Complete Reduction for Derivatives in a Primitive Tower*

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Abstract

A complete reduction ϕ for derivatives in a differential field is a linear operator on the field over its constant subfield. The reduction enables us to decompose an element f as the sum of a derivative and the remainder $\phi(f)$. A direct application of ϕ is that f is in-field integrable if and only if $\phi(f) = 0$.

In this paper, we present a complete reduction for derivatives in a primitive tower algorithmically. Typical examples for primitive towers are differential fields generated by (poly-)logarithmic functions and logarithmic integrals. Using remainders and residues, we provide a necessary and sufficient condition for an element from a primitive tower to have an elementary integral, and discuss how to construct telescopers for non-D-finite functions in some special primitive towers.

CCS Concepts

• **Computing methodologies → Algebraic algorithms**.

Keywords

Additive decomposition, Complete reduction, Elementary integral, Symbolic integration, Telescoper

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1 Introduction

Let V be a linear space and U a subspace of V. A linear operator ϕ on V is called a *complete reduction* for U if $v - \phi(v) \in U$ for all $v \in V$ and $U = \ker(\phi)$ by [23, Definition 5.67]. Such an operator ϕ is an idempotent and results in $V = U \oplus \operatorname{im}(\phi)$.

Let K be a differential field with derivation $'$ and C be the subfield of constants in K. For $L \subset K$, $L' := \{l' \mid l \in L\}$. Then L' is a C-subspace. For a complementary subspace R for K', the projection from K to R is a complete reduction for K'. So there always exist complete reductions for K'. It remains

(1) to fix a complementary subspace R for K', and
(2) to develop an algorithm that, for every $f \in K$, computes $g \in K$ and $r \in R$ such that $f = g' + r$.

In general, both K' and R are infinite-dimensional.

Example 1.1. *Let C be a field of characteristic zero, and $'$ be the usual derivation d/dx on $C(x)$. A complementary subspace R for $C(x)'$ is the set of proper rational functions with squarefree denominators. For every $f \in C(x)$, the Hermite-Ostrogradsky reduction on [8, page 40] computes $(g, r) \in C(x) \times R$ such that $f = g' + r$. The projection from $C(x)$ to R is a complete reduction for $C(x)'$.*

Our work is motivated by reduction-based creative telescoping (see [23, §5.6] and [31, §15]) and integration (summation) in finite terms (see [8, 22, 28, 29, 32, 33]). Both need preprocessors to split an integrand (summand) as the sum of an integrable (summable) part and a possibly non-integrable (non-summable) part.

A commonly-used preprocessor in reduction-based creative telescoping is also known as an additive decomposition, which can be described in terms of linear algebra below.

Let V and U be the same as those in the first paragraph. For an element $v \in V$, an additive decomposition for U computes $u \in U$ and $r \in V$ such that $v = u + r$, where r is minimal in some sense. And $v \in U$ if and only if $r = 0$. It is proposed for constructing minimal telescopers in [2–4, 24], in which V is the $C(x, y)$-subspace spanned by a hypergeometric term in x and y, and U is the C-subspace $\{g(x, y + 1) - g(x, y) \mid g \in V\}$. Additive decompositions also appear in [11, 19], in which V is a primitive tower of some special kinds, and U consists of all derivatives in V.

A complete reduction is interpreted as an additive decomposition in [21, §1.2] as follows. Let ϕ be a complete reduction for U on V, G be a basis of U, and H be a basis of $\operatorname{im}(\phi)$. Then $G \cup H$ is a basis of V. For every $v \in V$, $v = \sum_{w \in G \cup H} c_w w$ with $c_w \in C$. Define $\operatorname{supp}(v) = \{w \in G \cup H \mid c_w \neq 0\}$. For $v_1, v_2 \in V$, we say that v_1 is not higher than v_2 if $\operatorname{supp}(v_1) \subseteq \operatorname{supp}(v_2)$. If $v = u + r = \tilde{u} + \tilde{r}$ for some $u, \tilde{u} \in U$, $r \in \operatorname{im}(\phi)$ and $\tilde{r} \in V$, then $\operatorname{supp}(r) \subseteq \operatorname{supp}(\tilde{r})$ by an easy linear-algebra argument. Thus, r is not higher than $\tilde{r}$.

Additive decompositions do not always induce linear maps. So they are not necessarily complete reductions. Since linearity brings a lot of convenience into both theory and practice, it is worthwhile to seek complete reductions. So far complete reductions have been developed for hyperexponential functions [5], algebraic functions [12, 15], fractions of differential polynomials [7], fuchsian D-finite functions [16] and D-finite functions [6, 13, 35].

A classical topic in symbolic integration is to compute elementary integrals of transcendental Liouvillian functions (see [8, 17, 26, 28, 34]). Results about this topic are usually described in monomial extensions (see [8, §3.4]).

Let K and C be given in the second paragraph, and t be a monomial over K (see [8, Definition 3.4.1]). The monomial extension $K(t)$ contains three C-subspaces highly relevant to integration. They are: $K(t)'$ consisting of all derivatives in $K(t)$, S_t consisting of proper fractions whose denominators are normal polynomials, and W_t consisting of elements whose denominators are coprime with every normal polynomial (see [8, Definition 3.4.2]). Algorithm HermiteReduce in [8, §5.3] decomposes an element f of $K(t)$ as the sum of a derivative, an element s of S_t and an element w of W_t.

Assume further that t is either primitive or hyperexponential (see [8, Definition 5.1.1]) and that $K(t)$ and K have the same constants. One tries to integrate s by the residue criterion [27, Theorem 3], and w by solving parametric Risch equations [29, Main Theorem] and the parametric logarithmic derivative problem [8, §7.3]. This approach results in an algorithm for deciding in-field integrability in arbitrary primitive towers (see Definition 4.1). The algorithm may be turned into an additive decomposition.

To develop a complete reduction, we take a different approach to handling elements in W_t. The approach proceeds in three steps:

1. Define an auxiliary subspace A such that $W_t = W_t' + A$.
2. Determine a basis of $W_t' \cap A$.
3. Fix a complement of W_t' in W_t by the above basis.

The projection from W_t to the complement is a complete reduction for W_t', which, together with Algorithm HermiteReduce, leads to a complete reduction for derivatives in $K(t)$.

We prefer to work out all the details for the case, in which t is a primitive monomial, although our approach is likely applicable to other cases (see [10, 21]). This is because the approach for primitive monomials does not lead to any complicated case distinction, which seems unavoidable in other cases (see [5, Section 4.1]).

The auxiliary reduction (Algorithm 3.4) developed in step 1 and construction of a basis for $W_t' \cap A$ in step 2 benefit from the way of using integration by parts to reduce polynomial integrands in [11, 19], while the key lemma (Lemma 3.6) for step 2 is based on not only integration by parts but also the fact that the parametric Risch equation in our case is of the form $y' = ct' + a$, where $a \in K$ is given, and $(y, c) \in K \times C$ is to be determined. If a complete reduction $\phi : K \to K$ for K' is available, then $t' = u' + \phi(t')$ and $a = v' + \phi(a)$ for some $u, v \in K$. An application of ϕ to the above equation yields $c\phi(t') + \phi(a) = 0$. Thus, c is determined, and y can be taken as $cu + v$ when $\phi(a)\phi(t')^{-1}$ is a constant. There is no need to solve any limited integration problem [8, §7.2]. Algorithm 3.12 developed in step 3 is a dual technique for representing a subspace by the intersection of kernels of linear functions.

In this paper, we develop a complete reduction for derivatives in primitive towers by the above approach. The reduction leads to an algorithm for determining in-field integrability (see Examples 4.5 and 4.6), and can be applied to compute elementary integrals over such towers (see Example 5.4). We also construct telescopers for some non-D-finite functions by the reduction (see Example 5.7).

The rest of this paper is organized as follows. In Section 2, we specify notation and present several algorithms to be used in the sequel. Basic constructions in the above three steps are described in Section 3. The constructions yield an algorithm for our complete reduction, as soon as the notion of primitive towers is introduced in Section 4. Some applications of the complete reduction are presented in Section 5. Concluding remarks are given in Section 6.

2 Preliminaries

This section has three parts. In Section 2.1, we introduce some basic notions concerning symbolic integration and fix notation to be used. Effective bases are defined and constructed in Section 2.2. They allow us to apply a dual technique in linear algebra. In Section 2.3, we review an algorithm in the proof of [26, Theorem 3.9], which helps us compute elementary integrals in Section 5.

2.1 Notation and rudimentary notions

Throughout the paper, $G^\times$ denotes $G \setminus \{0\}$ for an additive group $(G, +, 0)$. For $n \in \mathbb{N}$, the sets $\{1, \ldots, n\}$ and $\{0, 1, \ldots, n\}$ are denoted by $[n]$ and $[n]_0$, respectively. The transpose of a matrix is denoted by $(\cdot)^\tau$. Comments in an algorithm are placed between (* $\cdots$ *).

All fields are of characteristic zero in the paper. Let K be a field. We denote its algebraic closure by $\overline{K}$. For a univariate polynomial p over K, its degree and leading coefficient are denoted by $\deg(p)$ and $\operatorname{lc}(p)$, respectively, when the indeterminate is clear from context. In particular, $\deg(0) := -\infty$ and $\operatorname{lc}(0) := 0$. Similarly, a univariate rational function is said to be *proper* if the degree of its numerator is less than that of its denominator. A rational function r can be uniquely written as the sum of a polynomial and a proper rational function, which are denoted by $\operatorname{poly}(r)$ and $\operatorname{proper}(r)$, respectively.

A map $' : K \to K$ is called a *derivation* on K if $(a+b)' = a' + b'$ and $(ab)' = ab' + a'b$ for all $a, b \in K$. A *differential field* is a field equipped with a derivation. Let $(K, ')$ be a differential field. An element c of K is called a *constant* if $c' = 0$. All constants in K form a subfield. A differential field (E, δ) is called a *differential field extension* of $(K, ')$ if K is a subfield of E and $'$ is the restriction of δ to K. We still use $'$ to denote δ when there is no confusion.

Assume that t belongs to a differential field extension of K. If t is transcendental over K and $t' \in K[t]$, then t is called *a monomial* over K and $K(t)$ is called a *monomial extension* of K.

Let t be a monomial over K. A polynomial $p \in K[t]^\times$ is said to be *normal* if $\gcd(p, p') = 1$. An element f of $K(t)$ is said to be *simple* if it is proper and has a normal denominator. The subset consisting of all simple elements is denoted by S_t, which is a K-subspace. Note that f is simple if it has a normal denominator in [8, Definition 3.5.2]. We further require that f is proper for the uniqueness of s in

(1) given below. We call t a *primitive monomial* over K if $t' \in K \setminus K'$. A primitive monomial extension $K(t)$ has no new constant other than the constants in K by [8, Theorem 5.1.1].

Let t be a primitive monomial over K. For every $f \in K(t)$, there exists $g \in K(t)$, $p \in K[t]$ and a unique $s \in S_t$ such that

$$f = g' + p + s. \tag{1}$$

The uniqueness of s is due to [11, Lemma 2.1]. Algorithm HermiteReduce in [8, §5.3] computes a triplet $(g, p, s) \in K(t) \times K[t] \times S_t$ such that the above equation holds.

Example 2.1. *Let $K = C(x)$, $t = \log(x)$ and*

$$f = \frac{(x+1)t^2 + (x^2+2x+2)t + x + 1}{x(t+1)} \in K(t).$$

HermiteReduce(f) *finds $(g, p, s) \in K(t) \times K[t] \times S_t$ such that* (1) *holds, where $g = 0$, $p = \frac{x+1}{x}t + \frac{x^2+x+1}{x}$, and $s = -\frac{x}{t+1}$. Unfortunately, the algorithm does not extract any in-field integrable part from f. It will be shown that $p \in K(t)'$ in Example 3.15.*

The next lemma presents two properties concerning decomposition and contraction in primitive monomial extensions. They play an important role in the proof of our main result (Theorem 3.13).

Lemma 2.2. *If t is a primitive monomial over K, then*

(i) $K(t) = (K(t)' + K[t]) \oplus S_t$, *and*
(ii) $K(t)' \cap K[t] = K[t]'$.

Proof. (i) holds by (1), and (ii) holds because the derivative of a proper element of $K(t)$ remains proper. □

2.2 Effective bases

This section is a preparation for a dual technique to be used in Sections 3 and 4.

Definition 2.3. *Let E be a field with a subfield F, Θ be an F-linear basis of E, $\theta \in \Theta$ and $a \in E$. Then*

(i) *θ^* stands for the F-linear function on E that maps θ to 1 and any other element of Θ to 0.*
(ii) *θ is said to be* effective *for a if $\theta^*(a) \neq 0$.*
(iii) *Θ is called an* effective F-basis *if there are two algorithms :*
  - *one finds $\theta \in \Theta$ effective for a if $a \neq 0$; and*
  - *the other computes $\theta^*(a)$.*

Let F be a field and $F(y)$ the field of rational functions in y. Set $Y = \left\{y^i \mid i \in \mathbb{N}\right\}$ and Q to be the set consisting of monic and irreducible polynomials with positive degrees. Then

$$\Theta = Y \cup \left\{\frac{y^i}{q^j} \mid q \in Q, 0 \le i < \deg(q), j \in \mathbb{Z}^+\right\} \tag{2}$$

is an effective F-basis of $F(y)$ by the irreducible partial fraction decomposition. The two algorithms required in Definition 2.3 (iii) are given below. Their correctness is evident.

Algorithm 2.4. BasisElement
Input: *$a \in F(y)^\times$* Output: *$(\theta, c) \in \Theta \times C^\times$ with $c = \theta^*(a)$*

1. $p \leftarrow \text{poly}(a)$, $r \leftarrow \text{proper}(a)$, $d \leftarrow$ *the denominator of r*
2. if $p \neq 0$ then return $\left(y^{\deg(p)}, \text{lc}(p)\right)$ end if
3. $q \leftarrow$ *a factor of d in Q*, $m \leftarrow$ *the multiplicity of q in d*
4. $h \leftarrow$ *the coefficient of q^{-m} in the q-adic expansion of r*
5. return $\left(y^{\deg(h)}/q^m, \text{lc}(h)\right)$

Remark 2.5. *There is no obvious rule for choosing an irreducible factor q of d in step 3 of Algorithm 2.4. For example, let $f = \frac{1}{y(y+1)}$. One may set q to be either y or $y+1$. Then θ obtained in step 5 may be either $\frac{1}{y}$ or $\frac{1}{y+1}$. So the algorithm does not guarantee that the same output will be returned when it is applied to the same input twice.*

In practice, we choose q to be the first member in the list of irreducible factors of d computed by a factorization algorithm.

Algorithm 2.6. Coefficient
Input: *$(b, \theta) \in F(y) \times \Theta$* Output: *$\theta^*(b)$*

1. $p \leftarrow \text{poly}(b)$, $r \leftarrow \text{proper}(b)$
2. *Write $\theta = y^k/q^m$ for some $k, m \in \mathbb{N}$, $q \in Q$, $\gcd(y, q) = 1$*
3. if $m = 0$ then return *the coefficient of y^k in p* end if
4. $h \leftarrow$ *the coefficient of q^{-m} in the q-adic expansion of r*
5. return *the coefficient of y^k in h*

Remark 2.7. *Let F and E be given in Definition 2.3 and C a subfield of F. Assume that F has an effective C-basis Θ_0 and that E has an effective F-basis Θ. Then $\{\theta_0\theta \mid \theta_0 \in \Theta_0, \theta \in \Theta\}$ is an effective C-basis of E by a straightforward recursive argument.*

2.3 Constant residues

Let $(K, ')$ be a differential field with constant subfield C, and t be a monomial over K. For $f \in S_t$ and $\alpha \in \overline{K}$, an element $\beta \in \overline{K}$ is the *residue* of f at α if and only if $f = g + \beta\frac{(t-\alpha)'}{t-\alpha}$ for some $g \in \overline{K}(t)$ whose denominator is coprime with $t - \alpha$. The residue of f at α is nonzero if and only if α is a root of its denominator.

Below is a minor variant of an algorithm described in the proof of [26, Theorem 3.9]. In its pseudo-code, D_t stands for the derivation on $K(t)$ that maps every element of K to 0 and t to 1, and κ for the coefficient-lifting derivation from $(K, ')$ to $K(t)$ (see [8, §3.2]).

Algorithm 2.8. ConstantMatrix
Input: *$f, g_1, \cdots, g_l \in S_t$*
Output: *$M \in C^{k\times l}$ and $\mathbf{v} \in C^k$ such that all residues of $f - \sum_{i=1}^l c_i g_i$ belong to $\overline{C}$ if and only if $M\left(c_1, \cdots, c_l\right)^\tau = \mathbf{v}$*

1. $h \leftarrow f - c_1 g_1 - \cdots - c_l g_l$,
   where $c_1, \ldots, c_l$ are constant indeterminates
2. $p \leftarrow$ *the numerator of h*, $q \leftarrow$ *the denominator of h*
3. $(u, v) \leftarrow$ *the respective inverses of $(q', D_t(q))$ mod q*
   $w \leftarrow \kappa(pu) - D_t(pu) \cdot v \cdot \kappa(q)$
   $r \leftarrow$ *remainder of w on division by q*
4. $(M, \mathbf{v}) \leftarrow$ *an augmented matrix of the linear system in $c_1, \ldots, c_l$ obtained by setting $r = 0$*
5. return $M, \mathbf{v}$

To see its correctness, we note that q obtained from step 2 is normal and free of $c_1, \ldots c_l$. Then $\gcd(q', q) = \gcd(D_t(q), q) = 1$. Therefore, both u and v can be computed in step 3. Let α be a root of q. Then $\alpha' = -v(\alpha) \cdot \kappa(q)(\alpha)$ by [8, Theorem 3.2.3]. On the other hand, the residue β of h at α is equal to $(pu)(\alpha)$ by [8, Lemma 4.4.2]. Then $\beta' = w(\alpha)$, where w is also computed in step 3. Therefore, $r = 0$ if and only if all residues of h belong to $\overline{C}$. The system obtained in step 4 is linear because $c_1, \ldots, c_l$ appear linearly in the coefficients of r.

3 Basic constructions

In this section, we let $(K, ')$ be a differential field and C be the subfield of its constants. Assume that there exists a complete reduction ϕ on K for K', and an algorithm that, for every $f \in K$, computes $g \in K$ such that $f = g' + \phi(f)$. We call $\phi(f)$ the *remainder* of f and $(g, \phi(f))$ a *reduction pair* of f (with respect to ϕ). A reduction pair will be abbreviated as an R-pair in the sequel.

Let t be a primitive monomial over K. We are going to define a complete reduction ψ on $K(t)$ for $K(t)'$. It suffices to construct a complementary subspace of $K[t]'$ in $K[t]$ by Lemma 2.2.

As a matter of notation, the C-subspace $\bigoplus_{i\in\mathbb{N}} V \cdot t^i$ for some C-subspace V of K is denoted by $V \otimes C[t]$ in virtue of the C-isomorphism $v \otimes t^i \mapsto vt^i$ from $V \otimes_C C[t]$ to $\bigoplus_{i\in\mathbb{N}} V \cdot t^i$.

First, we decompose $K[t]$ as the sum of $K[t]'$ and $\mathrm{im}(\phi) \otimes C[t]$.

Lemma 3.1. *Let $p \in K[t]$ with $\deg(p) = d$. There exists $q \in K[t]$ with $\deg(q) \le d$ and $r \in \mathrm{im}(\phi) \otimes C[t]$ with $\deg(r) \le d$ such that*

$$p = q' + r. \tag{3}$$

Proof. If $p = 0$, then set $q = r = 0$. Assume that p is nonzero with degree d and leading coefficient l.

Let $(g, \phi(l))$ be an R-pair of l, and $h = p - lt^d$. With integration by parts, we have

$$p = g't^d + \phi(l)t^d + h = \left(gt^d\right)' + \phi(l)t^d + h - (dgt')t^{d-1}. \tag{4}$$

Since $\phi(l)t^d \in \mathrm{im}(\phi) \otimes C[t]$ and $d > \deg\left(h - (dgt')t^{d-1}\right)$, the lemma follows from an induction on d. □

Definition 3.2. *The C-subspace $\mathrm{im}(\phi) \otimes C[t]$, denoted by A, is called the* auxiliary subspace *for $K[t]'$ in $K[t]$.*

Corollary 3.3. *$K[t] = K[t]' + A$.*

Proof. It is immediate from Lemma 3.1. □

The next algorithm is direct from the proof of Lemma 3.1.

Algorithm 3.4. AuxiliaryReduction

Input: $p \in K[t]$

Output: $(q, r) \in K[t] \times A$ *such that* (3) *holds*

1. $\tilde{p} \leftarrow p, q \leftarrow 0, r \leftarrow 0$
2. while $\tilde{p} \neq 0$ do
   $d \leftarrow \deg(\tilde{p})$, $l \leftarrow \mathrm{lc}(\tilde{p})$, *compute an R-pair* $(g, \phi(l))$ *of* l
   $q \leftarrow q + gt^d$, $r \leftarrow r + \phi(l)t^d$, $\tilde{p} \leftarrow \tilde{p} - lt^d - (dgt')t^{d-1}$
   end do
3. return (q, r)

Next, let us construct a C-basis of $K[t]' \cap A$. To this end, we fix an R-pair $(\lambda_t, \phi(t'))$ of t' and call it *the first pair associated to* $K(t)$.

Remark 3.5. *The remainder $\phi(t') \in K[t]'$, because it is equal to $(t - \lambda_t)'$. Moreover, $\phi(t') \neq 0$ because t is a primitive monomial.*

For all $i \in \mathbb{Z}^+$, we calculate

$$\phi(t')t^i = t't^i - \lambda_t't^i = \left(\frac{t^{i+1}}{i+1} - \lambda_t t^i\right)' + (i\lambda_t t')t^{i-1}. \tag{5}$$

There exists a pair $(q_i, r_i) \in K[t] \times A$ such that $(i\lambda_t t')t^{i-1} = q_i' + r_i$ and $\deg(r_i) \le i - 1$ by Lemma 3.1. It follows that

$$\phi(t')t^i - r_i = \left(\frac{t^{i+1}}{i+1} - \lambda_t t^i + q_i\right)'. \tag{6}$$

Lemma 3.6. *Let $v_0 = \phi(t')$ and v_i be the left-hand side of* (6). *Then*

(i) $\deg(v_i) = i$ *and* $\mathrm{lc}(v_i) = \phi(t')$ *for all* $i \in \mathbb{N}$.
(ii) *The set $\{v_0, v_1, \ldots\}$ is a C-basis of $K[t]' \cap A$.*

Proof. (i) holds because $\phi(t') \neq 0$ and r_i in (6) has degree $< i$.

(ii) Set $I = K[t]' \cap A$. Then $v_0 \in I^\times$ by Remark 3.5 and Definition 3.2. For $i > 0$, $v_i \in K[t]'$ by (6). It is in A because $\phi(t')t^i, r_i \in A$. Thus, $v_i \in I$ for all $i \in \mathbb{N}$. The v_i's are C-linearly independent by (i).

Assume that $p \in I$. Then $p \in K(t)' \cap K[t]$. It follows from [11, Lemma 2.3] that $\mathrm{lc}(p) = ct' + b'$ for some $c \in C$ and $b \in K$. On the other hand, $p \in A$ implies that $\mathrm{lc}(p) \in \mathrm{im}(\phi)$. Hence, applying ϕ to $\mathrm{lc}(p) = ct' + b'$ yields $\mathrm{lc}(p) = c\phi(t')$, because ϕ is an idempotent and $\phi(b') = 0$. Let $i = \deg(p)$ and $q = p - cv_i$. Then $q \in I$ with $\deg(q) < i$. Thus, p is a C-linear combination of $v_0, \ldots, v_i$ by a straightforward induction on i. □

Remark 3.7. *[11, Lemma 2.3] cited in the above proof combines two results in [8] into one statement. A referee of our manuscript points out that the two results are given in equation (5.13) and the second paragraph on page 176, respectively.*

Calculations in (5) and (6) lead to a naive algorithm to construct the basis $\{v_0, v_1, \ldots\}$ of $K[t]' \cap A$ in Lemma 3.6 (ii) up to a given degree d. A more elaborated algorithm suggested by a referee minimizes R-pairs to be computed. To present the algorithm, we need some notation and a technical lemma.

Let $(\lambda_t, \phi(t'))$ be the first pair associated to $K(t)$. Set $\mu_0 = \lambda_t$ and (μ_{k+1}, ν_{k+1}) to be an R-pair of $\mu_k t'$ with respect to ϕ for all $k \in \mathbb{N}$. Furthermore, we define two families of differential operators:

$$L_{i,j} = \sum_{k=1}^{i} (-1)^{k+1}\mu_{j+k} D_t^k \quad \text{and} \quad M_{i,j} = \sum_{k=1}^{i} (-1)^{k+1}\nu_{j+k} D_t^k \tag{7}$$

for all $i \in \mathbb{Z}^+$ and $j \in \mathbb{N}$, where D_t is the same as in Section 2.3.

Lemma 3.8. *With the notation just introduced, we have*

$$i\mu_j t' t^{i-1} = \left(L_{i,j}(t^i)\right)' + M_{i,j}(t^i)$$

for all $i \in \mathbb{Z}^+$ and $j \in \mathbb{N}$.

Proof. We proceed by induction on i and regard j as an arbitrary nonnegative integer. For $i = 1$, $\mu_j t' = \mu_{j+1}' + \nu_{j+1}$ by definition. So $\mu_j t' = L_{1,j}(t)' + M_{1,j}(t)$ by (7). The conclusion holds for $i = 1$.

Assume that $i > 1$ and the conclusion holds for values lower than i and every $j \in \mathbb{N}$. We calculate

$$\begin{aligned}
\mu_j t' t^{i-1} &= \left(\mu_{j+1}' + \nu_{j+1}\right) t^{i-1} \\
&= \left(\mu_{j+1} t^{i-1}\right)' - \mu_{j+1}\left(t^{i-1}\right)' + \nu_{j+1} t^{i-1} \\
&= \left(\mu_{j+1} t^{i-1}\right)' - (i-1)\mu_{j+1} t' t^{i-2} + \nu_{j+1} t^{i-1} \\
&= \left(\mu_{j+1} t^{i-1} - L_{i-1,j+1}(t^{i-1})\right)' + \nu_{j+1} t^{i-1} - M_{i-1,j+1}(t^{i-1}) \\
&= i^{-1}\left(L_{i,j}(t^i)\right)' + i^{-1} M_{i,j}(t^i),
\end{aligned}$$

in which the first equality holds because $\mu_j t' = \mu_{j+1}' + \nu_{j+1}$; the second is derived from integration by parts, the third is by a direct calculation; the fourth is due to the induction hypothesis

$$(i-1)\mu_{j+1} t' t^{i-2} = \left(L_{i-1,j+1}(t^{i-1})\right)' + M_{i-1,j+1}(t^{i-1});$$

and the last holds by replacing $iD_t^k(t^{i-1})$ with $D_t^k(t^i)$ and shifting the index k to $k+1$ in (7). □

Corollary 3.9. *For all $i \in \mathbb{Z}^+$, the element v_i in Lemma 3.6 can be taken as $\phi(t')t^i - M_{i,0}(t^i)$, which is equal to $\left(\frac{t^{i+1}}{i+1} - \lambda_t t^i + L_{i,0}(t^i)\right)'$.*

Proof. It follows from Remark 3.5 that

$$\phi(t')t^i = \left(\frac{t^{i+1}}{i+1} - \lambda_t t^i\right)' + i\lambda_t t' t^{i-1}.$$

Setting $j = 0$ in Lemma 3.8 and noticing $\mu_0 = \lambda_t$, we see that

$$\phi(t')t^i = \left(\frac{t^{i+1}}{i+1} - \lambda_t t^i + L_{i,0}(t^i)\right)' + M_{i,0}(t^i).$$

Since $M_{i,0}(t^i)$ belongs to A and is of degree less than i, the corollary holds when we set $q_i = L_{i,0}(t^i)$ and $r_i = M_{i,0}(t^i)$ in (6). □

Algorithm 3.10. Basis
Input: *$d \in \mathbb{N}$ and the first pair $(\lambda_t, \phi(t'))$ associated to $K(t)$*
Output: *$(u_0, v_0), \ldots, (u_d, v_d)$, where $(u_0, v_0) = (t - \lambda_t,\ \phi(t'))$ and, for all $i \in [d]$, $(u_i, v_i) = \left(\frac{t^{i+1}}{i+1} - \lambda_t t^i + L_{i,0}(t^i),\ \phi(t')t^i - M_{i,0}(t^i)\right)$ with $L_{i,0}$ and $M_{i,0}$ given in (7).*

We skip the pseudo-code for the above algorithm, because both $L_{i,0}(t^i)$ and $M_{i,0}(t^i)$ can be easily constructed iteratively according to (7). In the algorithm, one computes R-pairs $(\mu_i, \phi(\mu_{i-1}t'))$ for every $i \in [d]$, whereas the number of R-pairs needed is proposional to d^2 if one constructs $v_1, \ldots, v_d$ by Algorithm 3.4 directly.

Now, we turn the sum in Corollary 3.3 to a direct one by constructing a subspace of A that is a complement of $K[t]'$. To proceed, we need to assume further that K has an effective C-basis, which is denoted by Θ. Then there exists a pair $(\theta, c) \in \Theta \times C^\times$ such that $c = \theta^*(\phi(t'))$. We fix such a pair and call it the *second pair associated to* $K(t)$. A complementary subspace consists of polynomials in A whose coefficients are free of θ. In other words, the subspace is equal to $(\mathrm{im}(\phi) \cap \ker(\theta^*)) \otimes C[t]$.

Lemma 3.11. *Let (θ, c) be the second pair associated to $K(t)$. Then*
(i) *$A = (K[t]' \cap A) \oplus A_\theta$, where $A_\theta = (\mathrm{im}(\phi) \cap \ker(\theta^*)) \otimes C[t]$;*
(ii) *$K[t] = K[t]' \oplus A_\theta$.*

Proof. (i) Similar to the proof of Lemma 3.6, we set $I = K[t]' \cap A$.

First, we show $A = I + A_\theta$. Since $I \subset A$ and $A_\theta \subset A$, it suffices to show $A \subset I + A_\theta$. Let $\{v_0, v_1, \ldots\}$ be the basis of I in Lemma 3.6 (ii), and $p \in A$. Set $d = \deg(p)$, $l = \mathrm{lc}(p)$ and $z = \theta^*(l)$. By Lemma 3.6 (i),

$$p - c^{-1}zv_d = gt^d + h, \tag{8}$$

where $g = l - c^{-1}z\phi(t')$ and $h \in K[t]$ with $\deg(h) < d$. Since $p \in A$, we have that $l \in \mathrm{im}(\phi)$, and, thus, $g \in \mathrm{im}(\phi)$ by its definition. Furthermore, $\theta^*(g) = \theta^*(l) - c^{-1}z\theta^*(\phi(t')) = z - z = 0$. Hence, $g \in \ker(\theta^*)$. Consequently, $g \in \mathrm{im}(\phi) \cap \ker(\theta^*)$. We conclude that $gt^d \in A_\theta$. It follows from (8) that $h \in A$ and $p - h \in I + A_\theta$, which allow us to carry out an induction on d as follows.

If $d = 0$, then $h = 0$. So $p \in I + A_\theta$. Assume that $d > 0$ and that all elements of A with degree lower than d are in $I + A_\theta$. Then $h \in I + A_\theta$. Hence, $p \in I + A_\theta$.

Second, we show that $I \cap A_\theta = \{0\}$. Assume that $q \in I \cap A_\theta$. Then q is a C-linear combination of the v_i's. So $\mathrm{lc}(q)$ is the product of a constant and $\phi(t')$ by Lemma 3.6 (i). Since $\mathrm{lc}(q) \in \ker(\theta^*)$ and $\phi(t') \notin \ker(\theta^*)$, the constant is equal to zero, and so is $\mathrm{lc}(q)$. Accordingly, $q = 0$.

(ii) By Corollary 3.3 and (i), $K[t] = K[t]' + A_\theta$. Since $A_\theta \subset A$, we have that $K[t]' \cap A_\theta = (K[t]' \cap A) \cap A_\theta$, which is equal to $\{0\}$ by (i). So (ii) holds. □

The C-subspace A_θ in Lemma 3.11 is called the *θ-complement* of $K[t]'$ in $K[t]$ in the rest of this section. The next algorithm projects an element of A to $K[t]'$ and the θ-complement, respectively. It is correct by the proof of Lemma 3.11 (i).

Algorithm 3.12. Projection
Input: *$r \in A$, the first and second pairs $(\lambda_t, \phi(t'))$ and (θ, c) associated to $K(t)$*
Output: *$(u, v) \in K[t] \times A_\theta$ such that*

$$r = u' + v \tag{9}$$

1. $u \leftarrow 0$, $v \leftarrow r$, $d \leftarrow \deg(r)$
2. $B \leftarrow \textsc{Basis}(d, \lambda_t, \phi(t'))$ (**Algorithm 3.10**)
3. for i from 0 to d do
   $a \leftarrow$ *the coefficient of* t^{d-i} *in* v, $b \leftarrow \theta^*(a)$
   $(\tilde{u}, \tilde{v}) \leftarrow$ *the element of* B *with* $\deg(\tilde{v}) = d - i$,
   $\tilde{c} \leftarrow c^{-1}b$, $u \leftarrow u + \tilde{c}\tilde{u}$, $v \leftarrow v - \tilde{c}\tilde{v}$
   end do
4. return (u, v)

We are ready to present the main result of this section.

Theorem 3.13. *Let (θ, c) be the second pair associated to $K(t)$, and A_θ be the θ-complement of $K[t]'$. Then $K(t) = K(t)' \oplus A_\theta \oplus S_t$. Moreover, the projection ψ_θ from $K(t)$ to $A_\theta \oplus S_t$ with respect to the above direct sum is a complete reduction for $K(t)'$.*

Proof. By Lemma 2.2 (i) and Lemma 3.11,

$$K(t) = (K(t)' + A_\theta) \oplus S_t.$$

By Lemma 2.2 (ii) and $A_\theta \subset K[t]$, we have $K(t)' \cap A_\theta = K[t]' \cap A_\theta$, which is trivial by Lemma 3.11 (ii). So $K(t) = K(t)' \oplus A_\theta \oplus S_t$. It follows that ψ_θ is a complete reduction for $K(t)'$. □

Below is an algorithm for the complete reduction given in the above theorem.

Algorithm 3.14. CompleteReduction
Input: *$f \in K(t)$, the first and second pairs $(\lambda_t, \phi(t'))$ and (θ, c) associated to $K(t)$*
Output: *an R-pair of f with respect to ψ_θ in Theorem 3.13*

1. $(g, p, s) \leftarrow \textsc{HermiteReduce}(f)$ (**[8, §5.3]**)
   if $p = 0$ then return (g, s) end if
2. $(q, r) \leftarrow \textsc{AuxiliaryReduction}(p)$ (**Algorithm 3.4**)
   if $r = 0$ then return $(g + q, s)$ end if
3. $(u, v) \leftarrow \textsc{Projection}(r, \lambda_t, \phi(t'), \theta, c)$ (**Algorithm 3.12**)
   return $(g + q + u, s + v)$

Example 3.15. *Let $K(t)$ and f be given in Example 2.1, and Θ be the C-basis given in (2) with $F = C$ and $y = x$. The first and second associated pairs are $(0, x^{-1})$ and $(x^{-1}, 1)$, respectively. The above algorithm computes an R-pair of f as follows.*

1. $(g, p, s) = \left(0,\ \frac{x+1}{x}t + \frac{x^2+x+1}{x},\ -\frac{x}{t+1}\right)$ *by Example 2.1.*

2. *Algorithm 3.4 finds* $(q, r) = \left(xt + \frac{x^2}{2}, \frac{t+1}{x}\right) \in K[t] \times A$ *such that* (3) *holds, where* $A = S_x \otimes C[t]$.
3. *Algorithm 3.12 finds* $(u, v) = \left(\frac{t^2}{2} + t, 0\right)$ *such that* (9) *holds.*

Thus, $p = (q + u)'$ *and* $(g + q + u, s)$ *is an R-pair of* f*. Algorithm 3.14 finds* $s = -\frac{x}{t+1}$ *as a "minimal" non-in-field integrable part.*

At last, we describe the restriction of ψ_θ to K.

Corollary 3.16. *Let* $\phi : K \to K$ *be a complete reduction for* K'*,* (θ, c) *be the second pair associated to* $K(t)$ *and* ψ_θ *be the complete reduction given in Theorem 3.13. Then, for every* $f \in K$*, we have that* $\psi_\theta(f) = \phi(f) + \tilde{c}\phi(t')$*, where* $\tilde{c} = -\theta^*(\phi(f))\, c^{-1}$.

Proof. Since $f \in K$, we have $f \equiv \phi(f) \mod K'$. By Remark 3.5, $f \equiv \phi(f) + \tilde{c}\phi(t') \mod K(t)'$. Note that $\phi(f) + \tilde{c}\phi(t')$ belongs to the θ-complement. Applying ψ_θ to the above congruence, we conclude that $\psi_\theta(f) = \phi(f) + \tilde{c}\phi(t')$, because $K(t)' = \ker(\psi_\theta)$ and the restriction of ψ_θ to A_θ is the identity map. □

4 Complete reduction

In this section, we define primitive towers and remove the assumptions made in the first paragraph of Section 3.

Definition 4.1. *Let* K_0 *be a differential field whose subfield of constants is denoted by* C*. A* primitive tower *over* K_0 *is*

$$\begin{array}{ccccccc} K_0 & \subset & K_1 & \subset & \cdots & \subset & K_n \\ & & \| & & & & \| \\ & & K_0(t_1) & & & & K_{n-1}(t_n), \end{array} \tag{10}$$

where t_i *is a primitive monomial over* K_{i-1} *for all* $i \in [n]$.

Note that C is the subfield of constants in a primitive tower K_n.

Theorem 4.2. *Let* K_n *be a primitive tower as in* (10)*, and* Θ_0 *be an effective* C*-basis of* K_0*. Assume that* $\phi_0 : K_0 \to K_0$ *is a complete reduction for* K_0'*, and that there is an algorithm to compute an R-pair of every element in* K_0*. Then, for every* $i \in [n]_0$*,* K_i *has an effective* C*-basis* Θ_i *and a complete reduction* $\phi_i : K_i \to K_i$ *for* K_i'*. Moreover, there is an algorithm to compute an R-pair of every element in* K_i*.*

Proof. We proceed by induction on n. If $n = 0$, then the conclusion clearly holds. Assume that $n > 0$ and that there exists an effective C-basis Θ_{n-1} of K_{n-1}, a complete reduction ϕ_{n-1} for K_{n-1}' on K_{n-1} and an algorithm to compute an R-pair of every element in K_{n-1}. The first and second pairs $(\lambda_n, \phi_{n-1}(t_n'))$ and (θ_n, c_n) associated to K_n can be constructed by ϕ_{n-1} and Θ_{n-1}, respectively.

The tower K_n has an effective C-basis Θ_n by Remark 2.7. Replacing K with K_{n-1}, t with t_n, ϕ with ϕ_{n-1}, and θ with θ_n in Theorem 3.13, we find a complete reduction ψ_{θ_n} for K_n' on K_n. Doing the same replacements in Algorithms 3.4, 3.10, 3.12 and 3.14, we have an algorithm to compute an R-pair of every element in K_n with respect to ψ_{θ_n}. The induction is completed by setting $\phi_n = \psi_{\theta_n}$. □

Corollary 4.3. *Let* K_n *be a primitive tower as in* (10) *and* ϕ_i *be the complete reduction constructed in the proof of the above theorem. Then, for every* $i \in [n-1]_0$ *and* $f \in K_i$*,* $\phi_n(f) - \phi_i(f)$ *is a* C*-linear combination of* $\phi_i(t_{i+1}'), \ldots, \phi_{n-1}(t_n')$*, and belongs to* K_n'*.*

Proof. For every $j \in [n-1]_0$, $\phi_{j+1}(f) - \phi_j(f) = c_j \phi_j(t_{j+1}')$ for some $c_j \in C$ by Corollary 3.16. Summing up these equalities from i to $n-1$, we see that $\phi_n(f) - \phi_i(f)$ is a C-linear combination of $\phi_i(t_{i+1}'), \ldots, \phi_{n-1}(t_n')$. It belongs to K_n' by Remark 3.5. □

To perform complete reductions in practice, we assume further that $[K_0 : C(x)] < \infty$ and that K_0 contains no new constant. Complete reductions on $C(x)$ and its finite algebraic extensions are given in Example 1.1 and [15], respectively. Improvements on the reduction for algebraic functions can be found in [12]. Algorithms 2.4 and 2.6 show that $C(x)$ has an effective C-basis. So does K_0 by Remark 2.7. Consequently, a complete reduction for K_n' on K_n is available by Theorem 4.2.

Let us make a notational convention so that we can illustrate computations and proofs through a primitive tower concisely.

Convention 4.4. *Let* K_n *be a primitive tower as in* (10)*, and* ϕ_0 *be a complete reduction for* K_0' *on* K_0*. Let* Θ *be the effective* C*-basis of* K_n *obtained from a repeated use of Remark 2.7. For all* $i \in [n]$*,*

- $\phi_i : K_i \to K_i$ *stands for the complete reduction for* K_i' *in the proof of Theorem 4.2,*
- $(\lambda_i, \phi_{i-1}(t_i'))$ *and* (θ_i, c_i) *for the first and second pairs associated to* K_i*, respectively,*
- S_i *for the set of simple elements in* K_i *with respect to* t_i*, and*
- A_i *for the auxiliary subspace in* $K_{i-1}[t_i]$*.*

All associated pairs are constructed once and for all. So the possible ambiguity mentioned in Remark 2.5 will never occur.

Example 4.5. *Let* $K_0 = C(x)$*,* $t_1 = \log(1-x)$*, and* $t_2 = \operatorname{polylog}(2, x)$*, which is equal to* $-\int \frac{\log(1-x)}{x}$*. Then* $K_2 = K_0(t_1, t_2)$ *is a primitive tower. We associate* $(\lambda_1, \phi_0(t_1')) = \left(0, \frac{1}{x-1}\right)$*,* $(\theta_1, c_1) = \left(\frac{1}{x-1}, 1\right)$ *and* $(\lambda_2, \phi_1(t_2')) = \left(0, -\frac{t_1}{x}\right)$*,* $(\theta_2, c_2) = \left(\frac{t_1}{x}, -1\right)$ *to* K_1 *and* K_2*, respectively. Let us compute respective R-pairs of*

$$f = \frac{\left((x-1)^2\, t_1 + x\right) t_2^3 + x\,(x-1)\, t_1}{x^2\,(x-1)\, t_2^2} \quad \text{and} \quad \tilde{f} = t_2^2.$$

First, HermiteReduce(f) *finds* $(g, p, s) \in K_1(t_2) \times K_1[t_2] \times S_2$ *such that* (1) *holds, where* $g = \frac{1}{t_2}$*,* $p = \frac{(x-1)^2 t_1 + x}{x^2(x-1)} t_2$ *and* $s = 0$*.*

Second, AuxiliaryReduction(p) *yields* $(q, r) \in K_1[t_2] \times A_2$ *such that* (3) *holds, where* $q = \frac{t_1}{x} t_2 + \frac{x-1}{x} t_1^2$ *and* $r = \frac{t_1}{x} t_2 - \frac{2t_1}{x}$*.*

At last, we project r *to* $K_1[t_2]'$ *and the* θ_2*-complement by* Projection*. The respective projections are* u' *and* 0*, where* $u = -\frac{t_2^2}{2} + 2t_2$*. So* f *has an R-pair* $(g + q + u, 0)$*. Consequently,* $\int f = g + q + u$*.*

In the same vein, an R-pair of $\tilde{f}$ *is* $(\tilde{g}, \tilde{r})$*, where*

$$\tilde{g} = x t_2^2 + (2t_1 x - 2t_1 - 2x)\, t_2 + 2t_1^2 x - 2t_1^2 - 6t_1 x + 6t_1 + 6x$$

and $\tilde{r} = -\frac{2t_1^2}{x}$*. So* $\tilde{f}$ *does not have any integral in* K_2*. The remainder* $\tilde{r}$ *is "simpler" than* $\tilde{f}$ *in the sense that* $\tilde{r}$ *is of degree* 0 *in* t_2*.*

Example 4.6. *Let* $K_0 = C(x, y)$ *with* $y^3 - xy + 1 = 0$*. Set* $t_1 = \log(y)$*. Then* $K_1 = K_0(t_1)$ *is a primitive tower. Two associated pairs of* K_1 *are* $(\lambda_1, \phi_0(t_1')) = \left(\frac{2xy}{3}, -y\right)$ *and* $(\theta_1, c_1) = (y, -1)$*, respectively. We compute an R-pair of* $f = y(2 - 3t_1)$*.*

HermiteReduce(f) *finds a triplet* (g, p, s) *in* $K_0(t_1) \times K_0[t_1] \times S_1$ *such that* (1) *holds, where* $g = 0$*,* $p = -3yt_1 + 2y$ *and* $s = 0$*.*

Since $\phi_0(t_1') = -y$*, we see that* $y \in \operatorname{im}(\phi_0)$*. Then* $p \in A_1$*. So* (3) *holds by setting* $q = 0$ *and* $r = p$*.*

Projection$(r, \lambda_1, \phi_0(t_1'), \theta_1, c_1)$ *yields* $u = \frac{3}{2}t_1^2 - (2xy)t_1 + 2xy$ *and* $v = 0$ *such that* (9) *holds. Thus, an R-pair of* f *is* $(u, 0)$*. Consequently,* u *is an integral of* f*.*

We have compared our preliminary implementation of the complete reduction given in Theorem 4.2 with the Maple function `int` and Algorithm AddDecompInField in [19, page 150] for in-field integration. Empirical results are given in the appendix.

5 Applications of remainders

This section contains two applications: computing elementary integrals over K_n with $K_0 = C(x)$, and constructing telescopers for some non-D-finite functions. Convention 4.4 is kept in the sequel.

5.1 Elementary integrals

Let $f \in K_n$. Then f has an elementary integral over K_n if and only if its remainder $\phi_n(f)$ has one. Two properties of remainders allow us to apply Algorithm 2.8 directly to compute elementary integrals. To describe the properties, we need three C-subspaces of K_n. Let

$$P = \sum_{i\in[n]} t_i K_{i-1}[t_i], \quad S = \sum_{i\in[n]} S_i,$$

and T be the C-subspace spanned by $\phi_0(t_1'), \phi_1(t_2'), \ldots, \phi_{n-1}(t_n')$. Note that $\sum_{i\in[n]} t_i K_{i-1}[t_i]$, $\sum_{i\in[n]} S_i$ and $K_0 + P + S$ are all direct.

Proposition 5.1. $\mathrm{im}(\phi_n) \subset K_0 \oplus P \oplus S$.

Proof. The conclusion holds for $n = 0$ because $\mathrm{im}(\phi_0) \subset K_0$. Assume that $n > 0$ and that the conclusion holds for $n-1$. By Theorem 3.13, $\mathrm{im}(\phi_n) \subset A_n + S_n$. Since $A_n \subset \mathrm{im}(\phi_{n-1}) + t_n K_{n-1}[t_n]$, we see that $\mathrm{im}(\phi_n) \subset \mathrm{im}(\phi_{n-1}) + t_n K_{n-1}[t_n] + S_n$. The proposition then follows from the induction hypothesis. □

Proposition 5.2. *If* $h \in K_0 \oplus S$*, then* $h - \phi_n(h) \in K_0' + T$*.*

Proof. Assume $h = h_0 + \sum_{i\in[n]} s_i$, where $h_0 \in K_0$ and $s_i \in S_i$. Then $s_i = \phi_i(s_i)$ by Theorem 3.13, and $\phi_i(s_i) \equiv \phi_n(s_i) \bmod T$ by Corollary 4.3. Hence, $s_i \equiv \phi_n(s_i) \bmod T$, which, together with the application of ϕ_n to h, implies $h - \phi_n(h) \equiv h_0 - \phi_n(h_0) \bmod T$. By Corollary 4.3 again, $h - \phi_n(h) \equiv h_0 - \phi_0(h_0) \bmod T$. The proposition is proved by noting that $h_0 - \phi_0(h_0) \in K_0'$. □

An element s of S can be uniquely written as $\sum_{i\in[n]} s_i$, where $s_i \in S_i$. We say that all residues of s are constants if all residues of s_i as an element in $K_{i-1}(t_i)$ belong to $\overline{C}$ for every $i \in [n]$.

Theorem 5.3. *Let* K_n *be a primitive tower as in* (10) *with* $K_0 = C(x)$*. Assume that* C *is algebraically closed. Then* $f \in K_n$ *has an elementary integral over* K_n *if and only if*

(i) *there exists* $s \in S$ *such that* $\phi_n(f) \equiv s \mod K_0 + T$*, and*
(ii) *all residues of* s *belong to* C*.*

Proof. Assume that both (i) and (ii) hold. By (ii) and [18, Proposition 3.3], s has an elementary integral over K_n. Every element of K_0 has an elementary integral over K_0 because $K_0 = C(x)$. By Remark 3.5, $T \subset K_n'$. It follows from (i) that $\phi_n(f)$ has an elementary integral over K_n, and so does f.

Conversely, assume that f has an elementary integral over K_n. Then there exists a C-linear combination h of logarithmic derivatives in K_n such that $f \equiv h \mod K_n'$ by [8, Theorem 5.5.2]. Since $\phi_n(f) = \phi_n(h)$, it suffices to show that $\phi_n(h)$ satisfies both (i) and (ii). By the logarithmic derivative identity, $h \equiv s \bmod K_0$ for some $s \in S$, which has merely constant residues. Then $h \equiv \phi_n(h) \bmod K_0 + T$ by Proposition 5.2. Hence, $\phi_n(h) \equiv s \bmod K_0 + T$ by the above two congruences. Both (i) and (ii) hold. □

Next, we outline an algorithm for computing elementary integrals over K_n. Let $f \in K_n$.

1. Compute an R-pair $(g, \phi_n(f))$. If $\phi_n(f) = 0$, then $\int f = g$ and we are done.
2. Assume that $\phi_n(f) \neq 0$. By Proposition 5.1, we can write $\phi_n(f) = r + p + s$ and $\phi_{i-1}(t_i') = r_i + p_i + s_i$, where $i \in [n]$, $r, r_i \in K_0$, $p, p_i \in P$ and $s, s_i \in S$.
3. Let $z_1, \ldots, z_n$ be constant indeterminates.
   - Use ConstantMatrix (Algorithm 2.8) to compute a matrix $M \in C^{k\times n}$ and $\mathbf{v} \in C^k$ such that $s - \sum_{i\in[n]} z_i s_i$ has merely constant residues if and only if the linear system given by the augmented matrix $(M, \mathbf{v})$ is consistent.
   - Compute $N \in C^{l\times n}$ and $\mathbf{w} \in C^l$ such that $p = \sum_{i\in[n]} z_i p_i$ if and only if the linear system given by the augmented matrix $(N, \mathbf{w})$ is consistent.
   - Solve the linear system $\begin{pmatrix} M \\ N \end{pmatrix} (z_1, \ldots, z_n)^\tau = \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}$.
4. If the above system has no solution, then f has no elementary integral over K_n by Theorem 5.3. Otherwise, let $\tilde{c}_1, \ldots, \tilde{c}_n$ be such a solution. Set $\tilde{r} = r - \sum_{i\in[n]} \tilde{c}_i r_i$ and $\tilde{s} = s - \sum_{i\in[n]} \tilde{c}_i s_i$. Then $\int f = g + \int \tilde{r} + \int \tilde{s} + \sum_{i\in[n]} \tilde{c}_i (t_i - \lambda_i)$. Note that $\int \tilde{r}$ is elementary because $\tilde{r} \in C(x)$, and that $\int \tilde{s}$ is elementary over K_n by Theorem 5.3.

Example 5.4. *We follow the above outline to integrate*

$$f = \frac{x + (x-1)t_2}{(x-1)t_1} + \frac{t_2 + t_3(1 - t_1)}{x},$$

1. *By* ϕ_3*, we find an R-pair* $(t_2 t_3, \phi_3(f))$*, where* $\phi_3(f) = \frac{x}{(x-1)t_1}$*.*
2. *Compute* $\phi_{i-1}(t_i') = r_i + p_i + s_i$*, where*

i	1	2	3
(r_i, p_i, s_i)	$\left(\frac{1}{x-1}, 0, 0\right)$	$\left(\frac{1}{x}, -\frac{t_1}{x}, 0\right)$	$\left(\frac{1}{x}, 0, \frac{1}{t_1}\right)$

3. *By step 3 in the above outline, we have*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

   It has a solution $z_1 = z_2 = 0$ *and* $z_3 = 1$*.*
4. *Computing the residues yields* $\int f = t_2 t_3 + t_3 + \log\left(\frac{t_1}{x}\right)$*.*

Neither `int()` command in Maple 2021 nor `Integrate[]` command in Mathematica 14.1 found an elementary integral for f. The Axiom-based computer algebra system FriCAS 1.3.10 (see [20]) returned a correct integral. Comprehensive tests are given in [1] for elementary integration in current computer algebra systems.

5.2 Telescopers

General connections between symbolic integration and creative-telescoping are described in [26, Chapter 1]. Examples in [11, §7] illustrate that additive decompositions help us detect the existence of telescopers for elements in some primitive towers. We present two propositions for the same purpose by remainders and residues.

Let $K = C(x,y)$ be the field of rational functions in x and y equipped with the usual partial derivatives D_x and D_y. Differential fields related to integration for several derivations can be found in [9, 30]. Let t be an element in some partial differential field extension of K such that t is transcendental over K, $D_y D_x(t) = D_x D_y(t)$, $D_x(t) \in K[t]$ with degree less than two, and $D_y(t) \in K \setminus D_y(K)$. Then t is a primitive monomial over K with respect to D_y. The extended derivatives are still denoted by D_x and D_y, respectively.

Every element of $K[t]$ is D-finite over K. But $K(t)$ contains non-D-finite elements. For instance, t^{-1} is not D-finite over K, because t^{i+1} is the monic denominator of $D_y^i(t^{-1})$ for all $i \in \mathbb{N}$.

For $f \in K(t)$, a differential operator $L \in C(x)[D_x]^{\times}$ is called a *telescoper* for f if $L(f) \in D_y(K(t))$.

Proposition 5.5. *Let $\phi : K(t) \to K(t)$ be the complete reduction for $D_y(K(t))$ given in Convention 4.4 with $K = K_0$ and $\phi = \phi_1$. For $f \in K(t)$ and $m \in \mathbb{N}$, f has a telescoper of order no more than m if and only if there exist $l_0 \ldots, l_m \in C(x)$, not all zero, such that*

$$\sum_{i \in [m]_0} l_i \phi(D_x^i(f)) = 0. \tag{11}$$

Proof. Let $L = \sum_{i \in [m]_0} l_i D_x^i$ with $l_0, \ldots, l_m \in C(x)$, not all zero. Then $\phi(L(f)) = \sum_{i \in [m]_0} l_i \phi(D_x^i(f))$, because ϕ is $C(x)$-linear. Assume that (11) holds. Then L is a telescoper for f with order no more than m. Conversely, assume that L is a telescoper for f with order no more than m. Then $\phi(L(f)) = 0$ because ϕ is a complete reduction. Hence, (11) holds. □

Below is a sufficient condition on the existence of telescopers.

Proposition 5.6. *Let $f \in K(t)$. Then there exists a unique element $s \in S_t$ such that $\phi(f) \equiv s \bmod K[t]$. If all residues of s with respect to D_y are in $\overline{C(x)}$, then f has a telescoper.*

Proof. There exists a unique pair (q, s) in $K[t] \times S_t$ such that $\phi(f) = q + s$ by Proposition 5.1. Since q is D-finite over K, it has a telescoper by [36, Lemma 4.1] or [25, Lemma 3].

It remains to prove that s has a telescoper by [14, Remark 2.3]. Let $s = \frac{a}{b}$, where $a, b \in K[t]$, b is monic with respect to t and $\gcd(a, b) = 1$. Assume that $\alpha_1, \ldots, \alpha_k$ are the distinct roots of b. By [18, Lemma 3.1 (i)], we have that

$$s = \sum_{j \in [k]} \beta_j \frac{D_y(t - \alpha_j)}{t - \alpha_j}, \tag{12}$$

where $\beta_j \in \overline{K}$ is the residue of f at α_j with respect to D_y. Since each β_j is assumed to be in $\overline{C(x)}$, there exists $L \in C(x)[D_x]$ annihilating all of them by [23, Theorem 3.29 (3)]. By the commutativity of applying derivations and taking logarithmic derivatives, we have

$$D_x\left(\gamma \frac{D_y(u)}{u}\right) = D_y\left(\gamma \frac{D_x(u)}{u}\right) + D_x(\gamma)\frac{D_y(u)}{u}.$$

for all $\gamma \in \overline{C(x)}$ and $u \in K(t)$. A repeated application of the above equality to (12) yields $g \in \overline{C(x)}(y, t)$ such that

$$L(s) = D_y(g) + \sum_{j \in [k]} L\left(\beta_j\right) \frac{D_y(t - \alpha_j)}{t - \alpha_j} = D_y(g)$$

Moreover, g is symmetric in $\alpha_1, \ldots \alpha_k$ over $K(t)$ so that g actually belongs to $K(t)$. □

Example 5.7. *Let $K = \mathbb{C}(x,y)$ and $t = \log(x+y)$. We try to construct respective telescopers for*

$$f = \frac{2x}{(x+y)(t^2 - x)} \quad \text{and} \quad \tilde{f} = y \frac{D_y(t-y)}{t-y}.$$

Note that f is simple. So $\phi(f) = f$. Its nonzero residues are $\pm\sqrt{x}$ by [8, Theorem 4.4.3]. By Proposition 5.6, f has a telescoper. Using the notation in Proposition 5.5, we have $2x\phi(D_x(f)) = f$. Thus, the minimal telescoper for f is $2xD_x - 1$.

Again, $\tilde{f}$ is simple. So $\phi(\tilde{f}) = \tilde{f}$. Since $\tilde{f}$ has a nonzero residue y, Proposition 5.6 is not applicable. Let $g = \frac{D_y(t-y)}{t-y}$ and $\gamma = \frac{D_x(t-y)}{D_y(t-y)}$. Then $\tilde{f} = yg$ and $\gamma = (1-x-y)^{-1}$. For $\omega \in C(x,y)$, we calculate

$$\begin{aligned} D_x(\omega g) &= D_x(\omega) g + \omega D_x(g) = D_x(\omega) g + \omega D_y\left(\frac{D_x(t-y)}{t-y}\right) \\ &\equiv D_x(\omega) g - D_y(\omega)\frac{D_x(t-y)}{t-y} \mod D_y(K(t)) \\ &\equiv \left(D_x(\omega) - \gamma D_y(\omega)\right) g \mod D_y(K(t)). \end{aligned}$$

Then $\phi(D_x(\omega g)) = \left(D_x(\omega) - \gamma D_y(\omega)\right) g$ because g is simple. Set $\gamma_0 = y$ and $\gamma_i = D_x(\gamma_{i-1}) - \gamma D_y(\gamma_{i-1})$ for $i \geq 1$. It follows from the above calculation that $\phi(D_x^i(\tilde{f})) = \gamma_i g$. Moreover, the denominator of γ_i has degree $2i - 1$ in y for $i \geq 1$ by a straightforward induction. Therefore, $\phi(\tilde{f})$, $\phi(D_x(\tilde{f}))$, $\phi(D_x^2(\tilde{f}))$, ... are linearly independent over $C(x)$. Consequently, $\tilde{f}$ has no telescoper by Proposition 5.5.

6 Conclusions

In this article, we have developed a complete reduction for derivatives in a primitive tower. The reduction algorithm decomposes an element of such a tower as the sum of a derivative and a remainder, where the derivative is unique up to an additive constant and the remainder is unique. The algorithm can be applied to compute elementary integrals over primitive towers and to construct telescopers for some non-D-finite functions. The work is a step forward in the development of complete reductions for derivatives in transcendental Liouvillian extensions.

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A Empirical results

We present some empirical results about in-field integration obtained by our complete reduction (CR), Algorithm AddDecompInField in [19, page 150] (AD), and the Maple function `int`. Experiments were carried out with Maple 2021 on a computer with imac CPU 3.6GHZ, Intel Core i9, 16G memory. Maple scripts of CR and AD are available at http://mmrc.iss.ac.cn/~zmli/ISSAC2025.html.

Every integrand in experimental data was a derivative in the primitive tower $\mathbb{Q}(x)(t_1, t_2, t_3)$, where $t_1 = \log(x)$, $t_2 = \log(x+1)$ and $t_3 = \log(t_1)$. So CR, AD and `int` are all applicable and have the same output, which is an integral of the input in the same tower. Three integrands in the form p_i' were generated for each i, where p_i was a dense polynomial in some selected generators. Below is a summary of the average timings (in seconds).

In the first suite of data, we set $p_i \in \mathbb{Q}(x, t_1, t_2)[t_3]$ such that $\deg_{t_3}(p_i) = i$ and all coefficients of p_i are rational functions whose numerators and denominators are both sparse random polynomials in $\mathbb{Q}[x, t_1, t_2]$ with total degree 5.

i	1	2	3	4	5	6
CR	1.42	8.32	37.01	122.55	1085.04	>3600
AD	0.96	10.42	47.36	149.02	>3600	>3600
`int`	1.15	4.52	23.30	53.43	166.27	346.29

In the second suite, $p_i \in \mathbb{Q}(x, t_1, t_2)[t_3]$ with degree i in t_3. Its coefficients are quotients of linear polynomials in $\mathbb{Q}[x, t_1, t_2]$.

i	6	8	10	12	14	16
CR	0.90	2.09	7.05	12.56	30.35	62.11
AD	1.23	4.29	12.31	31.08	57.67	170.70
`int`	3.83	17.46	31.61	66.22	144.70	322.19

In the third suite, $p_i \in \mathbb{Q}(x)[t_1, t_2, t_3]$ whose total degree is equal to i and whose coefficients are quotients of random polynomials in $\mathbb{Q}[x]$ with degree 5.

i	1	2	3	4	5	6
CR	0.35	0.19	0.59	4.02	21.32	88.51
AD	0.39	0.51	3.48	30.53	614.90	1453.61
`int`	0.53	0.63	4.68	51.82	154.31	1255.49

In the last suite, $p_i \in \mathbb{Q}[x, t_1, t_2, t_3]$ with total degree i. The Maple function `int` returned expressions involving unevaluated integrals for some inputs. Whenever this happened, the corresponding entry is marked by $\int$.

i	5	10	15	20	25	30
CR	0.39	0.25	0.81	1.98	4.32	8.71
AD	0.45	1.06	6.69	32.83	141.09	280.47
int	0.49	$\int$	$\int$	7.09	$\int$	$\int$

The timings reveal that CR outperformed AD, and was more efficient than `int` except for the integrands in the first suite. There are also examples for which `int` took more than one hour without any output, but both CR and AD returned correct results.

We also observe that HermiteReduce and AuxiliaryReduction were much more time-consuming than Projection in the complete reduction (see Algorithm 3.14).