1. 设 $\boldsymbol{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix}0\\2\\-1\end{pmatrix}$.

(1) 计算 $V = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ 的一组基和 $\mathbb{Q}^3/V$ 的维数.

(2) 令 $\boldsymbol{w} = \begin{pmatrix}3\\2\\2\end{pmatrix}$, 判断 $\boldsymbol{w}$ 是否为 $\boldsymbol{v}_1, \boldsymbol{v}_2$ 的线性组合.

1. (1) $\begin{pmatrix}1 & 1 & 0\\2 & 0 & 2\\0 & 1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 0\\0 & -2 & 2\\0 & 1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 0\\0 & -2 & 2\\0 & 0 & 0\end{pmatrix}$

因为 $\operatorname{rank}(A)=2$，所以 $\dim(V)=2$，又因为 $\vec{v}_1$ 与 $\vec{v}_2$ 线性无关，所以 $\vec{v}_1, \vec{v}_2$ 是 V 的一组基

$\dim(\mathbb{Q}^3/V) = \dim(\mathbb{Q}^3) - \dim(V) = 3-2=1.$

(2) 设存在 $\alpha_1, \alpha_2 \in \mathbb{Q}$，使得

$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$

则 $\begin{pmatrix}3\\2\\2\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\2\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\Rightarrow \begin{cases} 3 = \alpha_1 + \alpha_2 \\ 2 = 2\alpha_1 \\ 2 = \alpha_2 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 2 \end{cases}$

所以 $\vec{w} = \vec{v}_1 + 2\vec{v}_2 \qquad \vec{w} = (\vec{v}_1, \vec{v}_2)\begin{pmatrix}1\\2\end{pmatrix}$ $\square$

2. 设在 $\mathbb{Q}^3$ 中, $\boldsymbol{\alpha}_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}, \boldsymbol{\alpha}_2=\begin{pmatrix}2\\1\\1\end{pmatrix}, \boldsymbol{\alpha}_3=\begin{pmatrix}1\\1\\1\end{pmatrix},$

$$\boldsymbol{\beta}_1=\begin{pmatrix}0\\1\\1\end{pmatrix}, \boldsymbol{\beta}_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}, \boldsymbol{\beta}_3=\begin{pmatrix}1\\2\\1\end{pmatrix} \text{ 且 } \boldsymbol{\gamma}=\begin{pmatrix}2\\5\\3\end{pmatrix}.$$

(1) 求基 $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$ 到基 $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3$ 的转换矩阵.

(2) 求 $\boldsymbol{\gamma}$ 分别在这两个基下的坐标.

解:(1) $(\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_3)=(\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_3)P$

$$\underbrace{\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}}_{B}=\underbrace{\begin{pmatrix}1&2&1\\0&1&1\\-1&1&1\end{pmatrix}}_{A}P$$

$$P=\begin{pmatrix}1&2&1\\0&1&1\\-1&1&1\end{pmatrix}^{-1}\begin{pmatrix}0&-1&1\\1&1&2\\1&0&1\end{pmatrix}.$$

求 A^{-1}:

$$\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\-1&1&1&0&0&1\end{pmatrix}$$

$$\longrightarrow\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\0&3&2&1&0&1\end{pmatrix}$$

$$\longrightarrow\begin{pmatrix}1&2&1&1&0&0\\0&1&1&0&1&0\\0&0&-1&1&-3&1\end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{pmatrix}$$

所以 $A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix}$

所以 $P = A^{-1} \cdot B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 1 \\ -1 & -3 & -2 \\ 2 & 4 & 4 \end{pmatrix}$$

(2) 设 $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} 2 = x_1 + 2x_2 + x_3 \\ 5 = x_2 + x_3 \\ 3 = -x_1 + x_2 + x_3 \end{cases}$$

增广矩阵:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ -1 & 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -1 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -8 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 10 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 2 \\ x_2 = -5 \\ x_3 = 10 \end{matrix}$$

$$\Rightarrow y = 2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 5\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 10\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

设 $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = y_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + y_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} 2 = -y_2 + y_3 \\ 5 = y_1 + y_2 + 2y_3 \\ 3 = y_1 + y_3 \end{cases} \Rightarrow \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad y_1 = 1 \quad y_2 = 0 \quad y_3 = 2$$

$$\Rightarrow \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{D}$$

3.

3. (1) 设 V 是域 $\mathbb{F}$ 上线性空间, $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n$ 是 V 的一组基, $\boldsymbol{w}_i=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\boldsymbol{x}_i$, $\boldsymbol{x}_i\in\mathbb{F}^n, i=1,\dots,k$. 证明: $\boldsymbol{w}_1,\dots,\boldsymbol{w}_k$ 线性相关 $\Longleftrightarrow$ $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k$ 线性相关.

(2) 设 $\boldsymbol{w}_1=\boldsymbol{v}_1+2\boldsymbol{v}_2$, $\boldsymbol{w}_2=\boldsymbol{v}_1+\boldsymbol{v}_3$, $\boldsymbol{w}_3=2\boldsymbol{v}_2-\boldsymbol{v}_3$. 求 $W=\langle\boldsymbol{w}_1,\boldsymbol{w}_2,\boldsymbol{w}_3\rangle$ 的一组基.

证明: ($\Rightarrow$): $\vec{w}_1,\dots,\vec{w}_k$ 线性相关, 所以 $\exists\, c_1,c_2\cdots c_k\in\mathbb{F}$, 不全为零, s.t.

$$c_1\vec{w}_1+c_2\vec{w}_2+\cdots+c_k\vec{w}_k=\vec{0}$$

$$\Rightarrow c_1(\vec{v}_1,\cdots,\vec{v}_n)\vec{x}_1+c_2(\vec{v}_1,\cdots,\vec{v}_n)\vec{x}_2+\cdots+c_k(\vec{v}_1,\cdots,\vec{v}_n)\vec{x}_k=\vec{0}$$

$\Rightarrow (\vec{v}_1,\cdots,\vec{v}_n)(c_1\vec{x}_1+c_2\vec{x}_2+\cdots+c_k\vec{x}_k)=\vec{0}$, $(\vec{v}_1,\cdots,\vec{v}_n)$ 满秩

$(\vec{v}_1,\cdots,\vec{v}_n)\neq 0$ 不够!!

$$\Rightarrow c_1\vec{x}_1+c_2\vec{x}_2+\cdots+c_k\vec{x}_k=\vec{0}$$

$\Rightarrow \vec{x}_1,\vec{x}_2,\cdots\vec{x}_k$ 线性相关.

($\Leftarrow$): $\vec{x}_1,\cdots,\vec{x}_k$ 线性相关, $\Rightarrow \exists\, c_1,\cdots,c_k\in\mathbb{F}$ 不全为0, s.t.

$$c_1\vec{x}_1+c_2\vec{x}_2+\cdots+c_k\vec{x}_k=\vec{0}$$

$$\Rightarrow (\vec{v}_1,\cdots,\vec{v}_n)(c_1\vec{x}_1+\cdots+c_k\vec{x}_k)=\vec{0}$$

$$\Rightarrow c_1(\vec{v}_1\cdots\vec{v}_n)\vec{x}_1+\cdots+c_k(\vec{v}_1,\cdots,\vec{v}_n)\vec{x}_k=\vec{0}$$

$$\Rightarrow c_1\vec{w}_1+\cdots+c_k\vec{w}_k=\vec{0}$$

$\Rightarrow \vec{w}_1,\cdots,\vec{w}_k$ 线性相关.

(2) $\vec{w}_1=(\vec{v}_1,\vec{v}_2,\vec{v}_3)\begin{pmatrix}1\\2\\0\end{pmatrix}$ $\vec{w}_2=(\vec{v}_1,\vec{v}_2,\vec{v}_3)\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\vec{w}_3=(\vec{v}_1,\vec{v}_2,\vec{v}_3)\begin{pmatrix}0\\2\\-1\end{pmatrix}$

$$\begin{pmatrix}1&1&0\\2&0&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&0\\0&-2&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&0\\0&-2&2\\0&0&0\end{pmatrix}$$

所以 $\dim W=2$ 又因为 $\begin{pmatrix}1\\2\\0\end{pmatrix}$ & $\begin{pmatrix}1\\0\\1\end{pmatrix}$ 线性无关 所以 $\vec{w}_1$ & $\vec{w}_2$ 线性无关

可以作为W的一组基. $\square$

4. 设 $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$, 其中 $\mathbb{F}$ 是域, 证明:

(1) $\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

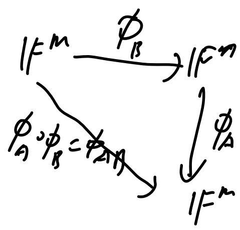
(2) $m - \operatorname{rank}(E_m - AB) = n - \operatorname{rank}(E_n - BA)$.

4. 证: (1) 设 $\phi_A: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ $\quad$ $\phi_B: \mathbb{F}^m \longrightarrow \mathbb{F}^n$

$\vec{x} \longmapsto A\vec{x}$ $\quad$ $\vec{x} \longmapsto B\vec{x}$

则 $\phi_A \circ \phi_B: \mathbb{F}^m \longrightarrow \mathbb{F}^m$

$\vec{x} \longmapsto AB\vec{x}$



故 $\phi_A \circ \phi_B = \phi_{AB}$

其中 $\operatorname{rank}(AB) = \dim(\operatorname{im}\phi_{AB})$ $\quad$ $\operatorname{rank}(A) = \dim(\operatorname{im}\phi_A)$

$\operatorname{rank}(B) = \dim(\operatorname{im}\phi_B)$

$\operatorname{im}\phi_{AB} \subseteq \operatorname{im}\phi_A$ 所以 $\operatorname{rank}(AB) \le \operatorname{rank}(A)$ ①

And $\ker(B) \subseteq \ker(AB)$

又因为 $\dim(\ker(B)) = m - \dim(\operatorname{im}(\phi_B)) = m - \operatorname{rank}(B)$

$\dim(\ker(AB)) = m - \dim(\operatorname{im}(\phi_{AB})) = m - \operatorname{rank}(AB)$

$m - \operatorname{rank}(B) \le m - \operatorname{rank}(AB)$

$\Longrightarrow \operatorname{rank}(AB) \le \operatorname{rank}(B)$ ②

①+② $\Longrightarrow \operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

(2) $\phi_{E_m-AB}: \mathbb{F}^m \longrightarrow \mathbb{F}^m$ $\quad$ $\phi_{E_n-BA}: \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$\vec{x} \longmapsto (E_m - AB)\cdot\vec{x}$ $\quad$ $\vec{x} \longmapsto (E_n - BA)\vec{x}$

$$m-\text{rank}(E_m-AB)=\dim(\ker\varphi_{E_m-AB})$$

$$n-\text{rank}(E_n-BA)=\dim(\ker\varphi_{E_n-BA})$$

定义映射 $\psi: \ker(\varphi_{E_m-AB}) \longrightarrow \ker(\varphi_{E_n-BA})$

$$\vec{x} \longmapsto B\vec{x}.$$

断言: ψ 是同构映射

①首先,验证 ψ 良定义, i.e. $\forall \vec{x}\in\ker(\varphi_{E_m-AB})$, $B\vec{x}\in\ker(\varphi_{E_n-BA})$

因为 $\vec{x}\in\ker(\varphi_{E_m-AB})$, 所以 $\varphi_{E_m-AB}(\vec{x})=\vec{x}-AB\vec{x}=\vec{0} \Rightarrow \vec{x}=AB\vec{x}$

$$\begin{aligned}\varphi_{E_n-BA}(B\vec{x}) &= (E_n-BA)\cdot(B\vec{x}) \\ &= B\vec{x}-BAB\vec{x} \\ &= B\vec{x}-B\vec{x}=\vec{0}\end{aligned}$$

良定义

② 证 ψ 既单又满

单: 设 $\exists\, \vec{x}\in\ker(\varphi_{E_m-AB})$, s.t. $\psi(\vec{x})=\vec{0}$ 则 $B\vec{x}=\vec{0}$ ①

又因为 $\varphi_{E_m-AB}(\vec{x})=\vec{x}-AB\vec{x}=\vec{0} \Rightarrow \vec{x}=AB\vec{x}$ ②

①+② $\Rightarrow AB\vec{x}=\vec{0}=\vec{x} \Rightarrow \vec{x}=\vec{0}$, 所以 ψ 单

满: $\forall \vec{y}\in\ker(\varphi_{E_n-BA})$, $\varphi_{E_n-BA}(\vec{y})=\vec{y}-BA\vec{y}=\vec{0}$

$\Rightarrow \vec{y}=BA\vec{y}=B(A\vec{y})$, 下证 $A\vec{y}\in\ker(\varphi_{E_m-AB})$

$$\varphi_{E_m-AB}(A\vec{y})=A\vec{y}-ABA\vec{y}=A\vec{y}-A\vec{y}=\vec{0}$$

$\Rightarrow \vec{y}\in \text{im}(\psi)$ 所以 ψ 满.

③ ψ 为线性映射　$\forall \alpha, \beta \in \mathbb{F}$,　$\vec{x}, \vec{y} \in \ker(\phi_{E_m-AB})$

$$\psi(\alpha\vec{x}+\beta\vec{y}) = B(\alpha\vec{x}+\beta\vec{y})$$
$$= \alpha B\vec{x} + \beta B\vec{y}$$
$$= \alpha\psi(\vec{x}) + \beta\psi(\vec{y}).$$

$\Longrightarrow$ ψ 是线性空间上的同构.

$\Longrightarrow \dim(\ker(\phi_{E_m-AB})) = \dim(\ker(\phi_{E_n-BA}))$.

$\square$

法二:
$$\operatorname{rank}(AB) = \dim(\operatorname{im}(\phi_{AB}))$$
$$= \dim(\phi_A(\operatorname{im}(\phi_B)))$$
$$\le \dim(\operatorname{im}(\phi_B))$$
$$= \operatorname{rank}(B)$$

Prop: 若 $v_1, \cdots, v_k$ 是 $\operatorname{im}(\phi_B)$ 中的一组基 则 $\phi_A(v_1), \cdots, \phi_A(v_k)$ 是 $\phi_A(\operatorname{im}(\phi_B))$ 中的一组生成元.

类似地
$$\operatorname{rank}(AB) = \dim(\phi_A(\operatorname{im}(\phi_B)))$$
$$\le \dim(\phi_A(\mathbb{F}^n)) \quad (\operatorname{im}\phi_B \subseteq \mathbb{F}^n)$$
$$= \dim(\operatorname{im}(\phi_A)) \quad (\phi_A(\mathbb{F}^n) = \operatorname{im}(\phi_A))$$
$$= \operatorname{rank}(A).$$

$\square$

(2): 设 $M = \begin{pmatrix} E_m - AB & 0 \\ 0 & E_n \end{pmatrix} \in M_{m+n}(\mathbb{R})$.

我(ii)计算 $N := \underbrace{\begin{pmatrix} E_m & A \\ 0 & E_n \end{pmatrix}}_{C_1} \cdot M = \begin{pmatrix} E_m - AB & A \\ 0 & E_n \end{pmatrix}$

$$P := N \underbrace{\begin{pmatrix} E_m & 0 \\ B & E_n \end{pmatrix}}_{C_2} = \begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix}$$

$$Q := \underbrace{\begin{pmatrix} E_m & 0 \\ -B & E_n \end{pmatrix}}_{C_3} \cdot P = \begin{pmatrix} E_m & A \\ 0 & E_n - BA \end{pmatrix}$$

$$R := Q \underbrace{\begin{pmatrix} E_m & -A \\ 0 & E_n \end{pmatrix}}_{C_4} = \begin{pmatrix} E_m & 0 \\ 0 & E_n - BA \end{pmatrix}$$

注意到 $C_1, C_2, C_3, C_4 \in M_{m+n}(\mathbb{R})$，且

$$\operatorname{rank}(C_i) \geq \operatorname{rank}(E_m) + \operatorname{rank}(E_n) = m+n$$

故 C_1, C_2, C_3, C_4 都满秩，$\Longrightarrow \operatorname{rank}(M) = \operatorname{rank}(R)$

$\Longrightarrow \operatorname{rank}(E_m - AB) + n = \operatorname{rank}(E_n - BA) + m$

$\square$

注意：若矩阵具有以下四种分块形式之一。

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \quad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \begin{pmatrix} C & A \\ B & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

则 $\operatorname{rank}(M) \geq \operatorname{rank}(A) + \operatorname{rank}(B)$ 且当 $C=0$ 时等号成立.

作业中错误：$M = \begin{pmatrix} E_n & B \\ A & E_m \end{pmatrix} \longrightarrow \begin{pmatrix} E_n & B \\ 0 & E_m - AB \end{pmatrix}$ 则 $\operatorname{rank}(M) = \operatorname{rank}(E_m - AB) + n$ ✗ 只有 $\geq$

雷健俊结化

5. 5. 设 $V = \mathbb{R}[x]^{(n)}$. 对 $i = 0, 1, \ldots, n$, 定义:

$$\phi_i \colon V \longrightarrow \mathbb{R}$$
$$f \longmapsto \frac{1}{i!}\frac{d^i f}{dx^i}(0).$$

验证 $\phi_0, \phi_1, \ldots, \phi_{n-1}$ 是 $1, x, \ldots, x^{n-1}$ 的对偶基.

证明: 对 $V = \mathbb{R}[x]^{(n)}$, 它的一组基为 $\{1, x, \cdots, x^{n-1}\}$

由定理6.1的证明, 只须验证 $\phi_i(x^j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$, $\forall\ 0 \leq j \leq n-1$

当 $j=i$ 时 $\phi_i(x^i) = \frac{1}{i!}\frac{d^i x^i}{dx^i}(0) = \frac{i!}{i!}(0) = 1$

当 $j>i$ 时 $\phi_i(x^j) = \frac{1}{i!}\frac{d^i x^j}{dx^i}(0)$

$$= \frac{1}{i!} \cdot j(j-1)\cdots(j-i+1) \cdot x^{j-i}(0)$$

$$= 0$$

当 $j<i$ 时 $\frac{d^i x^j}{dx^i} = 0$ 所以此时也有 $\phi_i(x^j) = 0$

$\square$

补充(秩不等式): (Frobenius) $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times s}$, $C \in \mathbb{F}^{s \times t}$, 则

$r(ABC) \geq r(AB) + r(BC) - r(B)$ where $r := \text{rank}$

Pf: $\phi_{AB} : \mathbb{F}^s \longrightarrow \mathbb{F}^m$ $\quad \phi_{BC} : \mathbb{F}^t \longrightarrow \mathbb{F}^n$

$\phi_{ABC} : \mathbb{F}^t \longrightarrow \mathbb{F}^m$ $\quad \phi_B : \mathbb{F}^s \longrightarrow \mathbb{F}^n$

$$\gamma(ABC) = t - \dim(\ker(\varphi_{ABC}))$$

$$\gamma(AB) = s - \dim(\ker(\varphi_{AB}))$$

$$\gamma(BC) = t - \dim(\ker(\varphi_{BC}))$$

$$\gamma(B) = s - \dim(\ker \varphi_B)$$

$$t - \dim(K_{ABC}) \geq s - \dim(K_{AB}) + (t - \dim(K_{BC})) - (s - \dim(K_B))$$

$$\Rightarrow t - \dim(K_{ABC}) \geq t - \dim(K_{AB}) - \dim(K_{BC}) + \dim(K_B)$$

$$\Rightarrow \dim(K_{AB}) - \dim(K_B) \geq \dim(K_{ABC}) - \dim(K_{BC})$$

$K_B \subseteq K_{AB}$　　$K_{BC} \subseteq K_{ABC}$

定义映射：ϕ: $K_{ABC}/K_{BC} \longrightarrow K_{AB}/K_B$

$$\alpha + K_{BC} \longmapsto \phi_C(\alpha) + K_B, \quad \alpha \in K_{ABC}$$

良定义：$\alpha \in K_{ABC}$　$\varphi_{ABC}(\alpha) = 0 \Rightarrow \phi_A \circ \phi_B \circ \phi_C(\alpha) = 0$

$$\Rightarrow \phi_A \circ \phi_B(\phi_C(\alpha)) = 0 \Rightarrow \phi_C(\alpha) \in K_{AB}$$

若 $\bar{\alpha} = \bar{\beta} \in K_{ABC}/K_{BC}$　i.e. $\alpha + K_{BC} = \beta + K_{BC}$, $\alpha - \beta \in K_{BC}$

$\Rightarrow \phi_{BC}(\alpha - \beta) = \phi_B \circ \phi_C(\alpha - \beta) = 0$, 也就是 $\phi_C(\alpha - \beta) \in \ker \phi_B = K_B$

则 $\phi(\bar{\alpha}) - \phi(\bar{\beta}) = \phi_C(\alpha) + K_B - (\phi_C(\beta) + K_B)$

$$= (\phi_C(\alpha) - \phi_C(\beta)) + K_B$$

$$= \varphi_C(\alpha-\beta)+K_B$$
$$= \bar{0}+K_B$$

$\Rightarrow \varphi(\bar{\alpha})=\varphi(\bar{\beta})$ 在 K_{AB}/K_B 中 所以 φ 良定义.

易证 φ 线性映射.

$\forall \bar{\alpha} \in \ker\varphi$, $\varphi(\bar{\alpha})=\varphi_C(\alpha)+K_B = 0+K_B$

$\Rightarrow \varphi_C(\alpha) \in K_B \Rightarrow \varphi_{BC}(\alpha)=0$

$\Rightarrow \alpha \in K_{BC} \Rightarrow \bar{\alpha}=\alpha+K_{BC}=\bar{0}$

故 φ 为单射.

从而 $\dim(K_{AB}/K_B) \geq \dim(K_{ABC}/K_{BC})$

故 $r(ABC) \geq r(AB)+r(BC)-r(B)$. □

法二:(分块矩阵法):

$$M=\begin{pmatrix} ABC & 0 \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} ABC & AB \\ 0 & B \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & AB \\ -BC & B \end{pmatrix}$$

所以 $\operatorname{rank}(M)=\operatorname{rank}(ABC)+\operatorname{rank}(B) \geq \operatorname{rank}(AB)+\operatorname{rank}(BC)$

$\Rightarrow \operatorname{rank}(ABC) \geq \operatorname{rank}(AB)+\operatorname{rank}(BC)-\operatorname{rank}(B)$. □