1. 运用二项式定理, 对 n 做归纳证明, 若 p 是素数, 则 $n^p - n$ 对任意 $n \in \mathbb{Z}$ 可以被 p 整除.

pf: $n=0, 1$ ✓.

若对 $n=k$ 成立.

$n=k+1$ 时.
$$(k+1)^p-(k+1)$$
$$=\sum_{i=0}^{p} C_p^i k^i - k - 1$$
$$=\sum_{i=1}^{p-1} C_p^i k^p + k^p - k$$

$p \mid C_p^i = \frac{p!}{i(p-i)!}$ 由归纳 $p \mid k^p - k$

$\Rightarrow p \mid (k+1)^p-(k+1)$

□.

$n<0$ 时
$$n^p - n = (-1)^p(-n)^p + (-n)$$
$$= (-1)^p\left((-n)^n + (-1)^p(-n)\right)$$

$p=2$ ✓.

$p>2$ $(-1)^p=-1$

由 $n\geq 0$ 情形得知.

2. 在 $\mathbb{R}^4$ 中, 判断向量 β 能否由向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性表出; 若能, 则写出它的一种表出方式.

(1)
$$\alpha_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \\ -5 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 7 \\ -3 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -4 \\ 1 \\ -2 \\ 6 \end{pmatrix}, \beta = \begin{pmatrix} 8 \\ 3 \\ -1 \\ -25 \end{pmatrix},$$

(2)
$$\alpha_1 = \begin{pmatrix} -2 \\ 7 \\ 1 \\ 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 3 \\ -5 \\ 0 \\ -2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -5 \\ -6 \\ 3 \\ -1 \end{pmatrix}, \beta = \begin{pmatrix} -8 \\ -3 \\ 7 \\ -10 \end{pmatrix}.$$

(1) $\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$

i.e.
$$\left(\begin{array}{ccc|c} -1 & 2 & -4 & 8 \\ 3 & 0 & 1 & 3 \\ 0 & 7 & -2 & -1 \\ -5 & -3 & 6 & -25 \end{array}\right) \to \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 6 & -11 & 27 \\ 0 & 7 & -2 & -1 \\ 0 & -13 & 26 & -65 \end{pmatrix} \to$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -2 & 5 \\ 0 & 6 & -11 & 27 \\ 0 & 7 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 12 & -36 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{有解.}$$

$x_3 = -3 \quad x_2 = -1 \quad x_1 = 2$

$\beta = 2\alpha_1 - \alpha_2 - 3\alpha_3$

(2) $\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$

$$\left(\begin{array}{ccc|c} -2 & 3 & -5 & -8 \\ 7 & -5 & -6 & -3 \\ 1 & 0 & 3 & 7 \\ 3 & -2 & -1 & -10 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & 3 & 1 & 6 \\ 0 & -5 & -27 & -52 \\ 0 & -2 & -10 & -31 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & 3 & 1 & 6 \\ 0 & 1 & -25 & -40 \\ 0 & 1 & -9 & -25 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & -9 & -25 \\ 0 & 0 & 28 & 81 \\ 0 & 0 & -16 & -15 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & -9 & -25 \\ 0 & 0 & 28 & 81 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

无解，不能表示

3. 在 $\mathbb{R}^4$ 中, 判断下列向量组是线性相关还是线性无关. 如果线性相关, 试找出其中一个向量, 使得它可以由其余向量线性表出, 并且写出它的一种表出式.

(1)

$$\alpha_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 3 \end{pmatrix}.$$

(2)

$$\alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 4 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 6 \end{pmatrix}.$$

(1) $$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 5 & 0 \\ -4 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 5 & -4 \\ 0 & 2 & 11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 31 \\ 0 & 0 & 25 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{线性无关.}$$

(2) $\begin{pmatrix} -2 & 1 & 3 & 2 \\ 1 & -3 & 0 & -2 \\ 0 & 2 & 2 & 4 \\ 3 & 4 & -1 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 0 & -2 \\ 0 & -5 & 3 & -2 \\ 0 & 2 & 2 & 4 \\ 0 & 13 & -1 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & -14 & -14 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & -3 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ 线性相关

$x_3 = -x_4$

$x_2 = -x_4$

$x_1 = -x_4$

令 $x_4 = 1$.

$\alpha_1 = -\alpha_2 - \alpha_3 + \alpha_4$

4. 在数域 K 上的线性空间 V 中, 设向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性无关,

(1) 判断向量组 $5\alpha_1 + 2\alpha_2, 7\alpha_2 + 5\alpha_3, 7\alpha_1 - 2\alpha_3$ 是否线性无关,

(2) 向量组 $a_1\alpha_1 + b_2\alpha_2, a_2\alpha_2 + b_3\alpha_3, \overset{b_1}{\cancel{a_3}}\alpha_1 + \overset{a_3}{\cancel{b_1}}\alpha_3$ 线性无关的充分必要条件是: $a_1a_2a_3 \neq -b_1b_2b_3$.

(2) $x_1(a_1\alpha_1 + b_2\alpha_2) + x_2(a_2\alpha_2 + b_3\alpha_3) + x_3(a_3\alpha_1 \; b_1\alpha_3) = 0$

$\begin{cases} a_1x_1 + b_1x_3 = 0 \\ b_2x_1 + a_2x_2 = 0 \\ b_3x_2 + a_3x_3 = 0 \end{cases}$ $A = \begin{bmatrix} a_1 & 0 & b_1 \\ b_2 & a_2 & 0 \\ 0 & b_3 & a_3 \end{bmatrix}$ $\det(A) = a_1 \begin{vmatrix} a_2 & \\ b_3 & a_3 \end{vmatrix} + b_1 \begin{vmatrix} b_2 & a_2 \\ 0 & b_3 \end{vmatrix}$

$= a_1a_2a_3 + b_1b_2b_3$

(1) 由(2) 取 $a_1 = 5 \quad a_2 = 7 \quad a_3 = -2$

$b_1 = 7 \quad b_2 = 2 \quad b_3 = 5$.

$a_1a_2a_3 + b_1b_2b_3 = 0$.

$\Leftrightarrow$ 线性相关. $\square$

5. 设 $\beta, \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}^n$ 且 $\alpha_1, \alpha_2, \cdots, \alpha_k$ 线性无关. 再设

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k,$$

其中 $a_1, a_2, \cdots, a_k \in \mathbb{R}$, 且 $a_1 \neq 0$. 证明:$\beta, \cancel{\alpha_1}, \alpha_2, \cdots, \alpha_k$ 线性无关.

若 $x_1\beta + x_2\alpha_2 + \cdots + x_k\alpha_k = 0$

$$a_1x_1\alpha_1 + (x_2 + x_1a_2)\alpha_2 + \cdots + (x_k + x_1a_k)\alpha_k = 0$$

$$\Rightarrow a_1x_1 = 0 \quad \Rightarrow x_1 = 0$$

$$\Rightarrow x_2\alpha_2 + \cdots + x_k\alpha_k = 0$$

$$\Rightarrow x_2 = \cdots = x_k = 0$$

$$\Rightarrow x_1 = \cdots = x_k = 0. \quad \square$$

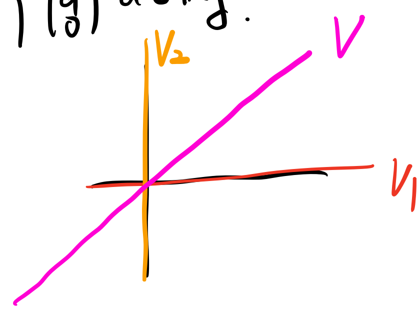
6. 设 V, V_1, V_2 是 $\mathbb{R}^n$ 的子空间. 举例说明

$$V \cap (V_1 + V_2) = V \cap V_1 + V \cap V_2$$

一般不成立.

$\mathbb{R}^3$中 取 $V_1 = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$ $\quad V_2 = \left\{ \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} a \in \mathbb{R} \right\}$.

$$V = \left\{ \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} \in \mathbb{R} \right\}.$$



则 $V \cap (V_1 + V_2) = V$

$V \cap V_1 = \{0\}$

$V \cap V_2 = \{0\} \Rightarrow (V \cap V_1) + (V \cap V_2) = 0 \neq V = V \cap (V_1 + V_2).$

行列式与线性方程组的解.

① 3×3 行列式性质: $A=(a_{ij})\in M_{3\times 3}$.

回顾: $\det(A)$

$$= \sum_{\sigma\in S_3} \epsilon_\sigma\, a_{1\sigma(1)}\cdots a_{3\sigma(3)}$$

① 交换两行 行列式值异号. eg: 交换 1,2行.

$$A' = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{11} & a_{11} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A' = \sum_{\sigma\in S_n} \epsilon_\sigma\, a_{2\sigma(1)} a_{1\sigma(2)} a_{3\sigma(3)}$$

取 $\tau=(1,2)$ 则 $(\sigma\tau)(1)=\sigma(2)$

$(\sigma\tau)(2)=\sigma(1)$

$(\sigma\tau)(3)=\sigma(3)$

$$\det A' = \sum_{\sigma\in S_n} \epsilon_\sigma\, \underbrace{a_{1\sigma\tau(1)} a_{2\sigma\tau(2)} a_{3\sigma\tau(3)}}$$

$$= \sum_{\sigma\in S_n} -\epsilon_{\sigma\tau}\,\underline{\qquad\qquad}$$

$$(\rho=\sigma\tau) = -\sum_{\rho\in S_n} \epsilon_\rho\, a_{1\rho(1)} a_{2\rho(2)} a_{3\rho(3)}$$

$$= -\det(A).$$

② 某行乘以非0 k倍 行列式值乘k倍.

③ 某行乘以k倍 加到另一行, 行列式值不变.

④ $\det(A)=\det(A^t)$ (第一次作业)

⑤ "初等列变换"变化同上.

⑥ $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33}\end{bmatrix} = a_{11}a_{12}a_{13}$.

⑦ 线性方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{cases}$

有非零解 $\Leftrightarrow \det(A)=0$

proof: 有非零解 $\Leftrightarrow$ A经初等行变换后化为的阶梯形矩阵B 对角线有0

$\Updownarrow$

$\det(A)=0 \Leftrightarrow \det(B)=0$ □

② 行列式定义的等价性.

Recall: $A=(a_{ij})_{n\times n}$

$$\det(A)=\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j})$$

$$d(A)=\sum_{\sigma\in S_n}\epsilon_\sigma\, a_{1\sigma(1)}\cdot\cdots\cdot a_{n\sigma(n)}$$

命题: $\det(A)=d(A)$.

证明: 对 n 归纳.

$n=1,2,3.$ ✓

若对 $n-1$ 成立 n 时

$$\det(A)=\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j})$$

归纳 + 置换符号可由逆序数计算

$$=\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\sum_{(j_2\cdots j_n)}(-1)^{N(j_2\cdots j_n)}a_{2j_2}\cdots a_{2j_n}\qquad \{j_2,\dots,j_n\}=\{1,\dots,n\}\setminus\{j\}$$

$$=\sum_{j=1}^{n}\sum_{(j_2\cdots j_n)}(-1)^{1+j}\cdot(-1)^{N(j_2\cdots j_n)}a_{1j}a_{2j_2}\cdots a_{nj_n}$$

$$N(j\,j_2\cdots j_n)=N(j_2\cdots j_n)+j-1$$

$$=\sum_{j=1}^{n}\sum_{(j_2\cdots j_n)}(-1)^{1+j}\cdot(-1)^{N(j\,j_2\cdots j_n)+j-1}a_{1j}a_{2j_2}\cdots a_{nj_n}$$

$$=\sum_{j=1}^{n}\sum_{(j_2\cdots j_n)}(-1)^{N(j\,j_2\cdots j_n)}a_{1j}a_{2j_2}\cdots a_{nj_n}$$

$$=\sum_{j=1}^{n}\sum_{\substack{\sigma\in S_n\\ \sigma(1)=j}}\epsilon_\sigma\, a_{1\sigma(1)}\cdots a_{n\sigma(n)}$$

$$=d(A).\qquad\square$$

③ n阶行列式性质.

- 关于行变换同3阶
- 仍有 $\det(A^t)=\det(A)$ $A^t=(a'_{ij})_{n\times n}$ $a'_{ij}=a_{ji}$

$$\det(A^t)=\sum_{\sigma\in S_n}\epsilon_\sigma\, a'_{1\sigma(1)}\cdots a'_{n\sigma(n)}$$

$$=\sum_{\sigma\in S_n}\epsilon_\sigma\, a_{\sigma(1)1}\cdots a_{\sigma(n)n}$$

$\sigma(k)=i \Rightarrow k=\sigma^{-1}(i)$

$$=\sum_{\sigma\in S_n}\epsilon_\sigma\, a_{1\sigma^{-1}(1)}\cdots a_{n\sigma^{-1}(n)}$$

$$=\sum_{\sigma\in S_n}\epsilon_{\sigma^{-1}}\, a_{1\sigma^{-1}(1)}\cdots a_{n\sigma^{-1}(n)}$$

双射
$S_n\longrightarrow S_n$
$\sigma\longmapsto\sigma^{-1}$

$$=\sum_{\sigma^{-1}\in S_n}\epsilon_{\sigma^{-1}}\, a_{1\sigma^{-1}(1)}\cdots a_{n\sigma^{-1}(n)}.$$

$$=\det(A)$$

- 关于列变换性质可由行变换得出. (A 中的行 为 A^t中列) $\det(A^t)=\det(A)$

- 方程

$$\begin{cases} a_{11}x_1+\cdots+a_{1n}x_n=0\\ \quad\vdots \\ a_{n1}x_1+\cdots+a_{nn}x_n=0\end{cases}$$

有非零解 $\Longleftrightarrow \det(A)=0.$