1.

解：$\vec{v_1}=\begin{pmatrix}1\\2\\0\end{pmatrix}$，$\vec{v_2}=\begin{pmatrix}1\\0\\1\end{pmatrix}$，$\vec{v_3}=\begin{pmatrix}0\\2\\-1\end{pmatrix}$.

$$A=\begin{pmatrix}1&1&0\\2&0&2\\0&1&-1\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&-2&2\\0&1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&0\\0&1&-1\\0&0&0\end{pmatrix}$$ （此处均做初等行变换）

从而 $\mathrm{rank}(A)=2$，故 $\dim(\langle v_1,v_2,v_3\rangle)=2$，容易看到 A的前两列线性无关，故 V的一组基为 v_1,v_2.

$\mathrm{rank}\left(\begin{pmatrix}1&1&0\\2&0&0\\0&1&1\end{pmatrix}\right)=3$，故 $\vec{v_1},\vec{v_2}$，$\begin{pmatrix}0\\0\\1\end{pmatrix}$ 构成 $\mathbb{Q}^3$的一组基，由引理4.13及其证明过程（第一章第二次讲义）

可知，$\mathbb{Q}^3/V$ 的一组基为 $\begin{pmatrix}0\\0\\1\end{pmatrix}+V$.

2. 解：$(\vec{v_1},\vec{v_2},\vec{v_3})=(\vec{u_1},\vec{u_2},\vec{u_3})\begin{pmatrix}1&1&0\\2&1&1\\2&1&2\end{pmatrix}$，则由基 $\vec{u_1},\vec{u_2},\vec{u_3}$ 到基 $\vec{v_1},\vec{v_2},\vec{v_3}$ 的转换矩阵为 $\begin{pmatrix}1&1&0\\2&1&1\\2&1&2\end{pmatrix}$，记为A.

$\vec{w}=\begin{pmatrix}1\\2\\3\end{pmatrix}=(\vec{u_1},\vec{u_2},\vec{u_3})\begin{pmatrix}1\\2\\3\end{pmatrix}=(\vec{v_1},\vec{v_2},\vec{v_3})A^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}$，　$A^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-1&2&-1\\2&-2&1\\0&-1&1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$

故 $\vec{w}$ 在 $\vec{v_1},\vec{v_2},\vec{v_3}$ 下的坐标为 $\begin{pmatrix}0\\1\\1\end{pmatrix}$

3.(i) 一般结论：设 $S\subset F[x]\setminus\{0\}$，如果S中的元素两两次数不同，则 S是F上的线性无关.

Pf：设 $f_1,\cdots,f_k\in S$，不妨设 $\deg(f_1)<\deg(f_2)<\cdots<\deg(f_k)$. 设 $\alpha_1,\cdots,\alpha_k\in F$, s.t

$$\alpha_1f_1+\cdots+\alpha_{k-1}f_{k-1}+\alpha_kf_k=0$$

若 $\alpha_k\neq0$，则 等式左侧是次数为 $\deg(f_k)$的多项式，矛盾，于是 $\alpha_k=0$

同理可得 $\alpha_{k-1}=\cdots=\alpha_2=0$.

$\Rightarrow \alpha_1f_1=0$

$\Rightarrow \alpha_1=0\ (f_1\neq0)$

$\Rightarrow f_1\cdots f_k$ 线性无关.

由上述结论可知 $\{1,x,x(x-1),\cdots,x(x-1)\cdots(x-d+1)\}$是 V中的线性无关集，

又由 $\dim(F[x]^{(d+1)})=d+1$，上述无关集正好是含有 $d+1$ 个元素，故 $\{1,x,x(x-1),\cdots x(x-1)\cdots(x-d+1)\}$

是 V的一组基.

(ii) 设 $\varepsilon_{n,i}$ 为n元对称多项式中的第i个基本对称多项式.

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)=x^n-\varepsilon_{n,1}(\alpha_1,\cdots,\alpha_n)x^{n-1}+\cdots+(-1)^n\varepsilon_{n,n}(\alpha_1,\cdots,\alpha_n)$$

对于矩阵P而言，P的第i列所有的元素对应着 $x(x-1)\cdots(x-i+2)$ 的所有系数，

$$x(x-1)\cdots(x-i+2)=x^{i-1}-\varepsilon_{i-1,1}(0,1,\cdots,i-1)x^{i-2}+\cdots\cdots+(-1)^{i-j}\varepsilon_{i-1,i-j}(0,1,\cdots,i-1)x^{j-1}$$

$i\cdots\cdots+0$

即第i列为
$$\begin{array}{c}\\ \\ \\ j\\ \\ \\ \end{array}\begin{pmatrix}0\\ \vdots\\ \vdots\\ (-1)^{i-j}\varepsilon_{i-1,i-j}(0,1,\cdots,i-1)\\ \vdots\\ 0\\ \vdots\end{pmatrix}$$

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$$(1,\ x,\ x(x-1),\ \dots,\ x(x-1)\cdots(x-d+1)) = (1,\ \dots,\ x^d)\begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ & 1 & -1 & \cdots & \cdots & \Sigma_{d,d+1}(0,1,\dots,d-1) \\ & & 1 & -3 & & \vdots \\ & & & \ddots & \ddots & -\frac{d(d-1)}{2} \\ & & & & & 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ & 1 & -1 & \cdots & \cdots & -\Sigma_{d,d+1}(0,1,\dots,d-1) \\ & & 1 & & & \vdots \\ & & & \ddots & \ddots & -\frac{d(d-1)}{2} \\ & & & & & 1 \end{pmatrix}$$

(iii) $\Delta(1) = 1-1 = 0$

$\Delta(x) = x+1-x = 1$

$\Delta(x(x-1)) = x(x+1) - x(x-1) = 2x$

$\vdots$

$\Delta(x(x-1)\cdots(x-i)) = (x+1)x\cdots(x-i+1) - x(x-1)\cdots(x-i) = x(x-1)\cdots(x-i+1)(x+1-x+i)$

$= (i+1)x(x-1)\cdots(x-i+1)$, $\quad i = 1,2,\dots,d-1$

$\Rightarrow \Delta(1,\ x,\ x(x-1),\ \dots,\ x(x-1)\cdots(x-d+1)) = (1,\ x,\ \dots,\ x(x-1)\cdots(x-d+1))\,P$

其中，

$$P = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & d \\ & & & & 0 \end{pmatrix}_{(d+1)\times(d+1)}$$

(iv) $\dim(\mathrm{im}\,\Delta) = \mathrm{rank}(P) = d$

$\dim(\ker\Delta) = d+1-\dim(\mathrm{im}\,\Delta) = d+1-d = 1$

4. pf: (i) $\forall \vec{x} \in V$, $\vec{x} = E\vec{x} = (\pi_1 + \dots + \pi_k)\vec{x} = \pi_1\vec{x} + \dots + \pi_k\vec{x}$

$\in \mathrm{im}(\pi_1) + \mathrm{im}(\pi_2) + \dots + \mathrm{im}(\pi_k)$

$\Rightarrow V \subseteq \mathrm{im}(\pi_1) + \mathrm{im}(\pi_2) + \dots + \mathrm{im}(\pi_k)$

又 $\because \mathrm{im}(\pi_1), \dots, \mathrm{im}(\pi_k) \subseteq V$

$\therefore \mathrm{im}(\pi_1) + \dots + \mathrm{im}(\pi_k) \subseteq V$

$\Rightarrow V = \mathrm{im}(\pi_1) + \mathrm{im}(\pi_2) + \dots + \mathrm{im}(\pi_k)$

$\forall \vec{x} \in \mathrm{im}(\pi_i) \cap \sum_{j\neq i} \mathrm{im}(\pi_j)$

$\exists \vec{x_i} \in V$, s.t. $x = \pi_i(\vec{x_i}) = \sum_{j\neq i} \pi_j(\vec{x_j})$

$\vec{x} = \pi_i(\vec{x_i}) = \pi_i(\pi_i(\vec{x_i})) = \sum_{j\neq i} \pi_i\pi_j(\vec{x_j}) = \vec{0}$

$\Rightarrow \mathrm{im}(\pi_i) \cap \sum_{j\neq i} \mathrm{im}(\pi_j) = \{0\}$ $\quad \Rightarrow V = \mathrm{im}(\pi_1) \oplus \mathrm{im}(\pi_2) \oplus \dots \oplus \mathrm{im}(\pi_k)$

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(ii) $\forall \vec{x} \in V$, $\vec{x} = \varepsilon\vec{x} = \pi_1(\vec{x}) + \cdots + \pi_k(\vec{x})$ (1)

由(i)知，$V = \mathrm{im}(\pi_1) \oplus \mathrm{im}(\pi_2) \oplus \cdots \oplus \mathrm{im}(\pi_k)$

故对于$\vec{x}$的(1)分解为唯一的，从而 $P_i(\vec{x}) = \pi_i(\vec{x})$.

由$\vec{x}$的任意性可知，$P_i = \pi_i$.

5. 解：$k=1$, $\dim(V_1) > 0$, $\Rightarrow V_1 \neq \{0\}$

法一：$k=2$, $\dim(V_1) + \dim(V_2) > n$，$\dim(V_1) + \dim(V_2) = \dim(V_1+V_2) + \dim(V_1 \cap V_2) > n$

$$\Rightarrow \dim(V_1 \cap V_2) > n - \dim(V_1+V_2) \geq n - n = 0$$

$$\Rightarrow V_1 \cap V_2 \neq \{0\}.$$

一般情况，证明存在$(k-1)$个子空间 $W_1, \cdots, W_{k-1}$，使得

$$\dim(V_1) + \cdots + \dim(V_k) = \dim(W_1) + \cdots + \dim(W_{k-1}) + \dim(V_1 \cap \cdots \cap V_k) \quad (1)$$

对k归纳. 当 $k=1$时，✓.

设 $k>1$且 (1)对 $(k-1)$个子空间成立，则存在子空间 $W_1, \cdots, W_{k-2}$使得

$$\dim(V_1) + \cdots + \dim(V_{k-2}) + \dim(V_{k-1}) = \dim(W_1) + \cdots + \dim(W_{k-2}) + \dim(V_1 \cap \cdots \cap V_{k-1})$$

于是

$$\dim(V_1) + \cdots + \dim(V_{k-2}) + \dim(V_{k-1}) + \dim(V_k)$$

$$= \dim(W_1) + \cdots + \dim(W_{k-2}) + \dim(V_k) + \dim(V_1 \cap \cdots \cap V_{k-1})$$

$$= \dim(W_1) + \cdots + \dim(W_{k-2}) + \dim(\underbrace{V_k + (V_1 \cap \cdots \cap V_{k-1})}_{\parallel \atop W_{k-1}}) + \dim(V_1 \cap \cdots \cap V_{k-1} \cap V_k).$$

从而(1)对 k个子空间成立.

故 (1)对任意的k 均成立.

因为 $\dim(W_i) \leq n$，故 $\underline{\dim(V_1) + \cdots + \dim(V_k)} \leq n(k-1) + \dim(V_1 \cap \cdots \cap V_k)$

又由题设知 $\dim(V_1) + \cdots + \dim(V_k) > n(k-1)$.

$$\Rightarrow n(k-1) < n(k-1) + \dim(V_1 \cap \cdots \cap V_k).$$

$$\Rightarrow \dim(V_1 \cap \cdots \cap V_k) > 0$$

$$\Rightarrow V_1 \cap \cdots \cap V_k \neq \{0\}.$$

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法二：任何一个 n 维线性空间线性同构于 F^n，从而 $V_1, V_2, \cdots, V_k$ 可看做 F^n 中的子空间．
进一步，V_i，$i=1,2,\cdots,k$．线性同构于齐次方程组的解空间

即：$V_i \cong \{\vec{X}_{n\times1} \mid A\vec{X}_{n\times1} = \vec{O}_n\}$

$\dim(V_i) = n - \operatorname{rank}(A_i)$

$\dim(V_1) + \cdots + \dim(V_k) = nk - (\operatorname{rank}(A_1) + \operatorname{rank}(A_2) + \cdots + \operatorname{rank}(A_k)) > n(k-1)$

$\Rightarrow \operatorname{ran}(A_1) + \cdots + \operatorname{rank}(A_k) < n$

$V_1 \cap \cdots \cap V_k \cong \left\{\vec{X}_{n\times1} \,\middle|\, \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} \vec{X}_{n\times1} = \vec{O}_n \right\}$

记 $A := \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}$，$\operatorname{rank}(A) \le \operatorname{ran}(A_1) + \cdots + \operatorname{rank}(A_k) < n$．

$\Rightarrow A\vec{X}_{n\times1} = \vec{O}_n$ 有非零解

$\Rightarrow V_1 \cap \cdots \cap V_k \ne \{\vec{0}\}$

比如 $\{\vec{u}_1, \cdots, \vec{u}_s\}$ 为 F^n 中子空间的一组基底，

构造齐次方程组 $\begin{pmatrix} \vec{u}_1^t \\ \vec{u}_2^t \\ \vdots \\ \vec{u}_s^t \end{pmatrix}_{s\times n} X_{n\times1} = 0_{s\times1}$，上述方程组总有解．

设 $\{\vec{v}_1, \cdots, \vec{v}_t\}$ 为其解空间的一组基底．

从而 $\begin{pmatrix} \vec{u}_1^t \\ \vec{u}_2^t \\ \vdots \\ \vec{u}_s^t \end{pmatrix}_{s\times n} (\vec{v}_1, \cdots, \vec{v}_t)_{n\times t} = \vec{O}_{s\times t}$

$\Rightarrow \left(\begin{pmatrix} \vec{u}_1^t \\ \vec{u}_2^t \\ \vdots \\ \vec{u}_s^t \end{pmatrix} (\vec{v}_1, \cdots, \vec{v}_t)\right)^t = \vec{O}_{t\times s}$

$\begin{pmatrix} \vec{v}_1^t \\ \vdots \\ \vec{v}_t^t \end{pmatrix} (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_s) = \vec{O}_{t\times s}$

故 $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_s\}$ 为 $\begin{pmatrix} \vec{v}_1^t \\ \vdots \\ \vec{v}_t^t \end{pmatrix} X = \vec{O}_{t\times s}$ 解空间的一组基底

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行3 线性空间的笛卡尔积.

设 设 V 和 W 是 F 上的线性空间. 在 $V\times W$ 上定义

$$\forall \vec{v_1},\vec{v_2}\in V,\ \vec{w_1},\vec{w_2}\in W,\ (\vec{v_1},\vec{w_1})+(\vec{v_2},\vec{w_2})=(\vec{v_1}+\vec{v_2},\vec{w_1}+\vec{w_2})$$

和

$$\forall \alpha\in F,\ \alpha(\vec{v_1},\vec{w_1})=(\alpha\vec{v_1},\alpha\vec{w_1})$$

引 验证 $V\times W$ 是 F 上的线性空间, 其零向量是 $(\vec{0}_V,\vec{0}_W)$. 由此我们可以诱导出4个自然的线性映射.

两个自然嵌入是

$$\phi_V: V\to V\times W,\ \vec{v}\mapsto(\vec{v},\vec{0}_W) \quad 和 \quad \phi_W: W\to V\times W,\ \vec{w}\mapsto(\vec{0}_V,\vec{w})$$

两个自然投射是

$$\psi_V: V\times W\to V,\ (\vec{v},\vec{w})\mapsto\vec{v} \quad 和 \quad \psi_W: V\times W\to W,\ (\vec{v},\vec{w})\mapsto\vec{w}$$

定理 设 V 和 W 都是有限维线性空间, 证明: $\dim(V\times W)=\dim(V)+\dim(W)$.

解 法一 设 $\dim V=n$, $\dim W=m$, $\{\vec{v_1},\cdots,\vec{v_n}\}$ 为 V 的一组基, $\{\vec{w_1},\cdots,\vec{w_m}\}$ 为 W 的一组基.

下证 $(\vec{v_1},\vec{0}_W),\cdots,(\vec{v_n},\vec{0}_W),(\vec{0}_V,\vec{w_1}),\cdots,(\vec{0}_V,\vec{w_m})$ 为 $V\times W$ 的基.

设 $\exists\alpha_1,\cdots,\alpha_{n+m}\in F$, s.t. $\sum_{i=1}^{n}\alpha_i(\vec{v_i},\vec{0}_W)+\sum_{j=1}^{m}\alpha_{n+j}(\vec{0}_V,\vec{w_j})=\vec{0}$

$$\Rightarrow \sum_{i=1}^{n}\alpha_i\vec{v_i}=0,\ \sum_{j=1}^{m}\alpha_{n+j}\vec{w_j}=0$$

$$\Rightarrow \alpha_i=0,\ i=1,2,\cdots,n+m$$

故 $(\vec{v_1},\vec{0}_W),\cdots,(\vec{v_n},\vec{0}_W),(\vec{0}_V,\vec{w_1}),\cdots,(\vec{0}_V,\vec{w_m})$ 线性无关.

$\forall(\vec{v},\vec{w})\in V\times W$, 有 $\vec{v}\in V$, $\exists\ \alpha_1,\cdots,\alpha_n\in F$, s.t. $\vec{v}=\sum_{i=1}^{n}\alpha_i\vec{v_i}$,
有 $\vec{w}\in W$, $\exists\ \beta_1,\cdots,\beta_m\in F$, s.t. $\vec{w}=\sum_{j=1}^{m}\beta_j\vec{w_j}$.

$$\Rightarrow(\vec{v},\vec{w})=(\vec{v},\vec{0}_W)+(\vec{0}_V,\vec{w})$$
$$=\sum_{i=1}^{n}\alpha_i(\vec{v_i},\vec{0}_W)+\sum_{j=1}^{m}\beta_j(\vec{0}_V,\vec{w_j})$$

故 $V\times W$ 中任何一个向量都可以通过这组向量线性表出.

从而 $\dim(V\times W)=n+m=\dim(V)+\dim(W)$.

法二: 考虑自然投射 ψ_V, $\ker(\psi_V)=\{0_V\}\times W$, $\operatorname{im}(\psi_V)=V$.

$$\Rightarrow \dim(V\times W)\overset{\text{I}}{=}\dim(\ker(\psi_V))+\dim(\operatorname{im}(\psi_V))=\dim(W)+\dim(V)$$

$\boxed{\text{I. } \phi: V\to W \text{ 线性映射, 则 } \dim(\ker(\phi))+\dim(\operatorname{im}(\phi))=\dim(V).}$

⑤

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法二：不妨设 $k>1$；考虑线性映射

$$\phi: V_1\times V_2\times\cdots\times V_k \longrightarrow \underbrace{V\times\cdots\times V}_{k-1}$$

$$(\vec{v_1},\vec{v_2},\cdots,\vec{v_k}) \longmapsto (\vec{v_1}-\vec{v_k},\cdots,\vec{v_{k-1}}-\vec{v_k})$$

易证 ϕ 是线性的.

$$\dim(V_1\times V_2\times\cdots\times V_k)=\dim(V_1)+\cdots+\dim(V_k).$$

$$\dim(\ker(\phi))+\dim(\mathrm{im}(\phi))=\dim(V_1\times V_2\times\cdots\times V_k).$$

$$\dim(\ker(\phi)) = \dim(V_1\times V_2\times\cdots\times V_k)-\dim(\mathrm{im}(\phi))$$

$$\geq \dim(V_1\times V_2\times\cdots\times V_k)-\dim(V\times\cdots\times V)$$

~~于是，存在非零向量 $(\vec{v}$~~ $\quad \geq \dim(V_1)+\dim(V_2)+\cdots+\dim(V_k)-(k-1)\cdot n$

$$>0.$$

于是，存在非零向量 $(\vec{v_1},\cdots,\vec{v_{k-1}},\vec{v_k})\in\ker(\phi)$，由 ϕ 的定义，$\vec{v_i}=\vec{v_k}$，$i=1,2,\cdots,k-1$.

即 $\vec{v_k}\in V_i$，$i=1,2,\cdots,k-1$，且 $\vec{v_k}\neq\vec{0}$. 特别地 $\vec{v_k}\in V_1\cap V_2\cap\cdots\cap V_k$.

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行列相伴变换.

设 F_{ij} 是n阶第一类初等矩阵, $i,j\in\{1,\dots,n\}$, (交换 E_n中 第i行和第j行),

$F_{ij}(\lambda)$ 是第二类初等矩阵, 其中 $i,j\in\{1,\dots,n\}$, $i\neq j$, $\lambda\in F$ (把 E_n中第j行遍乘 λ 加到第i行)

引理1. 设 $A=(a_{k,l})\in SM_n(F)$, $B=F_{ij}^t A F_{ij}$ 是对称矩阵且

$$B=\begin{array}{c} \begin{array}{cccc} & i & j & \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,i} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & a_{j,j} & \cdots & a_{j,i} & \cdots & a_{j,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,i} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,i} & \cdots & a_{n,n} \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \to i \\ \\ \to j \\ \\ \\ \end{array}$$

证: $$F_{ij}=\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} \Rightarrow F_{ij}^t=F_{ij}.$$

$$B^t=(F_{ij}^t A F_{ij})^t=F_{ij}^t A^t F_{ij}=F_{ij}^t A F_{ij}=B$$

故B为对称矩阵.

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1,i} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,i} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & a_{j,i} & \cdots & a_{j,j} & \cdots & a_{j,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,j} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\text{交换A的第i行和第j行}]{A_1=F_{ij}^t A} \begin{pmatrix} a_{11} & \cdots & a_{1,i} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & a_{j,i} & \cdots & a_{j,j} & \cdots & a_{j,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,i} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,j} & \cdots & a_{nn} \end{pmatrix}$$

$$\xrightarrow[\text{j列和i列}]{\text{交换}A_1\text{的第}} \begin{pmatrix} a_{11} & \cdots & a_{1,j} & \cdots & a_{1,i} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & \boxed{a_{j,j}} & \cdots & a_{j,i} & \cdots & a_{j,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & \boxed{a_{i,i}} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,i} & \cdots & a_{nn} \end{pmatrix}$$

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引理2. 设 $A=(a_{k,l})\in SM_n(F)$，$B=F_{i,j}(\lambda)^t A F_{i,j}(\lambda)$ 是对称矩阵且

$$B=\begin{array}{c}\begin{array}{ccccccc} & & i & & j & & \end{array}\\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,j}+\lambda a_{1,i} & \cdots & a_{1,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{i,1} & \cdots & a_{i,i} & \cdots & a_{i,j}+\lambda a_{i,i} & \cdots & a_{i,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{j,1}+\lambda a_{i,1} & \cdots & a_{j,i}+\lambda a_{i,i} & \cdots & \boxed{a_{j,j}+2\lambda a_{i,j}+\lambda^2 a_{i,i}} & \cdots & a_{j,n}+\lambda a_{i,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,j}+\lambda a_{n,i} & \cdots & a_{n,n}\end{pmatrix}\begin{array}{l} \\ \\ \to i\\ \\ \to j\\ \\ \\ \end{array}\end{array}$$

Pf:

$$F_{i,j}(\lambda)=\begin{array}{c}\begin{array}{ccc} i & & j\end{array}\\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & \lambda \\ & & & \ddots & \vdots \\ & & & & 1\end{pmatrix}\begin{array}{l} \\ \\ \to i\\ \\ \to j\end{array}\end{array},\quad F_{i,j}(\lambda)^t=\begin{array}{c}\begin{array}{ccc} i & & j\end{array}\\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & \lambda & \cdots & 1\end{pmatrix}\begin{array}{l} \\ \\ \to i\\ \\ \to j\end{array}\end{array}$$

，相当于把 E_n 的第 i 行乘 λ 倍加到第 j 行

$$A=\begin{array}{c}\begin{array}{ccccccc} & & i & & j & & \end{array}\\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,j} & \cdots & a_{1,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{i,1} & \cdots & a_{i,i} & \cdots & a_{i,j} & \cdots & a_{i,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{j,1} & \cdots & a_{j,i} & \cdots & a_{j,j} & \cdots & a_{j,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,j} & \cdots & a_{n,n}\end{pmatrix}\end{array}\xrightarrow[\text{第}i\text{行乘}\lambda\text{加到第}j\text{行}]{F_{i,j}(\lambda)^tA}$$

$$\begin{array}{c}\begin{array}{ccccccc} & & i & & j & & \end{array}\\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,j} & \cdots & a_{1,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{i,1} & \cdots & a_{i,i} & \cdots & a_{i,j} & \cdots & a_{i,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{j,1}+\lambda a_{i,1} & \cdots & a_{j,i}+\lambda a_{i,i} & \cdots & a_{j,j}+\lambda a_{i,j} & \cdots & a_{j,n}+\lambda a_{i,n}\\ \vdots & & \vdots & & \vdots & & \vdots\\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,j} & \cdots & a_{n,n}\end{pmatrix}\begin{array}{l} \\ \\ \to i\\ \\ \to j\\ \\ \\ \end{array}\end{array}\xrightarrow[\text{入加到第}j\text{列}]{\text{第}i\text{列乘}} B$$

对对称矩阵做有限次上述两个引理中的操作得到的矩阵称为通过（初等）行列相伴变换得到的矩阵。

引理3 设域的特征不等于2，$A\in SM_n(F)$ 如果 A 中对角线上元素都等于0但 $A\neq 0$，则我们可以通过行列相伴变换把 A 变成对称矩阵 $B=(b_{ij})$ 使得 $b_{1,1}\neq 0$。特别地，$A\sim_c B$.

Pf: 设 $A=(a_{k,l})_{n\times n}$，其中某个 $a_{i,j}\neq 0$，且 $i\neq j$，则 $F_{i,j}(1)^t A F_{i,j}(1)$ 在第 j 行 j 列处元素为

$$a_{jj}+2a_{i,j}+a_{i,i}=2a_{i,j}\neq 0 \quad (\because 2\neq 0,\ \text{引理}2).$$

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再由引理3，令 $B=F_{i,j}\left(F_{i,j}(1)^t A F_{i,j}(1)\right)F_{i,j}$，即为所求

定理4 设域 F 的特征不等于2，$A\in SM_n(F)$，则我们可以通过初等行列变换得到对角矩阵.

证：对 n 归纳. ①当 $n=1$ 时，A 是对角阵，定理显然成立.

②设 $n>1$ 且定理对 $n-1$ 成立，我们考虑 n 阶对称矩阵 A，

若 $A=O$，则 A 为对角矩阵；下面设 $A=(a_{ij})\neq O$

由引理3，我们可以进一步假设 $a_{1,1}\neq 0$. 由引理2.

$$F_{1,n}\left(-\frac{a_{n,1}}{a_{1,1}}\right)^t\cdots F_{1,2}\left(-\frac{a_{2,1}}{a_{1,1}}\right)^t A\,\overbrace{\boxed{F_{1,2}\left(-\frac{a_{2,1}}{a_{1,1}}\right)\cdots F_{1,n}\left(-\frac{a_{n,1}}{a_{1,1}}\right)}}^{M}=\begin{pmatrix} a_{1,1} & O_{1\times(n-1)} \\ O_{(n-1)\times 1} & B\end{pmatrix}$$

其中 $B\in SM_{n-1}(F)$. 由归纳假设存在 $Q\in GL_n(F)$（为第一类和第三类初等矩阵乘积），使得 Q^tBQ 为对角矩阵. 令

$$P=\begin{pmatrix} 1 & O_{1\times(n-1)} \\ O_{(n-1)\times 1} & Q\end{pmatrix},$$

则 $(MP)^tAMP$ 为对角阵.

例 设

$$A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0\end{pmatrix}\in SM_3(\mathbb{R}),$$

利用行列相伴变换把 A 化成对角阵 B，并计算 $P\in GL_3(\mathbb{R})$，使得 $B=P^tAP$.

解：

$$(A|E)=\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1\end{array}\right)\xrightarrow{r_1+r_2}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1\end{array}\right)\xrightarrow{C_1+C_2}\left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow{r_2-\frac{1}{2}r_1}\left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1\end{array}\right)\xrightarrow{C_2-\frac{1}{2}C_1}\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 \\ 2 & 0 & 0 & 0 & 0 & 1\end{array}\right)\xrightarrow{r_3-r_1}\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow{C_3-C_1}\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & -2 & 0 & 0 & 1\end{array}\right)$$

令 $P=\begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1\end{pmatrix}$，$B=\begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2\end{pmatrix}$. 即可.

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