# 第十次习题课.

- **定理 8.2** 设 $A, B \in \mathbb{R}^{m\times n}$. 则 $A \sim_e B \Longleftrightarrow \text{rank}(A) = \text{rank}(B)$.

- 初等矩阵: $F_{i,j}^{(n)} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ $\quad F_i^{(n)}(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$

  $F_{i,j}^{(n)}(\alpha) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & \alpha \\ & & & \ddots & \vdots \\ & & & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$

- **引理 8.5** *(打洞引理)* 设 $A \in \mathbb{R}^{m\times n}$. 则存在可逆矩阵 $P \in \text{M}_m(\mathbb{R})$ 和 $Q \in \text{M}_n(\mathbb{R})$, 其中 $P$ 和 $Q$ 都是初等矩阵的乘积, 使得

$$PAQ = \begin{pmatrix} E_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{pmatrix}, \quad \Longleftrightarrow \quad A = P^{-1}\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}$$

且 $\text{rank}(A) = r$.

- 推论: 可逆矩阵是若干初等矩阵之积.

- 矩阵求逆: ① $(A \mid E_n) \xrightarrow{\text{初等行变换}} (E_n \mid A^{-1})$

  ② $\exists\, \alpha_0, \alpha_1, \cdots, \alpha_k \in \mathbb{R}$ $\quad \alpha_k \neq 0$, s.t.

  $\alpha_k A^k + \cdots + \alpha_1 A + \alpha_0 E = O$

  $A$ 可逆 $\Longleftrightarrow \alpha_0 \neq 0 \qquad A^{-1} = -\frac{1}{\alpha_0}(\alpha_1 E + \cdots + \alpha_k A^{k-1})$

1. 求以下矩阵的逆

$$(1) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad (2) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(1) $\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

(2)

$$A^2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 4E \quad \Rightarrow -4E + A^2 = 0$$

$\alpha_0 = -4,\ \alpha_1 = 0,\ \alpha_2 = 1$

$$\Rightarrow A^{-1} = \frac{1}{4} A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

2. 证明: 任意一个 $n$ 阶方阵都可以写成一个 $n$ 阶对称矩阵和一个 $n$ 阶斜对称矩阵的和.

证: $\forall\ A \in M_n(\mathbb{R})$ 令 $B = \frac{1}{2}(A + A^t)$ $C = \frac{1}{2}(A - A^t)$

则 $A = B + C$ 且 $B^t = \frac{1}{2}(A^t + A) = B$ $C^t = \frac{1}{2}(A^t - A) = -C$

$\Rightarrow$ B 对称. C 斜对称.

3. 若 $A = A^t$, $B = B^t$, 证明:

$$\operatorname{tr}\left((AB)^2\right) \leq \operatorname{tr}\left(A^2 B^2\right).$$

证: 设 $C = AB - BA$

由 $A = A^t$, $B = B^t$, 则 $C^t = (AB - BA)^t = (AB)^t - (BA)^t = BA - AB$

则 $0 \leq tr(C \cdot C^t) = tr((AB - BA) \cdot (BA - AB))$

$= tr((AB^2)A - ABAB - (BAB)A + B(A^2B))$

$tr(MN) = tr(NM)$

$= tr(A^2B^2) - tr((AB)^2) - tr((AB)^2) + tr(A^2B^2)$

$\Rightarrow tr((AB)^2) \leq tr(A^2B^2)$

4. 设 $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = r$. 证明：$A$ 可以写成 $r$ 个秩为1的矩阵的和，且不能写成 $r-1$ 个秩为1的矩阵的和.（提示：利用打洞引理和秩不等式）

证：① 由打洞引理，$\exists\ P \in M_m(\mathbb{R})$ $Q \in M_n(\mathbb{R})$

s.t. $A = P\begin{pmatrix} E_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{pmatrix}Q = P\left(\underbrace{\begin{pmatrix} E_{11}^{(r)} & 0 \\ 0 & 0 \end{pmatrix}}_{A_1} + \cdots + \underbrace{\begin{pmatrix} E_{rr}^{(r)} & 0 \\ 0 & 0 \end{pmatrix}}_{A_r}\right)Q$

$\Rightarrow A = P\cdot A_1 Q + \cdots + P\cdot A_r Q.$

且 $\operatorname{rank}(PA_iQ) = 1 \quad \forall\ 1 \le i \le r.$

② 若 $A$ 可写成 $r-1$ 个秩为1的矩阵之和，设为 $B_1, \cdots, B_{r-1}$

则 $\operatorname{rank}(A) = \operatorname{rank}\left(\sum_{i=1}^{r-1} B_i\right) \le \sum_{i=1}^{r-1} \operatorname{rank}(B_i) = r-1$. 矛盾.

$\Rightarrow$ $A$ 不可写成 $r-1$ 个秩为1的矩阵之和

5. 设 $A \in \mathrm{M}_m(\mathbb{R})$ 可逆，$D \in \mathrm{M}_n(\mathbb{R})$ 可逆，证明： $X = \begin{pmatrix} A & B \\ O & D \end{pmatrix}$ 可逆，并求 $X^{-1}$.

① 证：$\because$ $A \in M_m(\mathbb{R})$ 可逆. $D \in M_n(\mathbb{R})$ 可逆.

$\therefore$ $\operatorname{rank}(A) = m \quad \operatorname{rank}(D) = n$.

则 $\operatorname{rank}(X) \ge \operatorname{rank}(A) + \operatorname{rank}(D) = m+n$

而另一方面 $X \in M_{m+n}(\mathbb{R}) \Rightarrow \operatorname{rank}(X) \le m+n$

$\Rightarrow \operatorname{rank}(X) = m+n$ $\quad X$ 满秩.

$\Rightarrow X$ 可逆.

② $\left(\begin{array}{cc|cc} A & B & E_m & O \\ O & D & O & E_n \end{array}\right) \longrightarrow \begin{pmatrix} E_m & A^{-1}B & A^{-1} & O \\ O & E_n & O & D^{-1} \end{pmatrix}$

$\longrightarrow \begin{pmatrix} E_m & O & A^{-1} & -A^{-1}BD^{-1} \\ O & E_n & O & D^{-1} \end{pmatrix} \Rightarrow X^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{pmatrix}$

6. 设 $A \in \mathrm{M}_m(\mathbb{R})$, $B \in \mathrm{M}_n(\mathbb{R})$, $C \in \mathbb{R}^{m\times n}$, $X = \begin{pmatrix} A & C \\ O_{n\times m} & B \end{pmatrix}$, 满足 $XX^t = X^tX$. 证明：

(1) 证明：$\mathrm{tr}(C^tC) = 0$;

(2) (选作) 证明：$C = O_{m\times n}$.

证：(1) $X = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ $X^t = \begin{pmatrix} A^t & 0 \\ C^t & B^t \end{pmatrix}$

$X \cdot X^t = \begin{pmatrix} AA^t + C\cdot C^t & CB^t \\ B\cdot C^t & B\cdot B^t \end{pmatrix}$ $X^t \cdot X = \begin{pmatrix} A^tA & A^tC \\ C^tA & C^t\cdot C + B^t\cdot B \end{pmatrix}$

$X\cdot X^t = X^t \cdot X \Rightarrow A\cdot A^t + C\cdot C^t = A^t\cdot A$.

$\Rightarrow C\cdot C^t = A^tA - A\cdot A^t$ 设 $A = (a_{ij})_{1\le i,j\le n}$

$\Rightarrow \mathrm{tr}(C^tC) = \mathrm{tr}(C\cdot C^t) = \mathrm{tr}(A^tA - A\cdot A^t)$

$= \mathrm{tr}(A^tA) - \mathrm{tr}(A\cdot A^t) = 0$ $\mathrm{tr}(A\cdot B) = \mathrm{tr}(BA)$.

(2). 由 $\mathrm{tr}(C^tC) = 0$

$\Rightarrow \sum_{i=1}^{n}\left(\sum_{k=1}^{n} C_{ki}C_{ki}\right) = 0$

$\Rightarrow \sum_{i=1}^{n}\sum_{k=1}^{n} C_{ki}^2 = 0$ $\Rightarrow C_{ki} = 0$ $(\forall\ 1\le i\le n,\ 1\le k\le n)$.

即 $C = O$.

## 补充内容:

补充: 证明任何一个方阵都能写成一个幂等矩阵和一个可逆矩阵的乘积. 其中, $A \in \mathrm{M}_n(\mathbb{R})$ 为幂等矩阵是指 $A$ 满足 $A^2 = A$.

*证明:* 对于矩阵的结构性问题, 我们自然想到用打洞引理. 设 $A \in \mathrm{M}_n(\mathbb{R})$, $\operatorname{rank}(A) = r$. 由打洞引理, 存在 $P, Q$ 为 $n$ 阶可逆方阵, 使得:

$$A = P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}Q.$$

则:

$$A = P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}Q = (P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}P^{-1})(PQ).$$

令 $B = P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}P^{-1}$, $C = PQ$. 则:

$$\begin{aligned} B^2 =& P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}P^{-1}P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}P^{-1} \\ =& P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}^2 P^{-1} \\ =& P\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}P^{-1} \\ =& B. \end{aligned}$$

$B$ 为幂等矩阵. 从而 $A = BC$ 是一个幂等矩阵和一个可逆矩阵的乘积. □

补充习题: 求 $n$ 阶方阵 $A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$ 的逆, $n \geq 2$.

*解:* 令 $B = A + E_n$, 则 $B$ 为所有位置均为 1 的矩阵. 很容易计算: $B^2 = nB$. 从而:

$$(A + E_n)^2 = n(A + E_n).$$

展开整理得:

$$A^2 - (n-2)A = (n-1)E_n.$$

故 $A^{-1} = \dfrac{1}{n-1}(A - (n-2)E_n)$.

**命题 3.1** 设 $A \in \mathrm{M}_n(\mathbb{R})$, $A^2 = A$. 则存在 $P \in \mathrm{M}_n(\mathbb{R})$ 可逆, 使得:

$$A = P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} P^{-1}.$$

其中, $r$ 为 $A$ 的秩.

*证明:* 由打洞引理, 存在 $P, Q$ 为 $n$ 阶可逆方阵使得 $A = P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} Q$. 则:

$$P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} QP \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} Q = A^2 = A = P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} Q.$$

由 $P, Q$ 可逆, 上式两侧消去 $P, Q$ 得:

$$\begin{pmatrix} E_r & O \\ O & O \end{pmatrix} QP \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} E_r & O \\ O & O \end{pmatrix}.$$

令 $T = QP$, 并给 $T$ 分块: $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. 其中 $T_1 \in \mathrm{M}_r(\mathbb{R})$, $T_4 \in \mathrm{M}_{n-r}(\mathbb{R})$. 计算:

$$\begin{pmatrix} E_r & O \\ O & O \end{pmatrix} QP \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1 & O \\ O & O \end{pmatrix}.$$

结合 (1) 式可知 $T_1 = E_r$. 而 $Q = TP^{-1}$, 故:

$$\begin{aligned} A =& P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} Q \\ =& P \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} TP^{-1} \\ =& P \begin{pmatrix} E_r & T_2 \\ O & O \end{pmatrix} P^{-1}. \end{aligned}$$

令 $H = \begin{pmatrix} E_r & T_2 \\ O & E_{n-r} \end{pmatrix}$, 不难看出(根据“初等行列变换”规律):

$$A = P \begin{pmatrix} E_r & T_2 \\ O & O \end{pmatrix} P^{-1} = PH^{-1} \begin{pmatrix} E_r & O \\ O & O \end{pmatrix} HP^{-1}$$

令 $P' = PH^{-1}$, 则 $P'$ 可逆, 结论得证. □