1. $\vec{u_1}=(1,0,1,0)^t$, $\vec{u_2}=(1,1,1,1)^t$. 计算 $\langle \vec{u_1},\vec{u_2}\rangle^\perp$ 的一组单位正交基.

解: 设 $\vec{e_1},\vec{e_2},\vec{e_3},\vec{e_4}$ 是 $\mathbb{R}^4$ 的标准基, $\forall x\in\mathbb{R}^4$.

$$x=x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3}+x_4\vec{e_4}.$$

$$\vec{x}\in U^\perp \iff (x|\vec{u_1})=0,\ (\vec{x}|\vec{u_2})=0$$

$$\iff \boxed{\begin{cases} x_1+x_3=0 \\ x_1+x_2+x_3+x_4=0 \end{cases}}$$

$X\in U^\perp$.

$$\begin{pmatrix}\vec{u_1}^t\\ u_2^t\end{pmatrix}X=0.$$

$$\Rightarrow U^\perp=\left\langle \underbrace{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}}_{\vec{w_1}},\ \underbrace{\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}}_{\vec{w_2}}\right\rangle$$

施密特正交化.

$$\vec{\varepsilon_1}=\frac{\vec{w_1}}{\|\vec{w_1}\|}=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$\vec{\varepsilon_2}'=\vec{w_2}-(\vec{w_2}|\vec{\varepsilon_1})\,\vec{\varepsilon_1}=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$\vec{\varepsilon_2}=\frac{\vec{\varepsilon_2}'}{\|\vec{\varepsilon_2}'\|}=\frac{\vec{w_2}}{\sqrt{2}}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$U^\perp$ 的一组单位正交基为 $\{\vec{\varepsilon_1},\vec{\varepsilon_2}\}$.

2. 设标准欧氏空间 $\mathbb{R}^4$ 中的子空间 U 是矩阵

$$A=\begin{pmatrix}0&1&1&1&0\\1&0&-1&0&2\end{pmatrix}$$

对应的齐次线性方程组的解空间, 计算 $U^\perp$ 的一组单位正交基.

解: 对 $\forall\vec{u}\in U$, $A^{(1)}\vec{u}=0$, $A^{(2)}\vec{u}=0$.

由正交补的定义可知, $\boxed{U^\perp=\langle (A^{(1)})^t,(A^{(2)})^t\rangle}$

$$=\left\langle \underbrace{\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}}_{\vec{w_1}},\ \underbrace{\begin{pmatrix}1\\0\\-1\\0\\2\end{pmatrix}}_{\vec{w_2}}\right\rangle$$

$$\vec{\varepsilon_1}=\frac{\vec{w_1}}{\|\vec{w_1}\|}=\frac{\sqrt{3}}{3}\begin{pmatrix}0\\1\\1\\ \vdots\end{pmatrix}$$

$$\vec{\varepsilon_2'} = \vec{w_2} - (\vec{w_2}|\varepsilon_1)\vec{\varepsilon_1} = \vec{w_2} - \frac{1}{3}(\vec{w_2}|\vec{w_1})\vec{w_1}$$

$$= \begin{pmatrix}1\\0\\-1\\2\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\1\\5\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\\2\end{pmatrix}$$

$$\vec{\varepsilon_2} = \frac{\vec{\varepsilon_2'}}{\|\vec{\varepsilon_2'}\|} = \frac{\sqrt{51}}{17}\begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\\2\end{pmatrix}$$

3. R^3，子空间 $W = \langle (1,0,0)^t, (1,2,1)^t \rangle$，再设 $\vec{v} = (1,1,1)^t$. 计算 $\vec{v}$ 到 W 的距离和 $\vec{v}$ 到 W 的夹角. ($\vec{w_1}$, $\vec{w_2}$)

设 $\vec{x} = \pi_W(\vec{v})$ $\vec{v}$ 到 W 的投影.

$d(\vec{v}, W) = \|\vec{v} - \vec{x}\|$

下求 $\vec{x}$.

法一：找 W 的一组单位正交基.

$\vec{\varepsilon_1} = \vec{w_1}$

$\vec{\varepsilon_2'} = \vec{w_2} - (\vec{w_2}|\vec{\varepsilon_1})\vec{\varepsilon_1} = \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix}$

$\vec{\varepsilon_2} = \frac{\vec{\varepsilon_2'}}{\|\vec{\varepsilon_2'}\|} = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\2\\1\end{pmatrix}$

$\vec{x} = \pi_W(\vec{v}) = (\vec{v}|\vec{\varepsilon_1})\vec{\varepsilon_1} + (\vec{v}|\vec{\varepsilon_2})\vec{\varepsilon_2} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{3}{5}\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}1\\\frac{6}{5}\\\frac{3}{5}\end{pmatrix}$

$d(\vec{v}, W) = \|\vec{x} - \vec{v}\| = \left\|\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\\frac{6}{5}\\\frac{3}{5}\end{pmatrix}\right\| = \left\|\begin{pmatrix}0\\-\frac{1}{5}\\\frac{2}{5}\end{pmatrix}\right\| = \sqrt{\frac{1}{25} + \frac{4}{25}} = \frac{\sqrt{5}}{5}$

夹角. $\arccos\left(\frac{(\vec{v}|\vec{x})}{\|\vec{v}\|\|\vec{x}\|}\right) = \arccos\left(\sqrt{\frac{14}{15}}\right)$

法二：设 $\vec{v}$ 在 W 上的正交投影是 $\vec{x} = \alpha_1\vec{w_1} + \alpha_2\vec{w_2}$，则

$(\vec{v} - \vec{x}) \perp W \Leftrightarrow \begin{cases}(\vec{v} - \vec{x}|\vec{w_1}) = 0\\(\vec{v} - \vec{x}|\vec{w_2}) = 0\end{cases} \Leftrightarrow \boxed{G(\vec{w_1}, \vec{w_2})\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}(\vec{v}|\vec{w_1})\\(\vec{v}|\vec{w_2})\end{pmatrix}}$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$\Rightarrow \alpha_1 = \frac{2}{5}, \quad \alpha_2 = \frac{3}{5}$

$\Rightarrow \vec{x} = \frac{2}{5}\vec{w_1} + \frac{3}{5}\vec{w_2} = (1, \frac{6}{5}, \frac{3}{5})^T$

$(\vec{v}|\vec{w_1}) = \alpha_1 (\vec{w_1}|\vec{w_1}) + \alpha_2 (\vec{w_2}|\vec{w_1})$

$(\vec{v}|\vec{w_2}) = \alpha_1 (\vec{w_1}|\vec{w_2}) + \alpha_2 (\vec{w_2}|\vec{w_2})$

$$\Leftrightarrow \begin{pmatrix} (\vec{w_1}|\vec{w_1}) & (\vec{w_1}|\vec{w_2}) \\ (\vec{w_2}|\vec{w_1}) & (\vec{w_2}|\vec{w_2}) \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} (\vec{v}|\vec{w_1}) \\ (\vec{v}|\vec{w_2}) \end{pmatrix}$$

---

4. U_1, U_2是V的子空间, 证明: $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$.

pf: claim1: $U \subset W$, $U^\perp \supset W^\perp$.

$\forall \vec{x} \in W^\perp$, 则 $(\vec{x}|\vec{u}) = 0$, $\forall \vec{u} \in U$.

$\Rightarrow \vec{x} \in U^\perp$.

$\Rightarrow W^\perp \subset U^\perp$.

Claim 2. $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

由于 $W_1 \subset W_1 + W_2$, 则 $W_1^\perp \supset (W_1 + W_2)^\perp$. (claim1)

同理 $W_2^\perp \supset (W_1 + W_2)^\perp$

则 $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$

$\forall \vec{x} \in W_1^\perp \cap W_2^\perp$, $\forall \vec{w} \in W_1 + W_2$, $\exists \vec{w_1} \in W_1$, $\vec{w_2} \in W_2$, s.t.

$\vec{w} = \vec{w_1} + \vec{w_2}$

$(\vec{x}|\vec{w}) = (\vec{x}|\vec{w_1} + \vec{w_2}) = (\vec{x}|\vec{w_1}) + (\vec{x}|\vec{w_2}) = 0$

$\Rightarrow \vec{x} \in (W_1 + W_2)^\perp$

$\Rightarrow W_1^{\perp} \cap W_2^{\perp} \subset (W_1+W_2)^{\perp}$

$\Rightarrow W_1^{\perp} \cap W_2^{\perp} = (W_1+W_2)^{\perp}$.

原命题证明：令 $W_1 = U_1^{\perp}$， $W_2 = U_2^{\perp}$

$(U_1^{\perp}+U_2^{\perp})^{\perp} = (U_1^{\perp})^{\perp} \cap (U_2^{\perp})^{\perp} = U_1 \cap U_2$.

$\Rightarrow \underbrace{((U_1^{\perp}+U_2^{\perp})^{\perp})^{\perp}}_{U_1^{\perp}+U_2^{\perp}} = (U_1 \cap U_2)^{\perp}$

法二：取 $U_1 \cap U_2$ 的一组单位正交基 $\vec{e_1}, \cdots, \vec{e_r}$.

扩充为 U_1 的一组基（单位正交） $\vec{e_1}, \cdots, \vec{e_r}, \vec{e_{r+1}}, \cdots, \vec{e_{r+s}}$

再扩充为 U_1+U_2 的一组基（单位正交） $\vec{e_1}, \cdots, \vec{e_r}, \vec{e_{r+1}}, \cdots, \vec{e_{r+s}}, \vec{e_{r+s+1}} \cdots, \vec{e_{r+s+t}}$.

最后扩充成 V 的一组单位正交基

（全空间）

$\vec{e_1}, \cdots, \vec{e_r}, \vec{e_{r+1}}, \cdots, \vec{e_n}$.

U_2 的一组单位正交基.

这里有问题

注：$\vec{e_{r+s+1}}, \cdots, \vec{e_{r+s+t}}$ 不一定在 U_2 中

$(U_1 \cap U_2)^{\perp} = \langle \vec{e_{r+1}}, \cdots, \vec{e_n} \rangle$.

$U_1^{\perp} = \langle \vec{e_{r+s+1}}, \cdots, \vec{e_n} \rangle$

$U_2 = \langle \vec{e_1}, \cdots, \vec{e_r}, \vec{e_{r+s+1}}, \cdots, \vec{e_{r+s+t}} \rangle$

$U_2^{\perp} = \langle \underline{\vec{e_{r+1}}, \cdots, \vec{e_{r+s}}}, \vec{e_{r+s+t+1}}, \cdots, \vec{e_n} \rangle$

$U_1^{\perp}+U_2^{\perp} = \langle \vec{e_{r+1}}, \cdots, \vec{e_{r+s}}, \vec{e_{r+s+1}}, \cdots, \vec{e_n} \rangle$.

$\Rightarrow U_1^{\perp}+U_2^{\perp} = \langle U_1 \cap U_2 \rangle^{\perp}$.

5. 证明：n 阶正交矩阵 A 的特征多项式 $\chi_A(t)$ 具有性质：$t^n \chi_A(\frac{1}{t}) = \pm \chi_A(t)$

证：$\boxed{AA^t = E}$

$t^n \chi_A(\frac{1}{t}) = t^n \left| \frac{1}{t}E - A \right| = t^n (\frac{1}{t})^n \left| E - tA \right|$

$|A| = \pm 1$

$A^t = A^{-1}$

$A \sim_s A^t$

$= |E - tA|$

$= |AA^t - tA|$

$= |A(A^t - tE)| = |A||A^t - tE| = \pm |tE - A^t|$

$= \pm \chi_{A^t}(t)$

$= \pm \chi_A(t)$

6. 设 V 是 n 维欧氏空间，$\vec{e_1}, \cdots, \vec{e_n} \in V$. 证明下列断言等价：

(i) $\vec{e_1}, \cdots, \vec{e_n}$ 是 V 的一组单位正交基

(ii) 对 $\forall \vec{x}, \vec{y} \in V$，$(\vec{x}|\vec{y}) = \sum_{i=1}^{n} (\vec{x}|\vec{e_i})(\vec{y}|\vec{e_i})$

(iii) 对任意 $\vec{x} \in V$，$\|\vec{x}\|^2 = \sum_{i=1}^{n} (\vec{x}|\vec{e_i})^2$.

Pf: (i)⇒(ii) $\forall \vec{x} \in V$，$x = x_1\vec{e_1} + \cdots + x_n\vec{e_n}$

$\exists x_i \in \mathbb{R}$. s.t.

$\Rightarrow (\vec{e_i}|\vec{x}) = x_i$.

$(\vec{x}|\vec{y}) = \left(\sum_{i=1}^{n} (x|\vec{e_i})\vec{e_i} \,\middle|\, \vec{y}\right) = \sum_{i=1}^{n} (x|\vec{e_i})(\vec{e_i}|\vec{y})$

$= \sum_{i=1}^{n} (x|\vec{e_i})(\vec{y}|\vec{e_i})$

(ii)⇒(iii). 令 $\vec{y} = \vec{x}$

(iii)⇒(i). 下证 $\|\vec{e_1}\| = 1$ 且 $(\vec{e_1}|\vec{e_i}) = 0$，$i = 2, 3, \cdots, n$.

设 $\vec{x} = \vec{e_1}$，则 $\boxed{\|\vec{e_1}\|^2 = \sum_{i=1}^{n} (\vec{e_1}|\vec{e_i})^2 = \|\vec{e_1}\|^4 + \sum_{i=2}^{n} (\vec{e_1}|\vec{e_i})^2} \geq \|e_1\|^4$

$\Rightarrow \|\vec{e_1}\|^4 \leq \|\vec{e_1}\|^2 \Rightarrow \|e_1\|^2 \leq 1 \Rightarrow \|\vec{e_1}\| \leq 1$ $\vee_0$

由 $\dim V = n$，故 $\exists \vec{v} \in \langle \vec{e_2}, \cdots, \vec{e_n} \rangle^{\perp} \setminus \{\vec{0}\}$. $\Rightarrow \|\vec{v}\| \neq 0$.

$\|\vec{v}\|^2 = \sum_{i=1}^{n} (\vec{v}|\vec{e_i})^2 = (\vec{v}|\vec{e_1})^2 \leq \|\vec{v}\|^2 \|\vec{e_1}\|^2$. $(\|\vec{v}\| > 0)$

$(\vec{v}|\vec{e_2}) = 0, \cdots, (\vec{v}|\vec{e_n}) = 0$

$\Rightarrow \|e_1\| \geq 1$.

$\Rightarrow$ 只是单位向量

由(*)可知, $1 = 1 + \sum_{i=2}^{n} (\vec{e_1}|\vec{e_i})^2$.

$\Rightarrow \sum_{i=2}^{n} (\vec{e_1}|\vec{e_i})^2 = 0$

$\Rightarrow (\vec{e_1}|\vec{e_i}) = 0,\ i = 2, \dots, n$.

同理可证 $\|\vec{e_2}\| = 1$, $(\vec{e_2}|\vec{e_i}) = 0,\ i = 1, 3, \dots, n$.

⋮

$\|\vec{e_n}\| = 1$, $(\vec{e_n}|\vec{e_i}) = 0,\ i = 1, 2, \dots, n-1$.

$\Rightarrow \vec{e_1}, \vec{e_2}, \dots, \vec{e_n}$ 是 V 的一组单位正交基.

设 q 是 $\mathbb{R}^n$ 上的二次型,

$$q: \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \vec{x} \longmapsto \vec{x}^t A \vec{x}. \qquad A \in SM_n(\mathbb{R})$$

$$P^t A P = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad P \in O_n(\mathbb{R})$$

$P = (\vec{P^{(1)}}, \dots, \vec{P^{(n)}})$

令 $\vec{y} = y_1 \vec{P^{(1)}} + \dots + y_n \vec{P^{(n)}}$

$q(\vec{y}) = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$.

$n=2$ 时, $q(\vec{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$.

假设 q 是正定的, 则 $\lambda_1, \lambda_2 > 0$

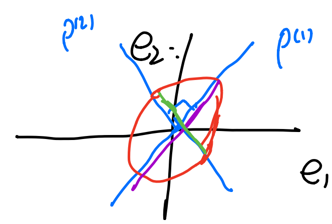
$$\frac{y_1^2}{\frac{1}{\lambda_1}} + \frac{y_2^2}{\frac{1}{\lambda_2}} = 1$$

? 何意义 ?

长轴长度 $2\sqrt{\frac{1}{\lambda_1}}$

短轴长度 $2\sqrt{\frac{1}{\lambda_2}}$

特征向量 代表 长轴与短轴 的方向



注: $\lambda_1 = \lambda_2$, 圆 (圆形)

在单位正交基变换下, 椭圆不会变成圆 (特征值不变).

$q(\vec{x}) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2 = 1$.

1. $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ \dots \end{pmatrix} \in SM_n(\mathbb{R})$, 计算正交矩阵 P 和对角矩阵 B, 使得 $P^t A P = B$.

$$\begin{pmatrix} & 1 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

解:

① $\chi_A = (t-2)(t+2)t^2$. $\Rightarrow \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -2$.

$\dim V^0 = 2$. $\operatorname{rank}(-A) = \operatorname{rank}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ * & & * & \end{pmatrix}$.

$\Rightarrow V^0$ 是 $\begin{cases} -x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases}$ 对应的解空间, $V^0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$.

$V^2 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$, $V^{-2} = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$.

施密特正交化只需对 V^{λ_i}.

$$P = \begin{pmatrix} \frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2} & 0 \\ 0 & \frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2} \\ 0 & \frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2} \\ \frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & -2 \end{pmatrix}$$

2. 设 V 是一个奇数维内积空间, $\mathcal{A}$ 为 V 上的正交变换, $\det(\mathcal{A}) = 1$.

证明: $\mathcal{A}$ 有特征值 1

Pf: $\mathcal{A}$ 为 V 的正交变换 则 $\exists\ \theta_1, \cdots, \theta_s \in (0,\pi) \cup (\pi, 2\pi)$ 和 $\lambda_{2s+1}, \cdots, \lambda_n$.

$\in \{-1, 1\}$. 使得 $\mathcal{A}$ 在 V 的某组单位正交基下的矩阵

$$A = \begin{pmatrix} N(\cos\theta, \sin\theta) & & & & \\ & \ddots & & & \\ & & N(\cos(\theta_s), \sin\theta_s) & & \\ & & & \lambda_{2s+1} & \\ & & & & \ddots \\ & & & & & \lambda_n \end{pmatrix}$$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\deg_t(\underset{\substack{\uparrow \\ R[x]}}{\chi_A}) = n$ 为奇数 $\Rightarrow$ χ_A 必有实根

, 2 复特征值 λ_i, 共轭出现 $\lambda_i, \bar\lambda_i$, $\lambda_i\bar\lambda_i = 1$.

$A=\prod_i \lambda_i$ $\qquad \lambda_i = \begin{cases} 1, \\ 2\cdot \boxed{-1} \\ 1, \end{cases}$ $\qquad -1\times -1$，成对出现

$\det(A)=1$

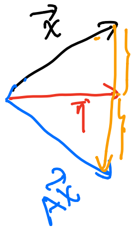
$\Rightarrow A$ 一定有特征值 1.

3. η 为 V 中的一个单位向量

$\mathcal{A}: V \to V$

$\vec{x} \mapsto \vec{x} - 2(\vec{x}|\eta)\eta$.

镜面反射，



$(\vec{x}|\vec{\eta})\vec{\eta}$，$x$ 在 $\langle\vec{\eta}\rangle$ 上投影

$\Rightarrow \mathcal{A}\vec{x} = \vec{x} - 2(\vec{x}|\vec{\eta})\vec{\eta}$.

(a) $\mathcal{A}$ 是正交变换

~~验证~~ $(\mathcal{A}\vec{x}|\mathcal{A}\vec{y}) = (\vec{x}|\vec{y})$

(b) $\mathcal{A}^2(\vec{x}) = \mathcal{A}(\vec{x} - 2(\vec{x}|\vec{\eta})\vec{\eta})$

$= \mathcal{A}\vec{x} - 2(\vec{x}|\vec{\eta})\mathcal{A}\vec{\eta}$

$= \vec{x} - 2(\vec{x}|\vec{\eta})\vec{\eta} - 2(\vec{x}|\vec{\eta})\underbrace{(\vec{\eta} - 2\underbrace{(\vec{\eta}|\vec{\eta})}_{1}\vec{\eta})}_{-\vec{\eta}}$

$= \vec{x} - 2(\vec{x}|\vec{\eta})\vec{\eta} + 2(\vec{x}|\vec{\eta})\vec{\eta}$

$= \vec{x}$.

(c). $\det(\mathcal{A}) = -1$.

取 $\vec{e_1} = \vec{\eta}$，扩充成 V 的一组单位正交基 $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}$.

$\mathcal{A}(\vec{e_1}) = \mathcal{A}\vec{\eta} = -\vec{\eta}$.

$\mathcal{A}(\vec{e_2}) = \vec{e_2} - \underbrace{(\vec{e_2}|\vec{e_1})}_{0}\vec{e_1} = \vec{e_2}$

$\vdots$

$\mathcal{A}\vec{e_n} = \vec{e_n}$

$\Rightarrow \mathcal{A}(\vec{e_1}, \cdots, \vec{e_n}) = (\vec{e_1}, \cdots, \vec{e_n}) \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$\Rightarrow \det(\mathcal{A}) = -1$.

4. A 斜对称且可逆，证明 $A+A^2$ 可逆.

A 斜对称，且可逆，

$\exists T \in O_n(\mathbb{R})$ s.t. $T^t A T = \begin{pmatrix} N(0,\beta_1) & & \\ & \ddots & \\ & & \end{pmatrix}$，$\beta_i \in \mathbb{R}$

$N(0,\beta_1) = \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix}$

$$\begin{pmatrix} & & \\ & \ddots & \\ & & N(0,\beta_k) \end{pmatrix}$$ 50%

A^2+A 可逆 $\Leftrightarrow$ $\underline{T^t(A^2+A)T}$ 可逆

$$= (T^tAT)^2 + T^tAT$$

$$= \begin{pmatrix} \underline{N^2(0,\beta_1)+N(0,\beta_1)} & & \\ & \ddots & \\ & & \underline{N^2(0,\beta_k)+N(0,\beta_k)} \end{pmatrix}$$

$\forall i$, $\underline{N^2(0,\beta_i)} + N(0,\beta_i) = \begin{pmatrix} -\beta_i^2 & -\beta_i \\ -\beta_i & -\beta_i^2 \end{pmatrix}$

$\Rightarrow \det(\sim) = \beta_1^2(\beta_1^2+1) \neq 0.$

5. 设 $A \in M_n(\mathbb{R})$，正定矩阵. 证明：$A+A^{-1}-2E_n$ 为半正定矩阵，并指出什么时候 $A+A^{-1}-2E_n$ 正定.

$\exists T \in O_n(\mathbb{R})$, s.t. $T^tAT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i > 0$

$$(T^tAT)^{-1} = T^tA^{-1}T = \begin{pmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{pmatrix}$$

$$T^t(A+A^{-1}-2E_n)T = \begin{pmatrix} \lambda_1+\frac{1}{\lambda_1}-2 & & \\ & \ddots & \\ & & \lambda_n+\frac{1}{\lambda_n}-2 \end{pmatrix}$$

$\forall i \in \{1,2,\cdots,n\}$, $\lambda_i + \frac{1}{\lambda_i} - 2 \geq 0$

$\Rightarrow$ $A+A^{-1}-2E$ 为半正定矩阵

正定，$\lambda_i+\frac{1}{\lambda_i}-2 \neq 0$，$\lambda_i \neq 1$

A 没有特征值 1

6.

$$f = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2$$

nE

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \qquad \chi_A = (t-n)t^{n-1}$$

$$\exists T \in O_n(\mathbb{R}),\ s.t.\ \ T^t A T = \begin{pmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$T^t(nE - A)T = \boxed{\begin{pmatrix} 0 & & & \\ & n & & \\ & & \ddots & \\ & & & n \end{pmatrix}}$$

$\lambda_1 = n$

$\lambda_2 = 0$

$(\lambda_1 E - A)x = 0$

$(\lambda_2 E - A)x = 0$

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