§1. 期中与作业　　　　M_i 期中第 i 题　H_i 第 9 次作业第 i 题.

$M1$. 四元数　$\mathbb{H}=\left\{\begin{pmatrix} u & v \\ -\bar{v} & \bar{u}\end{pmatrix} \middle| u,v\in\mathbb{C}\right\}$

令　$I=\begin{pmatrix} i & \\ & -i\end{pmatrix}$　$J=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$　$K=\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$

则 $\forall M\in\mathbb{H}$　$M=aE+bI+cJ+dK$.

$I^2=J^2=K^2=-E_2$.　$IJ=K$　$JK=I$　$KI=J$

令　$\mathbb{R}^4=\mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k$

$a+bi+cj+dk$

规定乘法为：$i^2=j^2=k^2=-1$　$ij=k$，$jk=i$，$ki=j$.

则. $(\mathbb{R}^4,\cdot)\simeq\mathbb{H}$.

$M2$，H_1，H_2，H_3 均考查线性映射相关知识

$H.1$　$V\ (e_1,e_2,e_3)$　　$W\ (\varepsilon_1,\varepsilon_2)$

$$\phi(e_1,e_2,e_3)\longrightarrow(\varepsilon_1,\varepsilon_2)\underbrace{\begin{pmatrix}1 & 1 & -2\\ -1 & 1 & 1\end{pmatrix}}_{A}$$

$\Rightarrow \operatorname{rk}\phi=2$　$\dim\ker\phi=1$.

$$\phi\left((e_1,e_2,e_3)\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)=0 \Leftrightarrow \begin{pmatrix}1 & 1 & -2\\ -1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$\Rightarrow (x_1,x_2,x_3)=k\begin{pmatrix}3\\1\\2\end{pmatrix}\Rightarrow \ker\phi=\langle 3e_1+e_2+e_3\rangle$$

$$(v_1,v_2,v_3)=(e_1,e_2,e_3)\cdot\underbrace{\begin{pmatrix}2 & 0 & 1\\ -1 & 2 & 0\\ 0 & -1 & 0\end{pmatrix}}_{P}\qquad (w_1,w_2)=\underbrace{\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}}_{Q}$$

$\phi(v_1, v_2, v_3) = (w_1, w_2) \cdot Q^{-1} \cdot A \cdot P$.

$Q^{-1} \cdot A \cdot P = \begin{pmatrix} 2 & \frac{3}{2} & 1 \\ -1 & \frac{5}{2} & 0 \end{pmatrix}$. $\square$

H2: $\mathscr{A}: M_n(F) \longrightarrow M_n(F)$, C固定,

$$X \longrightarrow C^{-1} X C$$

(i) (ii) 易证.

对于(iii), $\mathscr{A}(X) = 0 \Longleftrightarrow C^{-1} X C = 0$

$\Longleftrightarrow X = 0$

$\Rightarrow \ker \mathscr{A} = 0$. $\Rightarrow \mathscr{A}$ 为同构

$\Rightarrow \operatorname{rk} \mathscr{A} = \dim(M_n(F)) = n^2$

H3 $T^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, 设 $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \Rightarrow b = c, \ d = 0$$

T可逆 $\Rightarrow$ $T = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$ $b \neq 0$ 即可.

M3，M4 标准习题

M5：多项式 可约 $\overset{\Leftarrow}{\not\Leftrightarrow}$ 有根.

Eissenstein 判别法

素数 $p \mid \cdots$

$p \nmid \cdots$

则不可约.

3次以下多项式可约 $\Leftrightarrow$ 其有根

M6：利用线性映射的秩不等式，把问题中的秩条件全化为向量空间维数

证：$rk\begin{pmatrix}A\\B\end{pmatrix} = rk(A) + rk(B) \Leftrightarrow \dim(V_A + V_B) = n.$

$V_C = \{x \in F^n : \begin{pmatrix}A\\B\end{pmatrix}x = 0\}$ $\quad V_C = V_A \cap V_B.$

$rk\begin{pmatrix}A\\B\end{pmatrix} = n - \dim V_C \quad rk(A) = n - \dim V_A \quad rk(B) = n - \dim V_B.$

$\Rightarrow \quad rk\begin{pmatrix}A\\B\end{pmatrix} = rk(A) + rk(B)$

$\Leftrightarrow \quad n = \dim V_A + \dim V_B - \dim(V_A \cap V_B)$

$\Leftrightarrow \quad n = \dim(V_A + V_B)$

H4: $V \xrightarrow{\mathscr{A}} V \xrightarrow{\mathscr{B}} V$

(i) $\mathrm{rk}(\mathscr{A}) = \mathrm{rk}(\mathscr{B}\mathscr{A}) + \dim(\mathrm{im}\mathscr{A} \cap \ker\mathscr{B})$

(ii) $\dim(\mathrm{im}\mathscr{A}^{i-1} \cap \ker\mathscr{A}) = \dim\ker\mathscr{A}^{i} - \dim\ker\mathscr{A}^{i-1}$

(i) $\mathrm{rk}\mathscr{A} = n - \dim\ker\mathscr{A}$ $\quad \mathrm{rk}(\mathscr{B}) = n - \dim\ker\mathscr{B}$.

(i) $\Leftrightarrow$ $\dim(\ker\mathscr{B}\mathscr{A}) = \dim\ker\mathscr{A} + \dim(\mathrm{im}\mathscr{A} \cap \ker\mathscr{B})$

$\ker\mathscr{A} \subset \ker\mathscr{B}\mathscr{A}$

$$\ker(\mathscr{B}\mathscr{A})/\ker\mathscr{A} \longrightarrow \mathrm{im}\mathscr{A} \cap \ker\mathscr{B}$$

$$\bar{x} \longmapsto \mathscr{A}x$$

①良定义. ✓

②满 $\forall\ y \in \mathrm{im}\mathscr{A} \cap \ker\mathscr{B}$ $\quad y = \mathscr{A}x_0$, $x_0 \in V$

$0 = \mathscr{B}y = \mathscr{B}\mathscr{A}x_0 \quad \Rightarrow x_0 \in \ker\mathscr{B}\mathscr{A}$.

③单 $\mathscr{A}\bar{x} = \mathscr{A}x = 0 \quad \Rightarrow x \in \ker\mathscr{A}$.

$\Rightarrow \bar{x} = 0$.

$$\dim(\mathrm{im}\mathscr{A} \cap \mathrm{im}\mathscr{B}) = \dim(\ker\mathscr{B}\mathscr{A})/\ker\mathscr{A})$$

$$= \dim(\ker\mathscr{B}\mathscr{A}) - \dim(\ker\mathscr{A}$$

(ii) 取 $\mathscr{C} = \mathscr{A}^{i-1}$ $\quad \mathscr{D} = \mathscr{A}$.

由(i) $rk(\mathscr{C}) = rk(\mathscr{D}\circ\mathscr{C}) + \dim(\operatorname{im}\mathscr{C} + \ker\mathscr{D})$

i.e. $n - \dim(\ker\mathscr{A}^{i-1}) = n - \dim(\mathscr{A}^i) + \dim(\operatorname{im}\mathscr{A}^i + \ker\mathscr{A})$

$\Rightarrow \dim(\operatorname{im}\mathscr{A}^{i-1} \cap \ker\mathscr{A}) = \dim\ker\mathscr{A}^i - \dim(\ker\mathscr{A}^{i-1})$

M7 讲义内容.

M8: q: V上二次型　　$U \subset V$ 子空间

$q|_U$ 为二次型.

惯性: $q\ (k, l)$　　$q|_U\ (s, t)$, 证 $k \geq s$, $l \geq t$

法①: V的基 $e_1, \cdots, e_n$

s.t. $q = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+l}^2$

取 $W = \langle e_{k+1}, \cdots, e_n \rangle$

$\exists$ V的基 $\varepsilon_1, \cdots, \varepsilon_m$

s.t. $q|_U = y_1^2 + \cdots + y_s^2 - y_{s+1}^2 - \cdots - y_{s+t}^2$

取 $K = \langle \varepsilon_1, \cdots, \varepsilon_s \rangle$

则 $W \cap K = \{0\}$

若 $k < s$

则 $\dim W\cap K = \dim W + \dim K - \dim(W+K)$

$= l + s - \dim(W+K)$

$\geq n-k + s - n$

$= s-k > 0.$ 矛盾.

法②: U的基为 $(e_1, \cdots, e_m)$

设 $q|_U$ 矩阵为 $\begin{pmatrix} E_s & 0 & 0 \\ 0 & -E_t & 0 \\ 0 & 0 & 0 \end{pmatrix}$

扩充为 V的基 $(e_1, \cdots, e_n)$

q的矩阵形如

$$\left(\begin{array}{cc|c|c} E_s & 0 & 0 & \\ 0 & -E_t & 0 & C_1 \\ \hline 0 & 0 & 0 & D_1 \\ \hline \multicolumn{2}{c|}{C_1^{t}} & D_1^{t} & B_1 \end{array}\right) \xrightarrow{\text{行列相伴}} \left(\begin{array}{cc|c|c} E_s & 0 & 0 & 0 \\ 0 & -E_t & 0 & 0 \\ \hline 0 & 0 & 0 & D_2 \\ \hline 0 & 0 & D_2^{t} & B_2 \end{array}\right)$$

对于 $\begin{pmatrix} 0 & D_2 \\ D_2^{t} & B_2 \end{pmatrix} \xrightarrow{\text{行列相伴}} \begin{pmatrix} E_p & 0 & 0 \\ 0 & -E_q & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow{\text{行列相伴}} \left(\begin{array}{cc|ccc} E_s & 0 & 0 & 0 & 0 \\ 0 & -E_t & 0 & 0 & 0 \\ \hline 0 & 0 & E_p & 0 & 0 \\ 0 & 0 & 0 & -E_q & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow k = s+p$, $l = t+q$

$p, q \geq 0$, 命题得证.

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§2. (核核分解 II) 设 $\mathcal{A} \in L(V)$，则

$V = \ker\mathcal{A} \oplus \operatorname{im}\mathcal{A} \iff t^2 \nmid m_{\mathcal{A}}(t)$ $\quad m_{\mathcal{A}}(t)$ 为 $\mathcal{A}$ 的极小多项式.

Recall: $V = \ker\mathcal{A} \oplus \operatorname{im}\mathcal{A} \iff \operatorname{rk}\mathcal{A} = \operatorname{rk}\mathcal{A}^2 \iff \operatorname{im}\mathcal{A} = \operatorname{im}\mathcal{A}^2$

$\iff \ker\mathcal{A} = \ker\mathcal{A}^2$

Pf: 设 $K = \ker\mathcal{A}$, $I = \operatorname{im}\mathcal{A}$

"$\Rightarrow$" $V = K \oplus I$ 若 $K = 0$, $\mathcal{A}$ 可逆 $m_{\mathcal{A}}(0) \neq 0$.

$\Rightarrow t^2 \nmid m_{\mathcal{A}}$

若 $K = V$，则 $\mathcal{A} = 0$， $m_{\mathcal{A}} = t$ ， $t^2 \nmid m_{\mathcal{A}}$

若 $K \neq 0$, $I \neq 0$. $\mathcal{A}_K = \mathcal{A}|_K = 0$ $\quad m_{\mathcal{A}_K} = t$

$\mathscr{A}_I: \mathscr{A}|_I : I \longrightarrow I$

设 $v \in I$ 若 $\mathscr{A}(v) = \mathscr{A}|_I(v) = \mathscr{A}(v) = 0$

$v \in I \cap K$, $v = 0$.

$\Rightarrow \mathscr{A}_I$ 为双射. $\Rightarrow t \nmid \mu_{\mathscr{A}(I)}$

在 $K \oplus I$ 一组基下, $\mathscr{A}$ 的矩阵可写为 $\begin{pmatrix} 0 & 0 \\ 0 & A_I \end{pmatrix}$

$$\mu_{\mathscr{A}} = \mathrm{lcm}(\underset{\substack{\| \\ t}}{\mu_{\mathscr{A}K}}, \mu_{\mathscr{A}_I}) = t\,\mu_{\mathscr{A}_I}$$

$$\Rightarrow t^2 \nmid \mu_{\mathscr{A}}$$

$\Leftarrow$ 若 $t \nmid \mu_{\mathscr{A}}$, 则 $\mathscr{A}$ 可逆. $\Rightarrow V = K \oplus I$.

若 $\mu_{\mathscr{A}} = t$, 则 $\mathscr{A} = 0$.

若 $t | \mu_A$, $t^2 \nmid \mu_A$. $\mu_A = t \cdot p$ $\gcd(t, p) = 1$.

核核分解: $V \cong \ker \mathscr{A} \oplus \ker(p(\mathscr{A}))$

我们证明 $I = \ker(p(\mathscr{A}))$.

① $I \subset \ker p(\mathscr{A})$. 若 $y = \mathscr{A}x$,

$$p(\mathscr{A}) \cdot y = (p(\mathscr{A}) \cdot \mathscr{A})(y) = \mu_{\mathscr{A}}(\mathscr{A})(y) = 0$$

② $\ker p(\mathscr{A}) \subset I$ $p(t) = t^r + a_1 t^{r-1} + \cdots + a_r$, $x \in \ker(p(\mathscr{A}))$

$$p(\mathscr{A})x = (\mathscr{A}^r + \cdots + a_r) \cdot x = 0.$$

$$\Rightarrow x = \mathscr{A}(a_r^{-1}(\mathscr{A}^{r-1} + \cdots + a_{r-1})x)$$

$$\Rightarrow x \in I.$$

$\Rightarrow I = \ker(p(\mathscr{A}))$. $\square$