1、$\mathbb{Q}^3$

$$v_1=\begin{pmatrix}1\\2\\0\end{pmatrix}\quad v_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad v_3=\begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$$a_1v_1+a_2v_2+a_3v_3=0.$$

$$A=\begin{pmatrix}1&1&0\\2&0&2\\0&1&-1\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&-2&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&0\\0&1&-1\\0&0&0\end{pmatrix}$$

令 $\mathbb{Q}^3\xrightarrow{A}\langle v_1,v_2,v_3\rangle=V$ $\quad \dim V=\dim(\operatorname{im}A)=\operatorname{rank}A=2$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\longmapsto a_1v_1+a_2v_2+a_3v_3$$

$v_1-v_2-v_3=0 \qquad v_3=v_1-v_2$

v_1,v_2 构成 V 的一组基.

$\dim(\mathbb{Q}^3/V)=1$，任取 $v\in\mathbb{Q}^3$，$v\in V$，则 $\bar{v}$ 构成 $\mathbb{Q}^3/V$ 的一组基. 容易验证 $v=\begin{pmatrix}0\\0\\1\end{pmatrix}\notin V$. $\overline{\begin{pmatrix}0\\0\\1\end{pmatrix}}$ 为 $\mathbb{Q}^3/V$ 的基.

2、 $(v_1, v_2, v_3) = (u_1, u_2, u_3) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$\Rightarrow P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ $\quad (u_1, u_2, u_3) = (v_1, v_2, v_3) \cdot P^{-1}$

$$w = (u_1 u_2, u_3) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (v_1, v_2, v_3) \cdot P^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= (v_1, v_2, v_3) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ 坐标为 $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

3. 设 $\varepsilon_{n,i}$ 为 n元对称多次式中的第 i个基本对称多项式.

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$
$$= x^n - \varepsilon_{n,1}(\alpha_1, \cdots, \alpha_n) x^{n-1} + \cdots + (-1)^n \varepsilon_{n,n}(\alpha_1, \cdots, \alpha_n)$$

设 $a_{ij} = \varepsilon_{i,j}(0, 1, \cdots, i-1)$

则 $(x-0) \cdot \cdots \cdot (x-i+1) = x^i - a_{i1} x^{i-1} + a_{i2} x^{i-2} + \cdots + (-1)^i a_{ii}$

$a_{i1} = 1 + \cdots + i-1 = \frac{(i-1) \cdot i}{2}$

$a_{ii} = 0$

$$(i)+(ii) \ (1, x, x(x-1), \cdots, x \cdot (x-1)\cdots(x-d+1)) = (1, \cdots, x^d) \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ & 1 & -1 & a_{32} & & (-1)^{d-1} a_{d,d-1} \\ & & 1 & -3 & & (-1)^{d-2} a_{d,d-2} \\ & & & 1 & & \vdots \\ & & & & \ddots & -\frac{d(d-1)}{2} \\ & & & & & 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 \\ & 1 & -1 & -a_{32} & & (-1)^{d-1} a_{d,d-1} \\ & & 1 & -3 & & (-1)^{d-2} a_{d,d-2} \\ & & & 1 & \ddots & \\ & & & & \ddots & \frac{d(d-1)}{2} \\ & & & & & 1 \end{pmatrix}$$

注意到 P 可逆

$$(1, x, x(x-1), ---, x\cdot(x-1)\cdot \cdots (x-d+1)) = (1, x, ---, x^d)\cdot P^{-1}$$

$\Rightarrow 1, x, x\cdot(x-1), \cdots\cdots, x\cdot \cdots \cdot (x-d+1)$ 为一组基.

(iii) 记 $Q_d = x\cdot(x-1)\cdot --- (x-d+1)$

形式上, $Q_d = \binom{x}{d}\cdot d!$

$$\Delta Q_d = \left[\binom{x+1}{d} - \binom{x}{d}\right] d!$$

$$= \left[\binom{x}{d-1}\right]\cdot d!$$

$$= \frac{Q_{d-1}}{(d-1)!}\cdot d! = d\,Q_{d-1}$$

$$\Rightarrow \Delta(Q_0, \cdots, Q_d) = (Q_0, \cdots, Q_d) \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & 2 & 0 & \ddots & 0 \\ & & 0 & 3 & \ddots & \vdots \\ & & & 0 & \ddots & 0 \\ & & & & \ddots & d \\ & & & & & 0 \end{pmatrix}}_{\begin{matrix} \| \\ A \end{matrix}}$$

(iv) $\dim(\mathrm{Im}\,\Delta) = \mathrm{rank}\,\Delta = d$

$\dim(\ker\Delta) = d+1 - \dim(\mathrm{Im}\,\Delta) = 1.$ $\square$

4. (i) 由 (c) $\forall\ v \in V$

$$v = \mathcal{E}(v) = \pi_1(v) + \cdots + \pi_k(v), \quad \pi_i(v) \in \mathrm{Im}(\pi_i)$$

$$\Rightarrow \quad V = \mathrm{im} V_1 + \cdots + \mathrm{im} V_k$$

还需证明唯一分解.

即若 $v = \pi_1(v_1) + \cdots + \pi_k(v_k)$，则 $\pi_i(v_i) = \pi_i(v)$

$$\pi_i(v) = \pi_i \circ \pi_1(v_1) + \cdots + \pi_i^2(v_i) + \cdots + \pi_i \cdot \pi_k(v_k)$$

$$\overset{(a)+(b)}{=} 0 + \cdots + \pi_i^2(v_i) + \cdots + 0$$

$$= \pi_i(v_i).$$

(ii) $\forall v \in V$，设 $v = \pi_1(v_1) + \cdots + \pi_n(v_n)$ 为直和分解、

则 $\rho_i(v) = \pi_i(v_i) \in \mathrm{Im} V_i$

而 $\pi_i(v) = \pi_i(\pi_1(v_1) + \cdots + \pi_k(v_k)) = \pi_i(v_i)$.

$\Rightarrow \rho_i(v) = \pi_i(v)$，$\forall\ v \in V \Rightarrow \rho_i = \pi_i$ $\square$

5: $k=1$ ✓

$k=2$, $\dim V_1+\dim V_2=\dim(V_1\cap V_2)+\dim(V_1+V_2)$

$$\Rightarrow \text{din}(V_1\cap V_2)>n-\dim(V_1+V_2)\geq 0$$

$V_1\cap V_2\neq\{0\}$.

猜测由于维数原因，$V_1\cap\cdots\cap V_k\neq 0$

受此启发，我们证明：

$\exists$ $k-1$ 个子空间 $W_1,\cdots,W_{k-1}$

s.t. $\dim V_1+\cdots+\dim V_k=\dim W_1+\cdots+\dim W_{k-1}+\dim(V_1\cap\cdots\cap V_k)$

$$\Rightarrow \dim(V_1\cap\cdots\cap V_k)>0$$

$k=1$ 时，显然成立

$k=2$ 时，取 $W_1=V_1+V_2$ 即可.

若 $k-1$ 成立，k 时

$$\dim V_1+\cdots+\dim V_{k-1}>n(k-1)-\dim V_k\geq n(k-2)$$

$\Rightarrow\exists$ $W_1,\cdots,W_{k-2}$ s.t.

$$\dim V_1+\cdots+\dim V_{k-2}+\dim V_{k-1}$$
$$=\dim W_1+\cdots+\dim W_{k-2}+\dim(V_1\cap\cdots\cap V_{k-1})$$

则 $\dim V_k+\dim(V_1\cap\cdots\cap V_{k-1})=\dim(V_k+V_1\cap\cdots\cap V_{k-1})$
$+\dim(V_1\cap\cdots\cap V_k)$

$$\Rightarrow \dim V_1+\cdots+\dim V_{k-1}+\dim V_k$$
$$=\dim W_1+\cdots+\dim W_{k-2}+\dim(V_1\cap\cdots\cap V_{k-1})+\dim V_k$$
$$=\dim W_1+\cdots+\dim W_{k-2}+\dim(V_k+V_1\cap\cdots\cap V_{k-1})+\dim(V_1\cap\cdots\cap V_k)$$

取 $W_k=V_k+V_1\cap\cdots\cap V_{k-1}$ 即可. □

行列相伴变换

左乘

设 F_{ij} 为n阶第一类初等矩阵.（交换 i,j 行）

$F_{ij}(\lambda)$ 对$\lambda\in F$ 为第二类初等矩阵.（i行乘λ倍加到j行）

$$F_{ij}=\begin{pmatrix}1&&&&&&\\&\ddots&&&&&\\&&0&\cdots&1&&\\&&\vdots&\ddots&\vdots&&\\&&1&\cdots&0&&\\&&&&&\ddots&\\&&&&&&1\end{pmatrix}$$

(i,j)

(j,i)

$$F_{ij}(\lambda)=\begin{pmatrix}1&&&\\&1&&\\&\lambda&\ddots&\\&&&1\end{pmatrix}$$

(j,i)项

引理1：若 $A\in SM_n(F)$，则 $B=F_{ij}^t\cdot A\cdot F_{ij}$ 对称，且

$$B=\begin{array}{cc}\begin{array}{ccccccc}&&i&&j&&\end{array}&\\ \left(\begin{array}{ccccccc}a_{11}&\cdots&a_{1j}&\cdots&a_{1i}&\cdots&a_{1n}\\ \vdots&&\vdots&&\vdots&&\vdots\\ a_{j1}&\cdots&a_{jj}&\cdots&a_{ji}&\cdots&a_{jn}\\ \vdots&&\vdots&&\vdots&&\vdots\\ a_{i1}&\cdots&a_{ij}&\cdots&a_{ii}&\cdots&a_{in}\\ \vdots&&\vdots&&\vdots&&\vdots\\ a_{n1}&\cdots&a_{nj}&\cdots&a_{ni}&\cdots&a_{nn}\end{array}\right)&\begin{array}{c}\\ \\ i\\ \\ j\\ \\ \\ \end{array}\end{array}$$

Pf： 先交换ij行 再交换 i,j列. □

引理2： $A=(a_{ij})\in SM_n(F)$，$B=F_{ij}(\lambda)^t A\cdot F_{ij}(\lambda)$

$$B=\begin{pmatrix}1&&&&\\&\ddots&&&\\&&1&\lambda&\\&&&1&\\&&&&\ddots\\&&&&&1\end{pmatrix}A\cdot\begin{pmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&\lambda&\ddots&\\&&&&1\\&&&&&\ddots\\&&&&&&1\end{pmatrix}$$

(i,j)项

(j,i)项

$$= \begin{pmatrix} a_{11} & a_{1i}+\lambda a_{1j} & a_{1j} & a_{1n} \\ a_{i1}+\lambda a_{j1} & a_{ii}+2\lambda a_{ij}+\lambda^2 a_{jj} & a_{ij}+\lambda a_{jj} & a_{in}+\lambda a_{jn} \\ a_{j1} & a_{ji}+\lambda a_{jj} & a_{jj} & a_{jn} \\ a_{n1} & a_{ni}+\lambda a_{nj} & a_{nj} & a_{nn} \end{pmatrix}$$

上述引理中的变换称为"行列相伴变换"

Lem：若 $\mathrm{char}(F) \neq 2$，$A \in SM_n(F)$，则可通过行列相伴变换将 A 变为 B s.t. $b_{11} \neq 0$.

Pf：举例说明，一般情况同理

①对角线不全为0

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{交换1,2列}]{\text{交换1,2行后}} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

②对角线全为0

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[\text{第2列加第1列}]{\text{第2行加第1行后}} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

命题：若 $\mathrm{char}(F)\neq 2$，则可通过行列相伴变换得到对角矩阵.

证：对维数归纳.

$\dim=1$ ✓.

若 $\dim=n-1$ ✓.

$\dim=n$ 时：可假设 $A=(a_{ij})$ $a_{11}\neq 0$.

$$\underbrace{F_{n1}^{t}\left(-\frac{a_{1n}}{a_{12}}\right)\cdots F_{21}^{t}\left(-\frac{a_{12}}{a_{11}}\right)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{pmatrix} F_{21}\left(-\frac{a_{12}}{a_{11}}\right)\cdot\cdots\cdot F_{n1}\left(-\frac{a_{1n}}{a_{11}}\right)}_{\|}$$

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \ast & \\ 0 & & & \end{pmatrix}$$

由归纳即得.□

eg: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\xrightarrow{\text{右 } F_{12}(1)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{左 } F_{12}^{t}(1)} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{右 } F_{21}(-\frac{1}{2})} \begin{pmatrix} 2 & 0 & 1 \\ 1 & -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{\text{左 } F_{21}(-\frac{1}{2})^{t}} \begin{pmatrix} 2 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\xrightarrow{\text{右 } F_{31}(-\frac{1}{2})} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow{F_{31}^{t}(-\frac{1}{2})} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{F_{32}(1)} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{F_{32}^{t}(1)} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

可参考李老师2019-20年线代II习题课讲义4，
其中$F_{ij}(\lambda)$与本讲稿中有区别，需留意。