1. 解

$$f(x,y)=(x_1,x_2,x_3,x_4)\underbrace{\begin{pmatrix}1&0&0&0\\0&0&-3&0\\0&5&0&0\\2&0&0&-1\end{pmatrix}}_{A}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

f在标准基下的矩阵为A,

$$\because\ A\xrightarrow[r_4-2r_1]{r_2\leftrightarrow r_3}\begin{pmatrix}1&0&0&0\\0&5&0&0\\0&0&-3&0\\0&0&0&-1\end{pmatrix}\qquad \therefore\ \operatorname{rank}(f)=\operatorname{rank}(A)=4$$

2. 解: $A=\begin{pmatrix}0&1&1\\1&2&2\\1&2&2\end{pmatrix}$

(行列相伴)

$$\left(\begin{array}{ccc|ccc}0&1&1&1&0&0\\1&2&2&0&1&0\\1&2&2&0&0&1\end{array}\right)\xrightarrow{r_1\leftrightarrow r_2}\left(\begin{array}{ccc|ccc}1&2&2&1&0&0\\0&1&1&0&1&0\\1&2&2&0&0&1\end{array}\right)\xrightarrow{c_1\leftrightarrow c_2}\left(\begin{array}{ccc|ccc}2&1&2&0&1&0\\1&0&1&1&0&0\\2&1&2&0&0&1\end{array}\right)$$

$$\xrightarrow{r_2-\frac12 r_1}\left(\begin{array}{ccc|ccc}2&1&2&0&1&0\\0&-\frac12&0&1&0&0\\2&1&2&0&0&1\end{array}\right)\xrightarrow{c_2-\frac12 c_1}\left(\begin{array}{ccc|ccc}2&0&2&0&1&0\\0&-\frac12&0&1&-\frac12&0\\2&0&2&0&0&1\end{array}\right)\xrightarrow{r_3-r_1}\left(\begin{array}{ccc|ccc}2&0&2&0&1&0\\0&-\frac12&0&1&-\frac12&0\\0&0&0&0&0&1\end{array}\right)$$

$$\xrightarrow{c_3-c_1}\left(\begin{array}{ccc|ccc}2&0&0&0&1&0\\0&-\frac12&0&1&-\frac12&-1\\0&0&0&0&0&1\end{array}\right)$$

令 $P=\begin{pmatrix}0&1&0\\1&-\frac12&-1\\0&0&1\end{pmatrix}$, $B=\begin{pmatrix}2&0&0\\0&-\frac12&0\\0&0&0\end{pmatrix}$ 即为所求.

(降维法) $V=\mathbb{R}^3$, 设 $f: V\times V\to\mathbb{R}$ 双线性型, 在V中标准基 $\{\vec e_1,\vec e_2,\vec e_3\}$ 下矩阵为A.

令 $\vec\varepsilon_1=\vec e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, 则 $f(\vec\varepsilon_1,\vec\varepsilon_1)=(0,1,0)A\begin{pmatrix}0\\1\\0\end{pmatrix}=2\neq 0$

令 $U=\{\vec x\in V\mid f(\vec x,\vec\varepsilon_1)=0\}$.

设 $\vec x=x_1\vec e_1+x_2\vec e_2+x_3\vec e_3$, $f(\vec x,\vec\varepsilon_1)=(x_1,x_2,x_3)A\begin{pmatrix}0\\1\\0\end{pmatrix}=(x_1,x_2,x_3)\begin{pmatrix}1\\2\\2\end{pmatrix}=x_1+2x_2+2x_3=0$.

上述方程解空间一组基为 $\vec u_1=\begin{pmatrix}2\\-1\\0\end{pmatrix}$, $\vec u_2=\begin{pmatrix}2\\0\\-1\end{pmatrix}$, 即为U的一组基.

令 $g: U\times U\to\mathbb{R}$, 则 $g:=f|_{U\times U}$ 在 $\vec u_1,\vec u_2$ 下的矩阵为B

$$B=\begin{pmatrix}f(\vec u_1,\vec u_1)&f(\vec u_1,\vec u_2)\\f(\vec u_2,\vec u_1)&f(\vec u_2,\vec u_2)\end{pmatrix}=\begin{pmatrix}-2&-2\\-2&-2\end{pmatrix}$$

令 $\vec\varepsilon_2=1\vec u_1=\begin{pmatrix}2\\-1\\0\end{pmatrix}$, 则 $g(\vec\varepsilon_2,\vec\varepsilon_2)\neq 0$,

注: 此处应代表降维之后的: $\vec\varepsilon_2$ 在 $\{\vec u_1,\vec u_2\}$ 下的坐标

令 $\tilde U=\{\vec x\in U\mid g(\vec x,\vec\varepsilon_2)=0\}$. 令 $\vec x=x_1\vec u_1+x_2\vec u_2$

$$g(\vec x,\vec\varepsilon_2)=(x_1,x_2)\begin{pmatrix}-2&-2\\-2&-2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=-2(x_1+x_2)=0\qquad ①$$

其解空间的一组基为 $\begin{pmatrix}1\\-1\end{pmatrix}$，从而 $\tilde{U}$ 的基是 $(\vec{u}_1,\vec{u}_2)\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix}=\vec{\varepsilon}_3$.

令 $P=(\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{\varepsilon}_3)=\begin{pmatrix}0&2&0\\1&-1&-1\\0&0&+1\end{pmatrix}$，则 $P^tAP=\begin{pmatrix}f(\vec{\varepsilon}_1,\vec{\varepsilon}_1)&&\\&f(\vec{\varepsilon}_2,\vec{\varepsilon}_2)&\\&&f(\vec{\varepsilon}_3,\vec{\varepsilon}_3)\end{pmatrix}=\begin{pmatrix}2&&\\&-2&\\&&0\end{pmatrix}$

3. (i) Pf: (法一) 极化公式.

$$f: F^n\times F^n \longrightarrow F$$

$$\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\longmapsto (x_1,\cdots,x_n)A\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}.$$

由题意，$\forall \vec{x}\in F^n$，$\vec{x}^tA\vec{x}=0$，$f(\vec{x},\vec{y})=\frac{1}{2}(f(\vec{x}+\vec{y},\vec{x}+\vec{y})-f(\vec{x},\vec{x})-f(\vec{y},\vec{y}))$

$=\frac{1}{2}(0-0-0)=0$

$\Rightarrow A=0$ (由定理7.2结合例7.1知，给定 f，则对应的 A 唯一)

(法二) $\forall \vec{x},\vec{y}\in V$，$A$ 对称，有 $x^tAy=y^tAx$.

则 $(x+y)^tA(x+y)-x^tAx-y^tAy=y^tAx+x^tAy=2x^tAy=0$

$\Rightarrow x^tAy=0$，$\forall \vec{x},\vec{y}\in V$.

取 $x=(0,\cdots,\underset{i}{1},\cdots,0)^t$ $\quad y=(0,\cdots,\underset{j}{1},\cdots,0)^t$ $\quad x^tAy=a_{ij}=0$

$\Rightarrow A=0$

(ii) (法一) 取 $x=(0\cdots\underset{i}{1}\cdots0)^t$，$x^tAx=a_{ii}=0$.

取 $x=(0\cdots\underset{i}{1}\cdots\underset{j}{1}\cdots0)^t$，$x^tAx=a_{ij}+a_{ji}=0$ $(\because A=-A^t)$.

(法二) $f(\vec{x},\vec{y})=\vec{x}^tA\vec{y}=-\vec{x}^tA^t\vec{y}=-(\underbrace{\vec{y}^tA\vec{x}}_{F})^t=-\vec{y}^tA\vec{x}=-f(\vec{y},\vec{x})$.

$\Rightarrow f(\vec{x},\vec{x})=-f(\vec{x},\vec{x})$

$\Rightarrow 2f(\vec{x},\vec{x})=0$

$\Rightarrow f(\vec{x},\vec{x})=0$ $(2\neq0)$.

$\gcd(t^m, a-i(t^m)') = \gcd(t^m, a-imt^{m-1}\cdot it) = 1$

Pf:

4. 法一:

$\dim(U) = n - \mathrm{rank}(A)$

$\dim(V) = n - \mathrm{rank}\left(\begin{pmatrix} A \\ B \end{pmatrix}\right)$

$\Rightarrow \mathrm{rank}\begin{pmatrix} A \\ B \end{pmatrix} \le \mathrm{rank}(A) + \mathrm{rank}(B) \le \mathrm{rank}(A) + k$

$\Rightarrow \dim(V) \ge n - (\mathrm{rank}(A) + k) = n - \mathrm{rank}(A) - k = \dim(U) - k$

法二: 引理: 设 W_1, W_2 是 n 维线性空间 W 的子空间且 $\dim(W_2) \ge n-1$

$\dim(W_1 \cap W_2) \ge \dim(W_1) - 1$

Pf: $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \ge \dim(W_1) + n - 1 - n = \dim(W_1) - 1$

设 U_i 为以 B 中第 i 行为系数矩阵的齐次方程组的解空间, 则 $\dim(U_i) \ge n-1$, $i = 1, \cdots, k$.

$V = U \cap U_1 \cap \cdots \cap U_k$. 由上述不等式可得.

$\dim(U \cap U_1) \ge \dim(U) - 1$

$\Rightarrow \dim(U \cap U_1 \cap U_2) \ge \dim(U) - 1 - 1 = \dim(U) - 2$.

$\vdots$

$\Rightarrow \dim(U \cap U_1 \cap U_2 \cap \cdots \cap U_k) \ge \dim(U) - k$

5. Pf: (法一) 令 $M = \begin{pmatrix} E_m - AB & 0 \\ 0 & E_n \end{pmatrix}$, $\begin{pmatrix} E_m - AB & 0 \\ 0 & E_n \end{pmatrix} \xrightarrow[\begin{pmatrix} E_m & A \\ 0 & E_n \end{pmatrix}_{(m+n)\times(m+n)}]{\text{右乘}} \begin{pmatrix} E_m - AB & A \\ 0 & E_n \end{pmatrix}$

$\xrightarrow{\text{右乘} \begin{pmatrix} E_m & 0 \\ B & E_n \end{pmatrix}} \begin{pmatrix} E_m & A \\ B & E_n \end{pmatrix} \xrightarrow{\text{左乘} \begin{pmatrix} E_m & -A \\ 0 & E_n \end{pmatrix}} \begin{pmatrix} E_m & 0 \\ B & E_n - BA \end{pmatrix} \xrightarrow{\text{左乘} \begin{pmatrix} E_m & \\ -B & E_n \end{pmatrix}} \underbrace{\begin{pmatrix} E_m & 0 \\ 0 & E_n - BA \end{pmatrix}}_{N}$

由于上述(左乘或右乘)矩阵均为可逆矩阵, 故

$\mathrm{rank}(M) = \mathrm{rank}(E_m - AB) + n = \mathrm{rank}(N) = m + \mathrm{rank}(E_n - BA)$.

$\Rightarrow m - \mathrm{rank}(E_m - AB) = n - \mathrm{rank}(E_n - BA)$.

(法二) 1. 把矩阵解释为坐标空间的线性映射

设 $\phi_{E_m - AB}: F^m \to F^m$, $\vec{x} \mapsto (E_m - AB)\vec{x}$ $\quad \phi_{E_n - BA}: F^n \to F^n$, $\vec{x} \mapsto (E_n - BA)\vec{x}$.

2. 利用对偶定理 $\dim(\ker(\phi_A)) + \mathrm{rank}(A) = n$ 把秩转换为核的维数

$\dim(\ker(\phi_{E_m - AB})) + \mathrm{rank}(E_m - AB) = m$

$\dim(\ker(\phi_{E_n - BA})) + \mathrm{rank}(E_n - BA) = n$.

$\gcd(t^m, a-i(t^m)') = \gcd(t^m, a-imt^{m-1}l(t)) = 1$

要证 $m-\mathrm{rank}(E_m-AB) = n-\mathrm{rank}(E_n-BA)$.

即证 $\dim(\ker(\phi_{E_m-AB})) = \dim(\ker(\phi_{E_n-BA}))$

构造映射 φ: $\ker(\phi_{E_m-AB}) \longrightarrow \ker(\phi_{E_n-BA})$.

$$x \longmapsto Bx$$

$\forall x \in \ker(\phi_{E_m-AB})$, 则 $(E_m-AB)x=0$, 从而 $B(E_m-AB)x=0$

即: $Bx-BA(Bx)=0$

$\Rightarrow (E_n-BA)Bx=0$.

$\Rightarrow Bx \in \ker(\phi_{E_n-BA})$.

从而 φ 是良定义的.

容易验证 φ 是线性映射.

下证 φ 是单射.

$\forall \vec{x} \in \ker(\varphi)$, 则 $B\vec{x}=\vec{0}$, 且 $(E_m-AB)x=0 \Rightarrow \underset{x}{\underset{\shortparallel}{E_m x}}=0$.

下证 φ 是满射.

$\forall y \in \ker(\phi_{E_n-BA})$, 则 $(E_n-BA)y=0$,

$\Rightarrow y=BAy$

$\Rightarrow (E_m-AB)Ay = Ay-A(BAy) = Ay-Ay=0$.

$\Rightarrow Ay \in \ker(\phi_{E_m-AB})$.

$\Rightarrow$ 综上, $\ker(\phi_{E_m-AB}) \cong \dim(\phi_{E_n-BA})$ 故 $\dim(\ker(\phi_{E_m-AB})) = \dim(\ker(\phi_{E_n-BA}))$

6. 证明: 设 $\{\vec{u}_1,\dots,\vec{u}_d\}$ 为 U 的一组基, 将其扩充为 V 的一组基为

$$\vec{u}_1,\dots,\vec{u}_d,\vec{u}_{d+1},\dots,\vec{u}_n$$

设 $\{\vec{u}_1^*,\dots,\vec{u}_n^*\}$ 为 V^* 的一组基, 则 $\vec{u}_i^*(\vec{u}_j)=\delta_{ij}$.

故 $\vec{u}_i^*(\vec{u}_j)=0$ $(j=1,2,\dots,d,\ i=d+1,\dots,n)$.

对 $\forall f \in V^*$, 设 $f=\alpha_1 u_1^*+\dots+\alpha_n u_n^*$, $\alpha_i \in F$.

$f \in U^0 \Leftrightarrow f(\vec{u}_i)=0$, $\forall i=1,2,\dots,d$.

$$\underset{\sum_{k=1}^{n}\alpha_k \vec{u}_k^*(\vec{u}_i) = \alpha_i}{\shortparallel}$$

$\Leftrightarrow f=\alpha_{d+1}\vec{u}_{d+1}^*+\dots+\alpha_n\vec{u}_n^*$.

$\Leftrightarrow f \in \langle \vec{u}_{d+1}^*,\dots,\vec{u}_n^* \rangle$ 故 $\dim(U^0)=n-d$.

$\Leftrightarrow U^0 = \langle \vec{u}_{d+1}^*,\dots,\vec{u}_n^* \rangle$.

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$\frac{a}{t^m}$, $\gcd(a,t)=1$ is always differential-reduced.

$\gcd(t^m, a-i(t^m)') = \gcd(t^m, a-imt^{m-1}\cdot ut)=1$

V是域F上的n维向量空间,

定义(双线性型) $f: V\times V\to F$

$\forall \alpha,\beta\in F$, $\vec{x},\vec{y},\vec{z}\in V$, 有 $\begin{cases} f(\alpha\vec{x}+\beta\vec{y},\vec{z}) = \alpha f(\vec{x},\vec{z})+\beta f(\vec{y},\vec{z}) \\ f(\vec{x},\alpha\vec{y}+\beta\vec{z}) = \alpha f(\vec{x},\vec{y})+\beta f(\vec{x},\vec{z}) \end{cases}$

定义(对称双线型) 若f是V上的双线性型且满足 $f(\vec{x},\vec{y})=f(\vec{y},\vec{x})$

定义(二次型). $q: V\to F$, $\forall \vec{x},\vec{y}\in V$

满足 $\begin{cases} q(\vec{x})=q(-\vec{x}) \\ \boxed{f(\vec{x},\vec{y})}=\frac{1}{2}(q(\vec{x}+\vec{y})-q(\vec{x})-q(\vec{y})) \text{ 为 } V \text{上的}\overset{\boxed{\text{对称}}}{}\text{双线性型} \end{cases}$

称为q的配极.

给定V的一组基 $\{\vec{e_1},\dots,\vec{e_n}\}$.

$\mathcal{L}_2(V) := \{f \mid f\text{是}V\text{上的双线性型}\} \overset{\text{线性同构}}{\cong} M_n(F)$

$f \longmapsto (f(\vec{e_i},\vec{e_j}))_{n\times n}$

$f: V\times V\to F \longleftarrow A=(a_{ij})_{n\times n}$

$(\sum x_i\vec{e_i}, \sum y_i\vec{e_i}) \mapsto (x_1,\dots,x_n)A\begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix}$

$Q(V) := \{p \mid p\text{是}V\text{上的二次型}\} \cong \mathcal{L}_2^+(V) := \{f \mid f\text{是}V\text{上的对称双线性型}\} \cong SM_n(F)$

$p(\vec{x}) \longmapsto f(\vec{x},\vec{y})$ (f是p的配极)

$p(\vec{x}) := f(\vec{x},\vec{x}) \longleftarrow f(\vec{x},\vec{y})$

$SM_n(F) \cong \{F[x_1,\dots,x_n]\text{上的齐次二次多项式}\}$

$(a_{ij}) \longmapsto p(x_1,\dots,x_n)=\sum_{i\le j} a_{ij}x_ix_j$

$\longleftarrow p(x_1,\dots,x_n)=\sum_{i\le j} a_{ij}x_ix_j$

$=(x_1,\dots,x_n)B\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}$, 其中 $B=\begin{pmatrix} a_{11} & \frac{a_{12}}{2} & \cdots & \frac{a_{1n}}{2} \\ \frac{a_{12}}{2} & a_{22} & \cdots & \frac{a_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{1n}}{2} & \frac{a_{2n}}{2} & \cdots & a_{nn} \end{pmatrix}$

例 设 V 是域 F 上的 n 维线性空间，其中 F 的特征不等于 2. 设 q 是 V 上的二次型 且 $\mathrm{rank}(q)=n$.
再设 存在非零向量 $\vec{x}\in V$，使得 $q(\vec{x})=0$. 证明：q 是满射.

解：设 $x=\vec{e_1}$，由此扩充成 V 的一组基 $\vec{e_1},\vec{e_2},\cdots,\vec{e_n}$. 设 $A=(a_{ij})$ 是 q 在这组基下的矩阵.

由于 $q(\vec{e_1})=0$，设 f 是 q 的配极，$q(\vec{e_1})=f(\vec{e_1},\vec{e_1})=a_{11}=0$

由 $\mathrm{rank}(q)=n \Rightarrow \mathrm{rank}(A)=n$.

$\Rightarrow \exists\, j\in\{2,\cdots,n\}$, s.t. $a_{1j}\neq 0$

令 $v=\alpha\vec{e_1}+\beta\vec{e_j}$，$q(\vec{v})=f(\vec{v},\vec{v})=f(\alpha\vec{e_1}+\beta\vec{e_j},\alpha\vec{e_1}+\beta\vec{e_j})$

$$=\alpha^2 f(\vec{e_1},\vec{e_1})+\alpha\beta f(\vec{e_1},\vec{e_j})+\alpha\beta f(\vec{e_j},\vec{e_1})+\beta^2 f(\vec{e_j},\vec{e_j})$$

$$=2\alpha\beta a_{1j}+\beta^2 a_{jj}$$

$\forall\,\gamma\in F$，取 $\beta=1$，$\alpha=\dfrac{\gamma-a_{jj}}{2a_{1j}}$，

可得 $q(\vec{v})=\gamma$.

故证 q 是满射.

利用配方法 化二次型

$$p=x_1x_{2n}+x_2x_{2n-1}+\cdots+x_nx_{n+1}$$

令 $\begin{cases} x_1=y_1+y_{2n} \\ x_2=y_2+y_{2n-1} \\ \cdots\cdots \\ x_n=y_n+y_{n+1} \\ x_{n+1}=y_n-y_{n+1} \\ \cdots\cdots \\ x_{2n-1}=y_2-y_{2n-1} \\ x_{2n}=y_1-y_{2n} \end{cases}$，
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{pmatrix}=\underbrace{\begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & -1 \end{pmatrix}}_{P}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{2n-1} \\ y_{2n} \end{pmatrix}$$

$$p=(y_1+y_{2n})(y_1-y_{2n})+(y_2+y_{2n-1})(y_2-y_{2n-1})+\cdots+(y_n+y_{n+1})(y_n-y_{n+1})$$
$$=y_1^2-y_{2n}^2+y_2^2-y_{2n-1}^2+\cdots+y_n^2-y_{n+1}^2$$
$$=y_1^2+y_2^2+\cdots+y_n^2-y_{n+1}^2-\cdots-y_{2n-1}^2-y_{2n}^2$$

设 p 在标准基下的矩阵 $A=\begin{pmatrix} 0 & \cdots & & \frac{1}{2} \\ & & \frac{1}{2} & \\ & \iddots & & \\ \frac{1}{2} & & & \end{pmatrix}_{2n\times 2n}$，在另一组基下为 $B=\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}_{2n\times 2n}$

$$p=(x_1,\cdots,x_{2n})A\begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}=(y_1,y_2,\cdots,y_{2n})\underbrace{P^tAP}_{B}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \\ y_{2n} \end{pmatrix}$$

$\frac{a}{t}$, $\gcd(a,t)=1$ is always ...

$$P=\sum_{i=1}^{n} x_i^2+\sum_{1\le i<j\le n} x_i x_j$$

Pf:

令 $A_n(\alpha)=\begin{pmatrix}\alpha & \cdots & 1\\ \vdots & \alpha & \vdots\\ & \ddots & \\ 1 & \cdots & \alpha\end{pmatrix}_{n\times n}$

令 $P_1=\begin{pmatrix}1 & -\alpha^{-1} & \cdots & -\alpha^{-1}\\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\end{pmatrix}$, $P_1^t A_n(\alpha) P_1=\begin{pmatrix}\alpha & 0 & \cdots & 0\\ \vdots & \alpha-\alpha^{-1} & \cdots & 1-\alpha^{-1}\\ & \vdots & \ddots & \vdots\\ 0 & 1-\alpha^{-1} & \cdots & \alpha-\alpha^{-1}\end{pmatrix}$.

$$\begin{pmatrix}\alpha-\alpha^{-1} & \cdots & 1-\alpha^{-1}\\ \vdots & \ddots & \vdots\\ 1-\alpha^{-1} & \cdots & \alpha-\alpha^{-1}\end{pmatrix}=\frac{\alpha-1}{\alpha}\begin{pmatrix}\alpha+1 & 1 & \cdots & 1\\ \vdots & \ddots & & \vdots\\ 1 & \cdots & & \alpha+1\end{pmatrix}=\frac{\alpha-1}{\alpha}A_{n-1}(\alpha+1)$$

令 $P_2=\begin{pmatrix}1 & 0 & \cdots & & 0\\ 0 & 1 & -(\alpha+1)^{-1} & \cdots & -(1+\alpha)^{-1}\\ \vdots & & \ddots & & \\ 0 & & & & 1\end{pmatrix}$, $\frac{\alpha-1}{\alpha}P_2^t A_{n-1}(\alpha+1) P_2=\begin{pmatrix}\alpha & 0 & 0 & \cdots & 0\\ 0 & \frac{\alpha-1}{\alpha}(\alpha+1) & 0 & \cdots & 0\\ \vdots & 0 & (\alpha+1)-(\alpha+1)^{-1} & \cdots & 1-(\alpha+1)^{-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 1-(\alpha+1)^{-1} & \cdots & (\alpha+1)-(\alpha+1)^{-1}\end{pmatrix}$.

$$P_1P_2=\begin{pmatrix}1 & -\alpha^{-1} & \cdots & -\alpha^{-1}\\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\\ & 1 & -(\alpha+1)^{-1} \cdots & -(1+\alpha)^{-1}\\ & & \ddots & \\ & & & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -\alpha^{-1} & -(\alpha+1)^{-1} & \cdots & -(\alpha+1)^{-1}\\ & 1 & -(\alpha+1)^{-1} & \cdots & -(1+\alpha)^{-1}\\ & & \ddots & & \\ & & & & 1\end{pmatrix}$$

$$P_1P_2\cdots P_{n-1}=\begin{pmatrix}1 & -\alpha^{-1} & -(\alpha+1)^{-1} & -(\alpha+2)^{-1} & \cdots & -(\alpha+n-2)^{-1}\\ & 1 & -(\alpha+1)^{-1} & -(\alpha+2)^{-1} & & \vdots\\ & & 1 & & & \vdots\\ & & & 1 & & \vdots\\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}$$

故最终对角线元素依次为

α

$\frac{\alpha-1}{\alpha}(\alpha+1)$.

$\frac{\alpha-1}{\alpha}\cdot\frac{\alpha}{\alpha+1}\cdot(\alpha+2)=\frac{\alpha-1}{\alpha+1}(\alpha+2)$

$\vdots$

$\frac{\alpha-1}{\alpha+i-1}(\alpha+i)$

$\vdots$

$\frac{\alpha-1}{\alpha+n-2}(\alpha+n-1)$

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$\frac{a}{t}$, $\gcd(a,t)=1$ B always ...

命题.

$$A_n(\alpha) \sim_c \begin{pmatrix} \alpha & & & \\ & (\alpha+1)\beta_0(\alpha) & & \\ & & \ddots & \\ & & & (\alpha+n-1)\beta_{n-2}(\alpha) \end{pmatrix}$$

$$\beta_i(\alpha) = \prod_{j=0}^{i} (1 - (\alpha+j)^{-1}) = \frac{\alpha-1}{\cancel{\alpha}} \cdot \frac{\cancel{\alpha}}{\cancel{\alpha+1}} \cdots \frac{\alpha+i-1}{\alpha+i} = \frac{\alpha-1}{\alpha+i}$$

pf: $n=2$, $A_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix}$, 令 $P_1 = \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix}$, $P_1^t A_2(\alpha) P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\alpha-1}{\alpha}(\alpha+1) \end{pmatrix}$

假设命题对 $\le n-1$ 情况成立，下面来看 n 的情况.

$$A_n(\alpha) = \begin{pmatrix} \alpha & 1 & \cdots & 1 \\ \vdots & \alpha & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & \alpha \end{pmatrix}, \quad 令 P_1 = \begin{pmatrix} 1 & -\alpha^{-1} & \cdots & -\alpha^{-1} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$P_1^t A_n(\alpha) P_1 = \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ \vdots & \alpha-\alpha^{-1} & \cdots & 1-\alpha^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1-\alpha^{-1} & \cdots & \alpha-\alpha^{-1} \end{pmatrix} = (\alpha-\alpha^{-1}) \begin{pmatrix} \frac{\alpha^2}{\alpha-1} & \\ & A_{n-1}(\alpha+1) \end{pmatrix}$$

$$A_{n-1}(\alpha+1) \sim_c \underbrace{\begin{pmatrix} \alpha+1 & & & \\ & (\alpha+2)\beta_0(\alpha+1) & & \\ & & \ddots & \\ & & & (\underbrace{\alpha+1+n-1-1}_{\alpha+n-1})\beta_{n-3}(\alpha+1) \end{pmatrix}}_{B}, \quad \exists Q \in GL_n(\mathbb{R}). \; Q^t A_{n-1}(\alpha+1) Q = B.$$

令 $P_2 = \begin{pmatrix} 1 & \\ & Q \end{pmatrix}$, $\cancel{P_2^t A_{n-1}(\alpha+1)}$ $P_2^t P_1^t A_n(\alpha) P_1 P_2 = \frac{\alpha-1}{\alpha} \begin{pmatrix} \frac{\alpha^2}{\alpha-1} & & & & \\ & \alpha+1 & & & \\ & & (\alpha+2)\beta_0(\alpha+1) & & \\ & & & \ddots & \\ & & & & (\alpha+n-1)\beta_{n-3}(\alpha+1) \end{pmatrix}$

$$= \begin{pmatrix} \alpha & & & & \\ & \underbrace{\frac{\alpha-1}{\alpha}}_{\beta_0(\alpha)}(\alpha+1) & & & \\ & & \boxed{\frac{\alpha-1}{\alpha} \cdot \frac{\alpha}{\alpha+1}}(\alpha+2) & & \\ & & & \ddots & \\ & & & & (\alpha+n-1)\beta_{n-2}(\alpha) \end{pmatrix}$$