moreover, H_Σ and $\ker(\phi)$ are disjoint, since $\phi(\sigma^\ell(a_j)) = \theta^\ell(\phi(a_j)) \ne 0$. It follows that I_Σ is contained in $\ker(\phi)$. To show $\ker(\phi) \subset I_\Sigma$, assume $r \in \ker(\phi)$. By (3), $hr \equiv r_0 \mod [p_1, \ldots, p_n]$ for some $h \in H_\Sigma$ and $r_0 \in k[\mathbf{U}, \mathbf{x}]$. Thus, $\phi(r_0) = 0$, and so $r_0 = 0$ by the definition of ϕ. We have that $r \in I_\Sigma$. Consequently, $I_\Sigma = \ker(\phi)$, which, together with Theorem V in [8, page 71], implies that I_Σ is proper, prime and reflexive.

Conversely, assume that I_Σ is proper, prime and reflexive. It suffices to show that $\sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$ is algebraically independent over k by Lemma 2.3. Let p be a polynomial over k such that

$$p(\sigma(U), f_1, \ldots, f_n) = 0. \tag{4}$$

Denote $p(\sigma(U), \sigma(x_1), \ldots, \sigma(x_n))$ by r, which is an element of order less than two in $k[\sigma(U), \sigma(\mathbf{x})]$. By (3), $hr \equiv r_0 \mod (p_1, \ldots, p_n)$ for some h in $H_\Sigma \cap k[\mathbf{U}, \mathbf{x}]$ and r_0 in $k[\mathbf{U}, \mathbf{x}]$. Replacing $\sigma(x_j)$ by f_j, $j = 1, \ldots, n$, in the above congruence, we see that $r_0 = 0$ by (4). Consequently, r belongs to I_Σ. Since σ is an automorphism of k, there exists r' in $k[\mathbf{U}, \mathbf{x}]$ such that $\sigma(r') = r$. So r' is in I_Σ, since I_Σ is reflexive. By Lemma 3.2, r' is zero, so is r, and, hence, p is zero. The set $\sigma(\mathbf{U}) \cup \{f_1, \ldots, f_n\}$ is algebraically independent over k.

Finally, assume that Σ is submersive. Then the homomorphism ϕ defined in the first paragraph of the proof induces a difference isomorphism from $(K_\Sigma, \bar{\sigma})$ to $(k(\mathbf{U}, \mathbf{x}), \theta)$. $\square$

For a submersive system Σ, $(K_\Sigma, \bar{\sigma})$ is the difference field generated by $\mathbf{z} = (u_1 + I_\Sigma, \ldots, u_m + I_\Sigma, x_1 + I_\Sigma, \ldots, x_n + I_\Sigma)$, which is a generic zero of I_Σ ([8, page 76]). The vector $\mathbf{z}$ is also a solution of Σ, because any a_j does not vanish at $\mathbf{z}$ by Lemma 3.2. So we call $(K_\Sigma, \bar{\sigma})$ the *difference field associated with* Σ. Note that $(K_\Sigma, \bar{\sigma})$ is not inversive if $\mathbf{u}$ is nonempty. In the next section, we will construct the inversive closure of $(K_\Sigma, \bar{\sigma})$.

Remark 3.4 *For a submersive system Σ, there are two ways to determine the membership of I_Σ. One is to use successive pseudo-division, i.e. $r \in I_\Sigma$ iff $r_0 = 0$ by (3). The other is to use ϕ defined in the proof of Proposition 3.4, i.e. $r \in I_\Sigma$ iff $\phi(r) = 0$. The latter also yields a rational normal form function from $k[\mathbf{U}, \mathbf{X}]/I_\Sigma$ to $k(\mathbf{U}, \mathbf{x})$.*

4. INVERSIVE CLOSURES

Let J be a proper, prime and reflexive ideal in $k[\mathbf{U}, \mathbf{X}]$, K the quotient field of $k[\mathbf{U}, \mathbf{X}]/J$, and $\bar{\sigma} : K \to K$ the monomorphism induced by σ. In this section, we construct the inversive closure $(\widehat{K}, \hat{\sigma})$ of $(K, \bar{\sigma})$, which is an inversive field extension of K with the property that, for every $\hat{f} \in \widehat{K}$, there exists $\ell \in \mathbb{N}$ such that $\hat{\sigma}^\ell(\hat{f})$ belongs to K (see [8, page 66]). This inversive field will serve as a base field to analyze discrete-time systems in Section 7, when J is the ideal associated with a submersive system (equation).

Initially, we denote $k[\mathbf{U}, \mathbf{X}]$ by R_0, u_i by $u_{i,0}$, and x_j by $x_{j,0}$ for all i, j with $1 \le i \le m$ and $1 \le j \le n$. For every $\ell \in \mathbb{Z}^+$, we set $R_\ell = R_{\ell-1}[u_{1,-\ell}, \ldots, u_{m,-\ell}, x_{1,-\ell}, \ldots, x_{n,-\ell}]$ inductively. Extend σ from $R_{\ell-1}$ to R_ℓ by mapping $u_{i,-\ell}$ to $u_{i,-(\ell-1)}$ and $x_{j,-\ell}$ to $x_{j,-(\ell-1)}$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. This process yields an ascending chain: $R_0 \subset R_1 \subset \cdots \subset R_\ell \subset \cdots$ of difference rings, in which $\sigma(R_\ell) = R_{\ell-1}$.

Denote J by J_0, which is in R_0. Since $\sigma(R_1) = R_0$, $\sigma^{-1}(J_0)$ is well-defined in R_1. We set $J_1 = \sigma^{-1}(J_0)$. Define $J_\ell = \sigma^{-1}(J_{\ell-1})$ for all $\ell \in \mathbb{Z}^+$ recursively.

Lemma 4.1 *For every $\ell \in \mathbb{Z}^+$, $J_\ell \cap R_{\ell-1} = J_{\ell-1}$, and, moreover, J_ℓ is proper, prime and reflexive.*

Proof. Since $\sigma(J_0) \subset J_0$, $J_0 \subset J_1$ by definition. So $J_0 \subset J_1 \cap R_0$. For $r \in J_1 \cap R_0$, $\sigma(r) \in J_0$. Hence, r belongs to J_0, since J_0 is reflexive. This proves $J_0 = R_0 \cap J_1$. The ideal J_1 is proper, prime and reflexive by Lemma 3.1. If J_ℓ has the properties given in the statement of this lemma, so does $J_{\ell+1}$ by the same argument, in which J_0 is replaced by J_ℓ, and J_1 by $J_{\ell+1}$. $\square$

By Lemma 4.1, we have an ascending chain $J_0 \subset J_1 \subset \cdots \subset J_\ell \subset \cdots$, in which each J_ℓ is a proper, prime and reflexive ideal in R_ℓ, moreover, $J_{\ell+1} = \sigma^{-1}(J_\ell)$ and $J_{\ell+1} \cap R_\ell = J_\ell$.

The two chains allow us to form a ring $\widehat{R} = \cup_{\ell \in \mathbb{N}} R_\ell$ and an ideal $\widehat{J} = \cup_{\ell \in \mathbb{N}} J_\ell$, respectively. Extend σ to a map from $\widehat{R}$ to itself by mapping r to $\sigma(r)$ if $r \in R_\ell$ for some $\ell \in \mathbb{N}$. It is direct to verify that σ is a well-defined automorphism of $\widehat{R}$, and, therefore, $(\widehat{R}, \sigma)$ is an inversive difference ring. By Lemma 4.1, $J_\ell = \widehat{J} \cap R_\ell$ for all $\ell \in \mathbb{N}$, and $\widehat{J}$ is a proper, prime and reflexive ideal.

Proposition 4.2 *With the notation just introduced, let $(\widehat{R}/\widehat{J}, \hat{\sigma})$ be the induced ring, and $(\widehat{K}, \hat{\sigma})$ be its fraction field. Then $(\widehat{K}, \hat{\sigma})$ is the inversive closure of $(K, \bar{\sigma})$.*

Proof. The ring $(\widehat{R}/\widehat{J}, \hat{\sigma})$ is inversive, because $(\widehat{R}, \sigma)$ is inversive. Consequently, $(\widehat{K}, \hat{\sigma})$ is an inversive field.

Let $\tau_0 : R_0 \to \widehat{R}/\widehat{J}$ be defined by $r \mapsto r + \widehat{J}$. Then $\ker(\tau_0)$ is equal to J_0 by Lemma 4.1. Hence, $\tau : (R_0/J_0, \bar{\sigma}) \to (\widehat{R}/\widehat{J}, \hat{\sigma})$ defined by $r + J_0 \mapsto r + \widehat{J}$ for all $r \in R_0$, is a difference monomorphism. So τ can be extended to a difference monomorphism from K to $\widehat{K}$.

For every $r \in \widehat{K}$, there exist $\ell \in \mathbb{N}$ and $p_\ell, q_\ell \in R_\ell$ such that $\hat{r}$ equals $\frac{p_\ell + \widehat{J}}{q_\ell + \widehat{J}}$. Thus, $\hat{\sigma}^\ell(\hat{r})$ belongs to $\tau(K)$. By Theorem II in [8, page 66], $\widehat{K}$ is the inversive closure of $\tau(K)$, and so is that of K. $\square$

Assume now that J is a difference ideal associated with a submersive system Σ given by (1). By Propositions 3.4 and 4.2, the field K_Σ associated with Σ has an inversive closure, which is called the *inversive field associated with* Σ, and is denoted by $(\widehat{K}_\Sigma, \hat{\sigma})$.

The field $\widehat{K}_\Sigma$ may also be understood as the direct limit of the quotient fields of R_ℓ/J_ℓ for all $\ell \in \mathbb{N}$, each of which is isomorphic to a field of rational functions by Proposition 3.4.

There are two ways to determine, for $r \in \widehat{R}$, whether $r + \widehat{J}$ is equal to zero in $\widehat{K}_\Sigma$. Assume $r \in R_\ell$ for some $\ell \in \mathbb{N}$. First, Lemma 4.1 implies $r \in \widehat{J}$ iff $r \in J_\ell$. The latter is equivalent to that $\sigma^\ell(r)$ is in J, which can be determined by the methods described in Remark 3.4. Second, note that $J_\ell = [\sigma^{-\ell}(p_1), \ldots, \sigma^{-\ell}(p_n)] : \sigma^{-\ell}(H_\Sigma)$. Then both Lemma 3.2 and Proposition 3.4 hold for J_ℓ when we replace $\mathbf{u}$ by $\sigma^{-\ell}(\mathbf{u})$, and $\mathbf{x}$ by $\sigma^{-\ell}(\mathbf{x})$. So the methods in Remark 3.4 are applicable in the membership problem of J_ℓ.

Example 4.1 *Let Σ be $\left\{ \sigma(x_1) = \frac{x_2}{u}, \sigma(x_2) = \frac{x_1^2}{u} \right\}$, which is a submersive system. Let $p_1 = u\sigma(x_1) - x_2$ and $p_2 = u\sigma(x_2) - x_1^2$, and let H_Σ be the monoid generated by $u, \sigma(u), \ldots$. Then the ideal associated with Σ is $I_\Sigma = [p_1, p_2] : H_\Sigma$. Let $r = u_{-1}u_{-2}x_1 - x_{1,-2}^2$ be in $\widehat{R}$. We determine whether $r + \widehat{I}_\Sigma$ is zero in the inversive field associated with Σ. It suffices to determine whether $\sigma^2(r)$ is in I_Σ. By (3), we have that $u\sigma(u)\sigma^2(r) = u^2\sigma(u)\sigma(p_1) + u\sigma(u)p_2$. Hence $\sigma^2(r)$ is in I_Σ, and, consequently, $r + \widehat{I}_\Sigma$ is zero.*

5. KÄHLER DIFFERENTIALS AND TORSION ELEMENTS

We show that Kähler differentials of the inversive field associated with a submersive system form a module over a ring of skew polynomials, and determine its torsion submodule. Later, in Section 7, we point out that a submersive rational discrete-time system is accessible iff the corresponding module is torsion-free. This is in agreement with results known for nonlinear continuous-time systems [10], linear systems over Ore algebras [6], and linear time-delay systems [11].

In this section, all modules and linear spaces are left ones. Let Σ be a submersive system given by (1) whose associated difference ideal and inversive field are denoted by I_Σ and $(\widehat{K}_\Sigma, \hat{\sigma})$, respectively. Let $(k(\mathbf{U},\mathbf{x}),\theta)$ and $(K_\Sigma,\bar{\sigma})$ be the same as those in Proposition 3.4. Define ψ to be the k-homomorphism from $k(\mathbf{U},\mathbf{x})$ to K_Σ that maps z to $z + I_\Sigma$ for all $z \in \mathbf{U} \cup \mathbf{x}$. Then ψ is a difference k-isomorphism by the proof of Proposition 3.4. So ψ induces a difference k-monomorphism $\widehat{\psi}$ from $k(\mathbf{U},\mathbf{x})$ to $\widehat{K}_\Sigma$ that maps z to $z + \widehat{I}_\Sigma$ for all $z \in \mathbf{U} \cup \mathbf{x}$. This observation allows us to abridge $z + \widehat{I}_\Sigma$ in $\widehat{K}_\Sigma$ as z for all $z \in \mathbf{U} \cup \mathbf{x}$. Hence, we have, in $\widehat{K}_\Sigma$,

$$\hat{\sigma}(x_j) = f_j, \quad j = 1,\ldots,n. \tag{5}$$

Let $S_\Sigma = \widehat{K}_\Sigma[s;\hat{\sigma}]$ be the ring of $\hat{\sigma}$-shift operators over $\widehat{K}_\Sigma$, which is a special case for Ore algebras [27, 7, 4]. The commutation rule of S_Σ is $sr = \hat{\sigma}(r)s$ for all $r \in \widehat{K}_\Sigma$.

Let Ω_Σ be the vector space spanned by k-differentials over $\widehat{K}_\Sigma$. We view Ω_Σ as a module over S_Σ in the following manner. By Lemma 9.1.2 in [3], the automorphism $\hat{\sigma}$ of $\widehat{K}_\Sigma$ induces a unique skew-linear map σ^* from Ω_Σ to itself, that is, σ^* is additive, and $\sigma^*(r\omega) = \hat{\sigma}(r)\sigma^*(\omega)$ for all $r \in \widehat{K}_\Sigma$ and $\omega \in \Omega_\Sigma$. Moreover, the diagram

$$
\begin{array}{ccc}
\Omega_\Sigma & \xrightarrow{\ \sigma^*\ } & \Omega_\Sigma \\
{\scriptstyle d}\big\uparrow & & \big\uparrow{\scriptstyle d} \\
\widehat{K}_\Sigma & \xrightarrow{\ \hat{\sigma}\ } & \widehat{K}_\Sigma
\end{array}
\tag{6}
$$

is commutative. Define $s\omega$ to be $\sigma^*(\omega)$ for all ω in Ω_Σ. The skew-linearity of σ^* implies that $s(r\omega) = \hat{\sigma}(r)(s\omega)$ for all r in K_Σ and ω in Ω_Σ. Then Ω_Σ becomes a left module over S_Σ. Moreover, the commutativity of (6) implies $s(g\,dh) = \hat{\sigma}(g)d\hat{\sigma}(h)$ for all g,h in $\widehat{K}_\Sigma$.

An element ω of Ω_Σ is a *torsion element* if ω is annihilated by a nonzero polynomial in S_Σ, that is, the submodule generated by ω, denoted by $S_\Sigma\omega$, is a finite-dimensional linear subspace over $\widehat{K}_\Sigma$. The set of all torsion elements in Ω_Σ is denoted by $\mathrm{tor}(\Omega_\Sigma)$. The existence of the least common left multiples in S_Σ implies that $\mathrm{tor}(\Omega_\Sigma)$ is a submodule [7, page 46]. The module Ω_Σ is said to be *torsion-free* if $\mathrm{tor}(\Omega_\Sigma)$ equals $\{0\}$.

Let $\Omega_0 = \mathrm{span}_{\widehat{K}_\Sigma}\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$. Differentiating (5), we see that $s\,dz$ belongs to Ω_0 for all $z \in \mathbf{U} \cup \mathbf{x}$. So Ω_0 is a submodule of Ω_Σ. The next lemma will help us shift an element in Ω_Σ to Ω_0, and asserts that $\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$ is a $\widehat{K}_\Sigma$-basis of Ω_0.

Lemma 5.1 *(i) If ω is a nonzero element in Ω_Σ, then $s\omega$ is nonzero.*
(ii) In Ω_Σ, $\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$ is linearly independent over $\widehat{K}_\Sigma$.

Proof. (i) Let $\omega = \sum_i g_i dh_i$, where $g_i, h_i \in \widehat{K}_\Sigma$, and the set $\{dh_i\}$ is linearly independent over $\widehat{K}_\Sigma$. Suppose that $s\omega = 0$. Then $\{d\hat{\sigma}(h_i)\}$ is linearly dependent over $\widehat{K}_\Sigma$, since $s\omega = \sum_i \hat{\sigma}(g_i)d\hat{\sigma}(h_i)$. It follows from Lemma 2.1 that $\{\hat{\sigma}(h_i)\}$ is algebraically dependent over k, and so is $\{h_i\}$, because $\hat{\sigma}$ extends σ, which is an automorphism. So $\{dh_i\}$ is linearly dependent over $\widehat{K}_\Sigma$, a contradiction.

(ii) $\mathbf{U} \cup \mathbf{x}$, viewed as a subset of $k(\mathbf{U},\mathbf{x})$, is algebraically independent over k. So the assertion follows from the definition of $\widehat{\psi}$ given above and Lemma 2.1. □

We make an additional assumption in the rest of this section that $f_1, \ldots, f_n$ in (1) are in $k(\mathbf{u}, \mathbf{x})$ instead of $k(\mathbf{U}, \mathbf{x})$. Though this assumption helps to simplify the calculations in Ω_Σ, it is not restrictive and the same results can be derived in the general case. Denote the column vectors $(u_1,\ldots,u_m)^T$ and $(x_1,\ldots,x_n)^T$ by $\vec{u}$ and $\vec{x}$, respectively. Differentiating (5) yields

$$s\,d\vec{x} = A\,d\vec{x} + B\,d\vec{u}, \tag{7}$$

where $A = \frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}$ and $B = \frac{\partial(f_1,\ldots,f_n)}{\partial(u_1,\ldots,u_m)}$.

We use the method described in [2] to determine $\mathrm{tor}(\Omega_\Sigma)$. Let H_1 be the linear subspace spanned by $dx_1,\ldots,dx_n$ over $\widehat{K}_\Sigma$, and define $H_{\ell+1}$ to be $\{\omega \in H_\ell\,|\,s\omega \in H_\ell\}$ for all $\ell \in \mathbb{N}$. Then

$$H_1 \supseteq H_2 \supseteq \cdots \tag{8}$$

is a chain of finite-dimensional linear subspaces over $\widehat{K}_\Sigma$. Moreover, we let $H_\infty = \cap_{\ell \in \mathbb{Z}^+} H_\ell$. Since $sH_\infty \subset H_\infty$, H_∞ is a submodule.

Lemma 5.2 and Proposition 5.3 below show how the torsion submodule of Ω_Σ is related to H_∞, the submodule of autonomous one-forms described in [2] and Section 7.

Lemma 5.2 $H_\infty = \mathrm{tor}(\Omega_\Sigma) \cap \Omega_0$.

Proof. We claim that $\mathrm{tor}(\Omega_\Sigma) \cap \Omega_0 \subset H_1$. Assume that ω is in Ω_0. Then $\omega = \sum_{j=1}^n g_j dx_j + \sum_{i=1}^m P_i du_i$ for some g_j in $\widehat{K}_\Sigma$ and P_i in S_Σ, because $d\hat{\sigma}^\ell(u_i) = s^\ell du_i$ for all $\ell \in \mathbb{N}$. Suppose that ω is not in H_1. Then, without loss of generality, we assume that P_1 has a nonzero leading term $c_1 s^{d_1}$, where c_1 is in $\widehat{K}_\Sigma$ and d_1 in $\mathbb{N}$. Expanding $s^\ell \omega$ as a $\widehat{K}_\Sigma$-linear combination of $\{dz\,|\,z \in \mathbf{U} \cup \mathbf{x}\}$, we see that $s^\ell \omega$ involves the term $\hat{\sigma}^\ell(c_1)d\sigma^{d_1+\ell}(u_1)$ but is free of any $d\sigma^e(u_1)$, where e is greater than $d_1 + \ell$. Hence, $\{s^\ell\omega\,|\,\ell \in \mathbb{N}\}$ is linearly independent over $\widehat{K}_\Sigma$ by Lemma 5.1 (ii). So ω is not a torsion element. The claim is proved.

Let ω be an element of the submodule $\mathrm{tor}(\Omega_\Sigma) \cap \Omega_0$. Then $S_\Sigma\omega$ is a subset of $\mathrm{tor}(\Omega_\Sigma) \cap \Omega_0$. By the claim, $S_\Sigma\omega$ is a subset of H_1. By the definition of H_ℓ, $S_\Sigma\omega$ is contained in H_ℓ for all $\ell \in \mathbb{Z}^+$. It follows that $\omega \in H_\infty$. Hence, $\mathrm{tor}(\Omega_\Sigma) \cap \Omega_0 \subset H_\infty$. On the other hand, every element of H_∞ is a torsion element, because H_∞ is both a submodule over S_Σ and a finite-dimensional linear subspace over $\widehat{K}_\Sigma$. So $H_\infty \subset \mathrm{tor}(\Omega_\Sigma) \cap \Omega_0$. □

The next proposition describes $\mathrm{tor}(\Omega_\Sigma)$ by H_∞.

Proposition 5.3 *(i) $\mathrm{tor}(\Omega_\Sigma) = \{\omega \in \Omega_\Sigma\,|\,s^\ell\omega \in H_\infty$ for some $\ell \in \mathbb{N}\}$.*
(ii) Ω_Σ is torsion-free iff H_∞ is trivial.

Proof. (i) Assume that ω is in $\mathrm{tor}(\Omega_\Sigma)$. Then there exists a positive integer ℓ such that $s^\ell\omega$ belongs to Ω_0 by (7) and the definition of scalar multiplication. Hence, $s^\ell\omega$ is in $\mathrm{tor}(\Omega_\Sigma) \cap \Omega_0$, and so it is in H_∞ by Lemma 5.2. Conversely, let ω be in Ω_Σ such that $s^\ell\omega$ is in H_∞ for some $\ell \in \mathbb{N}$. Then $s^\ell\omega$ is in $\mathrm{tor}(\Omega_\Sigma)$ by Lemma 5.2, and so is ω by the definition of torsion elements.

(ii) It suffices to show that $H_\infty = \{0\}$ implies $\mathrm{tor}(\Omega_\Sigma) = \{0\}$ by Lemma 5.2. For $\omega \in \mathrm{tor}(\Omega_\Sigma)$, there exists $\ell \in \mathbb{N}$ such that $s^\ell\omega = 0$ by the first assertion and $H_\infty = \{0\}$. So $\omega = 0$ by Lemma 5.1 (i). □

To determine a $\widehat{K}_\Sigma$-basis of H_∞, we describe H_∞ without using any recursion.

Proposition 5.4 $H_\infty = \{\omega \in H_1\,|\,s^n\omega \in H_1\}$, *where n is the number of elements in $\mathbf{x}$ (also known as the order of the system Σ).*

Proof. By the definition of chain (8), we conclude that $H_i = H_\infty$ if $H_i = H_{i+1}$, and that $\dim_{\widehat{K}_\Sigma} H_i < \dim_{\widehat{K}_\Sigma} H_{i+1}$ if $H_{i+1} \subsetneqq H_i$. Therefore, $H_{n+1} = H_\infty$. It remains to show $H_{n+1} = \{\omega \in H_1 \mid s^n \omega \in H_1\}$. We shall prove that, for all $\ell \in \mathbb{Z}^+$, $H_\ell = H'_\ell$, where H'_ℓ stands for $\{\omega \in H_1 \mid s^{\ell-1}\omega \in H_1\}$. To this end we need:

Claim. Let ω be an element of Ω_Σ and ℓ be a positive integer. If both ω and $s^\ell \omega$ are in H_1, then $s^i \omega$ is in H_1 for all $i = 0, \ldots, \ell - 1$.

The claim clearly holds for $i = 0$. Assume $s^{i-1}\omega \in H_1$ for some i with $0 < i < \ell$. By (7) and Lemma 5.1, $s^i\omega$ can be uniquely written as $\omega_0 + \omega_1$ for some $\omega_0 \in H_1$ and $\omega_1 \in \mathrm{span}_{\widehat{K}_\Sigma}\{du_1, \ldots, du_m\}$. Thus, $s^\ell\omega = s^{\ell-i}s^i\omega = s^{\ell-i}\omega_0 + s^{\ell-i}\omega_1$. Observe that $s^{\ell-i}\omega_0$ is free of $d\hat{\sigma}^{\ell-i}(u_1), \ldots, d\hat{\sigma}^{\ell-i}(u_m)$, but $s^{\ell-i}\omega_1$ is a $\widehat{K}_\Sigma$-linear combination of $d\hat{\sigma}^{\ell-i}(u_1), \ldots, d\hat{\sigma}^{\ell-i}(u_m)$. It follows from $s^\ell\omega \in H_1$ that $s^{\ell-i}\omega_1 = 0$. By Lemma 5.1 (i), $\omega_1 = 0$. The claim is proved.

Clearly, $H_1 = H'_1$. Assume that $H_{\ell-1} = H'_{\ell-1}$. If ω is in H_ℓ, then $s\omega$ is $H_{\ell-1}$, and so $s\omega$ is in $H'_{\ell-1}$. Hence, $s^{\ell-1}\omega$ is in H_1. Consequently, ω is in H'_ℓ. Conversely, assume that ω belongs to H'_ℓ. Then both $s\omega$ and $s^{\ell-2}\omega$ are in H_1 by the claim. Then $\omega \in H'_{\ell-1}$ since $\omega, s^{\ell-2}\omega \in H_1$. Likewise, $s\omega \in H'_{\ell-1}$ since $s\omega, s^{\ell-1}\omega \in H_1$. So $\omega \in H_\ell$, because $\omega, s\omega \in H_{\ell-1}$ by the induction hypothesis. □

A $\widehat{K}_\Sigma$-basis of H_∞ can be determined by computing the left kernel of an $n \times mn$ matrix over $\widehat{K}_\Sigma$, as stated in the next proposition.

Proposition 5.5 *Let A and B be given by (7). Define $C_1 = \hat{\sigma}^{-1}(B)$ and $C_{i+1} = \hat{\sigma}^{-1}(A)\hat{\sigma}^{-1}(C_i)$ for all $i \in \mathbb{Z}^+$, and define C to be the $n \times mn$ matrix $(C_1, \ldots, C_n)$. Let $\omega = (h_1, \ldots, h_n)d\vec{x}$. Then ω is in H_∞ iff $(h_1, \ldots, h_n)$ is in the left kernel of C.*

Proof. By Proposition 5.4, ω belongs to H_∞ iff $s^n\omega$ is in H_1. Let A_0 be the $n \times n$ identity matrix, and define $A_\ell = \hat{\sigma}(A_{\ell-1})A$ for all $\ell \in \mathbb{Z}^+$. An easy induction shows that

$$A_\ell = \hat{\sigma}^{\ell-1}(A)A_{\ell-1} \quad \text{and} \quad \hat{\sigma}^\ell(C_\ell) = \hat{\sigma}(A_{\ell-1})B,$$

which, together with (7), imply $s^\ell d\vec{x} = A_\ell d\vec{x} + \sum_{i=1}^\ell \hat{\sigma}^\ell(C_i)d\hat{\sigma}^{\ell-i}(\vec{u})$ for all $\ell \in \mathbb{Z}^+$. So $s^n\omega$ is in H_1 iff $(\hat{\sigma}^n(h_1), \ldots, \hat{\sigma}^n(h_n))$ lies in the left kernel of $\hat{\sigma}^n(C)$ by Lemma 5.1 (ii). □

By Proposition 5.5, H_∞ is trivial iff C has full row rank, which is in agreement with the results in [23, 25].

6. RATIONAL DIFFERENCE EQUATIONS

The results, obtained in Sections 3, 4 and 5 for the systems described by state equations, that is, by a set of first-order rational difference equations, can be carried over to systems described by input-output equations, that is, by the higher-order rational difference equation.

Let $\mathbf{U}$ be the same as before and $\mathbf{Y} = \{y, \sigma(y), \sigma^2(y), \ldots\}$ be a set of new indeterminates. Extend the automorphism $\sigma : k \to k$ by mapping $\tilde{u}$ to $\sigma(\tilde{u})$ for all $\tilde{u} \in \mathbf{U}$, and $\tilde{y}$ to $\sigma(\tilde{y})$ for all $\tilde{y} \in \mathbf{Y}$. Then $(k[\mathbf{U}, \mathbf{Y}], \sigma)$ is a ring of difference polynomials. A rational difference equation is of the form

$$\sigma^n(y) = f \quad \text{where } f \in k\left(\mathbf{U}, y, \sigma(y), \ldots, \sigma^{n-1}(y)\right). \quad (9)$$

Such an equation is said to be *submersive* if $\left(\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_m}, \frac{\partial f}{\partial y}\right)$ is nonzero [19]. Write f in (9) as $\frac{d}{c}$, where c and d are coprime in $k\left[\mathbf{U}, y, \sigma(y), \ldots, \sigma^{n-1}(y)\right]$. Set $p = c\sigma^n(y) - d$ and H to be the monoid generated by $\sigma^\ell(c)$ for all $\ell \in \mathbb{N}$. We call $[p] : H$ the difference ideal associated with (9).

It is well-known that (9) is equivalent to the set of first-order rational difference equations

$$\{\sigma(y) = y_1, \ldots, \sigma(y_{n-2}) = y_{n-1}, \sigma(y_{n-1}) = \tilde{f}\}, \quad (10)$$

where $\tilde{f}$ is obtained by substituting y_i for $\sigma^i(y)$ in f for $i = 1, \ldots, n-1$. This equivalence enables us to translate Lemma 3.2, Corollary 3.3 and Proposition 3.4 into the next proposition.

Proposition 6.1 *Let I be the difference ideal associated with (9). (i) If I is proper, then $k[\mathbf{U}, y, \sigma(y), \ldots, \sigma^{n-1}(y)] \cap I = \{0\}$. (ii) I is prime. (iii) I is proper and reflexive iff (9) is submersive. (iv) If (9) is submersive, then $(K, \bar{\sigma})$ is isomorphic to*

$$\left(k\left(\mathbf{U}, y, \sigma(y), \ldots, \sigma^{n-1}(y)\right), \theta\right), \quad (11)$$

where K is the quotient field of $k[\mathbf{U}, \mathbf{Y}]/I$, $\bar{\sigma}$ is the map induced by σ, $\theta(z) = \sigma(z)$ for z free of $\sigma^{n-1}(y)$, and $\theta\left(\sigma^{n-1}(y)\right) = f$.

An algebraic description of the equivalence between (9) and (10), and a proof of Proposition 6.1 are given in [17].

If (9) is submersive, then we call $(K, \bar{\sigma})$ in Proposition 6.1 the *difference field associated with (9)*. The associated field K of a submersive equation (9) has an inversive closure $(\widehat{K}, \hat{\sigma})$, which is isomorphic to the inversive closure of the field associated with (10).

Next, we translate the results in Section 5 for submersive systems to a submersive equation. This translation leads to a criterion for the accessibility described by input-output equations.

Let (9) be submersive whose associated inversive field is denoted by $(\widehat{K}, \hat{\sigma})$. Let (Ω, d) be the $\widehat{K}$-linear space of k-differentials, which is a module over $S = \widehat{K}[s; \hat{\sigma}]$. Since $\widehat{K}$ is isomorphic to the inversive field associated with (10), Lemma 5.1 translates into

Lemma 6.2 *(i) If ω is a nonzero element in Ω, then $s\omega \neq 0$. (ii) In Ω, $\left\{dz \mid z \in \mathbf{U} \cup \{y, \sigma(y), \ldots, \sigma^{n-1}(y)\}\right\}$ is linearly independent over $\widehat{K}$.*

From now on, we assume that $m = 1$, i.e. $\mathbf{U} = \{u, \sigma(u), \sigma^2(u), \ldots\}$. Differentiating $\hat{\sigma}^n(y) = f$ yields

$$P\,dy = Q\,du, \quad (12)$$

where $P = s^n - \sum_{j=0}^{n-1} \frac{\partial f}{\partial \sigma^j(y)} s^j$ and $Q = \sum_i \frac{\partial f}{\partial \sigma^i(u)} s^i$. This compact form enables us to determine $\mathrm{tor}(\Omega)$ by Ore polynomials.

The next lemma reveals that (12) is the "minimal" linear relation among $\{dz \mid z \in \mathbf{U} \cup \mathbf{Y}\}$.

Lemma 6.3 *Let P and Q be given in (12). If there exist P' and Q' in S such that $P'dy = Q'du$ holds in Ω, then $P' = DP$ and $Q' = DQ$ for some D in S.*

Proof. The right-hand division yields $P' = DP + R$ with $\deg R$ less than n. It follows from $P'dy = Q'du$ and $DPdy = DQdu$ (see (12)) that $Rdy = (Q' - DQ)du$. Since $\deg R < n$. Lemma 6.2 (ii) implies that both $R = 0$ and $Q' = DQ$. Consequently, $P' = DP$. □

It is important to remark that the greatest common left divisors (abbreviated as: gcld) of P and Q is well-defined and computable by the left-hand Euclidean algorithm, because $\widehat{K}$ is inversive.

Lemma 6.4 *Let A_1, A_2, B_1, B_2 and L be nonzero elements in S. If $A_1 = LB_1, A_2 = LB_2$ and $\mathrm{gcld}(B_1, B_2) = 1$, then $L = \mathrm{gcld}(A_1, A_2)$.*

Proof. The (extended) left-hand Euclidean algorithm yields C_1 and C_2 in S such that $B_1C_1 + B_2C_2 = 1$. So $A_1C_1 + A_2C_2 = L$. It follows that $\mathrm{gcld}(A_1, A_2)$ is a left-hand divisor of L. □

The next proposition connects (12) with $\mathrm{tor}(\Omega)$.

Proposition 6.5 *Let P and Q be given in (12) and $G = \text{gcld}(P, Q)$. Put $P = G\tilde{P}$, $Q = G\tilde{Q}$ and $\tilde{\omega} = \tilde{P}dy - \tilde{Q}du$. Then*

(i) $\text{tor}(\Omega) = \{\omega \in \Omega \,|\, s^\ell \omega \in S\tilde{\omega} \text{ for some } \ell \in \mathbb{N}\}$.

(ii) Ω is torsion-free iff $G = 1$.

Proof. (i) Since (12) implies $G\tilde{\omega} = 0$, $S\tilde{\omega}$ is a finite-dimensional linear subspace over $\widehat{K}$. Consequently, $S\tilde{\omega}$ is a subset of $\text{tor}(\Omega)$, and, hence, $\{\omega \,|\, s^\ell \omega \in S\tilde{\omega} \text{ for some } \ell \in \mathbb{N}\}$ is a subset of $\text{tor}(\Omega)$.

Conversely, let ω be a nonzero torsion element. Putting $y_0 = y$ and $u_0 = u$, we may write ω as $M'dy_{-\ell} - N'du_{-\ell}$ for some $\ell \in \mathbb{N}$ and $M', N' \in S$. By Lemma 6.2, there are nonzero elements $M, N \in S$ such that $s^\ell \omega = Mdy - Ndu$. It remains to show that $s^\ell \omega$ is in $S\tilde{\omega}$. Since $s^\ell \omega$ is in $\text{tor}(\Omega)$, $Cs^\ell \omega = 0$ for some $C \in S$ with $C \neq 0$. So $CMdy = CNdu$. Then $CM = DP$ and $CN = DQ$ for some $D \in S$ by Lemma 6.3, which implies that

$$CM = DG\tilde{P} \quad \text{and} \quad CN = DG\tilde{Q}. \tag{13}$$

By Lemma 6.4 and $\text{gcld}(\tilde{P}, \tilde{Q}) = 1$, $DG = \text{gcld}(CM, CN)$. Therefore, $DG = C\tilde{C}$ for some $\tilde{C}$ in S, which, together with (13), implies that $M = \tilde{C}\tilde{P}$ and $N = \tilde{C}\tilde{Q}$, and so that $s^\ell \omega = \tilde{C}\tilde{\omega} \in S\tilde{\omega}$.

(ii) Note that $G = 1$ iff $\tilde{\omega} = 0$ by (12). The second assertion follows from the first one and Lemma 6.2 (i). $\square$

7. APPLICATIONS

We apply the results of previous sections to provide an algebraic condition of accessibility for a rational discrete-time nonlinear system, described either by a set of state equations or by a higher-order input-output difference equation. This enables us to implement the accessibility test in computer algebra systems such as Maple or Mathematica. In addition, we relate the algebraic accessibility condition to the reduced form of the transfer function of the system.

In this section, we let k be the field of real numbers, and σ the identity map on k. We extend σ to $k[\mathbf{U}, \mathbf{X}]$ (resp. $k[\mathbf{U}, \mathbf{Y}]$) by mapping z to $\sigma(z)$ for all $z \in \mathbf{U} \cup \mathbf{X}$ (resp. $k[\mathbf{U}, \mathbf{Y}]$).

7.1 State-space representations

We consider the rational control system of the form

$$\sigma(x_1) = f_1, \ldots, \sigma(x_n) = f_n \quad \text{and} \quad y = g \tag{14}$$

where $f_1, \ldots, f_n, g$ are in $k(\mathbf{u}, \mathbf{x})$. Let Σ stand for the rational difference system in (14), and call $y = g$ an output equation. We assume that Σ is submersive, and that Ω_Σ, H_∞ and the H_i are the same as those in Section 5.

Following [2], we define the *relative degree* of $\omega \in \Omega_\Sigma$ to be

$$r = \min\left\{\ell \geq 0 \,|\, s^\ell \omega \notin H_1\right\}.$$

If such an integer does not exist, then set $r = \infty$. An element $\omega \in \Omega_\Sigma$ is called an *autonomous one-form* if its relative degree is ∞. Such a one-form and its shifts by the powers of s do not depend on $d\hat{\sigma}^\ell(u_i)$ for all $\ell \in \mathbb{Z}$ and $i \in \{1, \ldots, m\}$.

Clearly, every element of H_∞ is autonomous. On the other hand, $\omega, s\omega, s^2\omega\ldots$ are all in H_1 if ω is autonomous. Hence, ω belongs to H_∞ by Proposition 5.4. Accordingly, H_∞ is precisely the set of autonomous elements.

We say that (14) is *accessible* if Ω_Σ does not contain any nontrivial autonomous one-form, that is, H_∞ is trivial (see [2, Theorem 4.5]). By Proposition 5.3, (14) is accessible iff Ω_Σ is torsion-free. The latter condition can be verified by computing the rank of the matrix C defined in Proposition 5.5.

Example 7.1 *Consider the rational control system of the form*

$$\sigma(x_1) = x_2 u, \ \sigma(x_2) = x_1 u \quad \text{and} \quad y = x_1^2 + x_1. \tag{15}$$

The difference field associated with (15) is isomorphic to

$$K = (k(x_1, x_2, u, \sigma(u), \ldots), \theta),$$

where $\theta(\sigma^i(u)) = \sigma^{i+1}(u)$, $\theta(x_1) = x_2 u$, and $\theta(x_2) = x_1 u$. The inversive closure of K is $\widehat{K} = (k(x_1, x_2, \ldots, \sigma^{-1}(u), u, \sigma(u), \ldots), \hat{\theta})$, where $\hat{\theta}^{-1}(x_1) = \frac{x_2}{\sigma^{-1}(u)}$, $\hat{\theta}^{-1}(x_2) = \frac{x_1}{\sigma^{-1}(u)}$, and $\hat{\theta}^{-1}(\sigma^i(u)) = \sigma^{i-1}(u)$ for all $i \in \mathbb{Z}$. Then (7) becomes

$$s\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} du. \tag{16}$$

The matrix C given in Proposition 5.5 is

$$\begin{pmatrix} x_1/\sigma^{-1}(u) & x_1/\sigma^{-2}(u) \\ x_2/\sigma^{-1}(u) & x_2/\sigma^{-2}(u) \end{pmatrix}.$$

The left kernel of C is spanned by $(x_2, -x_1)$. Hence, H_∞ is spanned by $x_2 dx_1 - x_1 dx_2$ over $\widehat{K}$. So (15) is not accessible.

Differentiating the equation $y = x_1^2 + x_1$ yields $dy = (2x_1 + 1)dx_1$, which, together with (16), enables us to express sdy and $s^2 dy$ as a linear combination of du, dx_1 and dx_2. Using the three linear relations to eliminate dx_1 and dx_2, we find that $Pdy = Qdu$ holds in Ω_Σ, where $P = s^2 - \frac{2x_1 u^2 \sigma(u) + u\sigma(u)}{2x_1 + 1}$ and

$$Q = (2x_1 u\sigma(u) + 1)x_1 us + (2x_1 u\sigma(u) + 1)x_1\sigma(u).$$

The skew-fraction $P^{-1}Q$ is called the transfer function of Σ ([15]). We find that $\text{gcld}(P, Q)$ has degree one by left-hand Euclidean algorithm in S_Σ. Hence, the reduced transfer function equals $\tilde{P}^{-1}\tilde{Q}$, where $\tilde{P} = s - \frac{x_2 u(2x_2 u + 1)}{x_1(2x_1 + 1)}$ and $\tilde{Q} = x_2 + 2ux_2^2$.

For this example, one can use algebraic Gröbner basis computation to get $x_1 = \frac{yu^2\sigma(u)^2 - \sigma^2(y)}{u\sigma(u)(u\sigma(u) - 1)}$ and $x_2 = \frac{\sigma(y)(\sigma(u))^2(\sigma^2(u))^2 - \sigma^3(y)}{u\sigma(u)\sigma^2(u)(\sigma(u)\sigma^2(u) - 1)}$. Therefore, the reduced transfer function can be expressed as a left-hand skew-fraction whose coefficients are rational functions in the input and output variables.

7.2 Input-output difference equations

Let $\mathbf{U} = \{u, \sigma(u), \ldots\}$ and $\mathbf{Y} = \{y, \sigma(y), \ldots\}$. For brevity, we put $\sigma^i(y) = y^{[i]}, \sigma^j(u) = u^{[j]}$ for $i, j \in \mathbb{Z}$. Consider a single-input single-output equation of the form

$$y^{[n]} = f\left(u^{[0]}, u^{[1]}, \ldots, u^{[n-1]}, y^{[0]}, y^{[1]}, \ldots, y^{[n-1]}\right). \tag{17}$$

We assume that (17) is submersive, and that $(\widehat{K}, \hat{\sigma})$ and (Ω, d) are the same as those in Section 6.

Following [19], we define the *relative degree* of $\omega \in \Omega$ to be

$$r = \min\left\{\ell \geq 0 \,|\, s^\ell \omega \notin G_1\right\},$$

where $G_1 = \text{span}_{\widehat{K}}\left\{dy^{[0]}, \ldots, dy^{[n-1]}, du^{[0]}, \ldots, du^{[n-1]}\right\}$. If such an integer does not exist, then set $r = \infty$. An element $\omega \in \Omega$ is called an *autonomous one-form* if its relative degree is ∞. Such a one-form and its shifts by the powers of s do not depend on $du^{[i]}$ for all $i \geq n$. An autonomous one-form ω is a torsion element, since the submodule $S\omega \subset G_1$ is a finite-dimensional linear space over $\widehat{K}$.

We say that (17) is *accessible* if Ω does not contain any nontrivial autonomous one-form, that is, Ω is torsion-free. The latter condition can be verified by determining whether P and Q given by (12) have a trivial gcld by Proposition 6.5.

The left-hand skew-fraction $P^{-1}Q$ is called the transfer function of (17) in [16]. To find its reduced form, one has to compute the gcld of P and Q. A Maple package is described in [26] for computing transfer functions. An idea for modular gcld-calculation in $\widehat{K}[s;\hat{\sigma}]$ is outlined in [22].

Example 7.2 *Consider a submersive equation*

$$y^{[4]} = y^{[0]}y^{[1]}u^{[2]} + u^{[0]}u^{[1]}y^{[3]} + u^{[0]}y^{[2]}u^{[3]} \qquad (18)$$

whose associated inversive field is denoted by $\widehat{K}$. Using (12), we find that $P\mathrm{d}y = Q\mathrm{d}u$ holds in the module Ω of Kähler differentials corresponding to the equation, where

$$
\begin{aligned}
P &= s^4 - u^{[0]}u^{[1]}s^3 - u^{[0]}u^{[3]}s^2 - y^{[0]}u^{[2]}s - y^{[1]}u^{[2]}, \\
Q &= u^{[0]}y^{[2]}s^3 + y^{[0]}y^{[1]}s^2 + u^{[0]}y^{[3]}s + u^{[1]}y^{[3]} + y^{[2]}u^{[3]}.
\end{aligned}
$$

To avoid using the left Euclidean algorithm to compute $\mathrm{gcld}(P,Q)$, which may cause intermediate expressions to swell, we use the notion of (right-hand side) Sylvester resultants. Let P_a and Q_a be the adjoint operators of P and Q, respectively (see, e.g. [1]). Note that P_a and Q_a are in $\widehat{K}[s;\hat{\sigma}^{-1}]$. Then $\mathrm{gcld}(P,Q) = 1$ iff the Sylvester resultant of P_a and Q_a is nonzero [21].

Form the Sylvester resultant D of the respective numerators of P_a and Q_a to get a 7×7 determinant whose entries are polynomials in $y^{[-6]}$, $y^{[-5]}$, $y^{[-4]}$, $y^{[-3]}$, $u^{[-6]}$, $u^{[-5]}$, ... over k. Computing the images of the entries of D modulo 2, and setting $u^{[i]} = 1$ and $y^{[j]} = 1$ for $i = -6, -5, \dots$, and $j = -6, \dots, -3$, we get a 7×7 nonzero determinant over $\mathbb{Z}_2$. So $\mathrm{gcld}(P,Q) = 1$, and (18) is accessible.

We cannot assert that D is zero when a homomorphic image of D vanishes. However, we may estimate the degree and coefficient bounds of D, and decide if it is zero using sufficiently many modular and evaluation homomorphisms.

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