1、 计算下面矩阵乘法:

$$(1)\ \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^n;\quad (2)\ \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}^n;\quad (3)\ \begin{pmatrix}0&1&&\\&0&\ddots&\\&&\ddots&1\\&&&0\end{pmatrix}^n_{n\times n}.$$

(1) $\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^n = \begin{pmatrix}1&n&\frac{n(n-1)}{2}\\0&1&n\\0&0&1\end{pmatrix}$

$n=1$ ✓. $n=k$ 成立, $n=k+1$ 时.

$$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^k = \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^k = \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&k&\frac{k(k-1)}{2}\\0&1&k\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&k+1&\frac{k(k-1)}{2}+k\\0&1&k+1\\0&0&1\end{pmatrix} = \begin{pmatrix}1&k+1&\frac{k(k+1)}{2}\\0&1&k+1\\0&0&1\end{pmatrix}$$

(2) $\begin{pmatrix}\cos\phi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}^n = \begin{pmatrix}\cos n\varphi & -\sin n\varphi\\ \sin n\varphi & \cos n\varphi\end{pmatrix}$

$n=1$ ✓

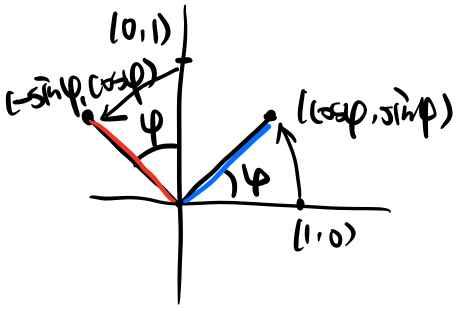
$n=k$ 成立

$n=k+1$ 时

$$\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos k\varphi & -\sin k\varphi\\ \sin k\varphi & \cos k\varphi\end{pmatrix}$$

$$= \begin{pmatrix}\cos\varphi\cos k\varphi - \sin\varphi\sin k\varphi & -\cos\varphi\sin k\varphi - \sin\varphi\cdot\cos k\varphi\\ \sin\varphi\cos k\varphi + \cos\varphi\sin k\varphi & -\sin\varphi\sin k\varphi + \cos\varphi\cos k\varphi\end{pmatrix}$$

$$= \begin{pmatrix}\cos(k+1)\varphi & -\sin(k+1)\varphi\\ \sin k\varphi & \cos k\varphi\end{pmatrix}$$



(3) $\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{n\times n}^{n} = 0$

2、设 A, B 为 $\mathbb{R}$ 上的 n 阶对称方阵 i.e. $A^t = A, B^t = B$, 其中 A^t 为 A 的转置. 证明: AB 是对称矩阵当且仅当 $AB = BA$.

解: $A = (a_{ij})$ $B = (b_{ij})$ $A \cdot B = (c_{ij}) = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

$A^t = (a'_{ij})$ $B^t = (b'_{ij})$ $(AB)^t = (c'_{ij})$

$a'_{ij} = a_{ji}$ $b'_{ij} = b_{ji}$ $c'_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} \cdot b_{ki}$

$c'_{ij} = \sum_{k=1}^{n} b'_{ik} \cdot a'_{kj} \Rightarrow (AB)^t = B^t \cdot A^t.$

AB 对称 $\Leftrightarrow B \cdot A = B^t \cdot A^t = (AB)^t = AB$ □

3、A, B, C 为 n 阶方阵, 若 $ABC = 0$, 证明:

$$\operatorname{rank}(A) + \operatorname{rank}(B) + \operatorname{rank}(C) \leq 2n,$$

并说明等号能否取到, 若能请举例.

解: $\operatorname{ran}(A) + r(B) - n \leq \operatorname{rank}(A \cdot B)$

$\operatorname{ran}(AB) + \operatorname{rank}(C) - n \leq \operatorname{rank}(ABC)$

$\operatorname{ran}(A) + \operatorname{ran}(B) \leq 2n + \operatorname{rank}(ABC) - \operatorname{rank}(C)$, $ABC = 0$

$\Rightarrow \operatorname{rank}(A) + \operatorname{rank}(B) + \operatorname{rank}(B) \leq 2n.$

若取 $A = B = E_n$, $C = 0$ 则取等号. □

4、 A, B 为 $\mathbb{R}$ 上的方阵，证明：

i) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

ii) $\operatorname{tr}(AA^t) \geq 0$, 且 $\operatorname{tr}(AA^t) = 0$ 当且仅当 $A = 0$;

iii) 若 $A = A^t, B = B^t$, 则

$$\operatorname{tr}\left((AB)^2\right) \leq \operatorname{tr}\left(A^2B^2\right).$$

Pf: (1) $A = (a_{ij})$, $B = (b_{ij})$ 为 n 阶方阵，则有 $A \cdot B = (c_{ij})$ $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

$B \cdot A = (d_{ij})$ $\quad d_{ij} = \sum_{k=1}^{n} b_{ik} \cdot a_{kj}$

$$\operatorname{tr}(A \cdot B) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} \cdot b_{ki}$$

$$\operatorname{tr}(B \cdot A) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik} \cdot a_{ki}$$

$$\Rightarrow \operatorname{tr}(AB) = \operatorname{tr}(BA).$$

(2) $\quad A = (a_{ij}) \quad A^t = (a'_{ij})$, $\quad a'_{ij} = a_{ji}$

$$\operatorname{tr}(A \cdot A^t) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} \cdot a'_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^2 \geq 0$$

$$\operatorname{tr}(A \cdot A^t) = 0 \Leftrightarrow a_{ik} = 0 \quad \forall i, k. \Leftrightarrow A = 0.$$

(3) 考虑 $C = AB - BA$ $\quad C^t = (AB - BA)^t = B^t A^t - A^t B^t = BA - AB$

则有 $\quad C \cdot C^t = (AB - BA)(BA - AB)$

$$= AB \cdot BA - AB \cdot AB - BA \cdot BA + B \cdot A \cdot A \cdot B$$

$$0 \underset{(2)}{\leq} \operatorname{tr}(C \cdot C^t) = \operatorname{tr}\left(AB^2A - (AB)^2 - B A B A + B A^2 B\right)$$

$$= \operatorname{tr}(AB^2A) - \operatorname{tr}((AB)^2) - \operatorname{tr}(BABA) + \operatorname{tr}(BA^2B)$$

$$= \operatorname{tr}(A^2B^2) - \operatorname{tr}((AB)^2) - \operatorname{tr}((AB)^2) + \operatorname{tr}(B^2A^2)$$

$$= 2\operatorname{tr}(A^2B^2) - 2\operatorname{tr}((AB)^2)$$

$$\Rightarrow \operatorname{tr}((AB)^2) \leq \operatorname{tr}(A^2B^2).$$

5、 (1) 证明秩为 r 的矩阵可以写成 r 个秩为 1 的矩阵之和, 但不能是小于 r 个秩为 1 矩阵之和.

(2) 设 A 为一个 $m \times n$ 阶矩阵, 秩为 1, 证明: 存在 C 为 $m \times 1$, D 为 $1 \times n$ 阶矩阵, 使得 $A = CD$.

(1)①设 A 为 $m\times n$ 阶矩阵. 不妨设其前 r 行线性无关. $\mathrm{rank}(A)=r$

其中 A_i 为 A 的第 i 行.

$$\Rightarrow A_i = b_{i1}A_1 + \cdots + b_{ir}A_r \quad i=1,\cdots,m,\ a_{ir} \text{ 唯一}.$$

取 $B_1 = \begin{pmatrix} b_{11}A_1 \\ b_{21}A_1 \\ \vdots \\ b_{m1}A_1 \end{pmatrix} \cdots\ B_r = \begin{pmatrix} b_{1r}A_r \\ b_{2r}A_r \\ \vdots \\ b_{mr}A_r \end{pmatrix}$ 事实上有 $b_{ii}=1$, $1\le i\le r$ 故而 $B_i \neq 0$. 且 B_i 秩为1.

则 $A = B_1 + \cdots + B_r$

②若 $A = C_1 + \cdots + C_s$ $\mathrm{rank}(C_i)=1$, $s<r$.

$$\mathrm{rank}(A) = \mathrm{rank}(C_1+\cdots+C_s) \le \mathrm{rank}(C_1)+\cdots+\mathrm{rank}(C_s) = s \ \lightning.$$

(2) 若 $\mathrm{rank}(A)=1$. 则其一定~~开~~形如 其中 $R=(r_1,\cdots,r_n)$

$$A = \begin{pmatrix} x_1\cdot R \\ \vdots \\ x_n \cdot R \end{pmatrix}$$ 且 $x_1,\cdots,x_n$ 不全为0.

$$A = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot R \qquad 取\ C = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad D = R \qquad \square$$

另一证明:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\varphi_A"] \arrow[rr, bend left, "\varphi_D"] \arrow[dr, "\varphi_A"'] & \overset{\dim=1}{\mathrm{Im}\,\varphi_A} \arrow[r, "h"] \arrow[d, "i"] & \mathbb{R} \arrow[dl, "\varphi_C"] \\
& \mathbb{R}^m &
\end{tikzcd}$$

φ_A 为 A 对应的线性映射

记 $i: \mathrm{Im}\,\varphi_A \longrightarrow \mathbb{R}^m$ 为含入映射

$h: \mathrm{Im}\,\varphi_A \longrightarrow \mathbb{R}$ 为线性双射

$$\varphi_A = \varphi_C \circ \varphi_D = \varphi_{C\cdot D} \qquad A = C\cdot D$$