1、 $f=x^3-3x+2$， $f'=3x^2-3$

$$\gcd(x^3-3x+2,3x^2-3)=x-1$$

$\Rightarrow$ f的无平方部分为 $\dfrac{f}{\gcd(f,f')}=x^2+x-2=(x-1)(x+2)$

2. 设 $f=\sum_I a_I x^I$ $\quad I=(i_1,\cdots,i_n)$

$x^I=x_1^{i_1}\cdot\cdots\cdot x_n^{i_n}$

$$f(\varepsilon_1,\cdots,\varepsilon_n)=\sum_I a_I\cdot\varepsilon_1^{i_1}\cdot\cdots\cdot\varepsilon_n^{i_n}$$

已知：① 两对称多项式积为对称多项式.

② 两对称多项式和为对称多项式.

$\Rightarrow$ $f(\varepsilon_1,\cdots,\varepsilon_n)$为对称多项式.

3. $|z|^2=z\cdot\bar{z}$.

$$\begin{aligned}|z_1+z_2|^2+|z_1-z_2|^2&=(z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)\\&=z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2\\&\quad+z_1\bar{z}_1-z_1\bar{z}_2-z_2\bar{z}_1+z_2\bar{z}_2\\&=2z_1\bar{z}_1+2z_2\bar{z}_2\\&=2|z_1|^2+2|z_2|^2\end{aligned}$$

4. $x=\frac{z+\bar z}{2}$, $y=\frac{z-\bar z}{2i}$

(i) $Ax+By+C=0 \Leftrightarrow A\frac{z+\bar z}{2}+B\frac{z-\bar z}{2i}+C=0$

$$\Rightarrow \frac{A-iB}{2}z+\frac{A+iB}{2}\bar z+C=0.$$

令 $\alpha=\frac{A+iB}{2}$, $c=C$. 则有 $\alpha\bar z+\bar\alpha z+c=0$.

(ii) $(x-a)^2+(y-b)^2=r^2 \Leftrightarrow |z-(a+ib)|^2=r^2$

$$\Leftrightarrow (z-(a+ib))(\bar z-(a-ib))=r^2$$

$$\Leftrightarrow z\cdot\bar z-(a+ib)\bar z-(a-ib)z+(a-ib)(a+ib)-r^2=0$$

令 $A=1$ $\beta=-(a+ib)$, $c=(a-ib)(a+ib)-r^2$
$=a^2+b^2-r^2$

$|\beta|^2=a^2+b^2>1\cdot(a^2+b^2-r^2)$.

注：① 此题逆命题也成立.
② "某种意义上"直线可视为特殊的圆.

5. $\xi_i = \xi_1^{i}$, $\qquad \xi_i^{-1} = (\xi_1^{\,i})^{-1} = \xi_1^{-i} = \xi_1^{\,n-i}$

$$\xi_i^{-j} = \xi_1^{(n-i)\cdot j}$$

题中矩阵记为

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \varepsilon_0 & & \varepsilon_{n-1} \\ \vdots & & \vdots \\ \varepsilon_0^{n-1} & & \varepsilon_{n-1}^{n-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & & 1 \\ \varepsilon_0^{-1} & & \varepsilon_{n-1}^{-1} \\ \vdots & & \vdots \\ \varepsilon_0^{-(n-1)} & & \varepsilon_{n-1}^{-(n-1)} \end{pmatrix}$$

令 $c_{ij} = (\varepsilon_0^{i-1}, \cdots, \varepsilon_{n-1}^{i-1}) \cdot \begin{pmatrix} 1 \\ \varepsilon_{j-1}^{-1} \\ \vdots \\ \varepsilon_{j-1}^{-(n-1)} \end{pmatrix} \Rightarrow A\cdot B = \frac{1}{n}(c_{ij})$.

$$c_{ij} = \sum_{k=0}^{n-1} \varepsilon_k^{i-1} \cdot \varepsilon_{j-1}^{-k} = \sum_{k=0}^{n-1} \varepsilon_1^{k(i-1)} \cdot \varepsilon_1^{(n-j-1))\cdot k}$$

$$= \sum_{k=0}^{n-1} \varepsilon_1^{k(i-j)}$$

$$= \sum_{k=0}^{n-1} \left(\varepsilon_1^{(i-j)}\right)^k \qquad \begin{array}{l} 1 \le i \le n \\ 1 \le j \le n \\ 1-n \le i-j \le n-1 \end{array}$$

$\Rightarrow \quad c_{ij} = n \quad$ if $\; i=j$.

if $\; i \ne j$, $\varepsilon_1^{i-j} \ne 1$, $\; c_{ij} = \dfrac{1-\varepsilon_1^{(i-j)\cdot}}{1-\varepsilon_1^{i-j}} = 0$. □

6、 字典序：查字典时 abandon 在 about 前面

由于 $a=a$

$b=b$

$a>o$

首项.

(1) 设 $f = \underline{a x_1^{i_1}\cdot \cdots \cdot x_n^{i_n}} + \cdots (+\lambda x_1^{t_1}\cdots x_n^{t_n})$

$g = \underline{b x_1^{j_1}\cdot \cdots \cdot x_n^{j_n}} + \cdots (+\mu x_1^{l_1}\cdots x_n^{l_n})$

首项.

则 $f\cdot g = a\cdot b\, x_1^{i_1+j_1}\cdot \cdots \cdot x_n^{i_n+j_n}$

$+ \quad (\lambda\cdot \mu \cdot x_1^{t_1+l_1}\cdot \cdots x_n^{t_n+l_n})$

$$(i_s+j_s)-(t_s+l_s) = (i_s-t_s)+(j_s-l_s)$$

令 $k=\min\{s \mid i_s\neq t_s,\ j_s\neq l_s\}$

则 $i_k+j_k > t_k+l_k$.

$\Rightarrow$ $f\cdot g$ 首项为 $a\cdot b x_1^{i_1+j_1}\cdot \cdots x_n^{i_n+j_n}$

(ii) ε_1的首项为 x_1

ε_2的首项为 $x_1 x_2$

$\vdots$

ε_n的首项为 $x_1 \cdots\cdots x_n$

由(i) $a\,\varepsilon_1^{i_1} \cdots\cdots \varepsilon_n^{i_n}$ 首项为

$$a \cdot x_1^{i_1} \cdot (x_1 x_2)^{i_2} \cdot \cdots\cdots (x_1 \cdots x_n)^{i_n}$$
$$= a\, x_1^{i_1+i_2+\cdots+i_n} \cdot x_2^{i_2+\cdots+i_n} \cdots \cdot x_n^{i_n}$$

(iii) $f = x_1^{i_1} \cdots x_n^{i_n} \qquad g = x_1^{j_1} \cdots\cdots x_n^{j_n}$

$f \neq g \Leftrightarrow (i_1, \cdots, i_n) \neq (j_1, \cdots, j_n)$

$f_\sigma = x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} \qquad g_\sigma = x_{\sigma(1)}^{j_1} \cdots\cdots x^{j_n}$

$f_\sigma \neq g_\sigma \Leftrightarrow (i_1, \cdots, i_n) \neq (j_1, \cdots, j_n)$

(iv) 若 f 为对称多项式．设其首项为 $a x_1^{i_1} \cdots x_n^{i_n}$

若 $i_1 \geqslant \cdots \geqslant i_n$ 不成立

设 $i_1 \geqslant \cdots \geqslant i_{k-1} < i_k$ $\quad 2 \leq k \leq n$

令 $\sigma = (k-1, k)$ 为对换．

则 $f = a_1 x^{i_1} \cdots x_n^{i_n} + \cdots$

$$f = f_\sigma = a_1 x_1^{i_1} \cdots x_{k-2}^{i_{k-2}} \cdot x_{k-1}^{i_k} \cdot x_k^{i_{k-1}} \cdot x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}$$
$$+ \cdots$$

由(iii) 在 σ 作用下，单项式不会抵消．

$\Rightarrow a_1 x_1^{i_1} \cdots x_{k-2}^{i_{k-2}} \cdot x_{k-1}^{i_k} \cdot x_k^{i_{k-1}} \cdot x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}$

为 f 的单项式

但 $(i_1, \cdots, i_n) < (i_1, \cdots, i_{k-2}, i_k, i_{k-1}, i_{k+1}, \cdots, i_n)$

与 $a x_1^{i_1} \cdots x_n^{i_n}$ 为首项矛盾． □

(v) f 首项为 $a x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}$.

由(iv)，$i_1 \geq \cdots \geq i_n$

令 $g = a_0 \sigma_1^{i_1 - i_2} \cdot \sigma_2^{i_2 - i_3} \cdot \cdots \cdot \sigma_{n-1}^{i_{n-1} - i_n} \sigma_n^{i_n}$

g 也为对称多项式，由(ii)，g 的首项为

$a_0 \cdot x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}$

$\Rightarrow$ f 与 g 的首项相同.

$f_1 = f - g$，由于 f 与 g 的首项被消去，

f_1 的首项按字典排序应位于 f 的首项之后.

$a_1 x_1^{i_1} \cdots x_{k-2}^{i_{k-2}} \cdot x_{k-1}^{i_k} \cdot x_k^{i_{k-1}} \cdot x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}$

设 f_1 的首项为 $a_1 x_1^{i_{11}} \cdot \cdots \cdot x_n^{i_{1n}}$

由(iv)知，f_1 的首项需满足 $i_{11} \geq \cdots \geq i_{1n}$

重复此操作，我们得到一列对称多项式

$f, f_1, f_2, \cdots, f_k, \cdots$

设 f_k 的首项为 $a_i \cdot x_1^{i_{k1}} \cdot \cdots x_n^{i_{kn}}$

则有 $(i_1, \cdots, i_n) > (i_{11}, \cdots, i_{1n}) > (i_{21}, \cdots, i_{2n}) > \cdots >$

$(i_{k1}, \cdots, i_{kn}) > \cdots$

满足：$i_{k1} \geq i_{k2} \geq \cdots \geq i_{kn}$

则 $\{(i_1,\cdots,i_n)\}\subset\{(a_1,\cdots,a_n)\mid a_i\in[0,i_1]\cap\mathbb{Z}\}$.

故亦有限.

$\Rightarrow$ 此操作有限步终止.

$\Rightarrow \exists\, \varphi\in R$ s.t. $\varphi(\varepsilon_1,\cdots,\varepsilon_n)=f$.

(vi) 若 $\varphi(\varepsilon_1,\cdots,\varepsilon_n)=0$，欲证 $\varphi=0$

~~如若~~不然，取 φ 中任意两不同单项式，

$a x_1^{i_1}\cdots x_n^{i_n}$，　$b x_1^{j_1}\cdots x_n^{j_n}$

由(ii)知，

$a\varepsilon_1^{i_1}\cdots\varepsilon_n^{i_n}=a\cdot x_1^{i_1+\cdots+i_n}\cdots x_n^{i_n}+\cdots$

$b\varepsilon_1^{j_1}\cdots\varepsilon_n^{j_n}=b\cdot x_1^{j_1+\cdots+j_n}\cdots x_n^{j_n}+\cdots$

首项不同，做加减运算不能抵消.

$\Rightarrow \varphi(\varepsilon_1,\cdots,\varepsilon_n)\neq 0$ 矛盾. $\square$