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Some remarks on Kähler differentials and ordinary differentials in nonlinear control theory

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ABSTRACT

In the algebraic approach to nonlinear control systems two similar notions, namely Kähler differentials and the formal vector space of differential one-forms having the properties of ordinary differentials, are frequently used to study the systems. This technical note explains that the formal vector space of differential one-forms is isomorphic to a quotient space (module) of Kähler differentials. These two modules coincide when they are modules over a ring of linear differential operators over the field of algebraic functions. Some remarks and examples demonstrating when the use of Kähler differentials might not be appropriate are included.

1. Introduction

Modern theory of nonlinear control systems is nowadays represented mainly by the systematic use of differential algebraic methods. A large part of such methods uses the so-called "tangent linear system" associated to the nonlinear one, or in other ways employs the differentials of system variables to study the systems. There exist two main representatives. The first is given by the Fliess' school where the notion of Kähler differentials is frequently used, being very natural from a purely algebraic point of view. See for instance [1] and a number of references therein. The second approach, however not purely algebraic, is given mainly by the works of Conte, Moog and Perdon and co-authors. See for instance [2] and, again, a number of references therein. In those works a similar structure to Kähler differentials having, however, the properties of "ordinary differentials" is introduced and called the formal vector space of differential one-forms. Naturally, the question can be asked whether those two structures i.e., Kähler differentials and ordinary differentials, coincide or not and under what conditions, forming the main scope of this technical note. A simple case of nonlinear continuous-time systems is chosen to demonstrate the construction of the associated module of Kähler differentials and the formal vector space of differential one-forms. Then, their basic properties are discussed and demonstrated, resulting in the answer under what conditions those two structures coincide. Finally, a short discussion demonstrating the case when Kähler differentials might not be appropriate to study the systems is given.

2. Nonlinear systems and differentials

To demonstrate the constructions of Kähler differentials and the formal vector space of differential one-forms we consider a simple case of nonlinear continuous-time systems of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ and elements of f and g are from a field K. It can be, for instance, the field of rational, algebraic or even meromorphic functions. However, this will play a key role in deciding whether Kähler differentials and ordinary differentials associated to the system coincide or not.

Remark 2.1. The systems of the form (1) are considered here only as an illustrative case, while the results given in what follows hold in general.

2.1. Kähler differentials

Kähler differentials represent a generalization of the notion of a differential to an arbitrary ring. For simplicity, we consider Kähler differentials over fields.

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Let $k \subset K$ be two fields. Let Ψ be the (left) K-linear space generated by the symbols $\mathrm{d}a$ for all $a \in K$

$$\Psi = \mathrm{span}_K\{\mathrm{d}a \mid a \in K\}, \tag{2}$$

and Φ the subspace spanned by

$$\{\mathrm{d}c \mid c \in k\} \cup \{\mathrm{d}(a+b) - \mathrm{d}a - \mathrm{d}b \mid a, b \in K\} \\ \cup \{\mathrm{d}(ab) - a\mathrm{d}b - b\mathrm{d}a \mid a, b \in K\}.$$

Let $\Omega = \Psi/\Phi$, which is a K-linear space. By a slight abuse of notation, we denote the equivalence class $\mathrm{d}a + \Phi$ by $\mathrm{d}a$. Thus, in Ω, we have

$$\begin{aligned} \forall a, b \in K,\ \mathrm{d}(a+b) &= \mathrm{d}a + \mathrm{d}b \\ \mathrm{d}(ab) &= a\mathrm{d}b + b\mathrm{d}a, \\ \forall c \in k, \qquad \mathrm{d}c &= 0. \end{aligned} \tag{3}$$

We call Ω the space of Kähler k-differentials of K.

Remark 2.2. This definition is from [3] and equivalent to the one defined in [4] which, however, involves tensors.

Let V be a linear space over K. A map $\delta : K \to V$ is called a k-derivation if δ is k-linear and $\delta(ab) = a\delta(b) + b\delta(a)$ for all $a, b \in K$. Note that a k-derivation annihilates every element of k. Usually, $\delta(a)$ is denoted by $\dot{a}$ and, more generally, $\delta^{\nu}(a)$ by $a^{(\nu)}$.

Define a map $\mathrm{d} : K \to \Omega$ by sending a to $\mathrm{d}a$ for all $a \in K$. The map is a k-derivation from K to Ω due to (3).

Remark 2.3. The definition of Ω is for all the fields, including differential or difference fields. There is even no requirement on the characteristic of k.

Now, a nonlinear system, modeled by equations of the form (1), can be understood as a finitely generated differential field extension K/k, see [5]. The differential field k is called the base (or ground) field and contains the coefficients of the model (1) and the differential field K is the system field; i.e., it involves all the variables and all algebraic expressions that can be constructed, as nicely explained in [6]. Then, the "tangent linear system" is the (left) $K[\delta]$-module Ω of Kähler differentials associated to the system K/k, see [5], and can be used to study the systems of the form (1).

2.2. The formal vector space of differential one-forms: ordinary differentials

Let $\mathcal{C}$ denote the infinite set of symbols which is, typically, for a nonlinear system of the form (1) defined as $\mathcal{C} = \{x_1, \ldots, x_n, u_j^{(i)}, j = 1, \ldots, m, i \geq 0\}$. Let $\mathcal{K}$ be a field containing the field $k(\mathcal{C})$ of rational functions. Furthermore, we assume that each partial derivative $\partial/\partial z$ on $k(\mathcal{C})$ can be extended to a derivation on $\mathcal{K}$, and that, for every $r \in \mathcal{K}$, $\partial r/\partial z$ is nonzero for merely a finite number of $z \in \mathcal{C}$. For example, $\mathcal{K}$ is the field of algebraic functions over $k(\mathcal{C})$.

As another example, we let ℓ be a positive integer, and the first ℓ elements of $\mathcal{C}$ be the coordinate variables in $\mathbb{R}^{\ell}$, and let $\mathcal{K}_{\ell}$ be the field of meromorphic functions in $\mathbb{R}^{\ell}$. Then there is a chain of fields: $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots$. Hence, the union $\mathcal{K} = \cup_{\ell=0}^{\infty} \mathcal{K}_{\ell}$ is such a field.

The system (1) corresponds to the differential field $(\mathcal{K}, \delta)$ where $\mathcal{K} = \mathbb{R}(\mathcal{C})\,(S)$ with S being the set consisting of the elements of f and g, and their partial derivatives with respect to every element of $\mathcal{C}$. In other words, $\mathcal{K}$ is the (partial) differential field generated by the elements of f and g with respect to the derivations $\partial/\partial z$ for all $z \in \mathcal{C}$ over the field $\mathbb{R}(\mathcal{C})$ of rational functions. Moreover, we may introduce a derivation on $\mathcal{K}$ according to (1). Discussions on this point will be carried out in Section 4.

Let $\mathcal{E}$ be the $\mathcal{K}$-space generated by the symbols $\mathrm{d}_0 z$ for all $z \in \mathcal{C}$

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}_0 z \mid z \in \mathcal{C}\}. \tag{4}$$

We define a map $\mathrm{d}_0 : \mathcal{K} \to \mathcal{E}$ by

$$\mathrm{d}_0 r = \sum_{z \in \mathcal{C}} \frac{\partial r}{\partial z} \mathrm{d}_0 z \quad \text{for all } r \in \mathcal{K}. \tag{5}$$

Clearly, d_0 is a k-derivation from $\mathcal{K}$ to $\mathcal{E}$. The $\mathcal{K}$-space $\mathcal{E}$ is called the formal vector space of differential one-forms, see [2, Section 1.3.1], and has the properties of ordinary differentials.

Note that from this point of view, in comparison to what is stated at the end of Section 2.1, a nonlinear system, modeled by equations of the form (1), is still understood as a set of equations. Then, to study the system, different sequences of subspaces of $\mathcal{E}$ might be associated to (1), see for instance [2,7].

3. Discussion

Notice now the formal similarity of (2), resulting in Kähler differentials $\Omega = \Psi/\Phi$, and (4). Hence, the question can be asked whether there is any significant difference between those two definitions. Roughly speaking, the only one is that to differentiate functions we have to follow the rule (3) when working over Ω while we have to follow the rule (5) when working over $\mathcal{E}$. Thus, is it necessary to have two different notions here; i.e., Kähler differentials and ordinary differentials? We shed some light on this in the next illustrative example.

Example 3.1. Let $k = \mathbb{R}$ and $K = \mathbb{R}(x, r, \exp(x))$ with $r = \sqrt{x^2+1}$. Then

$$\Omega = \mathrm{span}_K\{\mathrm{d}x, \mathrm{d}r, \mathrm{d}\exp(x)\}.$$

Since $r^2 = x^2 + 1$, $\mathrm{d}r^2 = 2r\mathrm{d}r = 2x\mathrm{d}x$ by (3). Consequently, $\mathrm{d}r = (x/r)\mathrm{d}x$. But $\mathrm{d}\exp(x) = \exp(x)\mathrm{d}x$ *does not* hold in Ω. Indeed, (3) only enables us to differentiate algebraic functions over $\mathbb{R}(x)$ in the usual way. On the other hand

$$\mathcal{E} = \mathrm{span}_K\{\mathrm{d}_0 x, \mathrm{d}_0 r, \mathrm{d}_0 \exp(x)\},$$

and (5) clearly enables us to compute

$$\mathrm{d}_0 \exp(x) = \exp(x)\mathrm{d}_0 x.$$

There are three useful diagrams for Kähler differentials.

1. If D is a k-derivation from K to a K-vector space V, then there exists a unique K-linear map $\widehat{D}$ such that $D = \widehat{D} \circ \mathrm{d}$. So we have a commutative diagram:

$$\begin{tikzcd} K \arrow[r, "D"] \arrow[d, "\mathrm{d}"'] & V \\ \Omega \arrow[ur, "\widehat{D}"'] & \end{tikzcd} \tag{6}$$

2. If σ is an injective homomorphism from K to itself with $\sigma(k) \subset k$, then there exists a unique skew-linear map σ^* such that $\sigma^* \circ \mathrm{d} = \mathrm{d} \circ \sigma$, that is,

$$\begin{tikzcd} \Omega \arrow[r, "\sigma^*"] & \Omega \\ K \arrow[u, "\mathrm{d}"] \arrow[r, "\sigma"] & K \arrow[u, "\mathrm{d}"'] \end{tikzcd} \tag{7}$$

is commutative.

3. If D is a derivation on K with $D(k) \subset k$, then there exists a unique additive map D^* from Ω to itself such that

$$D^*(a\omega) = D(a)\omega + aD^*(\omega) \tag{8}$$

for all $a \in K$ and $\omega \in \Omega$, and the diagram

$$\begin{tikzcd} \Omega \arrow[r, "D^*"] & \Omega \\ K \arrow[u, "\mathrm{d}"] \arrow[r, "D"] & K \arrow[u, "\mathrm{d}"'] \end{tikzcd} \tag{9}$$

is commutative.

The above three conclusions follow immediately from Lemmas 9.1.1–9.1.3 in [8], respectively. They lead to three consequences given below.

(i) If k is of characteristic zero, then diagram (6) implies that $a_1, \ldots, a_n \in K$ are algebraically independent over k if and only if $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent over K.

(ii) Diagram (7) turns Ω into a left module over $K[s;\sigma]$ with scalar multiplication given by

$$s(a\mathrm{d}b) = \sigma^*(a\mathrm{d}b) = \sigma(a)\mathrm{d}\sigma(b).$$

(iii) Similarly, diagram (9) turns Ω into a left module over $K[s;D]$ with scalar multiplication defined as

$$s(a\mathrm{d}b) = D^*(a\mathrm{d}b) = a\mathrm{d}D(b) + D(a)\mathrm{d}b.$$

In this case, submodules correspond to linear subspaces closed under D^* in [7].

Example 3.2. Let us revisit Example 3.1. Recall that $K = \mathbb{R}(x, r, \exp(x))$ with $r = \sqrt{x^2+1}$, which is algebraic over $\mathbb{R}(x)$, and $\mathrm{d}r = (x/r)\mathrm{d}x$. Since x and $\exp(x)$ are algebraically independent over $\mathbb{R}$, $\mathrm{d}x$ and $\mathrm{d}\exp(x)$ are linearly independent over $\mathbb{R}(x, \exp(x))$. So $\dim_K \Omega = 2$. Let

$$\omega = \mathrm{d}\exp(x) - \exp(x)\mathrm{d}x.$$

Put $\overline{\Omega} = \Omega/\mathrm{span}_K\{\omega\}$, that is, $\overline{\Omega}$ is the quotient space of Ω modulo $\mathrm{span}_K\{\omega\}$. Then $\bar{\mathrm{d}} = \pi \circ \mathrm{d} : K \to \overline{\Omega}$ is an $\mathbb{R}$-derivation, where π is the canonical map from Ω to $\overline{\Omega}$ defined by $\mathrm{d}a \mapsto \mathrm{d}a + \mathrm{span}_K\{\omega\}$. We have that

$$\bar{\mathrm{d}}\exp(x) = \exp(x)\bar{\mathrm{d}}x$$

as taught at college.

4. Formal vector space of differential one-forms as a quotient space of Kähler differentials

In this section, we assume that $\mathcal{K}$ is a field defined at the beginning of Section 2.2 and construct $\mathcal{E}$ from Kähler differentials. Let Ω be the $\mathcal{K}$-space of Kähler k-differentials. The k-derivation $\mathrm{d} : \mathcal{K} \to \Omega$ sends r to $\mathrm{d}r$ for all $r \in \mathcal{K}$. Note that (5) does not hold for d.

For every $r \in \mathcal{K}$, let

$$\omega_r = \mathrm{d}r - \sum_{z\in\mathcal{C}} \frac{\partial r}{\partial z}\mathrm{d}z \quad \text{and} \quad M = \mathrm{span}_{\mathcal{K}}\{\omega_r \mid r \in \mathcal{K}\}. \tag{10}$$

Lemma 4.1. *Consider the partial derivative $\frac{\partial}{\partial z}$ with $z \in \mathcal{C}$, which is a k-derivation from $\mathcal{K}$ to $\mathcal{K}$. Let $\widehat{D}_z : \Omega \to \mathcal{K}$ be the $\mathcal{K}$-linear map such that diagram (6) commutes, i.e. $\frac{\partial}{\partial z} = \widehat{D}_z \circ \mathrm{d}$. Then $\widehat{D}_z$ annihilates every element of M.*

Proof. Assume that ω_r is given in (10). It suffices to show that $\widehat{D}_z(\omega_r) = 0$ for all $r \in \mathcal{K}$. We compute

$$\begin{aligned}\widehat{D}_z(\omega_r) &= \widehat{D}_z \circ \mathrm{d}(r) - \sum_{y\in\mathcal{C}} \frac{\partial r}{\partial y}\widehat{D}_z \circ \mathrm{d}(y) \\ &= \frac{\partial r}{\partial z} - \sum_{y\in\mathcal{C}} \frac{\partial r}{\partial y}\frac{\partial y}{\partial z} = 0.\end{aligned}$$

This completes our proof. □

We now find a $\mathcal{K}$-basis of M.

Lemma 4.2. *Let T be a transcendence basis of $\mathcal{K}$ over $k(\mathcal{C})$. Then*

$$\mathcal{B}_T = \{\omega_t \mid t \in T\}$$

is a $\mathcal{K}$-basis of M.

Proof. First, we show that $\mathcal{B}_T$ is linearly independent over $\mathcal{K}$. If $\sum_{t\in T} a_t\omega_t = 0$ with $a_t \in \mathcal{K}$, then the definition of ω_t implies that

$$0 = \sum_{t\in T} a_t\mathrm{d}t - \sum_{z\in\mathcal{C}} b_z\mathrm{d}z \quad \text{for some } b_z \in \mathcal{K}.$$

Hence, all the a_t and b_z are equal to zero, because $\mathcal{C} \cup T$ are algebraically independent over k. This proves the linear independence of $\mathcal{B}_T$.

Next, we show that the elements of $\mathcal{B}_T$ span M over $\mathcal{K}$. It suffices to show that, for every $r \in \mathcal{K}$, ω_r is a linear combination of the elements in $\mathcal{B}_T$ over $\mathcal{K}$. Since r is algebraic over $k(\mathcal{C}, T)$, there is an irreducible polynomial $P(\mathcal{C}, T, x)$ in $k[\mathcal{C}, T, x]$ with $\deg_x P > 0$ such that $P(\mathcal{C}, T, r) = 0$, where x is a new indeterminate. Differentiating $P(\mathcal{C}, T, r)$ yields

$$0 = \mathrm{d}P(\mathcal{C}, T, r) = \frac{\partial P}{\partial x}(\mathcal{C}, T, r)\mathrm{d}r + \sum_{t\in T}\frac{\partial P}{\partial t}\mathrm{d}t + \sum_{z\in\mathcal{C}}\frac{\partial P}{\partial z}\mathrm{d}z.$$

This relation can be rewritten as

$$\mathrm{d}r + \sum_{z\in\mathcal{C}} \tilde{b}_z\mathrm{d}z = \sum_{t\in T}\tilde{a}_t\omega_t \quad \text{for some } \tilde{a}_t, \tilde{b}_z \in \mathcal{K}, \tag{11}$$

because $\frac{\partial P}{\partial x}(\mathcal{C}, T, r)$ is nonzero and $\mathrm{d}t - \omega_t$ is a $\mathcal{K}$-linear combination of the dz's by (10).

For $y \in \mathcal{C}$, let $\widehat{D}_y$ be the $\mathcal{K}$-linear map such that $\frac{\partial}{\partial y} = \widehat{D}_y \circ \mathrm{d}$ (see diagram (6)). Applying $\widehat{D}_y$ to (11) yields $\frac{\partial r}{\partial y} + \tilde{b}_y = 0$ by Lemma 4.1. So

$$\mathrm{d}r + \sum_{z\in\mathcal{C}} \tilde{b}_z\mathrm{d}z = \mathrm{d}r - \sum_{z\in\mathcal{C}}\frac{\partial r}{\partial z}\mathrm{d}z = \omega_r.$$

Consequently, (11) becomes $\omega_r = \sum_{t\in T}\tilde{a}_t\omega_t$. □

Let $\bar{\mathrm{d}} = \pi \circ \mathrm{d}$, where π is the canonical projection from Ω to Ω/M defined by $\omega \mapsto \omega + M$ for all $\omega \in \Omega$. Then $\bar{\mathrm{d}}$ is a k-derivation from $\mathcal{K}$ to Ω/M. Since $\mathcal{C} \cup T$ is a transcendence basis of $\mathcal{K}$, the set

$$\{\mathrm{d}w \mid w \in \mathcal{C} \cup T\}$$

is a $\mathcal{K}$-basis of Ω. It follows from Lemma 4.2 that $\mathcal{B}_{\mathcal{C}} := \{\bar{\mathrm{d}}z \mid z \in \mathcal{C}\}$ is a $\mathcal{K}$-basis of Ω/M.

Since d_0, defined in (5), is a k-derivation from $\mathcal{K}$ to $\mathcal{E}$, diagram (6) implies that there is a $\mathcal{K}$-linear map $\widehat{\mathrm{d}}_0$ from Ω to $\mathcal{E}$ such that $\mathrm{d}_0 = \widehat{\mathrm{d}}_0 \circ \mathrm{d}$. This map is clearly surjective. Its kernel contains M by (5). Hence, $\widehat{\mathrm{d}}_0$ induces a $\mathcal{K}$-linear map

$$\begin{array}{rccl} \phi: & \Omega/M & \to & \mathcal{E} \\ & \bar{\mathrm{d}}z & \mapsto & \mathrm{d}_0 z \quad \text{for all } z \in \mathcal{C}. \end{array} \tag{12}$$

The induced map is bijective because it maps a $\mathcal{K}$-basis of Ω/M onto a $\mathcal{K}$-basis of $\mathcal{E}$. Consequently, the kernel of $\widehat{\mathrm{d}}_0$ is equal to M. The above discussion leads to

Proposition 4.3. *Let Ω be the linear space of Kähler k-differentials, $\mathcal{E}$ the formal vector space of differential one-forms, and M defined in (10). Then ϕ defined in (12) is the unique $\mathcal{K}$-linear isomorphism from Ω/M to $\mathcal{E}$ such that the diagram*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\mathrm{d}_0"] \arrow[d, "\bar{\mathrm{d}}"'] & \mathcal{E} \\ \Omega/M \arrow[ur, "\phi"'] & \end{tikzcd} \tag{13}$$

is commutative, where $\bar{\mathrm{d}} = \pi \circ \mathrm{d}$ with π being the natural projection from Ω to Ω/M.

We say that ϕ in this proposition is the *canonical* $\mathcal{K}$*-linear isomorphism* from Ω/M to $\mathcal{E}$.

We now impose a module structure on $\mathcal{E}$. Assume that $\delta : \mathcal{K} \to \mathcal{K}$ is a derivation on $\mathcal{K}$ satisfying the following chain rule:

$$\delta(r) = \sum_{z\in\mathcal{C}} \frac{\partial r}{\partial z}\delta(z). \tag{14}$$

Many derivations on $\mathcal{K}$ satisfy this chain rule. For example, the partial derivative $\partial/\partial y$ for $y \in \mathcal{C}$ is equal to δ when we set $\delta(y) = 1$ and $\delta(z) = 0$ for all $z \neq y$ in (14).

Assume that the arguments of elements f and g in (1) are contained in a finite subset of $\mathcal{C}$. Let x and f be vectors defined in (1). According to (1), we define a derivation δ acting on $\mathcal{K}$ as $\delta x = f(x, u)$, and $\delta u^{(k)} = u^{(k+1)}$ for $k \geq 0$. It is straightforward to verify that δ is well defined. By (8) and (9), δ induces a unique map $\theta : \Omega \to \Omega$ such that

(a) $\theta(a\omega) = \delta(a)\omega + a\theta(\omega)$ for all $a \in \mathcal{K}$ and $\omega \in \Omega$; and
(b) $\theta \circ \mathrm{d} = \mathrm{d} \circ \delta$.

Lemma 4.4. *$\theta(M) \subset M$, where M is defined in (10).*

Proof. Using (14), we compute

$$\begin{aligned}
\theta(\omega_r) &= \theta\left(\mathrm{d}r - \sum_{z\in\mathcal{C}} \frac{\partial r}{\partial z}\mathrm{d}z\right) \\
&= \mathrm{d}\delta(r) - \sum_{z\in\mathcal{C}} \delta\left(\frac{\partial r}{\partial z}\right)\mathrm{d}z - \sum_{z\in\mathcal{C}} \frac{\partial r}{\partial z}\mathrm{d}\delta(z) \\
&= \sum_{z\in\mathcal{C}} \delta(z)\mathrm{d}\frac{\partial r}{\partial z} - \sum_{z\in\mathcal{C}} \delta\left(\frac{\partial r}{\partial z}\right)\mathrm{d}z \\
&= \sum_{z\in\mathcal{C}} \delta(z)\mathrm{d}\frac{\partial r}{\partial z} - \sum_{z\in\mathcal{C}} \left(\sum_{y\in\mathcal{C}} \frac{\partial^2 r}{\partial y\partial z}\delta(y)\right)\mathrm{d}z \\
&= \sum_{z\in\mathcal{C}} \delta(z)\mathrm{d}\frac{\partial r}{\partial z} - \sum_{z\in\mathcal{C}} \left(\sum_{y\in\mathcal{C}} \frac{\partial^2 r}{\partial y\partial z}\mathrm{d}y\right)\delta(z) \\
&= \sum_{z\in\mathcal{C}} \delta(z)\left(\mathrm{d}\frac{\partial r}{\partial z} - \sum_{y\in\mathcal{C}} \frac{\partial^2 r}{\partial y\partial z}\mathrm{d}y\right) \\
&= \sum_{z\in\mathcal{C}} \delta(z)\omega_{\frac{\partial r}{\partial z}}.
\end{aligned}$$

Hence, $\theta(\omega_r)$ is in M. $\square$

Let $\mathcal{S}$ be the ring $\mathcal{K}\langle\Delta\rangle$ with the commutation rule: $\Delta r = r\Delta + \delta(r)$ for all $r \in \mathcal{K}$. Then Ω is a module over $\mathcal{S}$ whose left multiplication is defined as

$$\Delta\omega = \theta(\omega) \quad \text{for all } \omega \in \Omega.$$

The two properties of θ listed above ensure that Ω is a well-defined left module over $\mathcal{S}$. By Lemma 4.4, M is a submodule over $\mathcal{S}$. Therefore, Ω/M is also a module over $\mathcal{S}$. Consequently, the left multiplication by Δ induces an additive map $L_\Delta : \Omega/M \to \Omega/M$ with $L_\Delta(\bar{\omega}) = \Delta\bar{\omega}$ for all $\bar{\omega} \in \Omega/M$. Moreover,

(a′) $L_\Delta(a\bar{\omega}) = \delta(a)\bar{\omega} + aL_\Delta(\bar{\omega})$; and
(b′) $L_\Delta \circ \bar{\mathrm{d}} = \bar{\mathrm{d}} \circ \delta$,

where $\bar{\mathrm{d}}$ is defined in Proposition 4.3.

Let $\delta^* = \phi \circ L_\Delta \circ \phi^{-1}$, where ϕ is the canonical $\mathcal{K}$-linear isomorphism from Ω/M to $\mathcal{E}$. Then the properties (a′) and (b′) imply that

(a″) $\delta^*(a\epsilon) = \delta(a)\epsilon + a\delta^*(\epsilon)$ for all $\epsilon \in \mathcal{E}$, and
(b″) $\delta^* \circ \mathrm{d}_0 = \mathrm{d}_0 \circ \delta$.

These two properties lead to a coordinate-wise formula:

$$\delta^*\left(\sum_{z\in\mathcal{C}} v_z\mathrm{d}_0 z\right) = \sum_{z\in\mathcal{C}} \left(\delta(v_z)\mathrm{d}_0 z + v_z\mathrm{d}_0\delta(z)\right),$$

where the coefficients v_z's are elements in $\mathcal{K}$ and almost all zeros.

The additive operator δ^* induces a left $\mathcal{S}$-module structure on $\mathcal{E}$ via $\Delta\epsilon = \delta^*(\epsilon)$ for all $\epsilon \in \mathcal{E}$. Therefore, ϕ is an $\mathcal{S}$-module isomorphism from Ω/M to $\mathcal{E}$ due to the following easy verification:

$$\phi(\Delta\bar{\omega}) = \phi \circ L_\Delta(\bar{\omega}) = \delta^* \circ \phi(\bar{\omega}) = \Delta\phi(\bar{\omega})$$

for all $\bar{\omega} \in \Omega/M$. This essential result is embodied in the following concluding theorem.

Theorem 4.5. *With the notation introduced in Proposition 4.3, assume that $\delta : \mathcal{K} \to \mathcal{K}$ is a derivation satisfying the chain rule (14). Then*

(i) *The canonical $\mathcal{K}$-linear isomorphism ϕ is an $\mathcal{S}$-module isomorphism from Ω/M to $\mathcal{E}$.*
(ii) *If we set $\mathcal{K}$ to be the field of algebraic functions over $k(\mathcal{C})$, then $\mathcal{E}$ and Ω are isomorphic as $\mathcal{S}$-modules.*

Proof. The first assertion follows from the above discussion. If $\mathcal{K}$ is the field of algebraic functions over $k(\mathcal{C})$, then $M = \{0\}$ because $\mathcal{B}_T$ defined in Lemma 4.2 is empty. The second conclusion follows. $\square$

5. Further remarks

First, the principal conclusion is the obvious one that Kähler differentials are not suited for systems that include transcendental functions. However, it does not mean they can only be defined for algebraic functions. In fact, Kähler differentials can be defined whenever there are two commutative rings $S \subset R$. In this technical note we restricted our attention to the case in which $k \subset K$ are two fields. Thus, the right question to ask is not whether they can or cannot be defined for more general fields (or even rings) but how they behave. It was shown that they behave like usual differential one-forms if $k = \mathbb{R}$ and K is a subfield of the field of algebraic functions over $\mathbb{R}$.

Remark 5.1. Note that if k is a finite field then the usual differentiation is not defined. Nevertheless, it would be expected that Kähler differentials might be considered as a proper tool even to analyze dynamic systems over finite fields. Hence, the Kähler differential can be understood as a universal derivation.

Second, it seems once we have a control system described by means of algebraic functions the choice is clear. However, in dealing with different control problems of nonlinear systems there might occur problems more involved than it could be seen at first sight. For instance, some suitable quotient modules of Kähler differentials are needed. We demonstrate this by the next example dealing with the realization problem.

Example 5.2. Consider the nonlinear system from [9] described by the algebraic equation $\ddot{y}^2 + \dot{y}^2\dot{u}^2 - \dot{u}^2 = 0$. In the realization problem our aim is to find, if possible, an observable state-space realization of the form (1). To solve the problem we can use the approach of [2] to the input–output equation, (locally) expressed as

$$\ddot{y} = \dot{u}\sqrt{1 - \dot{y}^2}. \tag{15}$$

One associates to (15) an extended system of the form (1), namely $\dot{z} = f(z, v)$, $y = z_1$ with the (extended) states $(z_1, z_2, z_3, z_4) = (y, \dot{y}, u, \dot{u})$ and the new input $v = \ddot{u}$. That is

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_4\sqrt{1-z_2^2}\\ \dot{z}_3 &= z_4\\ \dot{z}_4 &= v\\ y &= z_1.\end{aligned} \tag{16}$$

Note that the Eq. (15) (or alternatively, (16)) defines the field (K, δ) where

$$K = \mathbb{R}\left(y, \dot{y}, \sqrt{1-\dot{y}^2}, u, \dot{u}, \ddot{u}, \ldots\right)$$

and $\delta(\dot{y}) = \dot{u}\sqrt{1-\dot{y}^2}$. Let (Ω, d) be the K-vector space of Kähler $\mathbb{R}$-differentials. Then by diagram (9) δ induces an additive map $\theta : \Omega \to \Omega$ that satisfies (8). Let $\mathcal{E}$ be the corresponding K-vector space of one-forms. Then δ induces an additive map $\delta^* : \mathcal{E} \to \mathcal{E}$ that satisfies property (a″) in Section 4.

Now, the filtration of subspaces of $\mathcal{E}$ is defined as

$$\mathcal{H}_1 = \mathrm{span}_K\{\mathrm{d}_0 z_1, \ldots, \mathrm{d}_0 z_4\}$$

and

$$\mathcal{H}_{i+1} = \mathrm{span}_K\{\omega \in \mathcal{H}_i \mid \delta^*\omega \in \mathcal{H}_i\}.$$

Finally, there exists an observable state-space realization of the form (1) iff $\mathcal{H}_3$ is integrable, see [2] for more details. Thus, we get

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_K\{\mathrm{d}_0 y, \mathrm{d}_0\dot{y}, \mathrm{d}_0 u, \mathrm{d}_0\dot{u}\}\\ \mathcal{H}_2 &= \mathrm{span}_K\{\mathrm{d}_0 y, \mathrm{d}_0\dot{y}, \mathrm{d}_0 u\}\\ \mathcal{H}_3 &= \mathrm{span}_K\{\mathrm{d}_0 y, \mathrm{d}_0\dot{y} - \sqrt{1-\dot{y}^2}\mathrm{d}_0 u\}.\end{aligned}$$

Note that since the input–output equation is algebraic the sequence of the subspaces $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3$ is isomorphic to the sequence of subspaces of Kähler differentials $\Omega_1 \supset \Omega_2 \supset \Omega_3$ with $\Omega_1 = \mathrm{span}_K\{\mathrm{d}z_1, \ldots, \mathrm{d}z_4\}$ and $\Omega_{i+1} = \mathrm{span}_K\{\omega \in \Omega_i \mid \theta\omega \in \Omega_i\}$. However, even if $\mathcal{H}_3$ is (by Frobenius theorem) integrable there do not exist any algebraic functions r_1, r_2 such that $\mathcal{H}_3 = \mathrm{span}_K\{\mathrm{d}_0 r_1, \mathrm{d}_0 r_2\}$. Clearly, there only exist analytic functions $r_1 = y$, $r_2 = \arcsin\dot{y} - u$ and give us the state-space realization

$$\begin{aligned}\dot{x}_1 &= \sin(x_2 + u)\\ \dot{x}_2 &= 0\\ y &= x_1\end{aligned}$$

where again an analytic function appears. In addition, note that the system is not accessible (controllable) and, hence, the realization is not minimal. Since the system equations involve now analytic functions further analysis by means of "tangent linear system" would require the use of the formal vector space $\mathcal{E}$ of differential one-forms, respectively some suitable quotient modules of Kähler differentials.

Finally, note that the minimal realization of the system is

$$\begin{aligned}\dot{x} &= \sin u\\ y &= x\end{aligned} \tag{17}$$

and is related to the system discussed in [10].

Obviously, the differential field of algebraic functions (K, δ) is, by definition, closed under the derivation δ, but not under the operation of integration. Therefore, anytime the back integration is required as an in-between step in solution to a control problem the use of Kähler differentials might not be appropriate. For there is no guarantee of involving algebraic functions only anymore.

On the other side, for the reasons depicted in Example 5.2 systems like (17) are so-called "transformally algebraic". However, contrary to continuous-time case a system like $x(t+1) = \sin u(t)$ is not transformally algebraic, see [11], and in this respective case the use of the formal vector space of differential one-forms would be a natural choice. But this does not mean that to study the nonlinear discrete-time systems the formal vector space of differential one-forms is always necessary. If the attention is restricted to, for instance, rational difference systems the algebraic setting can naturally be built up by employing Kähler differentials only, see for instance [12].

6. Conclusions

In this technical note it was shown that the formal vector space of differential one-forms, having the properties of ordinary differentials, can be constructed from Kähler differentials. Namely, it is isomorphic to a quotient space (module) of Kähler differentials. These two spaces coincide when they are over the field of algebraic functions.

The existence of two different notions is, therefore, relevant in the study of nonlinear control systems. However, in case the attention is restricted to polynomial, rational or algebraic functions it is not necessary to introduce the formal vector space of differential one-forms, for it coincides with Kähler differentials which represent a natural choice in a purely algebraic setting. On the other side, the introduction of the formal vector space of differential one-forms is relevant, for instance, in case of meromorphic functions, or in cases the considered systems are not transformally algebraic. In addition, the use of the formal vector space of differential one-forms might also be relevant in cases the back integration is required as an in-between step in solving nonlinear control problems. This was demonstrated by the realization problem where the system described by the algebraic equation has the realization (either minimal or not) involving transcendental functions.

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