1、 $\mathbb{R}^4$ $\quad f(x,y) = x_1y_1 - 3x_2y_3 + 2x_4y_1 + 5x_3y_2 - x_4y_4$

$$f(x,y) = x^t \cdot A \cdot y \qquad a_{ij} = f(e_i, e_j)$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 5 & 0 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}$$

$$\det A = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 5 & 0 \end{vmatrix} = 15 \neq 0.$$

$\Rightarrow \operatorname{rank}(f) = 4.$

2. $\quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

$$A \xrightarrow{右F_{12}} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix} \xrightarrow{左F_{12}^t} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{右\begin{pmatrix} 1 & -\frac{1}{2} & \\ & 1 & \\ & & 1 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 2 \\ 1 & -\frac{1}{2} & 1 \\ 2 & 0 & 2 \end{pmatrix} \xrightarrow{左\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{\begin{pmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{1}{2} & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

3、(i) $\forall\ x, y \in V$, A 对称

$$2x^t A \cdot y = y^t \cdot Ax + x^t Ay$$
$$= (x+y)^t A(x+y) - x^t Ax - y^t Ay$$
$$= 0$$

$\mathrm{char}F \neq 2 \Rightarrow x^t Ay = 0$, $\forall\ x, y \in V$.

$x = e_i$, $y = e_j \Rightarrow a_{ij} = e_i^t \cdot A \cdot e_j = 0 \Rightarrow A = 0$.

(ii) $x = \sum_i x^i \cdot e_i$

$A^t = A$

$a_{ii} = 0\ \forall i$

$$x^t A x = \sum_{i,j} x^i\, e_i^t \cdot A e_j \cdot x^j$$
$$= \sum_{i,j} x^i x^j a_{ij}$$
$$= \sum_{i<j} x^i \cdot x^j (a_{ij} + a_{ji})$$
$$= 0$$

4. $A \in F^{m\times n}$ $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B \in F^{k\times n}$

$$\begin{pmatrix} A \\ B \end{pmatrix} \cdot X = 0 \iff A\cdot X = 0 \ \& \ B\cdot X = 0$$

$U = \{ x \in F^n \mid AX = 0 \}$ $V = \{ x \in F^n \mid AX = 0 \ \ BX = 0 \}$.

$W = \{ x \in F^n \mid BX = 0 \}$ $V = U \cap W$.

考虑线性 $A: F^n \longrightarrow F^m$

$B: F^n \longrightarrow F^k$

$x \longmapsto B\cdot x$

$W = \ker B$, $\dim W \geq n-k$

$$\dim V = \dim U\cap W = \dim U + \dim W - \dim(U+W)$$

$$\geq \dim U + n - k - \dim(U+W)$$

$$\geq \dim U - k$$

5.

M1: $A\in F^{m\times n}$, $B\in F^{n\times m}$

$$\phi_{E_m-AB}: F^m \longrightarrow F^m \qquad \phi_{E_n-BA}: F^n \longrightarrow F^n$$
$$x \longmapsto (E_m-AB)x \qquad x \longmapsto (E_n-BA)x$$

$$\dim\ker\phi_{E_m-AB} + rk(E_m-AB) = m$$

$$\dim\ker\phi_{E_n-BA} + rk(E_n-BA) = n$$

我们要证 $\dim\ker\phi_{E_m-AB} = \dim\ker\Phi_{E_n-BA}$.

令 $\varphi: k_1=\ker\phi_{E_m-AB} \longrightarrow \ker\phi_{E_n-BA} = k_2$
$$X \longmapsto BX$$

①: $X\in K_1$ $\quad (E_n-BA)BX = B\cdot(E_m-AB)X = 0$

$\Rightarrow \varphi$ 良好定义.

②: 若 $\varphi(x)=Bx=0$,

$x\in K_1 \Rightarrow 0=(E_m-AB)X = X-ABX=X$

$\Rightarrow \varphi$ 单射.

③: $\forall\ y\in k_2$, $(E_n-BA)y=0$

$\Rightarrow y = B\cdot Ay$

$(E_m-AB)\cdot Ay = A(E_n-BA)y = 0 \Rightarrow Ay\in K_1$

$y=\varphi(Ay) \Rightarrow \varphi$ 满射.

$\Rightarrow \dim K_1 = \dim K_2$. □

M2:

$$\begin{pmatrix} E_m & 0 \\ -B & E_n \end{pmatrix}\begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix} = \begin{pmatrix} E_m & A \\ 0 & E_n-BA \end{pmatrix}$$

$$\begin{pmatrix} E_m & A \\ 0 & E_n-BA \end{pmatrix}\begin{pmatrix} E_m & -A \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ 0 & E_n-BA \end{pmatrix}$$

同理

$$\begin{pmatrix} E_m & -A \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix} = \begin{pmatrix} E_m-AB & 0 \\ B & E_n \end{pmatrix}$$

$$\begin{pmatrix} E_m-AB & 0 \\ B & E_n \end{pmatrix}\begin{pmatrix} E_m & 0 \\ -B & E_n \end{pmatrix} = \begin{pmatrix} E_m-AB & 0 \\ 0 & E_n \end{pmatrix}$$

$$\operatorname{rk}\begin{pmatrix} E_m & 0 \\ 0 & E_n-BA \end{pmatrix} = \operatorname{rk}\begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix} = \operatorname{rk}\begin{pmatrix} E_m-AB & 0 \\ 0 & E_n \end{pmatrix}$$

$$\operatorname{rk}\begin{pmatrix} E_m & 0 \\ 0 & E_n-BA \end{pmatrix} = m+\operatorname{rk}(E_n-BA)$$

$$\operatorname{rk}\begin{pmatrix} E_m-AB & 0 \\ 0 & E_n \end{pmatrix} = \operatorname{rk}(E_m-AB)+E_n.$$

6.

证：设 $e_1,\cdots,e_d$ 为 U 的一组基，

补全为 $e_1,\cdots,e_d,e_{d+1},\cdots,e_n$ 为 V 的一组基

则记 $e^i=(e_i)^*$ 为 $e_1,\cdots,e_n$ 的对偶基.

即 $e^i(e_j)=\delta^i_j$.

我们证明 $e^{d+1},\cdots,e^{d+n}$ 为 U^0 的一组基.

① $\forall v\in U$，则 $v=\sum_{i=1}^{d}v^ie_i$

$$e^{d+j}=\sum_{i=1}^{d}v^i e^{d+j}(e_i)=0$$

$$\Rightarrow e^{d+1},\cdots,e^{d+n}\in U^0$$

② $\forall f\in V^*$，$f=\sum_{i=1}^{n}a_ie^i$，若 $f\in U^0$

$$\Rightarrow f(e_j)=\sum_{i=1}^{n}a_ie^i(e_j)=0 \quad \text{对于 } 1\le j\le d$$

$$\Rightarrow f(e_j)=a_j=0,\quad 1\le j\le d$$

$$\Rightarrow f=\sum_{i=d+1}^{n}a_i\cdot e^i.$$

$\Rightarrow e^{d+1},\cdots,e^n$ 为 U^0 的一组基.

$\Rightarrow \dim U^0=n-d.$

无穷维线性空间的二次对偶

设 V 为 F 上的线性空间，若 $\dim V<\infty$，则有

$$e: V \longrightarrow V^{**}$$
$$v \longmapsto e_v \quad , \quad e_v: f \longmapsto f(v)$$

为同构映射.

问：$\dim V=\infty$ 此命题是否成立？

否，但其仍然为单射.

Prop：$e: V \longrightarrow V^{**}$ 永远是单射

证：$v_0 \in \ker(e) \Longleftrightarrow f(v_0)=0$ 对 $\forall f \in Hom(V,F)$ 成立.

若 $v_0 \neq 0$，则将 v_0 补全为 V 的一组基.

设 $\langle v_0 \rangle \oplus W = V$. 令 $g: V \longrightarrow F$
$$kv_0+w \longmapsto k$$

则有 $g(v_0)\neq 0$，与 $0=e_{v_0}(g)=g(v_0)$ 矛盾.

Prop：若 $\dim V=\infty$. $e: V \longrightarrow V^{**}$ 不是满射.

证：设 $\{v_\lambda\}_{\lambda\in\Lambda}$ 为其极大线性无关向量组.

可以验证 $\{v_\lambda^*\}$ 线性无关

$\{v_\lambda\}$ 极大线性无关，$f \in V^*$ f 在 $\{v_\lambda\}$ 处的取值决定 f.

i.e. $f = \sum_{\lambda \in \Lambda} f_\lambda e_\lambda^*$，可以有无穷项相加.

$\Rightarrow$ V^* 极大线性无关组的势比 V 大　思考题：为什么？

同理，　V^{**} 极大线性无关组的势比 V^* 大.

$\Rightarrow$ V^{**} 极大线性无关组的势比 V 大.

$\Rightarrow$ e 不是满射. $\square$

对偶基的转移矩阵.

设 $(v_1, \cdots, v_n)$，$(e_1, \cdots, e_n)$ 为 V 的两组基.

$$(v_1, \cdots, v_n) = (e_1, \cdots, e_n) \cdot A$$

设 $e^i = e_i^*$，$v^i = v_i^*$ 为其对应的对偶基.

$$(v^1, \cdots, v^n) = (e^1, \cdots, e^n) \cdot B$$

问：$B = ?$

Einstein求和约定：若求和时指标既有上标又有下标，则省略求和符号，默认求和.

eg: $\sum_{i=1} a^i b_i$ 用Einstein求和约定记为 $a^i b_i$.

对于向量空间V 我们常用下标表示基，上标表示坐标、

如 $(e_1, \cdots, e_n)$ $\quad v=\sum_{i=1}^{n} x^i e_i$.

Einstein $\rightsquigarrow$ $v = x^i e_i$.

对于对偶基，常用上标表示基，下标表示坐标、

$$f=\sum_{i=1}^{n} y_i e^i, \quad \text{i.e.} \quad f = y_i e^i$$

转移矩阵记为 $A=(a^i_j)$，由此记号.

$$(v_1, \cdots, v_n) = (e_1, \cdots, e_n)\begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \vdots & & & \\ a^n_1 & a^n_2 & \cdots & a^n_n \end{pmatrix}$$

则 $v_i = a^j_i e_j$.

我们开始记算 B，令 $B^t = (b^i_j)$ s.t.

$$(v^1, \cdots, v^n) = (e^1, \cdots, e^n) \cdot B$$

$$(v^1, \cdots, v^n) = (e^1, \cdots, e^n) \begin{pmatrix} b^1_1 & b^2_1 & \cdots & b^n_1 \\ b^1_2 & b^2_2 & \cdots & b^n_2 \\ \vdots & \vdots & & \vdots \\ b^1_n & b^2_n & \cdots & b^n_n \end{pmatrix}$$

i.e. $v^i = b^i_j e^j$

我们知道 $v^i(v_j) = \delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$v^i(v_j) = (b^i_k \cdot e^k)(a^l_j \cdot e_l)$$
$$= b^i_k \cdot a^l_j \cdot \delta^k_l$$
$$= b^i_k \cdot a^k_j = \delta^i_j$$

这说明 $B^t \cdot A = (b^i_j) \cdot (a^i_j) = E$

$$\Rightarrow B^t = A^{-1}$$
$$\Rightarrow B = (A^t)^{-1} \quad \square$$