1、(i) $A, B \in M_n(F)$, A逆：则 $AB \sim_s BA$.

$$BA = (A^{-1}A)BA = A^{-1}(AB)A.$$

$\Rightarrow BA \sim_c AB$

(ii) $A, B \in SM_n(\mathbb{R})$，正定.

$A = P^t P \qquad B = Q^t \cdot Q$

$AB = P^t P Q^t Q \qquad\qquad P, Q$ 可逆

$\Rightarrow AB \sim_s (P^t)^{-1} P^t P Q^t Q \cdot P^t$

$= P Q^t Q \cdot P^t = (QP^t)^t \cdot (QP^t)$

$\Rightarrow AB$ 相似于正定矩阵.

2. $f = t^2 - 2 \in F[t] \qquad \mathscr{A}^2 = 2\mathscr{A} - \mathscr{E}$ ， $B = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$f(\mathscr{A}) = \mathscr{A}^2 - 2\mathscr{E} = 2\mathscr{A} - 3\mathscr{E}$

$f(B) = B^2 - 2E = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

$B \neq 0$ ，B 可逆 $\Rightarrow$ 计 μ_B ，B 的次数至少为2.

$B^2 + \alpha_1 B + \alpha_0 E = 0$

$\Rightarrow \begin{cases} \alpha_0 + \alpha_1 + 1 = 0 \\ \alpha_1 + 2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_0 = 1 \\ \alpha_1 = -2 \end{cases}$

$\mu_B = t^2 - 2t + 1.$

3. $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A = \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}_{4\times 4}$

$B = \begin{pmatrix} J_2 & 0 \\ 0 & E_2 \end{pmatrix}$ $C = \begin{pmatrix} J_2 & 0 \\ 0 & \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

J_2 不是数乘 $\Rightarrow \deg \mu_{J_2} \geq 2$. $\mu_{J_2} = t^2$

$J^2 = 0$.

$\mu_A = \operatorname{lcm}(\mu_{J_2}, \mu_0) = \operatorname{lcm}(t^2, t) = t^2$

$\mu_B = \operatorname{lcm}(\mu_{J_2}, \mu_{E_2}) = \operatorname{lcm}(t^2, t-1) = t^2(t-1)$

令 $M = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\mu_M = \operatorname{lcm}(t-2, t) = t(t-2)$

$\mu_C = \operatorname{lcm}(\mu_{J_2}, \mu_M) = \operatorname{lcm}(t^2, t(t-2)) = t^2(t-2)$.

4. $\mathcal{A}$, $\mathcal{A}^2 = 2\mathcal{A} + 3\mathcal{E}$. $\operatorname{char} F \neq 2$.

证: $\operatorname{rk}(\mathcal{A}+\mathcal{E}) + \operatorname{rk}(\mathcal{A}-3\mathcal{E}) = \dim V$.

$\mathcal{A}^2 - 2\mathcal{A} - 3\mathcal{E} = 0$

$f(t) = t^2 - 2t - 3 = (t-3)(t+1)$

$\Rightarrow (\mathcal{A}-3\mathcal{E})(\mathcal{A}+\mathcal{E}) = 0$

$\operatorname{char} F \neq 2$

$\gcd(t-3, t+1) = 1$.

$\Rightarrow \ker(\mathcal{A}-3\mathcal{E}) \oplus \ker(\mathcal{A}+\mathcal{E}) = V$

$\Rightarrow n = \dim V = n - \operatorname{rk}(\mathcal{A}-3\mathcal{E}) + n - \operatorname{rk}(\mathcal{A}+\mathcal{E})$

$\Rightarrow \operatorname{rk}(\mathcal{A}+\mathcal{E}) + \operatorname{rk}(\mathcal{A}-3\mathcal{E}) = n$.

5. (i) $\ker \mathcal{A}^i \subset \ker \mathcal{A}^{i+1}$

若 $v \in \ker \mathcal{A}^i$, $\mathcal{A}^i v = 0 \Rightarrow \mathcal{A}^{i+1} v = \mathcal{A}(\mathcal{A}^i v)$

$= \mathcal{A}(0)$

$= 0$

$\operatorname{im} \mathcal{A}^{i+1} \subset \operatorname{im} \mathcal{A}^i$

若 $v = \mathcal{A}^{i+1} w \in \operatorname{im} \mathcal{A}^{i+1} \Rightarrow v = \mathcal{A}^i(\mathcal{A} v) \in \operatorname{im} \mathcal{A}^i$

(ii) 令 $d_i = \dim \ker \mathcal{A}^i$

$e_i = \dim \operatorname{im} \mathcal{A}^i$

则 $0 = d_0 \le d_1 \le \cdots \le d_i \le \cdots \le n$

$n = e_0 \ge e_1 \ge e_2 \ge \cdots \ge e_i \ge \cdots \ge 0$.

$\Rightarrow \exists k$ s.t. $i \ge k$ $d_i = d_k$, $e_i = e_k$

i.e. $\ker \mathcal{A}^k = \ker(\mathcal{A}^{k+i})$ $\operatorname{im}(\mathcal{A}^k) = \operatorname{im}(\mathcal{A}^{k+i})$.

(iii) $\ker(\mathcal{A}^k) = \ker(\mathcal{A}^{2k})$

i.e. $rk(\mathcal{A}^k) = rk(\mathcal{A}^{2k})$

i.e. $\operatorname{im}(\mathcal{A}^k) = \operatorname{im}(\mathcal{A}^{2k})$

核像分解 $\Rightarrow$ $\operatorname{im}(\mathcal{A}^k) \oplus \ker(\mathcal{A}^k) = V$

线性映射在商空间上诱导的映射：

A：一般情形：

设 $\mathcal{A}: V \longrightarrow W$ 为一线性映射.

$V_0 \subset V$，$W_0 \subset W$ 为子空间.

问：① 什么时候 $\bar{\mathcal{A}}: V/V_0 \longrightarrow W/W_0$ 是良定义？

$\bar{v} \longmapsto \overline{\mathcal{A}v}$

② 在哪组基下 $\mathcal{A}$ 的矩阵表示良好？

①　$\bar{v}_1 = \bar{v}_2 \Longleftrightarrow v_1 - v_2 \in V_0$

$\bar{\mathcal{A}}(\bar{v}_1) = \bar{\mathcal{A}}(\bar{v}_2) \Longleftrightarrow \mathcal{A}(v_1) - \mathcal{A}(v_2) \in W \Leftrightarrow \mathcal{A}(v_1 - v_2) \subset W \Leftarrow v_1 - v_2 \in W.$

要想诱导出良定义的映射 $\bar{\mathcal{A}}$，只需 $\mathcal{A}(V_0) \subset W_0$

②　$\forall v \in V_0$，$\Rightarrow \mathcal{A}(v) \in W$

取 $(e_1, \cdots, e_k)$ 为 V_0 的基，扩充为 V 的基 $(e_1, \cdots, e_n)$

$(\varepsilon_1, \cdots, \varepsilon_l)$ 为 W_0 的基，扩充为 W 的基 $(\varepsilon_1, \cdots, \varepsilon_m)$

则 $\mathcal{A}(e_1, \cdots, e_k, \cdots, e_n) = (\varepsilon_1, \cdots, \varepsilon_l, \cdots, \varepsilon_m) \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}_{m \times n}$

A_1　$l \times k$ 矩阵

A_2　$l \times (n-k)$ 矩阵

A_3　$(m-l) \times n-k$

其中 $\mathcal{A}(e_1, \cdots, e_k) = (\varepsilon_1, \cdots, \varepsilon_l) \cdot A_1$

$$\mathscr{A}(e_{k+1},\cdots,e_n)=(\varepsilon_1,\cdots,\varepsilon_n)\begin{pmatrix}A_2\\A_3\end{pmatrix}$$

$$\Rightarrow\quad \bar{\mathscr{A}}(\bar{e}_{k+1},\cdots,\bar{e}_n)=(\bar{\varepsilon}_{k+1},\cdots,\bar{\varepsilon}_n)=A_3.$$

eg：设 $V_1,V_2\subset V$ 为子空间.

$$\underset{\substack{\parallel\\ id}}{\mathscr{A}}:\ V_1\longrightarrow V_1+V_2$$

$$V_1\cap V_2\subset V_2.$$

$$\rightsquigarrow\ \bar{\mathscr{A}}:\ V_1/V_1\cap V_2\longrightarrow V_1+V_2/V_2$$

取 $\varepsilon_1,\cdots,\varepsilon_s$ 为 $V_1\cap V_2$ 的基，

扩充为 $e_1,\cdots,e_s,f_1,\cdots,f_t$ 为 V_1 的基

$e_1,\cdots,e_s,g_1,\cdots,g_l$ 为 V_2 的基.

则 $\mathscr{A}(e_1,\cdots,e_s,f_1,\cdots,f_t)$

$$=(e_1,\cdots,e_s,g_1,\cdots,g_l,f_1,\cdots,f_t)\left(\begin{array}{c|c}E_s&0\\\hline 0&0\\\hline 0&E_t\end{array}\right)$$

$\bar{f}_1,\cdots,\bar{f}_t$ 为 $V_1/V_1\cap V_2$ 的基

同时也为 V_1+V_2/V_2 的基

$\Rightarrow \bar{\mathscr{A}}(\bar{f}_1,\cdots,\bar{f}_t)=(\bar{f}_1,\cdots,\bar{f}_t)\cdot E_t$

i.e. $\bar{\mathscr{A}}: V_1/V_1\cap V_2 \longrightarrow V_1+V_2/V_2$ 为同构.

B: $\mathscr{A}: V\longrightarrow V$ $V_0\subset V$ 为子空间

① $\bar{\mathscr{A}}: V/V_0 \longrightarrow V/V_0$

$\bar{v} \longmapsto \overline{\mathscr{A}v}$

为良定义 $\Leftrightarrow$ $\mathscr{A}(V_0)\subset V_0$ $\Leftrightarrow V_0$ 为 $\mathscr{A}$ 子空间

② 取 $(e_1,\cdots,e_k)$ 为 V_0 的基，扩充为 V 的基 $(e_1,\cdots,e_n)$

则 $\mathscr{A}(e_1,\cdots,e_k,\cdots,e_n)=(e_1,\cdots,e_k,\cdots,e_n)\begin{pmatrix}A & B\\ & C\end{pmatrix}$

其中 $\mathscr{A}(e_1,\cdots,e_k)=(e_1,\cdots,e_k)\cdot A$

$\mathscr{A}(e_{k+1},\cdots,e_n)=(e_1,\cdots,e_k)\cdot B+(e_{k+1},\cdots,e_n)\cdot C$

$\Rightarrow \bar{\mathscr{A}}(\bar{e}_{k+1},\cdots,\bar{e}_n)=(\bar{e}_{k+1},\cdots,\bar{e}_n)\cdot C$.

i.e. C 为 $\bar{\mathscr{A}}$ 在基 $\bar{e}_{k+1},\cdots,\bar{e}_n$ 下的矩阵.

二次型约化问题：

设 V 是一个向量空间， $V_0 \subset V$ 子空间

$f: V\times V \longrightarrow F$ 对称双线性型.

什么时候 $\bar{f}: V/V_0 \times V/V_0 \longrightarrow F$ 良定义?

$\bar{v}_1 \quad \bar{v}_2 \longmapsto f(v_1, v_2)$

$\bar{v}_1 = \bar{w}_1$ ， $\bar{v}_2 = \bar{w}_2$ 　　$v_1 - w_1 \in V_0$ ， $v_2 - w_2 \in V_0$

$\bar{f}(\bar{v}_1, \bar{v}_2) = \bar{f}(\bar{w}_1, \bar{w}_2) \Leftrightarrow f(v_1, v_2) = f(w_1, w_2)$

$\Leftrightarrow f(v_1, v_2) - f(w_1 v_2) + f(w_1 v_2) - f(w_1 w_2) = 0$

$\Leftrightarrow f(v_1 - w_1, v_2) + f(w_1, v_2 - w_2) = 0$

要想良定义，我们必须有 $f(v_0, v) = 0 \quad \forall v_0 \in V_0, v \in V$

i.e. 若取 $e_1, ---, e_k$ 为 V_0 的基， $e_1, ---, e_n$ 为 V 的基

f 在此基下矩阵须为 $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$

这时 $\bar{f}$ 在基 $\bar{e}_{k+1}, ---, \bar{e}_n$ 下矩阵为 B.