1. 上周作业

1.1 $M_n(\mathbb{R})$, $(A|B) = \mathrm{tr}\, A^t B$

注意到 $\mathrm{tr}\, A^t B = \sum_{i,j} a_{ij} b_{ij}$ $\Rightarrow$ 标准正交基 $\{E_{ij}\}$

1.2 $\mathbb{R}[x]^4$ $(f|g) = \int_0^1 fg\, dx$

基 $1, x, x^2, x^3$ 正交基 $1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2-x+\frac{1}{6}), 20\sqrt{7}(x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20})$

$e_1 = 1$

$e_2' = x - \frac{1}{2}$ $e_2 = 2\sqrt{3}(x-\frac{1}{2})$

$e_3' = x^2 - \left[\int_0^1 x^2 \cdot 2\sqrt{3}(x-\frac{1}{2})\right] 2\sqrt{3}(x-\frac{1}{2}) - \left[\int_0^1 x^2\right] 1 = x^2 - x + \frac{1}{6}$

$e_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6})$

$e_4' = x^3 - (x^3|e_3)e_3 - (x^3|e_2)e_2 - (x^3|e_1)e_1 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$

$e_4 = 20\sqrt{7}(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20})$

Rmk: 先单位化,再找正交. 套用 $e_k = e_k' - (e_k'|e_1)e_1 - \cdots - (e_k'|e_{k-1})e_{k-1}$

1.3. $V \subset \mathbb{R}^4$ 为 $\begin{cases} 2x_1 - x_2 - x_4 = 0 \\ x_1 + x_3 + x_4 = 0 \end{cases}$ 的子空间. 求 $V^\perp$ 正交基

解. $V^\perp$ 有基 $(2,-1,0,-1)$, $(1,0,1,1)$

$\Rightarrow$ 正交基 $\frac{1}{\sqrt{3}}(1,0,1,1)$, $\frac{1}{\sqrt{51}}(5,-3,-1,-4)$

或 $\frac{1}{\sqrt{6}}(2,-1,0,-1)$, $\frac{1}{\sqrt{102}}(4,1,6,7)$

1.4. $e_1, \cdots, e_m$ 为 m 个正交向量 $u \in V$

$\Rightarrow u = (u|e_1)e_1 + \cdots + (u|e_m)e_m + v$ $(v|e_i) = 0 \ \forall i$

$\Rightarrow (u|u) = \sum_i (u|e_i)^2 + (v|v) \geq \sum_i (u|e_i)^2$

等号成立 iff $v = 0$

1.5 $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$; $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$

pf $\Leftrightarrow U_1 \cap U_2 = (U_1^\perp + U_2^\perp)^\perp$

注意到 $(U_1^\perp + U_2^\perp)^\perp = (U_1^\perp)^\perp \cap (U_2^\perp)^\perp = U_1 \cap U_2$

~~设 $V = U_1 \oplus U_1^\perp$ 同时交 U_2 $U_2 = U_1 \cap U_2 \oplus U_1^\perp \cap U_2$ $\Rightarrow V = U_2 \oplus U_2^\perp = U_1 \cap U_2 \oplus U_1^\perp \cap U_2 \oplus U_2^\perp$~~

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1.6 λ 为 $A \in O_n(\mathbb{R})$ 的特征根 $\Rightarrow$ (i) $|\lambda|=1$ (ii) $u+vi$ 为特征向量 $\Rightarrow u^t u = v^t v,\ u^t v = 0$

证: $AX = \lambda X$ X 为特征向量 $\in \mathbb{C}^n$

$\Rightarrow \bar{\lambda} \bar{X}^t \lambda X = \bar{X}^t \bar{A}^t A X = \bar{X}^t X = |\lambda|^2 \bar{X}^t X \quad \Rightarrow |\lambda| = 1$

$(u+iv \mid u+iv) = (A(u+iv) \mid A(u+iv)) = \lambda^2 (u+iv \mid u+iv)$

$\lambda \neq \pm 1 \Rightarrow 0 = (u+iv \mid u+iv) = u^t u - v^t v + 2i\, u^t v$

$\Rightarrow u^t u = v^t v \quad u^t v = 0$

2. 设 V 为 n 维欧式空间. 则

(1) $\forall \alpha \in V$ $f_\alpha: V \to \mathbb{R}$ $\beta \mapsto (\alpha \mid \beta)$ 是 V 上线性函数 i.e. $f_\alpha \in V^*$

(2) $\sigma: V \to V^*$ $\alpha \to f_\alpha$ 为同构

(3) $\forall f \in V^*$, $\exists \alpha \in V$ s.t. $f(\beta) = (\alpha, \beta) = f_\alpha(\beta)$

证: (1) 直接验证

(2) $\alpha \in \ker \sigma \Leftrightarrow f_\alpha(\beta) = (\alpha \mid \beta) = 0 \ \forall \beta \Leftrightarrow \alpha = 0$

$\Rightarrow \ker \sigma = 0 \Rightarrow$ 同构

(3) 由(2)直接得

或 若 $f \neq 0$ 则 $\dim \ker f = n-1$ 令 $\langle x_0 \rangle = \ker f^{\perp}$ $\|x_0\| = 1$

$\Rightarrow x = (x, x_0) x_0 + y \quad y \in \ker f$

$\Rightarrow f(x) = (x \mid x_0) f(x_0) = (f(x_0) x_0 \mid x)$

Rmk.

$$\begin{tikzcd} y & V \arrow[r, "\sigma"] \arrow[d, "A^t"'] & V^* \arrow[d, "A^*"] & f_y \\ A^t y & V \arrow[r, "\sigma"] & V^* & \end{tikzcd}$$

$A^* f_y = f_{A^t y}$

$\because A^* f_y(x) = f_y(Ax) = (y \mid Ax)$

$f_{A^t y}(x) = (A^t y \mid x) = (y \mid Ax)$

3. 设 A 为正交矩阵, 特征值不为 -1. 则 $I + A$ 可逆

且 $K = (I-A)(I+A)^{-1}$ 反对称

$A = (I-K)(I+K)^{-1}$

pf: 特征值不为 -1 $\Rightarrow$ $\det(-I-A)\neq 0$ $\Rightarrow$ $I+A$ 可逆

$K^T = (I+A)^{-T}(I-A)^T = (I+A^{-1})^{-1}(I-A^{-1}) = (I+A^{-1})^{-1}A^{-1}(A-I) = -(A+I)^{-1}(I-A) = -K$

$AA^T = (I-K)(I+K)^{-1}(I+K)^{-T}(I-K)^T$

$= (I-K)(I+K)^{-1}(I-K)^{-1}(I+K)$

$= I$ （$I-K$, $I+K$, $(I+K)^{-1}$, $I-K^{-1}$ 都可交换）

4.（奇异值分解）$A\in M_{m\times n}(\mathbb{R})$ 则存在 $O_1\in O_m(\mathbb{R})$ $O_2\in O_n(\mathbb{R})$ s.t

$O_1AO_2 = \begin{pmatrix} u_1 & & & \\ & \ddots & & \\ & & u_r & \\ & & & 0_{\ddots 0}\end{pmatrix}$

pf: A^TA 为 n 阶对称矩阵，半正定

$\Rightarrow \exists O\in O_n(\mathbb{R})$ s.t $O^TA^TAO = \mathrm{diag}(\lambda_1,\cdots,\lambda_r,0\cdots 0)$ $\lambda_i>0$

$\Rightarrow$ 令 $D=\mathrm{diag}(\frac{1}{\sqrt{\lambda_1}},\cdots,\frac{1}{\sqrt{\lambda_r}},1\cdots 1)$ 则 $D^TO^TA^TAOD = \mathrm{diag}(1,\cdots,1,0\cdots 0)$

令 $B=AOD\in M_{m\times n}(\mathbb{R})$ 则 $B^TB=\mathrm{diag}(I_r,0)$

$\Rightarrow$ 设 $B=(\beta_1,\cdots,\beta_r,\cdots,\beta_n)$ β_i 为列向量

则 $\{\beta_i\}$ 中前 r 个两两正交，后 $n-r$ 个全为 0 向量

$\Rightarrow \exists \tilde{O}\in O_m(\mathbb{R})$ s.t $\tilde{O}B = \tilde{O}(\beta_1,\cdots,\beta_r,0\cdots 0) = (e_1,\cdots,e_r,0\cdots 0) = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$

$\Rightarrow \tilde{O}AOD = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$

$\Rightarrow \tilde{O}AO = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_r} & \\ & & & 0_{\ddots 0}\end{pmatrix}$

Rmk: $u_i=\sqrt{\lambda_i}$ 为 A^TA 的特征值的平方根，称作 A 的奇异值。

5. $A_{m\times n}$, $B_{n\times p}$ 实矩阵，则 $\mathrm{Tr}\,AB(AB)^T \le \mathrm{Tr}\,AA^T\max\lambda(BB^T)$

其中 $\max\lambda(BB^T)$ 为 BB^T 的最大特征值

pf: 奇异值分解 $\Rightarrow$ $B = O_1\,\mathrm{diag}(u_1,\cdots,u_r,0\cdots 0)\,O_2$

$\Rightarrow BB^T = O_1\,\mathrm{diag}(u_1^2,\cdots,u_r^2,0\cdots 0)\,O_1^T$

$\Rightarrow AB(AB)^T = ABB^TA^T = AO_1\,\mathrm{diag}(u_1^2,\cdots,u_r^2,0\cdots 0)(AO_1)^T$

令 $AO_1=(a_{ij})_{m\times n}$ $\Rightarrow$ $\mathrm{Tr}\,AB(AB)^T = \sum_{j=1}^{m}\sum_{i=1}^{r}u_i^2a_{ij}^2$

$\le \max\lambda(BB^T)\sum_j\sum_i a_{ij}^2$

$= \max\lambda(BB^T)\,\mathrm{Tr}\,AO_1(AO_1)^T$

$= \max\lambda(BB^T)\,\mathrm{Tr}\,AA^T$ □