HW16.

1、 $\mathbb{R}^4$. $u_1=(1,0,1,0)^t$ $u_2=(1,1,1,1)^t$

计算 $\langle u_1,u_2\rangle^\perp$ 的单位正交基.

解: $A=\begin{pmatrix}1&0&1&0\\1&1&1&1\end{pmatrix}$, $\langle u_1,u_2\rangle^\perp=\{X\mid AX=0\}$.

解方程有 $v_1=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$ $v_2=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$ 为 $\langle u_1,u_2\rangle^\perp$ 的一组基.

Gram-Schmidt 正交化.

取 $e_1=\frac{v_1}{\|v_1\|}=\begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\\0\end{pmatrix}$ $e_2=\frac{v_2}{\|v_2\|}=\begin{pmatrix}0\\\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix}$

e_1,e_2 就是 $\langle u_1,u_2\rangle^\perp$ 的一组基. □

2. $\mathbb{R}^5$ 中子空间 U,

$U=\{x\mid AX=0\}$. $A=\begin{pmatrix}0&1&1&1&0\\1&0&-1&0&2\end{pmatrix}$

取 $u_1=\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}$ $u_2=\begin{pmatrix}1\\0\\-1\\0\\2\end{pmatrix}$

对 $\forall$ $u\in U$

$A\cdot u=0 \Leftrightarrow u_1^t\cdot u=0$ 且 $u_2^t\cdot u=0$

$\Leftrightarrow (u_1|u)\ (u_2|u)=0$

$\Rightarrow u_1,u_2\in U^\perp$ $\dim U=3$, $\Rightarrow \dim U^\perp=2$

$\Rightarrow U^\perp=\langle u_1,u_2\rangle$

Gram-Schmidt 正交化:

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}$$

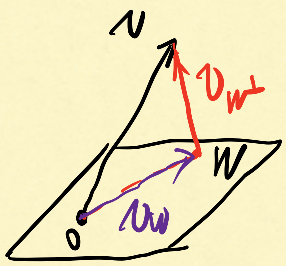
$$e_2' = u_2 - (u_2|e_1)\cdot e_1 = \begin{pmatrix}1\\\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\\6\end{pmatrix}$$

$$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{51}}\begin{pmatrix}3\\1\\-2\\1\\6\end{pmatrix}$$

e_1, e_2 为 $U^\perp$ 的单位正交基。 □

3. $\mathbb{R}^3$ $W = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \right\rangle$, $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

计算: v 到 W 的距离, v 和 W 的夹角.



解: 取 $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2' = \begin{pmatrix}1\\2\\1\end{pmatrix} - \left(\begin{pmatrix}1\\2\\1\end{pmatrix} \middle| \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix}$$

$$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\2\\1\end{pmatrix}$$

e_1, e_2 为 W 的单位正交基.

$$v_W = (v|e_1)\cdot e_1 + (v|e_2)e_2 = e_1 + \frac{3}{\sqrt{5}}e_2$$

$$= \begin{pmatrix}1\\\frac{6}{5}\\\frac{3}{5}\end{pmatrix}$$

$$v_{W^\perp} = v - v_W = \begin{pmatrix}0\\-\frac{1}{5}\\\frac{2}{5}\end{pmatrix}$$

$d(v, W) = \|v_{W^\perp}\| = \frac{\sqrt{5}}{5}$

夹角为 $\arccos\left(\frac{(v \mid v_W)}{\|v\| \cdot \|v_W\|}\right) = \arccos\left(\sqrt{\frac{14}{15}}\right)$. $\square$

4、U_1, U_2 为 V 子空间. $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$.

证:

引理1: 若 $U \subset W$, 则 $W^\perp \subset U^\perp$,

证: 设 $X \in W^\perp$, 对 $\forall\ u \in U$, $u \in W$

$(X \mid u) = 0 \Rightarrow X \in U^\perp$. $\square$

引理2: $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$

证: $U_1 \subset U_1 + U_2$ $\quad U_2 \subset U_1 + U_2$

$\Rightarrow (U_1 + U_2)^\perp \subset U_2^\perp \quad (U_1 + U_2)^\perp \subset U_1^\perp$

$\Rightarrow (U_1 + U_2)^\perp \subset U_1^\perp \cap U_2^\perp$

若 $y \in U_1^\perp \cap U_2^\perp$, 对 $\forall\ u_1 + u_2 \in U_1 \cap U_2$

$(y \mid u_1 + u_2) = (y \mid u_1) + (y \mid u_2) = 0$

$\Rightarrow U_1^\perp \cap U_2^\perp \subset (U_1 + U_2)^\perp$

$$(U_1^\perp + U_2^\perp)^\perp = (U_1^\perp)^\perp \cap (U_2^\perp)^\perp = U_1 \cap U_2$$

$$U_1^{\perp}+U_2^{\perp}=\left((U_1^{\perp}\cap U_2^{\perp})^{\perp}\right)^{\perp}=(U_1\cap U_2)^{\perp} \quad \square$$

5、A 正交 $A\in M_n(\mathbb{R})$，证：$t^n\chi_A(\frac{1}{t})=\pm\chi_A(t)$、

法1：
$$\begin{aligned}\chi_A(t)&=|tE-A|\\&=|A|\cdot|tA^{-1}-E|\\&=t^n(-1)^n\cdot|A|\cdot|t^{-1}E-A^{-1}|\\&=t^n\cdot(-1)^n\cdot|A|\,\chi_{A^{-1}}(\tfrac{1}{t})\end{aligned}$$

A 正交 $|A|=\pm1$. $\chi_A(t)=\pm t^n\chi_{A^{-1}}(\frac{1}{t})$

代入 $\frac{1}{t}$ 即可知

$A^t=A^{-1}$ $\chi_A=\chi_{A^t}\Rightarrow\chi_A(t)=\chi_{A^{-1}}(t)$

$$t^n\cdot\chi_A(\tfrac{1}{t})=\pm\chi_A(t) \qquad \square$$

法2：A 正交，$\exists\, T\in O(n)$ s.t.

$$T^tAT=\begin{pmatrix} N(\cos\theta_1,\sin\theta_1) & & & & \\ & \ddots & & & \\ & & N(\cos\theta_s,\sin\theta_s) & & \\ & & & \pm1 & \\ & & & & \ddots \\ & & & & & \pm1\end{pmatrix}$$

$$N(\cos\theta,\sin\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$\chi_A(t)=\chi_{T^tAT}(t)=\left(\prod_{i=1}^{s}\left(t^2-2\sin\theta_i t+1\right)\right)\cdot(t-1)^k(t+1)^l$$

$$2s+k+l=n$$

易验证 $\chi_A(t)$ 满足条件. $\blacksquare$

6. V 为 n 维欧式空间, $e_1,\cdots,e_n\in V$

(i) $\Rightarrow$ (ii) $x\in V$

$$x=(x|e_1)\cdot e_1+\cdots+(x|e_n)\cdot e_n$$

$$y=(y|e_1)e_1+\cdots+(y|e_n)\cdot e_n$$

$$(x|y)=\left(\sum_{i=1}^{n}(x|e_i)\cdot e_i \,\middle|\, \sum_{j=1}^{n}(y|e_j)\cdot e_j\right)$$

$$=\sum_{i=1}^{n}(x|e_i)(y|e_i)$$

(ii) $\Rightarrow$ (iii) 取 $y=x$ 即可.

(iii) $\Rightarrow$ (i) 若 对 $\forall$ $x\in V$, $\|x\|^2=\sum_{i=1}^{n}(x|e_i)^2$.

$$\|e_1\|^2=\underbrace{(e_1|e_1)^2}_{\|e_1\|^4}+\sum_{j=2}^{n}(e_1|e_j)^2$$

$$\Rightarrow \|e_1\|^4\le\|e_1\|^2 \Rightarrow \|e_1\|\le 1$$

$\dim V = n$, $\dim\langle e_2, \cdots, e_n\rangle \le n-1$

$\Rightarrow \exists\, v \neq 0,\ v \in \langle e_2, \cdots, e_n\rangle^{\perp}$

$\|v\|^2 = (v|e_1)^2 + \sum_{j=2}^{n} (v|e_j)^2$

$= (v|e_1)^2 \le \|v\|^2 \cdot \|e_1\|^2$

$\|v\| \neq 0$, $\Rightarrow \|e_1\|^2 \ge 1$.

$\Rightarrow \|e_1\| = 1$. 同理 $\|e_1\|, \cdots, \|e_n\| = 1$.

特别地 $\underbrace{\|e_1\|^2}_{1} = \underbrace{\|e_1\|^4}_{1} + \sum_{j=2}^{n} (e_1|e_j)^2$

从而 $(e_1|e_j) = 0$, $j \ge 2$.

同理 $(e_i, e_j) = 0$, $i \neq j$.

从而 $e_1, \cdots, e_n$ 为单位正交基. $\square$

HW17.

1.
$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

解:① $\chi_A(t) = |tE - A| = t^2(t-2)(t+2)$

② $V^0 = \{X \mid AX = 0\}$, 有一组基 $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ 和 $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$V^2 = \{X \mid (2E - A)X = 0\}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ 为 V^2 的基

$V^{-2}=\{X \mid (-2E-A)X=0\}$，$\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$ 为 V^{-2} 的基.

③ 正交化

取 $v_1=\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$ $v_2=\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$ $v_3=\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$ $v_4=\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix}$

④ $P=(v_1, v_2, v_3, v_4)$

$$P^t \cdot A \cdot P = \mathrm{diag}(0,0,2,-2)$$

$$=\begin{pmatrix}0&&&\\&0&&\\&&2&\\&&&-2\end{pmatrix}=B.$$ □

2、$\dim V=2k+1$，$\mathscr{A}\in \mathcal{L}(V)$ 正交变换，$\det(\mathscr{A})=1$. 证明 $\mathscr{A}$ 有特征值1.

证明：$\exists\ T\in O(n)$ s.t. $T^t \cdot A\cdot T=\begin{pmatrix}N(\cos\theta_1,\sin\theta_1)&&&&&\\&\ddots&&&&\\&&N(\cos\theta_s,\sin\theta_s)&&&\\&&&\pm1&&\\&&&&\ddots&\\&&&&&\pm1\end{pmatrix}$

$$\det(A)=\det(T^tAT)$$

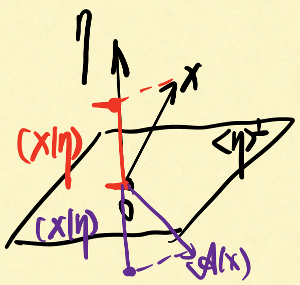
$$\det(N(\cos\theta,\sin\theta))=\det\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}=1$$

$$\det(\mathscr{A})=(-1)^{\#\text{标准型中对角线上的}-1\text{个}}=1$$

$\dim V=2k+1$ 奇数. $\Rightarrow$ 1 为 $\mathscr{A}$ 的特征值. □

3、 V，$(\ |\ )$ 内积． $\eta\in V$，$\|\eta\|=1$．

$$\mathscr{A}: V \longrightarrow V$$
$$x \longmapsto x-2(x|\eta)\cdot\eta$$



(i) $\forall\, x, y\in V$

$$(\mathscr{A}x, \mathscr{A}y)$$
$$=\left(x-2(x|\eta)\cdot\eta \;\middle|\; y-2(y|\eta)\cdot\eta\right)$$
$$=(x|y)-2(y|\eta)(x|\eta)$$
$$-2(x|\eta)\cdot(\eta|y)+4(x|\eta)(y|\eta)(\eta|\eta)$$
$$=(x|y)$$

(ii) $\mathscr{A}^2=\mathcal{E}$

对 $\forall\, x\in V$

$$\mathscr{A}^2(x)=\mathscr{A}(\mathscr{A}(x))=\mathscr{A}(x-2(x|\eta)\cdot\eta)$$
$$=\mathscr{A}(x)-2(x|\eta)\cdot\mathscr{A}(\eta)$$
$$\left(\mathscr{A}(\eta)=\eta-2(\eta|\eta)\cdot\eta=-\eta\right) \quad =x-2(x|\eta)\cdot\eta+2(x|\eta)\cdot\eta$$
$$=x.$$

(iii) $\det(\mathscr{A})$，取 $e_1=\eta$ 且 $e_2,\cdots,e_n$ 为 $\langle\eta\rangle^\perp$ 的单位正交基，

$$\mathscr{A}(e_1)=\mathscr{A}(\eta)=-\eta$$

$$\mathscr{A}(e_i)=e_i-2(e_i|\eta)\cdot\eta$$
$$=e_i$$

在基 $e_1,\cdots,e_n$ 下，$\mathscr{A}$ 矩阵为

$$\mathscr{A}(e_1,\cdots,e_n)=(e_1,\cdots,e_n)\begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&\ddots&\\&&&&1\end{pmatrix}$$

从而 $\det(\mathscr{A})=-1$.

4、A 斜对称，A 可逆。求证 $A+A^2$ 可逆.

证明：A 斜对称：　$\exists T\in O(n)$ s.t.

$$T^t\cdot A\cdot T=\begin{pmatrix}N(0,\beta_1)&&&\\&\ddots&&\\&&\ddots&\\&&&N(0,\beta_s)\end{pmatrix}\qquad N(0,\beta)=\begin{pmatrix}0&-\beta\\\beta&0\end{pmatrix}$$

A 可逆 $\Rightarrow \beta_1,\cdots,\beta_s$ 不为 0

$A+A^2$ 可逆 $\Leftrightarrow$ $T^t(A+A^2)T$ 可逆 $\Leftrightarrow$ $T^tAT+(T^tAT)^2$ 可逆.

从而只需对标准型斜对称矩阵证明，

进而，只需对形如 $N(0,\beta)$ 的矩阵证明即可，$\beta\neq 0$.

而 $C=N(0,\beta)+N(0,\beta)^2=\begin{pmatrix}0&-\beta\\\beta&0\end{pmatrix}+\begin{pmatrix}-\beta^2&0\\0&-\beta^2\end{pmatrix}=\begin{pmatrix}-\beta^2&-\beta\\\beta&-\beta^2\end{pmatrix}$

$\det(C) = \beta^4 + \beta^2 = \beta^2(\beta+1) \quad (\beta \neq 0)$

$\neq 0$ □

5、 $A \in M_n(\mathbb{R})$ 正定。 证明 $A + A^{-1} - 2E_n$ 半正定.

证： 正定 $\Rightarrow$ 对称. $\Rightarrow \exists\ T \in O(n)$

使 $T^t A \cdot T = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$

正定 $\Rightarrow \lambda_1, \cdots, \lambda_n > 0$.

$T^{-1} A^{-1} \cdot (T^t)^{-1} = \operatorname{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1})$

$T^t = T^{-1}$

$\Rightarrow T^t \cdot A^{-1} T = \operatorname{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1})$

$T^t(A + A^{-1} - 2E_n)T = \operatorname{diag}\left(\lambda_1 + \frac{1}{\lambda_1} - 2, \cdots, \lambda_n + \frac{1}{\lambda_n} - 2\right)$

$\lambda_i > 0 \Rightarrow \lambda_i + \frac{1}{\lambda_i} - 2 \geq 0$ "取=" $\lambda_i = 1$

$\Rightarrow$ $A + A^{-1} - 2E_n$ 半正定. 且 若 A 没有特征值 1, 则

$A + A^{-1} - 2E_n$ 正定. □