行列式的计算

基本性质回顾：

1、转置：$|{}^tA| = |A|$

2、初等变换：
$\underline{2.1}$ $|F_{s,t}A| = -|A|$

$\underline{2.2}$ $|F_{s,t}(\lambda)\cdot A| = |A|$

$\underline{2.3}$ $|F_s(\lambda)\cdot A| = \lambda|A|$

3. 乘法：$|AB| = |A|\cdot|B|$

4. 展开公式：$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} A_{ij}$ （按第i行展开）

5. 分块矩阵：$\begin{vmatrix} A & C \\ O & B \end{vmatrix} = |A|\cdot|B|$

6. 上三角矩阵的行列式：$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & O & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} = \prod_{i=1}^{n} a_{ii}$

7、Sylvester 等式：$A \in \mathbb{R}^{n\times m}$，$B \in \mathbb{R}^{m\times n}$

$$|E_n + AB| = |E_m + BA|.$$

例1.

$$\begin{cases} x+y+z=1 \\ ax+by+cz=d \\ a^2x+b^2y+c^2z=d^2 \end{cases} \qquad (*)$$

系数矩阵:
$$A=\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

$$|A|=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=(b-a)(c-a)(c-b)$$

若 $|A|\neq 0$, 即 a,b,c 互不相同, 则方程组(*)有解且唯一, 其解为

$$\begin{cases} x=\dfrac{\begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix}}{|A|}=\dfrac{(b-d)(c-d)(c-b)}{(b-a)(c-a)(c-b)}=\dfrac{(b-d)(c-d)}{(b-a)(c-a)} \\[2em] y=\dfrac{\begin{vmatrix} 1 & 1 & 1 \\ a & d & c \\ a^2 & d^2 & c^2 \end{vmatrix}}{|A|}=\dfrac{(d-a)(c-a)(c-d)}{(b-a)(c-a)(c-b)}=\dfrac{(d-a)(c-d)}{(b-a)(c-b)} \\[2em] z=\dfrac{\begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a & b & d^2 \end{vmatrix}}{|A|}=\dfrac{(b-a)(d-a)(d-b)}{(b-a)(c-a)(c-b)}=\dfrac{(d-a)(d-b)}{(c-a)(c-b)} \end{cases}$$

若 $|A|=0$, 则有 $\mathrm{rank}(A)<3$. 因为 $A_{(1)}=(1,1,1)$

$\mathrm{rank}(A)\geq 1$. 当 $\mathrm{rank}(A)=1$ 时, 有 $a=b=c$

(*) 有解 $\Leftrightarrow$ $\mathrm{rank}\left(A \,\middle|\, \begin{matrix} 1 \\ d \\ d^2 \end{matrix}\right)=\mathrm{rank}(A)=1$ 则有

$a=b=c=d$ 这时有解且不确定

当 $\mathrm{rank}(A)=2$ 时，则 A 的三列中必有一列是另外两列的线性组合且该两列线性无关. 不妨设 $A^{(1)}, A^{(2)}$ 线性无关, 则 (*) 有解 $\Longleftrightarrow$ $[1, d, d^2]$ 是 $A^{(1)}, A^{(2)}$ 的线性组合

$$\Longleftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a^2 & b^2 & d^2 \end{vmatrix} = 0$$

$$\Longleftrightarrow \quad d=a \text{ 或 } d=b.$$

通过行列式我们可以清楚地认识方程组 (*) 的解的结构.

注：利用克拉默法则 (Cramer's rule) 与范德蒙行列式结可以证明：设 $a_1, a_2 \cdots a_n$ 是 $\mathbb{R}$ 中互不相同的数, $b_1, \cdots b_n$ 是 $\mathbb{R}$ 中任给一组数, 则存在唯一的系数在 $\mathbb{R}$ 中的多项式 $f(x)=c_0x^{n-1}+\cdots+c_{n-1}$ 使 $f(a_i)=b_i$, $i=1,2,\cdots,n$.

<u>例 2</u>

$$d = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}_{n\times n}$$

$$d = \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ a+(n-1)b & b & b & \cdots & a \end{vmatrix} = (a+(n-1)b) \begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ 1 & b & \cdots & a \end{vmatrix}$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & \cdots & b \\ 0 & a-b & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{vmatrix}$$

$$= [a+(n-1)b]\,(a-b)^{n-1}$$

例3 由 $\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}_{n\times n} = 0$

可以说明：奇偶排列各为 $\frac{n!}{2}$.

例4

$$d = \begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix} \quad (n\geq 2)$$

$$d = x\begin{vmatrix} x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & x & y \\ 0 & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{1+n}y\begin{vmatrix} y & 0 & \cdots & 0 & 0 \\ x & y & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \end{vmatrix}$$

$$= x^n + (-1)^{1+n}y^n$$

例5

$$\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_0 a_1 \cdots a_n \begin{vmatrix} 1 & \frac{1}{a_0} & \frac{1}{a_0} & \cdots & \frac{1}{a_0} \\ \frac{1}{a_1} & 1 & 0 & \cdots & 0 \\ \frac{1}{a_2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{a_n} & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= a_0 a_1 \cdots a_n \begin{vmatrix} 1-\frac{1}{a_0}\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) & 0 & 0 & \cdots & 0 \\ \frac{1}{a_1} & 1 & 0 & \cdots & 0 \\ \frac{1}{a_2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{a_n} & & & & 1 \end{vmatrix}$$

$$= a_0 a_1 \cdots a_n \left(1-\frac{1}{a_0}\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)\right)$$

$$= a_1 \cdots a_n \left(a_0 - \sum_{i=1}^{n} \frac{1}{a_i}\right).$$

例6

$$f_n = \begin{vmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ 0 & -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & a_{n-2} \\ 0 & 0 & 0 & \cdots & -1 & x+a_{n-1} \end{vmatrix} = x \begin{vmatrix} x & 0 & \cdots & 0 & a_1 \\ -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & a_{n-2} \\ 0 & 0 & \cdots & -1 & x+a_{n-1} \end{vmatrix}$$

(归纳法可得). $+(-1)^{1+n} a_0 \begin{vmatrix} -1 & x & 0 & \cdots & 0 \\ 0 & -1 & x & \cdots & 0 \\ \vdots & & \ddots & & \\ & & & -1 & x \\ 0 & 0 & \cdots & & -1 \end{vmatrix}$

$$= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$(-1)^{1+n} \cdot (-1)^{n-1} \cdot a_0$

例7.

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}_{n-1\text{级}}$$

$$= \begin{vmatrix} -1 & 1 & \cdots & 1 & 1-n \\ -1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 1-n & & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & \cdots & 0 & -n \\ -1 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & & & \vdots \\ -1 & -n & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{(n-1)\times(n-1)} \qquad \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ n-1 & n-2 & \cdots & 1 \end{pmatrix} \quad (-1)^{\frac{(n-1)(n-2)}{2}}$$

$$= (-1)^{n-1}(-1)^{\frac{(n-1)(n-2)}{2}} n^{n-2} = (-1)^{\frac{n(n-1)}{2}} n^{n-2}$$

例8

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \\ 3 & 4 & 5 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 1 & 2 & \cdots & n-1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(1+n)}{2} & 2 & 3 & \cdots & n-1 & n \\ \frac{n(1+n)}{2} & 3 & 4 & \cdots & n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{n(1+n)}{2} & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

$$= \frac{n(1+n)}{2} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 3 & 4 & \cdots & n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

$$= \frac{n(1+n)}{2} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 1 & \cdots & 1 & 1-n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

由例7 $= \frac{n(1+n)}{2} (-1)^{\frac{n(n-1)}{2}} n^{n-2} = (-1)^{\frac{n(n-1)}{2}} \frac{n^{n-1}(1+n)}{2}.$

例9

证明 $$\sum_{\sigma\in S_n}\begin{vmatrix} a_{1\sigma(1)} & a_{1\sigma(2)} & \cdots & a_{1\sigma(n)} \\ a_{2\sigma(1)} & a_{2\sigma(2)} & \cdots & a_{2\sigma(n)} \\ \vdots & \vdots & & \vdots \\ a_{n\sigma(1)} & a_{n\sigma(2)} & \cdots & a_{n\sigma(n)} \end{vmatrix} = 0$$

例10

$$D_n = \begin{vmatrix} x & a & a & \cdots & a & a \\ -a & x & a & \cdots & a & a \\ -a & -a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & x \end{vmatrix}$$

当 $n=1$ 时 $D_1 = x$

下设 $n\geq 2$ 第2列 -1倍加到第1列

$$D_n = \begin{vmatrix} x-a & a & a & \cdots & a & a \\ -x-a & x & a & \cdots & a & a \\ 0 & -a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -a & -a & \cdots & -a & x \end{vmatrix}$$

按第1列展开

$$D_n = (x-a)\begin{vmatrix} x & a & \cdots & a & a \\ -a & x & \cdots & a & a \\ \vdots & \vdots & & \vdots & \vdots \\ -a & -a & \cdots & -a & x \end{vmatrix}$$

$$+ (x+a)\begin{vmatrix} a & a & \cdots & a & a \\ -a & x & \cdots & a & a \\ \vdots & \vdots & & \vdots & \vdots \\ -a & -a & \cdots & x & a \\ -a & -a & \cdots & -a & x \end{vmatrix}$$

$$= (x-a)D_{n-1} + (x+a)\begin{vmatrix} a & a & \cdots & a & a \\ 0 & x+a & \cdots & 2a & 2a \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x+a & 2a \\ 0 & 0 & \cdots & 0 & x+a \end{vmatrix}$$

$$= (x-a)D_{n-1} + a(x+a)^{n-1}$$

将 D_n 中 a 用 $-a$ 来代替可得转置矩阵行列式，即

$$D_n = (x+a)D_{n-1} + (-a)(x-a)^{n-1}$$

两式联立 D_n, D_{n-1} $\Longrightarrow$ $$D_n = \frac{1}{2}\left[(x+a)^n + (x-a)^n\right].$$

证明：

$$D=\begin{vmatrix} a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}$$

$$=\underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{|A|} + x\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}$$

证明：

$$D=\begin{vmatrix} a_{11} & a_{12}+x & \cdots & a_{1n}+x \\ \vdots & & & \\ a_{n1} & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix} + \begin{vmatrix} x & a_{12}+x & \cdots & a_{1n}+x \\ \vdots & \vdots & & \vdots \\ x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}+x \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}+x \end{vmatrix} + \begin{vmatrix} a_{11} & \cancel{a_{12}} & \cdots & a_{1n}+x \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & x & \cdots & a_{nn}+x \end{vmatrix}$$

$$+\begin{vmatrix} x & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ x & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \cdots = |A| + \sum_{j=1}^{n} \begin{array}{c} \\ \end{array}\begin{vmatrix} a_{11} & \cdots & x & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & x & \cdots & a_{nn} \end{vmatrix}$$

$$= |A| + x\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}$$