第五次习题课.

- 置换的奇偶性：

奇置换：可写成奇数个对换的乘积.

偶置换：可写成偶数个对换的乘积.

- 置换的阶：　$\sigma \in S_n$. 使得 $\sigma^k = e$ 的最小正整数称为 σ 的阶.

σ 写成互不相交的循环 $\tau_1 \tau_2 \cdots \tau_k$ 之积.

Cor 6.17：$\mathrm{ord}(\sigma) = \mathrm{lcm}(\mathrm{ord}(\tau_1), \cdots, \mathrm{ord}(\tau_k))$

1. 判断以下置换的奇偶性：

(1)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 8 & 7 & 1 & 6 & 4 & 3 \end{pmatrix}$$

(2)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}$$

(1) $\sigma = (125)(38)(47)(6) = (52)(51)(38)(47)$

$\Rightarrow$ 偶置换.

(2) $\sigma = (1\,n)(2\,n-1)(3\,n-2)\cdots(k,\,n+1-k)$

① 当 $n=4s$ 时　$k=2s$　$\Rightarrow$ 偶置换

② 当 $n=4s+1$ 时　$k=2s$　$\Rightarrow$ 偶置换

③ 当 $n=4s+2$ 时　$k=2s+1$　$\Rightarrow$ 奇置换

④ 当 $n=4s+3$ 时　$k=2s+1$　$\Rightarrow$ 奇置换.

2. 设 $\sigma \in S_n$ 为置换.

(1) 若 $\mathrm{ord}(\sigma)$ 为奇数，证明 σ 为偶置换；

(2) 当 $\mathrm{ord}(\sigma)$ 为偶数时，σ 是否一定为奇置换？给出证明或举出反例.

(1) 由 Cor 6.17，$ord(\sigma) = lcm(ord(\tau_1), \cdots, ord(\tau_k))$ 为奇数.

$\Rightarrow$ $ord(\tau_1), \cdots, ord(\tau_k)$ 都是奇数.

由 Cor 6.26，σ 的奇偶性与 $\sum_{i=1}^{k} \underbrace{(ord(\tau_i)-1)}_{\text{偶数}}$ 的相同.

$\Rightarrow$ σ 是偶置换

偶数相加仍是偶数.

(2) 不一定.

例：$\sigma = (12)(34)$ $\Rightarrow$ $ord(\sigma) = 2$ $\Rightarrow$ σ 偶置换.

3. (1) 求 $\gcd(35, 60)$, $\mathrm{lcm}(35, 60)$;

(2) 计算整数 u, v 使得 $60u + 35v = \gcd(35, 60)$ 成立 (写过程).

(1) 由扩展的辗转相除法：

设 $r_0 = 60$，$r_1 = 35$

$r_2 = rem(r_0, r_1) = rem(60, 35) = 25$

$r_3 = rem(r_1, r_2) = rem(35, 25) = 10$

$r_4 = rem(r_2, r_3) = rem(25, 10) = 5$

$r_5 = rem(r_3, r_4) = rem(10, 5) = 0$

$\Rightarrow gcd(60, 35) = r_4 = 5$

由命题 7.11 设 $m, n \in \mathbb{Z}^+$，则 $lcm(m,n) = \frac{mn}{gcd(m,n)}$

$lcm(35, 60) = \frac{35 \times 60}{5} = 420$

(2) $r_0 = 60$，$u_0 = 1$，$v_0 = 0$，$r_1 = 35$，$u_1 = 0$，$v_1 = 1$

$\Rightarrow$ $r_2 = rem(r_0, r_1) = 25$ $\qquad$ $r_3 = rem(r_1, r_2) = 10$

$q_2 = quo(r_0, r_1) = 1$ $\qquad$ $q_3 = quo(r_1, r_2) = 1$

$u_2 = u_0 - q_2 u_1 = 1$ $\qquad$ $u_3 = u_1 - q_3 u_2 = -1$

$v_2 = v_0 - q_2 v_1 = -1$ $\qquad$ $v_3 = v_1 - q_3 v_2 = 2$

$r_4 = rem(r_2, r_3) = 5$

$q_4 = quo(r_2, r_3) = 2$

$u_4 = u_2 - q_4 u_3 = 1 - 2\times(-1) = 3$

$v_4 = v_2 - q_4 v_3 = -1 - 2\times 2 = -5$

$r_5 = rem(r_3, r_4) = 0$

$\Rightarrow \quad 3\times 60 + (-5)\times 35 = 5 = \gcd(60, 35)$

4. 设 $n \in \mathbb{Z}^+$, $n \geq 3$, $a_1, a_2, \ldots, a_n \in \mathbb{Z}^+$. 证明:

(1) $\gcd(a_1, a_2, \ldots, a_n) = \gcd(\gcd(a_1, a_2, \ldots, a_{n-1}), a_n)$;

(2) 存在整数 $u_1, u_2, \ldots, u_n$，使得 $u_1a_1 + u_2a_2 + \cdots + u_na_n = \gcd(a_1, a_2, \ldots, a_n)$.

证: (1) 设 $g = \gcd(a_1, a_2, \cdots, a_n)$

$\Rightarrow g \mid \gcd(a_1, a_2, \cdots, a_{n-1})$

设 $\gcd(a_1, a_2, \cdots, a_{n-1}) = k\cdot g$ 且 $a_n = s\cdot g$

则 $(R.H.S.) = \gcd(kg, sg) = g\cdot\gcd(k, s)$

$\Rightarrow$ 只需证 $\gcd(k, s) = 1$

(反证法) 假设 $\gcd(k, s) = m > 1$

则 $mg \mid kg = \gcd(a_1, \cdots, a_{n-1})$ 和 $mg \mid sg = a_n$

$\Rightarrow$ 与 $g = \gcd(a_1, \cdots, a_n)$ 矛盾. 则 $m = 1$.

即 $\gcd(k, s) = 1$. 证毕!

(2). 由数学归纳法.

① 当 $n=2$ 时. 由扩展的辗转相除法. 成立.

$\exists\ u_1, u_2$ s.t. $u_1a_1 + u_2a_2 = \gcd(a_1, a_2)$

② 设 $n=k$ 时 成立 即 $\exists\ u_1, u_2, \cdots, u_k$ s.t.

$u_1a_1 + \cdots + u_ka_k = \gcd(a_1, \cdots, a_k)$

再由扩展的辗转相除法.

则 $n=k+1$ 时 $(u_1a_1 + \cdots + u_ka_k) + u_{k+1}a_{k+1} = \gcd(\gcd(a_1, \cdots, a_k), a_{k+1})$

由 (1) 可知 $(L.H.S.) = \gcd(a_1, \cdots, a_{k+1})$ 即成立.

- n 维列向量(坐标)空间:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \ x_1, x_2, \cdots, x_n \in \mathbb{R} \right\}$$

设 $w, v_1, v_2, \cdots, v_k \in \mathbb{R}^n$

- 线性组合: $\exists\ \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ s.t. $\alpha_1 v_1 + \cdots + \alpha_k v_k = w$
- 线性相关: $\exists$ 不全为零的 $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$. s.t. $\alpha_1 v_1 + \cdots + \alpha_k v_k = \vec{0}$
- 线性无关: $\alpha_1 v_1 + \cdots + \alpha_k v_k = \vec{0} \Rightarrow \alpha_1 = \cdots = \alpha_k = 0$.

5. 计算线性组合 $3\mathbf{v}_1 + 4\mathbf{v}_2 - 5\mathbf{v}_3$, 其中

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

$$3v_1 + 4v_2 - 5v_3 = 3\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + 4\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} - 5\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 17 \end{pmatrix}$$

6. 判断下述向量组是否线性无关

(1)

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}.$$

(1) 由于 $2v_1 - v_2 = v_3$ 即 $\exists$ 不全为零的 $s_1, s_2, s_3 \in \mathbb{R}$.

$s_1 v_1 + s_2 v_2 + s_3 v_3 = 0$

$\Rightarrow$ 线性相关.

(2)

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ -5 \\ 2 \\ 6 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 6 \\ -3 \\ 3 \\ 9 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 4 \\ -1 \\ 5 \\ 6 \end{pmatrix}.$$

(2) $A=(v_1,v_2,v_3,v_4)=\begin{pmatrix} 4 & 2 & 6 & 4 \\ -5 & -2 & -3 & -1 \\ 2 & 1 & 3 & 5 \\ 6 & 3 & 9 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & \frac{1}{2} & \frac{9}{2} & 19 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ 以A为系数矩阵的齐次线性方程组 $AX=0$ 有解. $\Rightarrow$ 线性相关.

7. 给定一个线性无关的向量组 $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$. 判断向量组

$b_1 = 3\alpha_1+2\alpha_2+\alpha_3+\alpha_4,\ b_2 = 2\alpha_1+5\alpha_2+3\alpha_3+2\alpha_4,\ b_3 = 3\alpha_1+4\alpha_2+2\alpha_3+3\alpha_4.$

是否线性相关.

$b_1=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ $\quad b_2=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix} 2 \\ 5 \\ 3 \\ 2 \end{pmatrix}$ $\quad b_3=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \end{pmatrix}$

$(b_1,b_2,b_3) = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix} 3 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

由 $A=\begin{pmatrix} 3 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 4 \\ 3 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 0 & -7 & -3 \\ 0 & -1 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad AX=0$ 只有平凡解

$\Rightarrow$ 线性无关.

8. 设 $\beta,\ \alpha_1,\ \alpha_2,\ldots,\alpha_k \in \mathbb{R}^n$ 且 $\alpha_1,\ \alpha_2,\ldots,\alpha_k$ 线性无关. 再设

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k,$$

其中 $a_1, a_2, \ldots, a_k \in \mathbb{R}$ 且 $a_1 \neq 0$. 证明: $\beta,\ \alpha_2,\ldots,\alpha_k$ 线性无关.

证：设 $u_1, u_2, \cdots, u_k \in \mathbb{R}$.

$$u_1\beta + u_2\alpha_2 + \cdots u_k\alpha_k = 0$$

又由 $\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k$.

$$\Rightarrow 0 = u_0 a_1 \alpha_1 + (u_0 a_2 + u_2)\alpha_2 + \cdots + (u_0 a_k + u_k)\alpha_k$$

由 $\alpha_1, \alpha_2, \cdots, \alpha_k$ 线性无关

$$\Rightarrow \begin{cases} u_1 a_1 = 0 \\ u_1 a_2 + u_2 = 0 \\ \vdots \\ u_1 a_k + u_k = 0 \end{cases}$$

由 $a_1 \neq 0 \Rightarrow u_1 = 0$

进而 $u_2 = \cdots = u_k = 0$ 即 $\beta, \alpha_2, \cdots, \alpha_k$ 线性无关.