第十次习题课

一、分块矩阵

左乘初等矩阵是做行变换，右乘初等矩阵做列变换

$$E_{2n}=\begin{pmatrix}E_n & O\\ O & E_n\end{pmatrix}\qquad M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

$$F_{ij}^{(n)}\longleftrightarrow \underbrace{\begin{pmatrix}O & E_n\\ E_n & O\end{pmatrix}}_{M_1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}C & D\\ A & B\end{pmatrix}\qquad \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}O & E_n\\ E_n & O\end{pmatrix}=\begin{pmatrix}B & A\\ D & C\end{pmatrix}$$

相当M两行互换　　　　相当于M两列互换

$$F_{ij}^{(n)}(\lambda)\longleftrightarrow \underbrace{\begin{pmatrix}E_n & P\\ O & E_n\end{pmatrix}}_{M_2}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A+PC & B+PD\\ C & D\end{pmatrix}$$

P不一定满秩

相当于M第二行左乘P加到第一行

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}E_n & P\\ O & E_n\end{pmatrix}=\begin{pmatrix}A & AP+B\\ C & CP+D\end{pmatrix}$$

相当于M第一列右乘P加到第二列

$$F_{i}^{(n)}(\lambda)\longleftrightarrow \underbrace{\begin{pmatrix}E_n & O\\ O & P'\end{pmatrix}}_{M_3}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A & B\\ P'C & P'D\end{pmatrix}$$

注：P'满秩

相当于M第二行左乘P.

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}E_n & O\\ O & P'\end{pmatrix}=\begin{pmatrix}A & BP'\\ C & DP'\end{pmatrix}$$

相当于M第二列右乘P.

注：① M_1, M_2, M_3 可逆

② 注意左、右乘，矩阵乘法不满足交换律.

eg1. $\begin{pmatrix} E_n & O \\ E_n & E_n \end{pmatrix}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} E_n & O \\ -E_n & E_n \end{pmatrix}$

$$= \begin{pmatrix} A & B \\ B+A & A+B \end{pmatrix}\begin{pmatrix} E_n & O \\ -E_n & E_n \end{pmatrix} = \begin{pmatrix} A-B & B \\ O & A+B \end{pmatrix}$$

hw4. $\operatorname{rank}(AB) + \operatorname{rank}(BC) \le \operatorname{rank}(ABC) + \operatorname{rank}(B)$

证: $\begin{pmatrix} E_m & -A \\ O & E_n \end{pmatrix}\begin{pmatrix} AB & O \\ B & BC \end{pmatrix} = \begin{pmatrix} O & -ABC \\ B & BC \end{pmatrix}$

$$\begin{pmatrix} O & -ABC \\ B & BC \end{pmatrix}\begin{pmatrix} E_p & -C_{p\times q} \\ O & E_q \end{pmatrix} = \begin{pmatrix} O & -ABC \\ B & O \end{pmatrix}$$

$$\operatorname{rank}\begin{pmatrix} O & -ABC \\ B & O \end{pmatrix} = \operatorname{rank}\begin{pmatrix} AB & O \\ B & BC \end{pmatrix} \ge \operatorname{rank}(AB) + \operatorname{rank}(BC)$$

$$\|$$

$$\operatorname{rank}(B) + \operatorname{rank}(ABC)$$

hw5. 证: $A = \begin{pmatrix} A_1 & A_2 \\ O & A_3 \end{pmatrix}$ $\quad A^t = \begin{pmatrix} A_1^t & O \\ A_2^t & A_3^t \end{pmatrix}$

$$A \cdot A^t = \begin{pmatrix} A_1 \cdot A_1^t + A_2 \cdot A_2^t & A_2 \cdot A_3^t \\ A_3 \cdot A_2^t & A_3 \cdot A_3^t \end{pmatrix}$$

$$A^t \cdot A = \begin{pmatrix} A_1^t A_1 & A_1^t A_2 \\ A_2^t A_1 & A_2^t A_3 + A_3^t A_3 \end{pmatrix}$$

$$A^t \cdot A = A \cdot A^t \implies A_1 \cdot A_1^t + A_2 \cdot A_2^t = A_1^t A_1$$

$$\implies tr(A_1 \cdot A_1^t + A_2 \cdot A_2^t) = tr(A_1^t A_1)$$

$$\implies tr(A_2 \cdot A_2^t) = 0$$

$$\implies A_2 = 0 \quad \text{(见之前习题)}$$

eg2. 设 $A \in M_n(\mathbb{R})$, $B \in M_m(\mathbb{R})$ 是非退化矩阵, C 是任意 $n \times m$ 矩阵. 利用矩阵分块乘法证明:

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

证: $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & E_m \end{pmatrix} = \begin{pmatrix} E_n & C \\ 0 & B \end{pmatrix}$, $\begin{pmatrix} E_n & C \\ 0 & B \end{pmatrix}\begin{pmatrix} E_n & -C \\ 0 & E_m \end{pmatrix} = \begin{pmatrix} E_n & 0 \\ 0 & B \end{pmatrix}$

$$\begin{pmatrix} E_n & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} E_n & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} E_n & 0 \\ 0 & E_m \end{pmatrix} = E_{n+m}$$

即 $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & E_m \end{pmatrix}\begin{pmatrix} E_n & -C \\ 0 & E_m \end{pmatrix}\begin{pmatrix} E_n & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} E_n & 0 \\ 0 & E_m \end{pmatrix}$

故 $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & E_m \end{pmatrix}\begin{pmatrix} E_n & -C \\ 0 & E_m \end{pmatrix}\begin{pmatrix} E_n & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$

二、多重线性函数

a. 定义

b. 特别地, 多重斜对称线性函数

$\Downarrow$

引出行列式的定义

$$\det: \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x_1, x_2, \cdots, x_n) \longmapsto \sum_{\sigma \in S_n} x_{1\sigma(1)} \cdots x_{n\sigma(n)}$$

hw 3. 证: $|A(t)| = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \Longrightarrow |A(t)|$ 可微

$$\begin{aligned}
\frac{d}{dt}|A(t)| &= \frac{d}{dt}\left(\sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}\right) \\
&= \sum_{\sigma \in S_n} \frac{d}{dt}\left(a_{1\sigma(1)} \cdots a_{n\sigma(n)}\right) \\
&= \sum_{\sigma \in S_n}\left(\sum_{i=1}^{n} a_{1\sigma(1)} \cdots \frac{d}{dt}a_{i\sigma(i)} \cdots a_{n\sigma(n)}\right) \\
&= \sum_{i=1}^{n}\left(\sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots \frac{d}{dt}a_{i\sigma(i)} \cdots a_{n\sigma(n)}\right) \\
&= \sum_{i=1}^{n} |A_i(t)|
\end{aligned}$$

$\square$

三、行列式的性质及计算

1) a、某行或某列公因子可以提出 (L1)、(L2)

b. 某行或某列的线性组合可以拆分 (L3)、(L4)

c、互换两行(列)位置变号 (S1)

d、某两行(列)相同 行列式值为0 (S2)

e、某两行(列)线性相关 (S3)

f、某一行(列)的倍式加到另一行值不变 (S4)

2) $\det(A) = \det(A^t)$

3) 余子式、代数余子式 按照某一行(列)展开

a、Vandermonde 行列式

b、高阶行列式

4) 分块矩阵的行列式

a. $\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B)$ 及推论.

b. $\det(AB) = \det(A)\det(B)$

hw1. (1)

$$|A| = \begin{vmatrix} -2 & 5 & -1 & 3 \\ 1 & -9 & 13 & 7 \\ 3 & -1 & 5 & -5 \\ 2 & 8 & -7 & -10 \end{vmatrix} = - \begin{vmatrix} 1 & -9 & 13 & 7 \\ -2 & 5 & -1 & 3 \\ 3 & -1 & 5 & -5 \\ 2 & 8 & -7 & -10 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & -9 & 13 & 7 \\ 0 & -13 & 25 & 17 \\ 0 & 26 & -34 & -26 \\ 0 & 26 & -33 & -24 \end{vmatrix} = - \begin{vmatrix} 1 & -9 & 13 & 7 \\ 0 & -13 & 25 & 17 \\ 0 & 0 & 16 & 8 \\ 0 & 0 & 17 & 10 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & -9 & 13 & 7 \\ 0 & -13 & 25 & 17 \\ 0 & 0 & 16 & 8 \\ 0 & 0 & 0 & \frac{3}{2} \end{vmatrix} = -1 \times (-13) \times 16 \times \frac{3}{2}$$

$$= 312$$

(2)

$$|B| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= -4 \times 1 \times (-2) \times (-2)$$

$$= -16$$

(3)

$$|C| = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 2 & 3 & 4 & \cdots & n \\ 3 & 3 & 3 & 4 & \cdots & n \\ 4 & 4 & 4 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n & n & n & n & \cdots & n \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$= (-1)^{n-1} \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \end{vmatrix} = (-1)^{n-1} n.$$

hw 2.

$$|A| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 3 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & n & 1 \\ 1 & 1 & 1 & \cdots & 1 & n+1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \end{vmatrix} = n!$$

hw 6.

$$|A| = \begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix} = \frac{1}{1000} \begin{vmatrix} 1798 & 7 & 9 & 8 \\ 2139 & 1 & 3 & 9 \\ 3255 & 2 & 5 & 5 \\ 4867 & 8 & 6 & 7 \end{vmatrix} = |B|$$

故 $|B| = 1000|A|$ $\quad 31 \mid |B| \quad 31 \nmid 1000 \Rightarrow 31 \mid |A|$.

若 p 是素数, $p \mid ab \Rightarrow p \mid a$ 或 $p \mid b$.

eg3. 设 X 是 $n\times k$ 矩阵而 Y 是 $k\times n$ 矩阵. 证明:

$$\det(E_n + XY) = \det(E_k + YX)$$

证: $\begin{pmatrix} E_k+YX & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k & Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n+XY \end{pmatrix}$

$$\begin{vmatrix} E_k+YX & 0 \\ X & E_n \end{vmatrix}\begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix} = \begin{vmatrix} E_k & Y \\ 0 & E_n \end{vmatrix}\begin{vmatrix} E_k & 0 \\ X & E_n+XY \end{vmatrix}$$

$$\Longrightarrow |E_k+YX||E_n|\,|E_k||E_n| = |E_k||E_n|\,|E_k||E_n+XY|$$

法二: $\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & E_n+XY \end{pmatrix}$

$$\Longrightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_n+XY|$$

$$\begin{pmatrix} E_k & -Y \\ 0 & E_n \end{pmatrix}\begin{pmatrix} E_k & Y \\ -X & E_n \end{pmatrix}\begin{pmatrix} E_k & 0 \\ X & E_n \end{pmatrix} = \begin{pmatrix} E_k+YX & 0 \\ 0 & E_n \end{pmatrix}$$

$$\Longrightarrow \begin{vmatrix} E_k & Y \\ -X & E_n \end{vmatrix} = |E_k+YX|.$$

$$\Longrightarrow |E_n+XY| = |E_k+YX|$$

eg4.

计算 $\Delta = \begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix}$

$$= \left| E_n + \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1 \cdots b_n) \right| = \left| E_1 + (b_1 \cdots b_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right|$$

$$= 1 + \sum_{i=1}^{n} a_i b_i$$

eg5. 证明:

$$B_n(s,t) = \begin{vmatrix} \binom{s}{t} & \binom{s}{t+1} & \cdots & \binom{s}{t+n-1} \\ \binom{s+1}{t} & \binom{s+1}{t+1} & \cdots & \binom{s+1}{t+n-1} \\ \vdots & \vdots & & \vdots \\ \binom{s+n-1}{t} & \binom{s+n-1}{t+1} & \cdots & \binom{s+n-1}{t+n-1} \end{vmatrix}$$

$$= \frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\cdots\binom{n}{n}}$$

证：不妨设 $s \geq t$.

第i行第j列元素为

$$\binom{s+i-1}{t+j-1} = \frac{(s+i-1)!}{(t+j-1)!\,(s+i-t-j)!}$$

$$= \frac{s+i-1}{t+j-1} \frac{(s+i-2)!}{(t+j-2)!\,(s+i-t-j)!}$$

$$= \frac{s+i-1}{t+j-1} \binom{s+i-2}{t+j-2}$$

第i行提出公因子 $s+i-1$

第j列提出公因子 $t+j-1$

$$B_n(s,t) = \frac{s(s+1)\cdots(s+n-1)}{t(t+1)\cdots(t+n-1)} \begin{vmatrix} \binom{s-1}{t-1} & \binom{s-1}{t} & \cdots & \binom{s-1}{t+n-2} \\ \binom{s}{t-1} & \binom{s}{t} & \cdots & \binom{s}{t+n-2} \\ \vdots & \vdots & & \vdots \\ \binom{s+n-2}{t-1} & \binom{s+n-2}{t} & \cdots & \binom{s+n-2}{t+n-2} \end{vmatrix}$$

$$\parallel$$

$$B_n(s-1,\,t-1)$$

$$\frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}} = \frac{\frac{(n+s-1)!}{n!(s-1)!}}{\frac{(n+t-1)!}{n!(t-1)!}} = \frac{s(s+1)\cdots(s+n-1)}{t(t+1)\cdots(t+n-1)}$$

故 $B_n(s,t) = \frac{\binom{n+s-1}{n}}{\binom{n+t-1}{n}} B_n(s-1,t-1) = \frac{\binom{n+s-1}{n}\binom{n+s-2}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}} B_n(s-2,t-2)$

$$= \frac{\binom{n+s-1}{n}\binom{n+s-2}{n}\binom{n+s-3}{n}\cdots\binom{n+s-t}{n}}{\binom{n+t-1}{n}\binom{n+t-2}{n}\binom{n+t-3}{n}\cdots\binom{n}{n}} \underbrace{B_n(s-t,0)}_{=1}$$