1、 $g, h \in V^*$， $f: V\times V \longrightarrow F$

$(x, y) \longmapsto g(x)\cdot h(y)$.

① $f(k_1x_1+k_2x_2, y) = g(k_1x_1+k_2x_2)h(y)$

$= (k_1 g(x_1)+k_2 g(x_2))\, h(y)$

$= k_1 g(x_1)h(y) + k_2 g(x_2)h(y)$

$= k_1 f(x_1, y) + k_2 f(x_2, y)$

② 设 e_i 为标准基 $e^i = e_i^*$ 为对偶基.

$g = \sum_i a_i e^i$ $\quad h = \sum_i b_i e^i$ $\quad x = \sum_i x^i \cdot e_i$ $\quad y = \sum_i y^i e_i$

$g(x) = \sum_i a_i x^i = (a_1, \cdots, a_n)\cdot \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$

$h(y) = \sum_i b_i y^i = (b_1, \cdots, b_n)\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}$

$f(x, y) = (x^1, \cdots, x^n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\cdot (b_1, \cdots, b_n)\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}$

$\Longrightarrow$ f在e_i下矩阵为 $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}(b_1, \cdots, b_n) \Rightarrow \operatorname{rank} f = \begin{cases} 0 & f=0 \text{ or } g=0 \\ 1 & f\neq 0 \text{ 且 } g\neq 0. \end{cases}$

(3) $q(kx) = g(kx)^2 = k^2 g(x)^2$

$f(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y)) = g(x)\cdot g(y)$ 为双线性函数

$\Rightarrow$ $q(x)$ 为二次型.

2. $q(x) = x_1^2 - x_1x_2 + 3x_3x_2 + 10x_4x_3 - 2x_4^2$

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & 0 & 5 \\ 0 & 0 & 5 & -2 \end{pmatrix}$$

$$A \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 3 & 0 \\ 0 & \frac{3}{2} & 0 & 5 \\ 0 & 0 & 5 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 9 & 5 \\ 0 & 0 & 5 & -2 \end{pmatrix}$$

$\Rightarrow rk(q) = rk(A) = 4.$

3、$q(x) = x_1^2 - 3x_3^2 - 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

① 配方法：令 $y_1 = x_1 - x_2 + x_3$

$y_2 = x_2$

$y_3 = x_3$

$$q(x) = (x_1 - x_2 + x_3)^2 - x_2^2 - x_3^2 + 2x_2x_3 - 3x_3^2 + 2x_2x_3$$

$$q(y) = y_1^2 - y_2^2 - y_3^2 + 4y_2y_3 - y_3^2$$

$$= y_1^2 - y_2^2 - 4y_3^2 + 4y_2y_3$$

令 $z_1 = y_1$

$z_2 = y_2 + 2y_3$

$z_3 = y_3$

$$q(y) = y_1^2 - (y_2+2y_3)^2$$

$$q(z) = z_1^2 + z_2^2$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

②行列相伴变换

$$q(x) = (x_1, x_2, x_3) \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix} \xrightarrow{\text{右}\cdot\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 2 \\ 1 & 2 & -4 \end{pmatrix} \xrightarrow{\text{左}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

$$\xrightarrow{\text{右}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{\text{左}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

令 $P = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad P^t \cdot \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix} \cdot P = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix} = (P^{-1})^t \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} P^{-1}$$

$P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$，令 $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ 则 $q(z) = z_1^2 - z_2^2$

4、 对于$\mathbb{C}$上二次型 p，其标型为 $A=\begin{pmatrix} E_r & 0 \\ & 0 \end{pmatrix}$

$r=\operatorname{rank} p$.

故而 $\operatorname{rank} p=\operatorname{rank} q \Rightarrow A\sim_c B$.

合同变换不改变矩阵秩，故 $A\sim_c B \Rightarrow \operatorname{rank} p=\operatorname{rank} q$.

5、 由于 $\dim V=\dim V^*$，我们证明 ϕ 单射即可.

设 $\phi(v)=f(-,v)=0$

设 f 在基 $e_1,\cdots,e_n$ 下矩阵为 A.

$\Rightarrow f(e_1,v)=\cdots=f(e_n,v)=0$.

$f(e_i,v)=0 \Leftrightarrow \underset{\substack{\uparrow \\ i\text{-th}}}{(0,\cdots,1,\cdots 0)} A\cdot v=0$

$\Rightarrow \quad E_n\cdot A\cdot v=0 \quad \Rightarrow A\cdot v=0$，$A$可逆 $\Rightarrow v=0$.

6. 令 $x=\sum_i x^i e_i$

设 $f_i(x)=(a_{i1},\cdots,a_{in})\cdot\begin{pmatrix}x^1\\ \vdots\\ x^n\end{pmatrix}$

则 $f_i^2(x)=(x^1,\cdots,x^n)\cdot\begin{pmatrix}a_{i1}\\ \vdots\\ a_{in}\end{pmatrix}\cdot(a_{i1},\cdots,a_{in})\begin{pmatrix}x^1\\ \vdots\\ x^n\end{pmatrix}$

令 $a_i=(a_{i1},\cdots,a_{in})$

称 a_i 为 f_i 在基 e_i 下的矩阵.

记 $A_i=a_i^t\cdot a_i$

则 $q(x)=x^t\cdot\left(\sum A_i\right)x$ 为 2次型.

$\operatorname{rank} q=\operatorname{rank}\left(\sum A_i\right)$,

为什么?

$\left\{\begin{array}{l}\text{不妨设 } f_1,\cdots,f_r \text{ 线性无关, 则 } \{a_1,\cdots,a_r\}\subset F^{1\times n} \text{ 线性无关.}\\ \text{且 } j>r \text{ 时 } a_j \text{ 可由 } a_1,\cdots,a_r \text{ 线性表示}\end{array}\right.$

于是 $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\sum A_i=\sum_{i,j=1}^{d} b_{ij}\,a_i^t\cdot a_j}$ $b_{ij}=b_{ji}$, 将 $\{a_1,\cdots,a_r\}$ 扩充为 $F^{1\times n}$ 的基 $\{c_1,\cdots,c_n\}$ $a_1=c_1,\cdots,a_r=c_r$.

$\Rightarrow$ $q(x)=\sum_{i,j=1}^{d} b_{ij} f_i(x) f_j(x)$

令 $P=\begin{pmatrix}c_1\\ \vdots\\ c_r\\ \vdots\\ c_n\end{pmatrix}=\begin{pmatrix}a_1\\ \vdots\\ a_r\\ c_{r+1}\\ \vdots\\ c_n\end{pmatrix}$

令 $(\varepsilon_1, \cdots, \varepsilon_n) = (e_1, \cdots, e_n) \cdot P^{-1}$ 为V的另一组基且

$$(e_1, \cdots, e_n)\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = x = (\varepsilon_1, \cdots, \varepsilon_n)\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}$$

有坐标变换

$$\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = P \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_r(x) \\ * \\ * \end{pmatrix}$$

在基 $\varepsilon_1, \cdots, \varepsilon_n$ 下，q 的标达式为

$$q(y) = \sum_{i,j=1}^{d} b_{ij} y_i y_j$$

故矩阵形如 $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ $B = (b_{ij})_{d \times d}$.

$\Rightarrow \operatorname{rank} q \leq r = \dim \langle f_1, \cdots, f_k \rangle$.

另一种证法：（基于田翎翔同学想法）

设 $f_1, \cdots, f_r$ 线性无关，为 $\langle f_1, \cdots, f_r \rangle$ 一组基.

将 $f_1, \cdots, f_r$ 补全为 V^* 一组基 $\langle g_1, \cdots, g_n \rangle$

$g_1 = f_1, \cdots, g_r = f_r$

取 $e_1, \cdots, e_n$ 为一组基.

$(e^1, \cdots, e^n)$ 为其对偶基.

设 $(g_1, \cdots, g_n) = (e^1, \cdots, e^n) \cdot A$ 为 V^* 基变换.

令 $(\varepsilon_1, \cdots, \varepsilon_n) = (e_1, \cdots, e_n) \cdot B$ 为 V 上基变换.

设 $\varepsilon^1, \cdots, \varepsilon^n$ 为 $\varepsilon_1, \cdots, \varepsilon_n$ 对偶基.

则由上周计算 $(\varepsilon^1, \cdots, \varepsilon^n) = (e^1, \cdots, e^n) \cdot (B^t)^{-1}$.

若取 $B = A^t$，则 $(\varepsilon^1, \cdots, \varepsilon^n) = (e^1, \cdots, e^n) \cdot A$

$\Rightarrow (g_1, \cdots, g_n) = (\varepsilon^1, \cdots, \varepsilon^n)$ 为 V^* 的基.

那么，$h(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$

$$= \sum_{i=1}^{k} f_i(x) f_j(x)$$

已知 $f_1 = \varepsilon^1, \cdots, f_r = \varepsilon^r$，$i > r$ f_i 可由 $\varepsilon^1, \cdots, \varepsilon^r$ 线性表示.

$\Rightarrow f_i(\varepsilon_j) = 0$ 对 $\forall j > r$ 成立.

$\Rightarrow$ q 在 $\varepsilon_1, \cdots, \varepsilon_n$ 下矩阵形如

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \qquad B = (b_{ij}) \quad b_{ij} = h(\varepsilon_i, \varepsilon_j)$$

$\Rightarrow \mathrm{rk}(q) = \mathrm{rk}(B) \le r = \dim\langle f_1, \cdots, f_r \rangle$. $\square$