1、上周作业

1.1 $\chi_A=(t-1)^4(t+1)^3t^2$，$\mu_A=(t-1)^3(t+1)^3t^2$

$\Rightarrow J_A=\mathrm{diag}(J_3(1),J_1(1),J_3(-1),J_2(0))$

1.2 $A=\begin{pmatrix}2&3&-3\\-1&10&-6\\-2&14&-8\end{pmatrix}$

$\chi_A=(t-1)^2(t-2)$

$(A-E)(A-2E)=\begin{pmatrix}3&-15&9\\3&-15&9\\4&-20&12\end{pmatrix}\neq 0\Rightarrow \mu_A=\chi_A\Rightarrow J_A=\begin{pmatrix}1&1&\\&1&\\&&2\end{pmatrix}$

$\mathrm{rank}(A-E)=\mathrm{rank}\begin{pmatrix}1&3&-3\\-1&9&-6\\-2&14&-9\end{pmatrix}=2\Rightarrow J_A=\begin{pmatrix}1&1&\\&1&\\&&2\end{pmatrix}$

$A=\begin{pmatrix}4&7&-3\\-2&-4&2\\-4&-10&4\end{pmatrix}$

$\chi_A=(t-2)(t^2-2t+2)=(t-2)(t-1+i)(t-1-i)\Rightarrow J_A=\begin{pmatrix}2&&\\&1-i&\\&&1+i\end{pmatrix}$

1.3 $\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}=\begin{pmatrix}a&b\\1&0\end{pmatrix}\begin{pmatrix}a_n\\a_{n-1}\end{pmatrix}$

$\Rightarrow \begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}=\begin{pmatrix}a&b\\1&0\end{pmatrix}^{n-1}\begin{pmatrix}a_2\\a_1\end{pmatrix}$

Let $A=\begin{pmatrix}a&b\\1&0\end{pmatrix}$ 计算 A 的标准型与过渡矩阵

$\chi_A=t^2-at-b\Rightarrow \lambda=\frac{a}{2}\pm\frac{1}{2}\sqrt{a^2+4b}$

• $a^2+4b\neq 0$ $\quad A\sim_s\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}$

$A\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow\begin{pmatrix}ax+by\\x\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow x=\lambda y\Rightarrow\begin{pmatrix}\lambda_1\\1\end{pmatrix}\begin{pmatrix}\lambda_2\\1\end{pmatrix}$ 为特征向量

$\Rightarrow A=\begin{pmatrix}\lambda_1&\lambda_2\\1&1\end{pmatrix}\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}\begin{pmatrix}\lambda_1&\lambda_2\\1&1\end{pmatrix}^{-1}$

$\Rightarrow A^{n-1}=\begin{pmatrix}\lambda_1&\lambda_2\\1&1\end{pmatrix}\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}^{n-1}\begin{pmatrix}\lambda_1&\lambda_2\\1&1\end{pmatrix}^{-1}$

• $a^2+4b=0$ $\quad A\sim\begin{pmatrix}\frac{a}{2}&1\\&\frac{a}{2}\end{pmatrix}$

$\alpha=\begin{pmatrix}\frac{a}{2}\\1\end{pmatrix}$ 为特征向量 $\quad A(\alpha_1,\alpha_2)=(\alpha_1,\alpha_2)\begin{pmatrix}\frac{a}{2}&1\\&\frac{a}{2}\end{pmatrix}$

$\Rightarrow A\alpha_2=\alpha_1+\frac{a}{2}\alpha_2\Rightarrow(A-\frac{a}{2}I)\alpha_2=\begin{pmatrix}\frac{a}{2}&-\frac{a^2}{4}\\1&-\frac{a}{2}\end{pmatrix}\alpha_2=\begin{pmatrix}\frac{a}{2}\\1\end{pmatrix}\Rightarrow\alpha_2=\begin{pmatrix}1\\0\end{pmatrix}$

$\Rightarrow A=\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{a}{2}&1\\&\frac{a}{2}\end{pmatrix}\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}^{-1}$

$\Rightarrow A^{n-1}=\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{a}{2}&1\\&\frac{a}{2}\end{pmatrix}^{n-1}\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}^{-1}$

$=\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}\begin{pmatrix}(\frac{a}{2})^{n-1}&(n-1)(\frac{a}{2})^{n-2}\\&(\frac{a}{2})^{n-1}\end{pmatrix}\begin{pmatrix}\frac{a}{2}&1\\1&0\end{pmatrix}^{-1}$

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1.4. $A \sim_s A^t$ $\quad A \in M_n(\mathbb{R})$

pf: 1. 复相似

$A \sim_s \mathrm{diag}(\cdots, J_k(\lambda), \cdots)$

$\Rightarrow A^t \sim_s \mathrm{diag}(\cdots, J_k^t(\lambda), \cdots)$

只需证明 $J_k(\lambda) \sim_s J_k(\lambda)^t$ 　 因 秩. $\mathrm{rank}(J_k(\lambda)-\lambda I)^i = \mathrm{rank}(J_k^t(\lambda)-\lambda I)^i$ $\forall i$

2. 因 基 $J(e_1, \cdots, e_n) = (e_1, \cdots, e_k) J$

$J^t(e_k, e_{k-1}, \cdots, e_1) = (e_k, \cdots, e_1) J^t$

2. 实相似 : $\exists\, P \in GL_n(\mathbb{R})$ s.t $A = PA^tP^{-1}$ 　 $\mathbb{R}$!

已知 $\exists\, P = P_1 + iP_2 \in GL_n(\mathbb{C})$ s.t $A = PA^tP^{-1}$

$\Rightarrow A(P_1+iP_2) = (P_1+iP_2)A^t$

$\Rightarrow \begin{cases} AP_1 = P_1B \\ AP_2 = P_2B \end{cases}$

P_1, P_2 可能不可逆 但是 $\exists\, c \in \mathbb{R}$ s.t $P_1 + cP_2 \in GL_n(\mathbb{R})$ 　 (考虑 $\det(P_1+cP_2)$ 为 c 的多项式)

$\Rightarrow A(P_1+cP_2) = (P_1+cP_2)A^t$

$\Rightarrow A = (P_1+cP_2)A^t(P_1+cP_2)^{-1}$

Remark: $A, B \in M_n(\mathbb{R})$ 　 A, B 实相似 $\Leftrightarrow$ A, B 复相似

1.5 (i) $A \in M_n(\mathbb{C})$ $\mathrm{tr}A^k = 0$ $k=1, \cdots, n$ $\Rightarrow$ A 幂零

(ii) $AB - BA \triangleq C$ $AC = CA$ $\Rightarrow$ C 幂零

pf (i) $A \sim_s \mathrm{diag}(\cdots, J_{n_i}(\lambda_i), \cdots)$

$\Rightarrow A^k \sim_s \mathrm{diag}(\cdots, J_{n_i}^k(\lambda_i), \cdots)$

$\Rightarrow \mathrm{tr}A^k = \sum_{\lambda_i \text{可能相同}} n_i \lambda_i^k = \sum_{\lambda_i \text{两两不同}} m_i \lambda_i^k = 0$

$\Rightarrow \begin{pmatrix} \lambda_1 & \cdots & \lambda_s \\ \vdots & & \vdots \\ \lambda_1^n & & \lambda_s^n \end{pmatrix}_{n\times s} \begin{pmatrix} m_1 \\ \vdots \\ m_s \end{pmatrix}_{s\times 1} = \begin{pmatrix} 0 \end{pmatrix}_{n\times 1}$

由 $\begin{pmatrix} \lambda_1 & \cdots & \lambda_s \\ \vdots & & \vdots \\ \lambda_1^s & & \lambda_s^s \end{pmatrix}$ 可逆 $(\lambda_i \neq \lambda_j)$ $\Rightarrow m_1 = \cdots = m_s = 0$ 也不可能

$\Rightarrow \lambda_i = 0$ $\forall i$

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iii) $\operatorname{tr} C = \operatorname{tr}(AB - BA) = \operatorname{tr} AB - \operatorname{tr} BA = 0$

$\operatorname{tr} C^2 = \cancel{\operatorname{tr}(AB-BA)(AB-BA)} = \operatorname{tr} CAB - \operatorname{tr} CBA = \operatorname{tr} ACB - \operatorname{tr} CBA$

$= \operatorname{tr} CBA - \operatorname{tr} CBA = 0$

$\operatorname{tr} C^k = \operatorname{tr} C^{k}AB - \operatorname{tr} C^{k}BA = \operatorname{tr} AC^{k}B - \operatorname{tr} C^{k}BA = \operatorname{tr} C^{k}BA - \operatorname{tr} C^{k}BA = 0$

$\Rightarrow C$ 幂零

2. $A \in M_n(\mathbb{C})$ $\quad e^A := \sum_{i=0}^{\infty} \frac{A^i}{i!}$

- $e^0 = I_n$
- $\det e^A = e^{\operatorname{tr} A}$
- $\operatorname{tr} e^A = e^{\lambda_1} + \cdots + e^{\lambda_n}$ $\quad \lambda_i$ 为 A 特征值（可以相同）

pf: $A = P\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$

$\Rightarrow A^k = P\begin{pmatrix} \lambda_1^k & & * \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} P^{-1}$

$\Rightarrow e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = P\left(\sum_{i=0}^{\infty} \frac{\begin{pmatrix} \lambda_1^i & & * \\ & \ddots & \\ & & \lambda_n^i \end{pmatrix}}{i!}\right) P^{-1} = P\begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} P^{-1}$

$\Rightarrow \det e^A = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\operatorname{tr} A}$

$\operatorname{tr} e^A = e^{\lambda_1} + \cdots + e^{\lambda_n}$

- $\frac{d}{dt} e^{tA} = A e^{tA}$
- $AB = BA \Rightarrow e^{A+B} = e^A \cdot e^B$
- $(e^A)^{-1} = e^{-A}$

3. $A \in GL(n, \mathbb{C})$ $\quad \exists\ \sigma: [0,1] \to GL(n, \mathbb{C})$ 连续 s.t. $\sigma(0) = A$ $\quad \sigma(1) = I_n$

pf: $A = P\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$

令 $\sigma_1(t) = P\begin{pmatrix} \lambda_1 & & (1-t)* \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$ $\quad \Rightarrow \sigma_1(0) = A$ $\quad \sigma_1(1) = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$

~~令 $\sigma_2(t) = P(\ldots)$ 为 $\mathbb{C}$ 中连接 $(\lambda_1, \ldots, \lambda_n)$ 与 $(1, \ldots, 1)$ 的曲线，且不经过原点~~

令 $\gamma_i(t)$ 为 $\mathbb{C}$ 中连接 λ_i 与 1 的曲线 $\quad \gamma_i(0) = \lambda_i$, $\quad \gamma_i(1) = 1$ $\quad \gamma_i(t) \neq 0 \quad \forall\ t \in [0,1]$

令 $\sigma_2(t) = P\begin{pmatrix} \gamma_1(t) & & \\ & \ddots & \\ & & \gamma_n(t) \end{pmatrix} P^{-1}$ $\quad \sigma_2(0) = P\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$ $\quad \sigma_2(1) = P\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} P^{-1} = I_n$

将 σ_1 σ_2 连接起来 令 $\sigma(t) = \begin{cases} \sigma_1(2t) & 0 \le t \le \frac{1}{2} \\ \sigma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$ 为所求