1、 给定实数域 $\mathbb{R}$ 上 n 阶方阵 A, B, 设

$$X = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

证明 $|X| = |A+B| \cdot |A-B|$.

解：$|X| = \begin{vmatrix} A-B & B-A \\ B & A \end{vmatrix} = \begin{vmatrix} A-B & 0 \\ 0 & E_n \end{vmatrix} \begin{vmatrix} E_n & -E_n \\ B & A \end{vmatrix}$

$= |A-B| \cdot \begin{vmatrix} E_n & 0 \\ B & B+A \end{vmatrix} = |A-B| \cdot |A+B|.$

2、 计算下面 n 阶矩阵的行列式:

$$A_n = \begin{pmatrix} x+y & xy & 0 & 0 & \dots & 0 & 0 \\ 1 & x+y & xy & 0 & \dots & 0 & 0 \\ 0 & 1 & x+y & xy & \dots & 0 & 0 \\ \vdots & \vdots & 1 & x+y & xy & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x+y & xy \\ 0 & 0 & 0 & 0 & \dots & 1 & x+y \end{pmatrix}.$$

$n=1$， $|A_1| = |x+y| = x+y$

$n=2$， $|A_2| = (x+y)^2 - xy = x^2+xy+y^2$

$n\geq 3$， $|A_n| = (x+y)|A_{n-1}| - xy|A_{n-2}|$

由高中技巧， $\exists$ a, b s.t.

$(|A_n| - a|A_{n-1}|) = b(|a_{n-1}| - a|A_{n-2}|)$

事实上 $(|A_n| - x|A_{n-1}|) = y(|A_{n-1}| - x|A_{n-2}|)$

$\Rightarrow |A_n| = \begin{cases} \dfrac{x^{n+1} - y^{n+1}}{(x-y)} & x \neq y \\ (n+1)x^n & x = y. \end{cases}$ $\square$

3、 证明伴随矩阵具有如下性质:

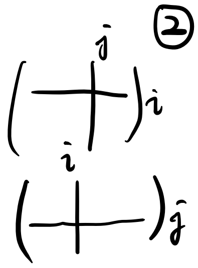
$$(\lambda A)^\vee = \lambda^{n-1}A^\vee; \quad (A^t)^\vee = (A^\vee)^t.$$

Pf 记 B_{ij} A去掉第 i行 j列剩下的矩阵. $A^\vee = (A_{ij})$

则 $A_{ij} = (-1)^{i+j}\det B_{ji}$

① $(\lambda A)^\vee_{ij} = (-1)^{i+j}\det(\lambda B_{ji}) = (-1)^{i+j}\lambda^{n-1}\det B_{ji}$
$= \lambda^{n-1}A_{ij}.$

② $(A^t)^\vee_{ij} = \det(B_{ij})^t = \det B_{ij} = A_{ji} = (A^\vee)^t_{ij}.$



4、 设 $A = (a_{ij})$ 为 n 阶矩阵, 且任取 $i \neq j$, 有 $(n-1)|a_{ij}| < |a_{ii}|$. 证明: $|A| \neq 0$.

Pf:若 $|A|\neq 0$, 考虑方程 $AX=0$ 则有非零解 $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

则 $\exists$ k s.t. $|x_k| \geq |x_i|$ $i\neq k$. $|x_k| \neq 0$.

$AX=0 \Rightarrow a_{kk}x_k + \sum_{j\neq k} a_{kj}x_j = 0$

$\Rightarrow |a_{kk}|\cdot|x_k| = |\sum_{j\neq k} a_{kj}x_j| \leq \sum_{j\neq k}|a_{kj}|\cdot|x_j|$

$< \sum_{j\neq i}\frac{1}{n-1}|a_{kk}||x_j| \leq |a_{kk}|\cdot|x_k|$

$\Rightarrow |a_{kk}|\cdot|x_k| < |a_{kk}|\cdot|x_k|$, 矛盾. □

5、 在平面直角坐标系内给定二次曲线

$$D:\quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0.$$

证明:

$$F = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{vmatrix}$$

是坐标变换

$$\begin{cases} x = \cos\theta \cdot x' - \sin\theta \cdot y' \\ y = \sin\theta \cdot x' + \cos\theta \cdot y' \end{cases}$$

的不变量.

(换句话说，坐标变换后，有曲线 $a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}{}'^2 + 2b'_1x' + 2b'_2y' + c' = 0$, 与 $F' = \begin{vmatrix} a'_{11} & a'_{12} & b'_1 \\ a'_{12} & a'_{22} & b'_2 \\ b'_1 & b'_2 & c' \end{vmatrix}$, 证明:$F' = F$).

Pf: D可表示为: $Q(x,y) = (x\ y)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (b_1\ b_2)\begin{pmatrix} x \\ y \end{pmatrix} + c$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$ 令 $T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x' \\ y' \end{pmatrix}$ $\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

变量替换后此方程变为: $(x'\ y')T^tAT\begin{pmatrix} x' \\ y' \end{pmatrix} + (b_1 b_2)\cdot T\begin{pmatrix} x' \\ y' \end{pmatrix} + c = 0$

$\Rightarrow D' = \left\{ Q'(x',y') = (x'\ y')T^tAT\begin{pmatrix} x' \\ y' \end{pmatrix} + (b_1 b_1)\cdot T\begin{pmatrix} x' \\ y' \end{pmatrix} + c \right\} = 0.$

$\Rightarrow A' = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = T^tAT \quad (b'_1, b'_2) = (b_1, b_2)T. \quad c' = c.$

令 $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $F = \begin{pmatrix} A & b \\ b^t & c \end{pmatrix}$

$F' = \begin{pmatrix} T^tAT & T^t\cdot b \\ b^t\cdot T & c \end{pmatrix} = \begin{pmatrix} T^t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & b \\ b^t & c \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}$

$\det F' = \det F.$
$\det T^t \cdot \det T$
$= \det F.$ □

6、证明伴随矩阵有性质 $(AB)^\vee = B^\vee \cdot A^\vee$.(提示: 考虑矩阵 $E_n + \epsilon AB$, ϵ 充分小.)

Pf: 由摄动法: $E_n+\varepsilon A$, $E_n+\varepsilon B$ 可逆, $|\varepsilon|<c$

若 A,B 可逆, 则 $(AB)^\vee\cdot(AB)=\det(AB)\cdot E_n$

$\Rightarrow (AB)^\vee = \det(AB)\cdot(AB)^{-1}$

$= (\det B)\cdot B^{-1}\cdot \det(A)\cdot A^{-1}$

$= B^\vee\cdot A^\vee$.

$$f(\varepsilon)=\left((E_n+\varepsilon A)(E_n+\varepsilon B)\right)^\vee$$

$$g(\varepsilon)=(E_n+\varepsilon B)^\vee(E_n+\varepsilon A)^\vee$$

$f(\varepsilon)$, $g(\varepsilon)$ 为每一项都关于 ε 的矩阵多项式,

且 $g(\varepsilon)=f(\varepsilon)$ 对于 $0\le|\varepsilon|<c$ 成立

注意到 若 α,β 为两多项式, $\alpha-\beta=0$ 有无穷多解.

则有 $\alpha\equiv\beta$.

$\Rightarrow g(\varepsilon)=f(\varepsilon)$ $\forall\varepsilon\in\mathbb{R}$,

$\Rightarrow \left((A+\frac{1}{\varepsilon}E_n)(B+\frac{1}{\varepsilon}E_n)\right)^\vee = (B+\frac{1}{\varepsilon}E_n)^\vee\cdot(A+\frac{1}{\varepsilon}E_n)^\vee$

考虑 $h(t)=\left((A+tE)(B+tE_n)\right)^\vee-(B+tE)^\vee\cdot(A+tE)^\vee$

则 $h(t)=0$, $t\neq 0$, $h(t)$ 关于 t 连续.

$\Rightarrow h(0)=0$. $\square$

Binet-Cauchy 公式.

设 A 为一矩阵.

$M_A\begin{pmatrix} i_1 \cdots i_m \\ j_1 \cdots j_m \end{pmatrix}$ 为 A 的 $i_1, \cdots, i_m$ 行，$j_1, \cdots, j_m$ 列构成的子式.

Lemma：设 $A=(a_{ij})\in M_{m\times n}(\mathbb{R})$， $m\leq n$.

固定 $1\leq i_1<\cdots<i_m\leq m$，

设 $j_1, \cdots, j_m$ 为 $\{i_1, \cdots, i_m\}$ 的一个排列.

则有 $$\begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ \vdots & \vdots & & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix} = (-1)^{N(j_1, \cdots, j_m)} \begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_m} \\ \vdots & \vdots & & \vdots \\ a_{mi_1} & a_{mi_2} & \cdots & a_{mi_m} \end{vmatrix}$$

Pf：考虑置换符号与逆序数的关系. □

定理 (Binet-Cauchy) 设 $A\in M_{m\times n}(\mathbb{R})$，$B\in M_{n\times m}(\mathbb{R})$.

则有 $$|A\cdot B| = \sum_{1\leq i_1<i_2<\cdots<i_m\leq n} M_A\begin{pmatrix} 1 \cdots m \\ i_1 \cdots i_m \end{pmatrix} M_B\begin{pmatrix} i_1 \cdots i_m \\ 1 \cdots m \end{pmatrix}$$

附：若 $m>n$，则 $\operatorname{rank}(A\cdot B)\leq n$，$|AB|=0$

此时 $M_A\begin{pmatrix}1 & \cdots & m\\ i_1 & \cdots & i_m\end{pmatrix}$ 无意义

故而设 $m\leq n$.

$AB=(c_{ij})\quad c_{ij}=\sum\limits_{k} a_{ik}\cdot b_{kj}$

$$|AB|=\begin{vmatrix}\sum\limits_{i_1}a_{1i_1}\cdot b_{i_11} & \sum\limits_{i_2}a_{1i_2}\cdot b_{i_22} & \cdots & \sum\limits_{i_m}a_{1i_m}\cdot b_{i_mm}\\ \sum\limits_{i_1}a_{2i_1}\cdot b_{i_11} & \sum\limits_{i_2}a_{2i_2}\cdot b_{i_22} & \cdots & \sum\limits_{i_m}a_{2i_m}\cdot b_{i_mm}\\ \\ \sum\limits_{i_1}a_{mi_1}b_{i_11} & \sum\limits_{i_2}a_{mi_2}\cdot b_{i_2} & \cdots & \sum\limits_{i_m}a_{mi_m}b_{i_mm}\end{vmatrix}$$

$$=\sum_{i_1}\cdots\sum_{i_m}\begin{vmatrix}a_{1i_1}\cdot b_{i_11} & a_{1i_2}\cdot b_{i_22} & \cdots & a_{1i_m}\cdot b_{i_mm}\\ a_{2i_1}\cdot b_{i_11} & a_{2i_2}\cdot b_{i_22} & \cdots & a_{2i_m}\cdot b_{i_mm}\\ \\ a_{mi_1}b_{i_11} & a_{mi_2}\cdot b_{i_2} & \cdots & a_{mi_m}b_{i_mm}\end{vmatrix}$$

$$=\sum_{i_1}\cdots\sum_{i_m}\begin{vmatrix}a_{1i_1} & a_{1i_2} & \cdots & a_{1i_m}\\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_m}\\ \vdots & \vdots & & \vdots\\ a_{mi_1} & a_{mi_2} & \cdots & a_{mi_m}\end{vmatrix} b_{i_11}\cdots b_{i_mm}$$

$$= \sum_{\{i_1,\dots,i_m\}\subset\{1,\dots,n\}} \left(\sum_{\substack{(j_1,\dots,j_m) \\ \{j_1,\dots,j_m\} \\ \| \\ \{i_1,\dots,i_m\}}} \begin{vmatrix} a_{1j_1} & \cdots & a_{1j_m} \\ \vdots & & \vdots \\ a_{mj_1} & \cdots & a_{mj_m} \end{vmatrix} \cdot b_{j_1 1} \cdots b_{j_m m} \right)$$

$$= \sum_{1\le i_1<\cdots<i_m\le n} \left(\begin{vmatrix} a_{1i_1} & \cdots & a_{1i_m} \\ \vdots & & \vdots \\ a_{mi_1} & \cdots & a_{mi_m} \end{vmatrix} \cdot \sum_{(j_1,\dots,j_m)} (-1)^{N(j_1\cdots j_m)} \cdot b_{j_1 1}\cdots b_{j_m m} \right)$$

$$= \sum_{1\le i_1<\cdots<i_m\le n} M_A\begin{pmatrix} 1 & \cdots & m \\ i_1 & \cdots & i_m \end{pmatrix} \cdot M_B\begin{pmatrix} i_1 & \cdots & i_m \\ 1 & \cdots & m \end{pmatrix}$$

应用示例：

证明 Cauchy 恒等式．

$$\left(\sum_{i=1}^{n} a_i \cdot c_i\right) \cdot \left(\sum_{i=1}^{n} b_i \cdot d_i\right) - \left(\sum_{i=1}^{n} a_i \cdot d_i\right) \cdot \left(\sum_{i=1}^{n} b_i \cdot c_i\right)$$

$$= \sum_{1 \le j < k \le n} (a_j b_k - a_k b_j) \cdot (c_j d_k - c_k d_j).$$

Pf：考虑 $A = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix}$ $B = \begin{pmatrix} c_1 & d_1 \\ \vdots & \vdots \\ c_n & d_n \end{pmatrix}$

$$\Rightarrow A \cdot B = \begin{pmatrix} \sum_i a_i \cdot c_i & \sum_i a_i d_i \\ \sum_i b_i \cdot c_i & \sum_i b_i \cdot d_i \end{pmatrix}$$

$$|A \cdot B| = \sum_{1 \le j < k \le n} \begin{vmatrix} a_j & a_k \\ b_j & b_k \end{vmatrix} \cdot \begin{vmatrix} c_j & d_j \\ c_k & d_k \end{vmatrix}$$

$\square$