1. 设 $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

(1) 计算 $V = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ 的一组基和 $\mathbb{Q}^3/V$ 的维数.

(2) 令 $\boldsymbol{w} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$, 判断 $\boldsymbol{w}$ 是否为 $\boldsymbol{v}_1, \boldsymbol{v}_2$ 的线性组合.

解: (1)

初等列变换

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

V 的一组基为 $\{(1\ 2\ 0)^t, (0\ -2\ 1)^t\}$.

$\mathbb{Q}^3/V$ 的维数为 1.

另: 作初等行(列)变换, 求得矩阵的秩等于 $\dim V$

又注意到 $\vec{v_1}, \vec{v_2}$ 是线性无关的 故 $\{\vec{v_1}, \vec{v_2}\}$ 构成 V 的一组基.

典型错误: 只做初等行变换, 取行向量构成基.

作初等行变换.
$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{-2(1)+(2)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[-\frac{1}{2}(2)]{\substack{-(2)+(1) \\ -(2)+(3)}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

因此. $\vec{v}_3 = \vec{v}_1 - \vec{v}_2$ 且 $\vec{v}_1$ 与 $\vec{v}_2$ 线性无关

V 的一组基是 $\{\vec{v}_1, \vec{v}_2\}$.

$\dim(\mathbb{Q}^3/V) = \dim\mathbb{Q}^3 - \dim V = 3-2 = 1$. $\mathbb{Q}^3/V$ 的维数是 1.

$\vec{v}_i \in \mathbb{F}^{m\times 1}$

$$(\vec{v}_1 \ \ \vec{v}_2 \ \cdots \ \vec{v}_n) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \vec{0}, \qquad M = (\vec{v}_1, \cdots, \vec{v}_n) \in \mathbb{F}^{m\times n}.$$

$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ 是齐次线性方程组 $M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ 的解.

对 M 作初等行变换 保持解空间不动.

(2). 设 $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2$, $x_1, x_2 \in \mathbb{Q}$. $\Leftrightarrow$ $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} x_2$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

故该非齐次线性方程组有唯一组解：$x_1 = 1$，$x_2 = 2$

即 $\vec{w} = \vec{v_1} + 2\vec{v_2}$.

2. 设在 $\mathbb{Q}^3$ 中, $\boldsymbol{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{\alpha}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\alpha}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$

$$\boldsymbol{\beta}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{\beta}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{\beta}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ 且 } \gamma = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}.$$

(1) 求基 $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ 到基 $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$ 的转换矩阵.

(2) 求 γ 分别在这两个基下的坐标.

解：(1) 设 $P \in GL_3(\mathbb{Q})$ 使得，

$$(\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3) = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3) \cdot P$$

典型错误：$P = BA^{-1}$.

令 $B = (\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3)$ $A = (\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3)$ 则转换矩阵 $P = A^{-1}B$.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix}$$

另：作初等行变换

$$(A|E) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{F_{3,1}(1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{F_{1,2}(-2)} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow[F_{3,2}(-3)]{F_3(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right)$$

$$\xrightarrow[F_{2,3}(-1)]{F_{1,3}(1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right)$$

$$P = A^{-1} \cdot B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -3 & -2 \\ 2 & 4 & 4 \end{pmatrix}$$

回顾：坐标变换：

$$\gamma = (\vec{\beta_1}\ \vec{\beta_2}\ \vec{\beta_3}) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (\vec{\alpha_1}\ \vec{\alpha_2}\ \vec{\alpha_3}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\vec{\alpha_1}\ \vec{\alpha_2}\ \vec{\alpha_3})\, P \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(2) 设. $\gamma = y_1\beta_1 + y_2\beta_2 + y_3\beta_3$. 即

$$B \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & -1 & 1 & 2 \\ 1 & 1 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$ 求得唯一一组解: $\begin{cases} y_1 = 1 \\ y_2 = 0 \\ y_3 = 2 \end{cases}$

γ 在基 $\{\beta_1, \beta_2, \beta_3\}$ 下的坐标为 $(1, 0, 2)^T$

γ 在基 $\{\alpha_1, \alpha_2, \alpha_3\}$ 下的坐标为 $P\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -3 & -2 \\ 2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 10 \end{pmatrix}$

另: 解另一非齐次线性方程组 $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

3. (1) 设 V 是域 $\mathbb{F}$ 上线性空间, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ 是 V 的一组基, $\boldsymbol{w}_i = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)\boldsymbol{x}_i$, $\boldsymbol{x}_i \in \mathbb{F}^n, i = 1, \ldots, k$. 证明: $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ 线性相关 $\Longleftrightarrow$ $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ 线性相关.

(2) 设 $\boldsymbol{w}_1 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2$, $\boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3$, $\boldsymbol{w}_3 = 2\boldsymbol{v}_2 - \boldsymbol{v}_3$. 求 $W = \langle \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \rangle$ 的一组基.

证明:(1) $\vec{w}_1,\cdots,\vec{w}_k$ 线性相关等价于存在不全为零的系数 $\lambda_1,\cdots,\lambda_k\in F$. s.t.

$$\underbrace{(w_1,\cdots,w_k)}_{1\times k}\underbrace{\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}}_{k\times 1}=0$$

又因为 $w_i=\underbrace{(v_1,\cdots,v_n)}_{1\times n}\underbrace{\vec{x}_i}_{n\times 1}$, $\vec{x}_i\in F^{n\times 1}$ 故 $\underbrace{(w_1,\cdots,w_k)}_{1\times k}=\underbrace{(v_1,\cdots,v_n)}_{1\times n}\underbrace{(\vec{x}_1,\cdots,\vec{x}_k)}_{n\times k}$

代入上式得:

$$(v_1,\cdots,v_n)(\vec{x}_1,\cdots,\vec{x}_k)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}=0 \quad (V\text{中的零元})$$

由于 $\{v_1,\cdots,v_n\}$ 是 V 的一组基.

$\{v_1,\cdots,v_n\}$ 是线性无关集 $(\vec{x}_1,\cdots,\vec{x}_k)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ 它等价于 $\vec{x}_1,\cdots,\vec{x}_k$ 线性相关.

(2)

初等行变换: $\begin{pmatrix}1&1&0\\2&0&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&0\\0&-2&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\end{pmatrix}$

$\vec{w}_3=\vec{w}_1-\vec{w}_2$. 且 $\vec{w}_1,\vec{w}_2$ 是线性无关的 故 W 的一组基是 $\{\vec{w}_1,\vec{w}_2\}$.

$V \cong \mathbb{F}^3$ 两个空间是同构的. 所有的线性性质都将被保持.

$\sigma: V \to \mathbb{F}^3$

$\vec{x} \mapsto (x_1, x_2, x_3)$

$\vec{x} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$.

典型错误: ① $\{(1\ 2\ 0)^t, (1\ 0\ 1)^t\}$ 是W的一组基.

② $\underbrace{(\lambda_1\vec{x}_1+\cdots+\lambda_k\vec{x}_k)}_{n\times 1}\underbrace{(v_1\cdots,v_n)}_{1\times n}=0$

注意矩阵的"形状". 实际上应该是 $(v_1\cdots v_n)(\lambda_1\vec{x}_1+\cdots+\lambda_k\vec{x}_k)=0$.

③ $(v_1,\cdots,v_n)(\vec{x}_1,\cdots,\vec{x}_k)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}=\vec{0}$

令 $V=(v_1,\cdots,v_n)$. 由于 $\{v_1,\cdots,v_n\}$ 是V的一组基. 故 $|V|\neq 0$

那么等式两边同时乘 V^{-1} 得 $(\vec{x}_1,\cdots,\vec{x}_k)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix}=\vec{0}$.

☆ 问题在于 v_i 是抽象的线性空间中的元素 不存在 $|V|$.

4. 设 $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$, 其中 $\mathbb{F}$ 是域, 证明:

(1) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

(2) $m - \operatorname{rank}(E_m - AB) = n - \operatorname{rank}(E_n - BA)$.

证明: (1) $\varphi_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ $\qquad$ $\varphi_B : \mathbb{F}^m \longrightarrow \mathbb{F}^n$

$\vec{x} \longmapsto A\vec{x}$ $\qquad$ $\vec{x} \mapsto B\cdot\vec{x}$.

$$\begin{tikzcd} \mathbb{F}^m \arrow[r, "\phi_B"] \arrow[dr, "\phi_{AB}=\phi_A\circ\phi_B"'] & \mathbb{F}^n \arrow[d, "\phi_A"] \\ & \mathbb{F}^m \end{tikzcd}$$

$\varphi_{A\cdot B} = \varphi_A \circ \varphi_B : \mathbb{F}^n \longrightarrow \mathbb{F}^m$

$\vec{x} \longmapsto AB\vec{x}$

$\operatorname{rank}(AB) = \dim_{\mathbb{F}} \operatorname{Im}\varphi_{AB}$ $\qquad$ $\dim_{\mathbb{F}} \operatorname{Im}\varphi_{AB} \leq \dim_{\mathbb{F}} \operatorname{Im}\varphi_A = \operatorname{rank} A$.

$\dim_{\mathbb{F}} \operatorname{Im}\varphi_{AB} = \dim_{\mathbb{F}} \varphi_A(\operatorname{Im}\varphi_B) \leq \dim_{\mathbb{F}} \operatorname{Im}\varphi_B = \operatorname{rank} B$ $\quad\square$.

行 $\quad R_{AB} \quad$ B $\quad$ 行 $\quad R_B$

另: 设 AB 的列向量生成的线性空间 V_{AB}. A 的列向量生成的线性空间为 V_A.

$A_i \in \mathbb{F}^{m\times 1}$ $\qquad AB = (A_1 \cdots A_n)\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm}\end{pmatrix}_{n\times m} = \left(\sum_{i=1}^n b_{i1}A_i, \cdots, \sum_{i=1}^n b_{im}A_i\right)$

$\operatorname{rank}(AB) = \dim V_{AB} \leq \dim V_A = \operatorname{rank} A$

$B_i \in \mathbb{F}^{m\times 1}$ $\qquad AB = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}_{m\times n}\begin{pmatrix} B_1 \\ \vdots \\ B_n\end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i}B_i \\ \vdots \\ \sum_{i=1}^n a_{mi}B_i\end{pmatrix}$

$\operatorname{rank}(AB) = \dim R_{AB} \le \dim R_B = \operatorname{rank} B$.

典型错误: $\ker(A) \subseteq \ker(AB)$ ✗　　$\ker(B) \subseteq \ker(AB)$ ✓.

(2) 设 $\sigma: \underset{\mathbb{F}^m}{\ker(E_m - AB)} \to \underset{\mathbb{F}^n}{\ker(E_n - BA)}$

$$\vec{x} \mapsto B\vec{x}.$$

验证 $\sigma \in \operatorname{Hom}(\mathbb{F}^m, \mathbb{F}^n)$. 任取 $\vec{x} \in \ker(E_m - AB)$. 那么 $AB\vec{x} = \vec{x}$.

$BA(B\vec{x}) = B\vec{x}$. 即 $BA\cdot\sigma(\vec{x}) = \sigma(\vec{x}) \Rightarrow \sigma(\vec{x}) \in \ker(E_n - BA)$.

设 $\mu: \ker(E_n - BA) \to \ker(E_m - AB)$

$$\vec{x} \mapsto A\vec{x}.$$

同理可得: $\mu \in \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$

$\forall \vec{x} \in \ker(E_m - AB)$, $\mu \circ \sigma(x) = A\cdot B\vec{x} = \vec{x}$.

$\forall \vec{x} \in \ker(E_n - BA)$　$\sigma \circ \mu(x) = B\cdot A\vec{x} = \vec{x}$.

$\Rightarrow \mu\circ\sigma = \sigma\circ\mu = id$.

满　单　满　单

故 σ 是线性同构. 即 $\ker(E_m - AB) \cong \ker(E_n - BA)$

$$m - \operatorname{rank}(E_m - AB) = n - \operatorname{rank}(E_n - BA)$$

另: 验证 σ 是单射且是满射. 设 $\vec{x} \in \ker(\sigma)$. 则 $B\vec{x} = 0$. 又由 $\vec{x} - AB\vec{x} = 0 \Rightarrow \vec{x} = 0$.

设 $\vec{y} \in \ker(E_n - BA)$ 则 $\vec{y} = BA\cdot\vec{y}$. 取 $\vec{x} = A\vec{y}$.　$A\vec{y} - AB\cdot(A\vec{y}) = \vec{0} \Rightarrow \vec{x} \in \ker(E_m - AB)$ 故 $\vec{y} = \sigma(\vec{x})$.

典型错误：　由σ：$\dim\ker(E_n-BA)\le\dim\ker(E_m-AB)$

像空间的维数小于原空间的维数，需要说明σ是满射。

矩阵解法：$A\in\mathbb{F}^{m\times n}$　$B\in\mathbb{F}^{n\times m}$

$$\begin{pmatrix} E_m-AB & 0 \\ 0 & E_n \end{pmatrix}_{(m+n)\times(m+n)} \xrightarrow{\text{列变换}} \begin{pmatrix} E_m-AB & 0 \\ B_{n\times m} & E_n \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix}$$

$$\xrightarrow{\text{列变换}} \begin{pmatrix} E_m & 0 \\ B & E_n-BA \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} E_m & 0 \\ 0 & E_n-BA \end{pmatrix}$$

5. 设 $V=\mathbb{R}[x]^{(n)}$. 对 $i=0,1,\ldots,n$, 定义:

$$\begin{aligned}\phi_i\colon V &\longrightarrow \mathbb{R}\\ f &\longmapsto \frac{1}{i!}\frac{d^i f}{dx^i}(0).\end{aligned}$$

验证 $\phi_0,\phi_1,\ldots,\phi_{n-1}$ 是 $1,x,\ldots,x^{n-1}$ 的对偶基.

验证：(1) $\forall i$, $\phi_i \in Hom(V, F)$

$\forall f, g \in \mathbb{R}[x]^{(n)} = \{ p \in \mathbb{R}[x] \mid deg(p) = 0, 1, \cdots, n-1 \}$

$\forall \alpha, \beta \in \mathbb{R}$. $\phi_i(\alpha \cdot f + \beta \cdot g) = \frac{1}{i!} \frac{d^i}{dx^i}(\alpha \cdot f + \beta \cdot g)\Big|_{x=0} = \alpha \cdot \frac{1}{i!} \frac{d^i f}{dx^i}(0) + \beta \cdot \frac{1}{i!} \frac{d^i g}{dx^i}(0)$

$= \alpha \cdot \phi_i(f) + \beta \cdot \phi_i(g)$.

(2) $\phi_i(x^j) = \delta_{i,j}$. $i, j = 0, 1, \cdots, n-1$.

$\frac{1}{i!} \frac{d^i}{dx^i}(x^j)\Big|_{x=0} = \delta_{i,j}$.

此定义可结合"基"的两条性质。见上周讲义 对偶空间简介 定理 6.1.

① 设 $\alpha_0, \cdots, \alpha_{n-1} \in \mathbb{R}$. s.t. $\alpha_0 \phi_0 + \cdots + \alpha_{n-1} \phi_{n-1} = 0^*$ (V^*的零元).

那么 $\alpha_0 = \cdots = \alpha_{n-1} = 0$.

② $\forall f \in V^*$. 那么取 $\beta_i = f(x^i)$ $i = 0, \cdots, n-1$.

$f = \beta_0 \phi_0 + \cdots + \beta_{n-1} \phi_{n-1}$

补充练习:

Frobenius不等式: $A\in\mathbb{F}^{m\times n}$, $B\in\mathbb{F}^{n\times s}$, $C\in\mathbb{F}^{s\times t}$ 则: $\mathrm{rank}(ABC)\geq \mathrm{rank}(AB)+\mathrm{rank}(BC)-\mathrm{rank}(B)$.

证明: $\mathrm{rank}(ABC)+\mathrm{rank}(B)\geq \mathrm{rank}(AB)+\mathrm{rank}(BC)$.

$$M:=\begin{pmatrix} ABC & 0 \\ 0 & B\end{pmatrix}\to\begin{pmatrix} ABC & AB \\ 0 & B\end{pmatrix}\to\begin{pmatrix} 0 & AB \\ -BC & B\end{pmatrix}:=N$$

$$\mathrm{rank}(ABC)+\mathrm{rank}(B)=\mathrm{rank}(M)=\mathrm{rank}(N)\geq \mathrm{rank}(AB)+\mathrm{rank}(BC)$$

思考: 从线性映射的角度解释Frobenius不等式.

$\phi_A:\mathbb{F}^n\to\mathbb{F}^m$ $\quad\phi_B:\mathbb{F}^s\to\mathbb{F}^n$ $\quad\phi_C:\mathbb{F}^t\to\mathbb{F}^s$.

$$\begin{tikzcd}
\mathbb{F}^t \arrow[r, "\phi_C"] \arrow[rr, bend right, "\phi_{BC}"'] \arrow[rrr, bend right=50, "\phi_{ABC}"'] & \mathbb{F}^s \arrow[r, "\phi_B"] \arrow[rr, bend right, "\phi_{AB}"'] & \mathbb{F}^n \arrow[r, "\phi_A"] & \mathbb{F}^m.
\end{tikzcd}$$

用$K(M)$代表$\ker(M)$的维数.

$$\cancel{m}-K(ABC)\geq\cancel{m}-K(AB)+\cancel{n}-K(BC)-\cancel{n}+K(B)$$

$$\Leftrightarrow\quad K(BC)+K(AB)\geq K(ABC)+K(B).$$

$$\Leftrightarrow\quad K(AB)-K(B)\geq K(ABC)-K(BC).$$

$\mathbb{F}^s\cong\ker(\phi_B)\oplus\mathrm{Im}(\phi_B)$, $\quad\mathrm{Im}(\phi_B)\cong(\mathrm{Im}(\phi_B)\cap\ker(\phi_A))\oplus\mathrm{Im}(\phi_{AB})$

$\mathbb{F}^t\cong\ker(BC)\oplus\mathrm{Im}(\phi_{BC})$, $\quad\mathrm{Im}(\phi_{BC})\cong(\mathrm{Im}(\phi_{BC})\cap\ker(\phi_A))\oplus\mathrm{Im}(\phi_{ABC})$

$$\mathrm{Im}(\phi_B)\cap\ker(\phi_A)\supseteq\mathrm{Im}(\phi_{BC})\cap\ker(\phi_A)$$

$$\therefore\quad K(AB)-K(B)\geq K(ABC)-K(BC).$$

令 $n=s$，$B=I_n$ 则.

Sylvester 不等式：$\operatorname{rank}(A\cdot C) \geq \operatorname{rank}(A)+\operatorname{rank}(C)-n$

$$\phi_A: \mathbb{F}^n \to \mathbb{F}^m \qquad \phi_C: \mathbb{F}^t \to \mathbb{F}^n$$

取 $\operatorname{Im}\phi_C \cap \ker\phi_A$ 的一组基 $\{u_1,\cdots,u_k\}$ 再由此扩充到 $\operatorname{Im}\phi_C$ 的一组基 $\{u_1,\cdots,u_k,v_1,\cdots,v_d\}$.

那么：$\operatorname{rank}(AC) = \dim(\phi_A(\operatorname{Im}\phi_C)) = d$.

$\operatorname{rank}(A) = n - \dim\ker(\phi_A) \leq n-k$.

$\operatorname{rank}(A)+\operatorname{rank}(C)-n \leq n-k+k+d-n = d$.

$$\begin{pmatrix} AC & 0 \\ 0 & I_n \end{pmatrix} \longrightarrow \begin{pmatrix} AC & A \\ 0 & I_n \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & A \\ -C & I_n \end{pmatrix}$$

复习矩阵秩(不)等式：

$$\operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{rank}(A)+\operatorname{rank}(B) \qquad \operatorname{rank}\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \geq \operatorname{rank}(A)+\operatorname{rank}(B).$$

$$\operatorname{rank}\begin{pmatrix} A_{11} & & \\ & A_{22} & \\ & & \ddots & \\ & & & A_{nn} \end{pmatrix} = \sum_{i=1}^{n} \operatorname{rank}(A_{ii}) \qquad \operatorname{rank}\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \geq \operatorname{rank}(A_{ij}) \quad \forall i,j.$$

$\mathrm{rank}(A+B) \leqslant \mathrm{rank}(A) + \mathrm{rank}(B)$.

$\mathrm{rank}(AB) \neq \mathrm{rank}(BA)$

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$