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μ-Bases and singularities of rational planar curves

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ABSTRACT

We provide a technique to detect the singularities of rational planar curves and to compute the correct order of each singularity including the infinitely near singularities without resorting to blow ups. Our approach employs the given parametrization of the curve and uses a μ-basis for the parametrization to construct two planar algebraic curves whose intersection points correspond to the parameters of the singularities including infinitely near singularities with proper multiplicity. This approach extends Abhyankar's method of t-resultants from planar polynomial curves to rational planar curves. We also derive the classical result that for a rational planar curve of degree n the sum of all the singularities with proper multiplicity is $(n-1)(n-2)/2$. Examples are provided to flesh out our results.

1. Introduction

Singularities are the most consequential points on rational planar curves. These critical points allow us to determine the geometry and topology of the curve, and to develop robust rendering algorithms based on the locations and types of the singularities. Over the years, a great deal of research has been devoted to the study of these singularities; see, for example, Chen and Sederberg (2002), Coolidge (1931), Fulton (1989), Hilton (1920), Perez-Diaz (2007), and Walker (1950).

While singularities on rational planar curves are not so difficult to detect, the correct order of each singularity, including the infinitely near singularities, is not so easy to compute. Often blow ups are required to resolve a singularity. The purpose of this paper is to detect the singularities of rational planar curves and to compute the correct order of each singularity including the infinitely near singularities without resorting to blow ups.

The technique we shall employ here uses the method of μ-bases. A μ-basis is a special basis for the syzygy module for the parametrization of a rational planar curve. Fortuitously, μ-bases are easy to compute. Using μ-bases, we shall reduce the problem of computing the singularities on a rational planar curve to the problem of computing the intersection points on two related planar algebraic curves. The parameters of each singularity correspond to the intersection points of these algebraic curves, and the multiplicity of the parameters, including all the infinitely near singularities, corresponds exactly to the multiplicity of the corresponding intersection points. Thus using μ-bases, we shall find all the singularities with their correct multiplicities without resorting to blow ups. This approach extends Abhyankar's method (Abhyankar, 1990) of t-resultants from planar polynomial curves to rational planar curves.

Nevertheless, the main contribution of this paper is not the development of better computational tools nor the proof of novel theoretical results, but rather the insights provided by the novel way in which these issues are addressed. In a sequel to this paper –

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see Jia and Goldman (2009) – we shall show how to apply our approach to prove an outstanding conjecture of Chen, Wang and Liu using the Smith form of the Bezout matrix of a μ-basis to compute both the singular points and the infinitely near singular points on a rational planar curve – see Chen et al. (2008). So far no other approach to the analysis of singularities has proved powerful enough to establish this conjecture. Thus the main purpose of this paper is insight not computation.

We begin in Section 2 by characterizing singularities and infinitely near singularities for rational planar curves. We also discuss the method of blow ups for rational planar curves. In Section 3 we review μ-bases and their properties. Section 4 is devoted to stating and proving our main result: using μ-bases to construct two planar algebraic curves whose intersection points correspond to the parameters of the singularities with proper multiplicity of a given rational planar curve. Based on the results of Section 4, we provide, in Section 5, two examples to flesh out these theorems. We close in Section 6 with a brief summary of our work and a few follow up problems for future research.

2. Singularities, infinitely near points and blow ups

Let $k[s,u]$ be the set of homogeneous polynomials in the homogeneous parameter $s:u$ with coefficients in the algebraically closed field k of characteristic zero. A degree n rational planar curve is usually written in homogeneous form as

$$\mathbf{P}(s,u) = \big(a(s,u), b(s,u), c(s,u)\big), \tag{1}$$

where $a(s,u)$, $b(s,u)$, $c(s,u)$ are degree n homogeneous polynomials in $k[s,u]$. To avoid the degenerate case where $\mathbf{P}(s,u)$ parameterizes a line, we shall assume that the three homogeneous polynomials $a(s,u)$, $b(s,u)$, $c(s,u)$ are relatively prime and linearly independent. Moreover, throughout this paper we will assume for simplicity that the parametrization $\mathbf{P}(s,u)$ is generically one-to-one.

2.1. Singularities

Every degree n rational planar curve $\mathbf{P}(s,u)$ as in Eq. (1) has an implicit polynomial representation. That is, up to a constant multiple, there is a unique irreducible homogeneous polynomial $f(x,y,w)$ of degree n such that

$$f\big(a(s,u), b(s,u), c(s,u)\big) \equiv 0.$$

Singularities are usually defined in terms of the vanishing of the partial derivatives of $f(x,y,w)$.

Definition 2.1. A point $\mathbf{Q}_0 = (x_0 : y_0 : w_0) \in \mathbb{P}_k^2$ is a singularity of multiplicity r on the algebraic curve $f(x,y,w) = 0$ if and only if

$$\frac{\partial^{r-1} f}{\partial x^i \partial y^j \partial w^k}(x_0, y_0, w_0) = 0, \quad i + j + k = r - 1, \tag{2}$$

and at least one rth partial derivative of f at $\mathbf{Q}_0$ does not vanish.

Definition 2.1 is a classical definition for the multiplicity of singularities of algebraic curves. But for rational planar curves, we have the following alternative characterization of the multiplicity of a singularity in terms of the number of parameter values corresponding to the singular point.

Definition 2.2. Let $\mathbf{Q}_0 = (x_0 : y_0 : w_0)$ be a point on the rational planar curve $\mathbf{P}(s,u) = (a(s,u), b(s,u), c(s,u))$. We say that there are exactly r parameters corresponding to $\mathbf{Q}_0$ if and only if the equation $\mathbf{P}(s,u) = \lambda \mathbf{Q}_0$ has exactly r solutions counting multiplicity, where λ is a nonzero constant.

Remark 2.1. Let $\mathbf{Q}_0 = (x_0 : y_0 : w_0)$ be a point on the rational planar curve $\mathbf{P}(s,u) = (a(s,u), b(s,u), c(s,u))$. Without loss of generality we can assume that $w_0 \neq 0$. Let $h(s,u) = \gcd(w_0 a(s,u) - c(s,u)x_0, w_0 b(s,u) - c(s,u)y_0)$. Then by Definition 2.2 there are exactly r parameters including multiplicity corresponding to the point $\mathbf{Q}_0$ if and only if $\deg(h) = r$. Moreover, the multiplicity of a parameter (s_0, u_0) corresponding to the singularity $\mathbf{Q}_0$ is the multiplicity of (s_0, u_0) as a root of the polynomial $h(s,u)$.

Proposition 2.1. *A point $\mathbf{Q}_0 = (x_0 : y_0 : w_0)$ is a singularity of multiplicity r on the rational planar curve $\mathbf{P}(s,u)$ if and only if there are exactly r parameters (s_i, u_i) including multiplicities corresponding to the point $\mathbf{Q}_0$.*

Proof. It is enough to show that if there are exactly r parameters including multiplicities corresponding to a point $\mathbf{Q}_0$ on a rational planar curve $\mathbf{P}(s,u)$, then the point $\mathbf{Q}_0$ is a singularity of order r. Without loss of generality we can assume that the singularity $\mathbf{Q}_0 = (0:0:1)$. Then the rational planar curve $\mathbf{P}(s,u)$ has the following parameterization:

$$\mathbf{P}(s,u) = \big(a(s,u)h(s,u), b(s,u)h(s,u), c(s,u)\big), \tag{3}$$

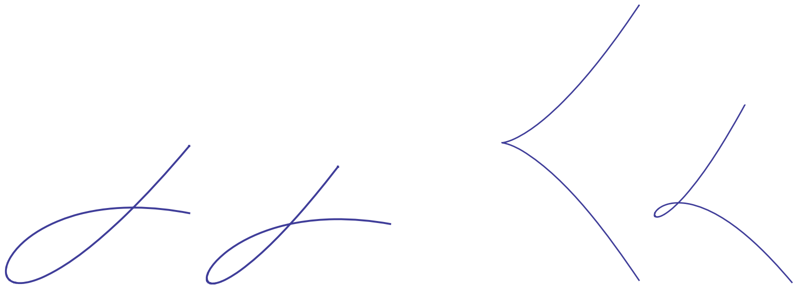


Fig. 1. Singularities without any infinitely near singular points.

where $\gcd(h,c)=1$, $\gcd(a,b)=1$, $\deg(h)=r$, and the roots of the polynomial $h(s,u)$ give all the parameters corresponding to the singular point $\mathbf{Q}_0$ on the rational planar curve $\mathbf{P}(s,u)$.

Intersect the parametric curve $\mathbf{P}(s,u)$ with a generic line $y=mx$, $m\neq 0$. By Bezout's Theorem we get $n=\deg(\mathbf{P})$ intersections which are roots of the polynomial

$$b(s,u)h(s,u)-ma(s,u)h(s,u)=h(s,u)\big(b(s,u)-ma(s,u)\big). \tag{4}$$

We can choose the slope m of the line $y=mx$ so that the roots of the degree $n-r$ polynomial $b(s,u)-ma(s,u)$ correspond to $n-r$ distinct points $\mathbf{Q}_i$, $i=1,\dots,n-r$, on the curve $\mathbf{P}(s,u)$. Hence the line intersects the curve $\mathbf{P}(s,u)$ at the points $\mathbf{Q}_i$, $i=0,1,\dots,n-r$.

On the other hand, let $F(x,y,w)$ be the implicit equation of the rational planar curve $\mathbf{P}(s,u)$. Suppose that the lowest degree term of $F(x,y,1)$ is of degree k. Then we can write $F(x,y,1)$ in the following form

$$F(x,y,1)=f_k(x,y)+f_{k+1}(x,y)+\cdots+f_n(x,y), \tag{5}$$

where $f_i(x,y)$ is a homogeneous polynomial of degree i in x,y. Intersecting the implicit curve $F(x,y,1)=0$ with the same line $y=mx$, we get a univariate polynomial

$$F(x,mx,1)=x^k\big(f_k(1,m)+xf_{k+1}(1,m)+\cdots+x^{n-k}f_n(1,m)\big). \tag{6}$$

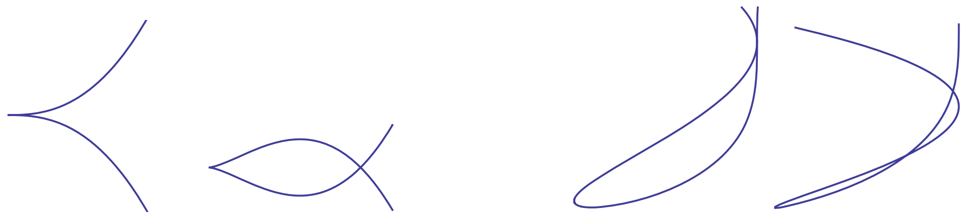
Since $F(x,y,1)=0$ and $\mathbf{P}(s,u)$ represent the same planar curve and $y=mx$ is a generic line, the x-coordinates of the $n-r$ distinct points $\mathbf{Q}_i$, $i=1,\dots,n-r$ must be all the distinct solutions for the degree $n-k$ factor $f_k(1,m)+xf_{k+1}(1,m)+\cdots+x^{n-k}f_n(1,m)$. Hence $k=r$. But by Definition 2.1, the point $\mathbf{Q}_0=(0:0:1)$ is a singularity of multiplicity r if and only if the lowest degree term of the polynomial $F(x,y,1)$ is of degree r. The proof is now complete. □

By Proposition 2.1 an intuitive geometric interpretation of the *multiplicity* r of a singular point on a rational planar curve is that the curve passes through the singularity exactly r times. Hence there are precisely r tangents at the singular point counted with multiplicities. If these r tangents are all distinct, we say that the singularity is *ordinary*. Otherwise, if some of the tangents are repeated, the singularity is *non-ordinary*. If we perturb the curve slightly, an ordinary singularity will be stable, while there might be extra singularities arising from non-ordinary singularities, which are usually called *infinitely near singular points*.

Example 2.1. Fig. 1(a) illustrates the rational cubic planar curve $\mathbf{P}(s,u)=(s(s-u)u, s^2(s-u), u^3)$ with a node at $\mathbf{Q}=(0:0:1)$. Since the two tangents at the singular point $\mathbf{Q}$ are distinct, the point $\mathbf{Q}$ is an ordinary singularity. The second picture in Fig. 1(a) is the curve $\mathbf{P}(s,u)$ under a small perturbation. We can see that the multiplicity of the singular point $\mathbf{Q}$ is stable under small perturbations. Therefore, there are no infinitely near singular points arising from the singular point $\mathbf{Q}$.

Example 2.2. Fig. 1(b) depicts the rational cubic planar curve $\mathbf{P}(s,u)=(s^2u, s^3, u^3)$ with a cusp at $\mathbf{Q}=(0:0:1)$. Since the tangent at the singular point $\mathbf{Q}$ is repeated twice at the parameter $(s,u)=(0,1)$, the point $\mathbf{Q}$ is a non-ordinary singularity. From the second picture in Fig. 1(b), we can see that the cusp becomes a node under a small perturbation of the curve, which preserves the multiplicity of the singularity. Therefore, once again there are no infinitely near singular points arising from the singular point $\mathbf{Q}$.

Example 2.3. Fig. 2(a) shows the degree 5 rational planar curve $\mathbf{P}(s,u)=(s^2u^3, s^5, u^5)$ with a tacnode at $\mathbf{Q}=(0:0:1)$, where the tangent is repeated twice at the same parameter $(s,u)=(0,1)$. From the second picture in Fig. 2(a), we can see that the tacnode becomes a node plus a cusp under a small perturbation of the curve, which means that there is an infinitely near double point $\mathbf{Q}^*$ arising from the singular point $\mathbf{Q}$.



(a) (s^2u^3, s^5, u^5) (b) $((s+u)^2(s-u)^3, (s+u)(s-u)u^3, u^5)$

Fig. 2. Singularities with infinitely near singular points.

Example 2.4. Fig. 2(b) depicts the degree 5 rational planar curve $\mathbf{P}(s,u) = ((s+u)^2(s-u)^3, (s+u)(s-u)u^3, u^5)$ with a double point at $\mathbf{Q} = (0:0:1)$, where the curve has parallel tangents at distinct parameters $(s,u) = (1,1)$ and $(s,u) = (-1,1)$. From the second picture in Fig. 2(b), we can see that the double point becomes two nodes under a small perturbation of the curve, which means that there is an infinitely near double point $\mathbf{Q}^*$ arising from the singular point $\mathbf{Q}$.

Remark 2.2. Infinitely near singularities of a singular point $\mathbf{Q}$ on a rational planar curve $\mathbf{P}(s,u)$ are caused by the following two conditions:

1. The curve $\mathbf{P}(s,u)$ has parallel tangents at different parameters corresponding to the point $\mathbf{Q}$.
2. The curve $\mathbf{P}(s,u)$ has an extra multiple tangent at a multiple parameter (s^*, u^*) of the point $\mathbf{Q}$, i.e., if $u^* \neq 0$, then the rate of change of the slope of the curve $\mathbf{P}(s,1)$ with respect to the variable s at the parameter $s = s^*/u^*$ is zero; and similarly if $u^* = 0$, then the rate of change of the slope of the curve $\mathbf{P}(1,u)$ with respect to the variable u at the parameter $u^* = 0$ is zero.

Let $\mathbf{Q}$ be a singularity on the rational planar curve $\mathbf{P}(s,u)$. To prepare for our subsequent results, translate the coordinate system so that the point $\mathbf{Q}$ is moved to the origin $(0:0:1)$. Then the parametrization of the curve $\mathbf{P}(s,u)$ becomes:

$$\mathbf{P}(s,u) = \big(a(s,u)h(s,u), b(s,u)h(s,u), c(s,u)\big), \tag{7}$$

where $\gcd(a,b) = 1$, $\gcd(h,c) = 1$, and the roots of h are the parameters of $\mathbf{Q}$ with proper multiplicity. We can also make sure that $\gcd(a,h) = 1$ by a coordinate transformation.

Lemma 2.3. *Let $\mathbf{P}(s,u)$ be a rational planar curve given by Eq. (7), and let (s^*, u^*) and (t^*, v^*) be two distinct parameters corresponding to the singularity $\mathbf{Q} = (0:0:1)$. Then*

1. *The curve $\mathbf{P}(s,u)$ has parallel tangents at (s^*, u^*) and (t^*, v^*) if and only if*
$$\frac{b(s^*,u^*)}{a(s^*,u^*)} = \frac{b(t^*,v^*)}{a(t^*,v^*)}. \tag{8}$$
2. *The curve $\mathbf{P}(s,u)$ has extra multiple tangents at a multiple parameter (s^*, u^*) corresponding to the point $\mathbf{Q}$ if and only if*
   (a) $\left(\frac{b(s,1)}{a(s,1)}\right)'\Big|_{s=s^*/u^*} = 0$ *if* $u^* \neq 0$;
   (b) $\left(\frac{b(1,u)}{a(1,u)}\right)'\Big|_{u=0} = 0$ *if* $u^* = 0$.

Proof. Let (s^*, u^*) be a parameter corresponding to the point $\mathbf{Q}$. If $u^* \neq 0$, then we dehomogenize $a(s,u)$, $b(s,u)$, $c(s,u)$ to $a(s,1)$, $b(s,1)$, $c(s,1)$ and let

$$x(s) \triangleq \frac{a(s,1)h(s,1)}{c(s,1)}, \qquad y(s) \triangleq \frac{b(s,1)h(s,1)}{c(s,1)}.$$

Now the slope of the tangent to the curve $\mathbf{P}(s,u)$ at the parameter $(s^*, u^*) = (s^*/u^*, 1)$ is:

$$\frac{dy}{dx}\bigg|_{s=s^*/u^*} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}}\bigg|_{s=s^*/u^*} = \frac{\left(\frac{b(s,1)h(s,1)}{c(s,1)}\right)'}{\left(\frac{a(s,1)h(s,1)}{c(s,1)}\right)'}\Bigg|_{s=s^*/u^*} = \frac{b(s^*/u^*,1)}{a(s^*/u^*,1)} = \frac{b(s^*,u^*)}{a(s^*,u^*)}. \tag{9}$$

Similarly if $u^* = 0$, then we dehomogenize $a(s,u)$, $b(s,u)$, $c(s,u)$ to $a(1,u)$, $b(1,u)$, $c(1,u)$ and let

$$x(u) \triangleq \frac{a(1,u)h(1,u)}{c(1,u)}, \qquad y(u) \triangleq \frac{b(1,u)h(1,u)}{c(1,u)}.$$

Now the slope of the tangent to the curve $\mathbf{P}(s,u)$ at the parameter $(s^*,0)=(1,0)$ is:

$$\left.\frac{dy}{dx}\right|_{u=0}=\left.\frac{\frac{dy}{du}}{\frac{dx}{du}}\right|_{u=0}=\left.\frac{\left(\frac{b(1,u)h(1,u)}{c(1,u)}\right)'}{\left(\frac{a(1,u)h(1,u)}{c(1,u)}\right)'}\right|_{u=0}=\frac{b(1,0)}{a(1,0)}. \tag{10}$$

Therefore, in either case the slope of the tangent to the curve $\mathbf{P}(s,u)$ at a parameter (s^*,u^*) corresponding to the point $\mathbf{Q}$ is equal to $\frac{b(s^*,u^*)}{a(s^*,u^*)}$. Hence the curve $\mathbf{P}(s,u)$ has parallel tangents at two distinct parameters (s^*,u^*) and (t^*,v^*) corresponding to the point $\mathbf{Q}$ if and only if

$$\frac{b(s^*,u^*)}{a(s^*,u^*)}=\frac{b(t^*,v^*)}{a(t^*,v^*)}.$$

On the other hand, for a parameter (s^*,u^*) corresponding to the point $\mathbf{Q}$ with $u^*\neq 0$, the curve $\mathbf{P}(s,u)$ has extra multiple tangents at the parameter $(s^*,u^*)=(s^*/u^*,1)$ if and only if $(\frac{dy}{dx})'|_{s^*/u^*}=0$. By Eq. (9) this means that

$$\left.\left(\frac{dy}{dx}\right)'\right|_{s=s^*/u^*}=\left.\left(\frac{b(s,1)}{a(s,1)}\right)'\right|_{s=s^*/u^*}=0.$$

Case 2(b) can be treated similarly. □

2.2. Blow ups

Infinitely near singularities to a singular point $\mathbf{Q}$ can be found by *blowing up* the rational planar curve at $\mathbf{Q}$. Most articles in algebraic geometry on blowing up planar curves generally deal with algebraic curves and start from their implicit equation (Hartshorne, 1977). Next we are going to translate the classical approach to blow ups into the language of parametrization (for related work, see Abhyankar, 1990).

To blow up the curve $\mathbf{P}(s,u)$ in Eq. (7) at the singularity $\mathbf{Q}=(0:0:1)$, we let $U=Y/X=(\frac{bh}{c})/(\frac{ah}{c})=b/a$. Then the dehomogenized form of the new curve is

$$(X,U)=\left(\frac{ah}{c},\frac{b}{a}\right). \tag{11}$$

Homogenizing (11), we get the new rational planar curve

$$\mathbf{P}^1(s,u)=\left(a^2h,bc,ac\right). \tag{12}$$

Certain singularities on the curve $\mathbf{P}^1(s,u)$ are related to the infinitely near singularities of the point $\mathbf{Q}$. To understand this relationship, we examine how the two conditions in Lemma 2.3 affect the curve in Eq. (11):

- For the first condition in Lemma 2.3, Eq. (8) implies that $U(s^*,u^*)=U(t^*,v^*)\triangleq U_0$. Since $h(s^*,u^*)=h(t^*,v^*)=0$, we also have $X(s^*,u^*)=X(t^*,v^*)\triangleq X_0$. Therefore, (X_0,U_0) is a singularity on the curve (X,U).
- For the second condition in Lemma 2.3, suppose that $u^*\neq 0$. Then we dehomogenize $X(s,u)$ and $U(s,u)$ to $X(s,1)$ and $U(s,1)$. Now 2(a) in Lemma 2.3 means that $U'(s^*/u^*,1)=0$. Also note that $h'(s^*/u^*,1)=0$ yields $X'(s^*/u^*,1)=0$. This means that the parameter $(s^*,u^*)=(s^*/u^*,1)$ is at least a double parameter corresponding to the point $(X_0,U_0)=(X(s^*,u^*),U(s^*,u^*))$. Therefore, (X_0,U_0) is a singularity on the curve (X,U). The case where $u^*=0$ can be treated similarly.

Hence to find all the infinitely near singularities in the first neighborhood of $\mathbf{Q}$, we just need to find the singularities related to the point $\mathbf{Q}$ on the curve $\mathbf{P}^1(s,u)$. Note that we say *in the first neighborhood* because we blow up the curve $\mathbf{P}(s,u)$ once to get the curve $\mathbf{P}^1(s,u)$. If we continue to blow up the curve $\mathbf{P}^1(s,u)$ to get $\mathbf{P}^2(s,u)$, the points on the curve $\mathbf{P}^2(s,u)$ related to the point $\mathbf{Q}$ are said to be in the second neighborhood of $\mathbf{Q}$, and so on. Therefore, it is appropriate for us to give the following definition:

Definition 2.3. We say that there is an infinitely near singular point of multiplicity m arising from the ith neighborhood of the point $\mathbf{Q}$, if there is a singularity $\mathbf{Q}^*$ of multiplicity m on the ith blow up curve $\mathbf{P}^i(s,u)$, whose corresponding parameters are a subset of all the parameters corresponding to the point $\mathbf{Q}$.

Remark 2.4. When we blow up a curve $\mathbf{P}(s,u)$ at a singularity $\mathbf{Q}$ to get a new curve $\mathbf{P}^1(s,u)$, we are interested only in those singularities that are related to the point $\mathbf{Q}$ on the curve $\mathbf{P}^1(s,u)$ – that is, the singularities whose parameters are a subset of the parameters of the point $\mathbf{Q}$ on the curve $\mathbf{P}(s,u)$.

Example 2.5. Let a rational planar curve $\mathbf{P}(s,u)$ be given by

$$\mathbf{P}(s,u)=\left(s^5u^3,s^8,u^8\right).$$

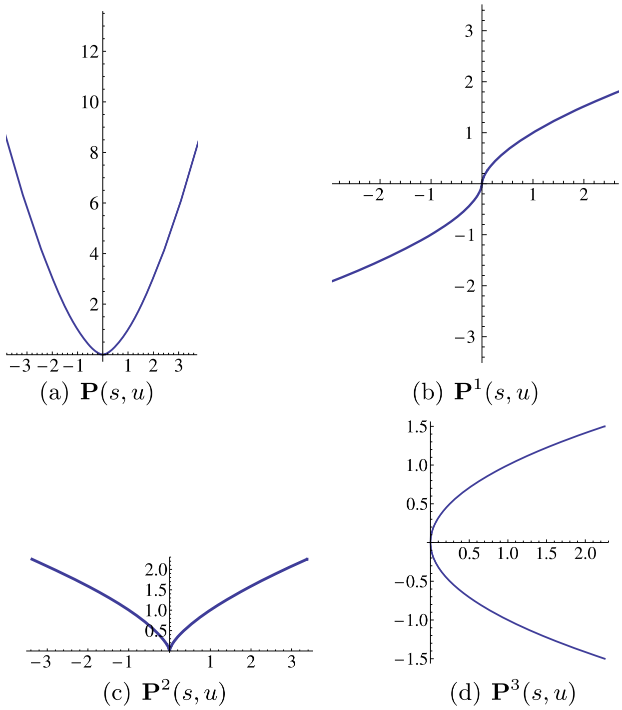


Fig. 3. The rational planar curve $\mathbf{P}(s,u) = (s^5u^3, s^8, u^8)$ and the first three blow ups $\mathbf{P}^1(s,u), \mathbf{P}^2(s,u), \mathbf{P}^3(s,u)$.

By Proposition 2.1 the point $\mathbf{Q} = (0:0:1)$ is a singularity of order 5 corresponding to the parameter $(0,1)$ on the curve $\mathbf{P}(s,u)$. To check whether there are any infinitely near singularities arising from the point $\mathbf{Q}$, we blow up the curve $\mathbf{P}(s,u)$ at the point $\mathbf{Q}$:

$$\mathbf{P}^1(s,u) = \left(s^5, s^3u^2, u^5\right).$$

Again by Proposition 2.1 the point $\mathbf{Q}^* = (0:0:1)$ is a triple point corresponding to the parameter $(0,1)$ on the curve $\mathbf{P}^1(s,u)$. Therefore, $\mathbf{Q}^*$ is an infinitely near triple point in the first neighborhood of $\mathbf{Q}$.

Next we blow up the curve $\mathbf{P}^1(s,u)$ at the point $\mathbf{Q}^*$. Note that for the curve $\mathbf{P}^1(s,u)$, $\gcd(a,h) = s^2$. Thus we first reflect the curve $\mathbf{P}^1(s,u)$ about the 45 degree line to get a new curve for which $\gcd(a,h) = 1$:

$$\widetilde{\mathbf{P}}^1(s,u) = \left(s^3u^2, s^5, u^5\right).$$

Now we blow up the curve $\widetilde{\mathbf{P}}^1(s,u)$ at the point $\mathbf{Q}^* = (0:0:1)$ and get:

$$\mathbf{P}^2(s,u) = \left(s^3, s^2u, u^3\right).$$

There is a double point $\mathbf{Q}^{**} = (0:0:1)$ related to the point $\mathbf{Q}$ on the curve $\mathbf{P}^2(s,u)$, which is an infinitely near double point in the second neighborhood of the point $\mathbf{Q}$.

Next we continue to blow up the curve $\mathbf{P}^2(s,u)$ at the point $\mathbf{Q}^{**}$. Again note that for the curve $\mathbf{P}^2(s,u)$, $\gcd(a,h) = s^2$. Thus again we reflect $\mathbf{P}^2(s,u)$ about the 45 degree line to get a new curve for which $\gcd(a,h) = 1$:

$$\widetilde{\mathbf{P}}^2(s,u) = \left(s^2u, s^3, u^3\right).$$

Now we blow up the curve $\widetilde{\mathbf{P}}^2(s,u)$ at the point $\mathbf{Q}^{**} = (0:0:1)$ and get:

$$\mathbf{P}^3(s,u) = \left(s^2, su, u^2\right).$$

Note that there are no singularities on the curve $\mathbf{P}^3(s,u)$. Therefore, we have found all the infinitely near singularities of the original singular point $\mathbf{Q}$. Fig. 3 shows the rational planar curve $\mathbf{P}^1(s,u)$ and the blow ups $\mathbf{P}^1(s,u), \mathbf{P}^2(s,u), \mathbf{P}^3(s,u)$.

Note that there is also a triple point $(0:1:0)$ on the curve $\mathbf{P}(s,u)$. One can use methods similar to those presented above to derive all the infinitely near singularities arising from the triple point $(0:1:0)$. Fig. 4 shows the singularity trees arising from the singular points $(0:0:1)$ and $(0:1:0)$.

3. Moving lines and μ-bases for rational planar curves

We are going to use moving lines and μ-bases to find all the singularities and all the infinitely near singularities of rational planar curves without resorting to blow ups.

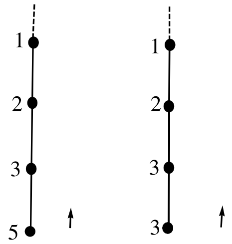


Fig. 4. Singularity trees of the singular points $(0:0:1)$ (left) and $(0:1:0)$ (right) of the rational planar curve $\mathbf{P}(s,u)=(s^5u^3, s^8, u^8)$.

A *moving line* $L(s,u;x,y,w)=A(s,u)x+B(s,u)y+C(s,u)w=0$ is a set of lines with each homogeneous parameter $s:u$ corresponding to a line and where $A(s,u)$, $B(s,u)$, $C(s,u)$ are homogeneous polynomials in $k[s,u]$. For convenience we shall also write a moving line in the form of a polynomial vector $\mathbf{L}(s,u)=(A(s,u),B(s,u),C(s,u))$.

A moving line $\mathbf{L}(s,u)=(A(s,u),B(s,u),C(s,u))$ is said to *follow* a rational planar curve $\mathbf{P}(s,u)=(a(s,u),b(s,u),c(s,u))$ if and only if

$$\mathbf{L}(s,u)\cdot\mathbf{P}(s,u)=A(s,u)a(s,u)+B(s,u)b(s,u)+C(s,u)c(s,u)\equiv 0. \tag{13}$$

Eq. (13) means that for every homogeneous parameter $s_0:u_0$, the line $A(s_0,u_0)x+B(s_0,u_0)y+C(s_0,u_0)w=0$ in the family of lines $L(s,u;x,y,w)$ passes through the point $\mathbf{P}(s_0,u_0)$ on the rational planar curve $\mathbf{P}(s,u)$.

The *syzygy module* of a rational planar curve $\mathbf{P}(s,u)$ consists of all the moving lines $\mathbf{L}(s,u)$ that follow the curve $\mathbf{P}(s,u)$. The syzygy module of a rational planar curve $\mathbf{P}(s,u)$ is known to be a free module with two generators (Chen and Wang, 2003). We denote the syzygy module of the curve $\mathbf{P}(s,u)$ by $\mathbf{M}_p$.

Definition 3.1. Two moving lines $\mathbf{p}(s,u)$ and $\mathbf{q}(s,u)$ are called a *μ-basis* for the rational planar curve $\mathbf{P}(s,u)$ if $\mathbf{p}$ and $\mathbf{q}$ form a basis for $\mathbf{M}_p$ i.e., every moving line $\mathbf{L}(s,u)\in\mathbf{M}_p$ can be written as

$$\mathbf{L}(s,u)=\alpha(s,u)\mathbf{p}(s,u)+\beta(s,u)\mathbf{q}(s,u), \tag{14}$$

where $\alpha(s,u),\beta(s,u)\in\mathbb{R}[s,u]$.

Note that since we are using homogeneous polynomials, Definition 3.1 implicitly implies the following degree constraint of the elements of a μ-basis (Cox et al., 1998):

$$\deg(\mathbf{p})+\deg(\mathbf{q})=\deg(\mathbf{P}).$$

Every rational planar curve has a μ-basis. Moreover, there is a fast algorithm for computing μ-bases based on Gaussian elimination (Chen and Wang, 2003).

μ-bases have many advantageous properties. For example, we can recover the parametrization of the rational planar curve $\mathbf{P}(s,u)$ from the outer product of a μ-basis:

$$\mathbf{p}(s,u)\times\mathbf{q}(s,u)=k\mathbf{P}(s,u), \tag{15}$$

where k is a nonzero constant. We can also retrieve the implicit equation $f(x,y,w)=0$ of the rational planar curve $\mathbf{P}(s,u)$ by taking the resultant of a μ-basis:

$$f(x,y,w)=\mathrm{Res}_{s,u}\big(\mathbf{p}(s,u)\cdot\mathbf{X},\mathbf{q}(s,u)\cdot\mathbf{X}\big), \tag{16}$$

where $\mathbf{X}=(x,y,w)$ (Chen and Wang, 2003). Later on we shall make use of the following result.

Proposition 3.1. *(See Chen and Wang, 2003.) Let $\mathbf{p}(s,u)$, $\mathbf{q}(s,u)$ be a μ-basis for the rational planar curve $\mathbf{P}(s,u)$. Then $\mathbf{p}(s,u)$ and $\mathbf{q}(s,u)$ are linearly independent for every parameter (s,u).*

Proof. This result follows from Eq. (15) and the fact that, by assumption, $\mathbf{P}(s,u)=(a(s,u),b(s,u),c(s,u))$ has no base points. □

4. Computing the singularities of rational planar curves

In this section we are going to show that the singularities of a rational planar curve are equivalent to the intersections of two implicit polynomial curves constructed from a parametrization and a μ-basis of the rational planar curve. The intersections of these two algebraic curves provide all the parameters of the singularities on the rational planar curve with correct multiplicities, including the infinitely near singularities arising from each singularity. Moreover, we shall also prove a classical theorem in algebraic geometry for counting all these singularities with proper multiplicity – that is, the multiplicity given by the δ-invariants of the singularities.

First we present a theorem that shows how to use a μ-basis to calculate all the parameters corresponding to a point $\mathbf{Q}$ on a rational planar curve $\mathbf{P}(s,u)$.

Theorem 4.1. *(See Chen et al., 2008.) Let $\mathbf{p}(s,u), \mathbf{q}(s,u)$ be a μ-basis of a rational planar curve $\mathbf{P}(s,u)$, and let $\mathbf{Q}$ be a point on $\mathbf{P}(s,u)$. Then the roots of the polynomial $\gcd(\mathbf{p}(s,u)\cdot\mathbf{Q}, \mathbf{q}(s,u)\cdot\mathbf{Q})$ give all the parameters corresponding to the point $\mathbf{Q}$ with the correct multiplicities.*

Theorem 4.1 shows that the μ-bases contain all the information concerning the parameters of the singularities on rational planar curves. Chen et al. (2008) use the Bezout matrix of the two μ-basis elements to compute all the singularities, except for the infinitely near singularities, of rational planar curves. Their method also allows them to distinguish ordinary from non-ordinary singularities. In addition, they present a conjecture on how to compute all the infinitely near singularities from the Smith form of the Bezout matrix. Next we shall present a different approach also based on μ-bases to compute the singularities of rational planar curves. In contrast to Chen et al. (2008), our approach provides not only the multiplicities of the singularities but also the δ-invariant for each singularity. These δ-invariants indicate whether there are infinitely near singularities arising from the singularities on the rational planar curve. Moreover, we can also provably detect all the parameters that correspond to parallel tangents or extra multiple tangents and so correspond to infinitely near singularities on rational planar curves. In a sequel to this paper – see Jia and Goldman (2009) – we shall also show how to apply our approach to prove the conjecture of Chen, Wang and Liu.

We begin by constructing two functions from a μ-basis $\mathbf{p}(s,u), \mathbf{q}(s,u)$ of the rational planar curve $\mathbf{P}(s,u)$:

$$\begin{aligned} F(s,u;t,v) &\triangleq \frac{\mathbf{p}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu},\\ G(s,u;t,v) &\triangleq \frac{\mathbf{q}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu}. \end{aligned} \tag{17}$$

Note that $F(s,u;t,v)$ and $G(s,u;t,v)$ are both polynomials. In fact, since $\mathbf{p},\mathbf{q}$ are a μ-basis,

$$\mathbf{p}(s,u)\cdot\mathbf{P}(s,u) = \mathbf{q}(s,u)\cdot\mathbf{P}(s,u) \equiv 0. \tag{18}$$

Thus

$$\begin{aligned} &(sv-tu)|\mathbf{p}(s,u)\cdot\mathbf{P}(t,v),\\ &(sv-tu)|\mathbf{q}(s,u)\cdot\mathbf{P}(t,v), \end{aligned} \tag{19}$$

so $F(s,u;t,v)$ and $G(s,u;t,v)$ are indeed polynomials.

Theorem 4.2. *A parameter pair $(s^*,u^*;t^*,v^*)$ is a common root of $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ if and only if the two parameters (s^*,u^*) and (t^*,v^*) correspond to the same singularity on the curve $\mathbf{P}(s,u)$.*

Proof. If $(s^*,u^*)\neq(t^*,v^*)$, then by Eq. (17)

$$F(s^*,u^*;t^*,v^*) = G(s^*,u^*;t^*,v^*) = 0$$

if and only if

$$\mathbf{p}(s^*,u^*)\cdot\mathbf{P}(t^*,v^*) = \mathbf{q}(s^*,u^*)\cdot\mathbf{P}(t^*,v^*) = 0.$$

By Proposition 3.1, the two vectors $\mathbf{p}(s^*,u^*)$ and $\mathbf{q}(s^*,u^*)$ are linearly independent. Hence since $\mathbf{p}(s^*,u^*)$ and $\mathbf{q}(s^*,u^*)$ are both perpendicular to $\mathbf{P}(t^*,v^*)$,

$$\mathbf{p}(s^*,u^*)\times\mathbf{q}(s^*,u^*) = k_1\mathbf{P}(t^*,v^*)$$

for some nonzero constant k_1. But by Eq. (15),

$$\mathbf{p}(s^*,u^*)\times\mathbf{q}(s^*,u^*) = k_2\mathbf{P}(s^*,u^*)$$

for some nonzero constant k_2. Thus

$$\mathbf{P}(t^*,v^*) = k\mathbf{P}(s^*,u^*)$$

for some nonzero constant k, so (s^*,u^*) and (t^*,v^*) correspond to the same singularity $\mathbf{Q}=\mathbf{P}(t^*,v^*)=k\mathbf{P}(s^*,u^*)$.

If $(s^*,u^*)=(t^*,v^*)$ and $u^*\neq 0$, then we dehomogenize the curve $\mathbf{P}(s,u)$ to $\mathbf{P}(s,1)$, and dehomogenize the μ-basis elements $\mathbf{p}(s,u),\mathbf{q}(s,u)$ to $\mathbf{p}(s,1),\mathbf{q}(s,1)$. Now by Eq. (17)

$$F\left(\frac{s^*}{u^*},1;\frac{s^*}{u^*},1\right)=\mathbf{p}'\left(\frac{s^*}{u^*}\right)\cdot\mathbf{P}\left(\frac{s^*}{u^*}\right)=0,$$
$$G\left(\frac{s^*}{u^*},1;\frac{s^*}{u^*},1\right)=\mathbf{q}'\left(\frac{s^*}{u^*}\right)\cdot\mathbf{P}\left(\frac{s^*}{u^*}\right)=0. \tag{20}$$

Let $\mathbf{Q}\triangleq\mathbf{P}(\frac{s^*}{u^*})$. By Theorem 4.1, the parameter $\frac{s^*}{u^*}$ is a root of the polynomial $\gcd(\mathbf{p}(s,1)\cdot\mathbf{Q},\mathbf{q}(s,1)\cdot\mathbf{Q})$. But by Eq. (20), the parameter $\frac{s^*}{u^*}$ is at least a double root of the polynomial $\gcd(\mathbf{p}(s,1)\cdot\mathbf{Q},\mathbf{q}(s,1)\cdot\mathbf{Q})$. Hence again by Theorem 4.1, the parameter $(\frac{s^*}{u^*},1)$ is at least a double parameter corresponding to the point $\mathbf{Q}$. Therefore, the point $\mathbf{Q}$ is a singular point on the curve $\mathbf{P}(s,u)$. The case that $u^*=0$ can be treated similarly. □

Remark 4.3. By Theorem 4.2 the parameter pair $(s^*,u^*;t^*,v^*)$ is an intersection point of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ if and only if the parameter pair $(t^*,v^*;s^*,u^*)$ is also an intersection point of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$.

Theorem 4.2 shows that the intersection points of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ correspond to the parameters of the singularities on the rational planar curve $\mathbf{P}(s,u)$. Thus the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ have no common components; otherwise there would be an infinite number of singularities on the rational planar curve, which is impossible.

Next we shall show that the intersection of the two curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ also gives the correct multiplicity for each singular point on the curve $\mathbf{P}(s,u)$.

Theorem 4.4. *Let $(s^*,u^*;t^*,v^*)$ be an intersection point of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$. Then the roots of the polynomial*

$$h(s,u)\triangleq(su^*-s^*u)\gcd\bigl(F(s,u;s^*,u^*),G(s,u;s^*,u^*)\bigr)$$

or

$$h(s,u)\triangleq(sv^*-t^*u)\gcd\bigl(F(s,u;t^*,v^*),G(s,u;t^*,v^*)\bigr)$$

give all the parameters corresponding to the singular point $\mathbf{Q}=\mathbf{P}(s^*,u^*)=\mathbf{P}(t^*,v^*)$ with the correct multiplicities.

Proof. Since $\mathbf{P}(s^*,u^*)=\mathbf{Q}$, we have

$$F(s,u;s^*,u^*)=\frac{\mathbf{p}(s,u)\cdot\mathbf{Q}}{su^*-s^*u},\qquad G(s,u;s^*,u^*)=\frac{\mathbf{q}(s,u)\cdot\mathbf{Q}}{su^*-s^*u}. \tag{21}$$

By Theorem 4.1, the roots of the polynomial $\gcd(\mathbf{p}(s,u)\cdot\mathbf{Q},\mathbf{q}(s,u)\cdot\mathbf{Q})$ give all the parameters corresponding to the point $\mathbf{Q}$ with the correct multiplicities. But by Eq. (21),

$$\gcd\bigl(\mathbf{p}(s,u)\cdot\mathbf{Q},\mathbf{q}(s,u)\cdot\mathbf{Q}\bigr)=(su^*-s^*u)\gcd\bigl(F(s,u;s^*,u^*),G(s,u;s^*,u^*)\bigr).$$

The proof is the same for (t^*,v^*). □

By Theorem 4.4 we directly have

Corollary 4.1. *Let $(s^*,u^*;t^*,v^*)$ be an intersection point of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$. Then the singular point $\mathbf{P}(s^*,u^*)=\mathbf{P}(t^*,v^*)=\mathbf{Q}$ is of multiplicity*

$$\deg\bigl(\gcd\bigl(F(s,u;s^*,u^*),G(s,u;s^*,u^*)\bigr)\bigr)+1$$

or

$$\deg\bigl(\gcd\bigl(F(s,u;t^*,v^*),G(s,u;t^*,v^*)\bigr)\bigr)+1.$$

Theorem 4.4 and Corollary 4.1 provide an efficient way to determine the multiplicity of a singular point together with all the parameters corresponding to the singular point directly from the intersection of the two algebraic curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ without actually computing the singular point.

The intersection of the two algebraic curves not only provides the multiplicity of the singularities on the rational planar curve, but also gives a very important index describing the multiplicity of the singularities including all the infinitely near singularities arising from the singularities.

Definition 4.1. Let $\mathbf{S}=(s^*,u^*;t^*,v^*)$ be a parameter pair. Then $I_{\mathbf{S}}(F,G)$ denotes the intersection multiplicity of the two curves $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ at the point $\mathbf{S}=(s^*,u^*;t^*,v^*)$.

Definition 4.2. Let $\mathbf{Q}$ be a singular point on the rational planar curve $\mathbf{P}(s,u)$, and let (s_i,u_i), $i=1,\dots,k$ be all the distinct parameters corresponding to the point $\mathbf{Q}$. Then we define the intersection multiplicity of $F(s,u;t,v)=0$ and $G(s,u;t,v)=0$ contributed by the singularity $\mathbf{Q}$ as

$$I_{\mathbf{Q}}(F,G) \triangleq \sum_{i,j} I_{\mathbf{S}_{ij}}(F,G), \tag{22}$$

where $\mathbf{S}_{ij}=(s_i,u_i;s_j,u_j)$, $i,j=1,\dots,k$.

Now we can state our main result.

Theorem 4.5. *Let $\nu_{\mathbf{Q}^*}$ denote the multiplicity of an infinitely near point $\mathbf{Q}^*$ of a singular point $\mathbf{Q}$ on the curve $\mathbf{P}(s,u)$. Then*

$$I_{\mathbf{Q}}(F,G)=\sum_{\mathbf{Q}^*}\nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*}-1),$$

where the sum is taken over all the infinitely near points $\mathbf{Q}^*$ of the point $\mathbf{Q}$ including $\mathbf{Q}$ itself.

In order to prove this theorem, we first introduce several lemmas. To proceed, note that the two polynomials $F(s,u;t,v)$ and $G(s,u;t,v)$ are constructed from a μ-basis $\mathbf{p},\mathbf{q}$, which is a basis for the syzygy module of the curve $\mathbf{P}(s,u)$. Now let $\widetilde{\mathbf{p}}(s,u)$ and $\widetilde{\mathbf{q}}(s,u)$ be any pair of syzygies that are linearly independent for the parameters corresponding to the point $\mathbf{Q}$ on the rational planar curve $\mathbf{P}(s,u)$. Then we can construct two polynomials in the same way as in Eq. (17):

$$\begin{aligned}\widetilde{F}(s,u;t,v) &\triangleq \frac{\widetilde{\mathbf{p}}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu},\\ \widetilde{G}(s,u;t,v) &\triangleq \frac{\widetilde{\mathbf{q}}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu}.\end{aligned} \tag{23}$$

Lemma 4.6.

$$I_{\mathbf{Q}}(\widetilde{F},\widetilde{G})=I_{\mathbf{Q}}(F,G). \tag{24}$$

Proof. Since $\widetilde{\mathbf{p}}(s,u)$ and $\widetilde{\mathbf{q}}(s,u)$ are syzygies of the curve $\mathbf{P}(s,u)$, and $\mathbf{p}(s,u)$ and $\mathbf{q}(s,u)$ are a μ-basis, there must be polynomials $\alpha(s,u)$, $\beta(s,u)$, $\gamma(s,u)$, $\delta(s,u)$ such that

$$\begin{aligned}\widetilde{\mathbf{p}}(s,u) &= \alpha(s,u)\mathbf{p}(s,u)+\beta(s,u)\mathbf{q}(s,u),\\ \widetilde{\mathbf{q}}(s,u) &= \gamma(s,u)\mathbf{p}(s,u)+\delta(s,u)\mathbf{q}(s,u).\end{aligned} \tag{25}$$

Therefore

$$\begin{aligned}\widetilde{F}(s,u;t,v) &= \alpha(s,u)F(s,u;t,v)+\beta(s,u)G(s,u;t,v),\\ \widetilde{G}(s,u;t,v) &= \gamma(s,u)F(s,u;t,v)+\delta(s,u)G(s,u;t,v).\end{aligned} \tag{26}$$

But $\alpha(s,u)\delta(s,u)-\beta(s,u)\gamma(s,u)=0$ if and only if the two moving lines $\widetilde{\mathbf{p}}(s,u)$ and $\widetilde{\mathbf{q}}(s,u)$ are linearly dependent, which by assumption never happens for parameters (s,u) corresponding to the point $\mathbf{Q}$. Therefore, by (26) we conclude that

$$I_{\mathbf{Q}}(\widetilde{F},\widetilde{G})=I_{\mathbf{Q}}(F,G). \qquad \square \tag{27}$$

By Lemma 4.6 in order to prove Theorem 4.5, we can turn to another pair of syzygies, which are much easier to study.

Without loss of generality, we can assume that the singularity $\mathbf{Q}=(0:0:1)$. Then the curve $\mathbf{P}(s,u)$ has the parametrization

$$\mathbf{P}(s,u)=\big(a(s,u)h(s,u),\,b(s,u)h(s,u),\,c(s,u)\big), \tag{28}$$

where $\gcd(h,c)=\gcd(a,b)=1$, and h gives all the parameters corresponding to the point $\mathbf{Q}$ with the correct multiplicities. Moreover, if $\gcd(a,h)\neq 1$, we can make a coordinate transformation so that $\gcd(a,h)=1$.

Now there are two obvious syzygies of the parametrization (28):

$$\mathbf{M}(s,u)=(-b,a,0), \qquad \mathbf{L}(s,u)=(c,0,-ah). \tag{29}$$

From these two syzygies we can construct two polynomials:

$$\begin{aligned}M(s,u;t,v) &= \frac{\mathbf{M}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu}=\frac{a(s,u)b(t,v)-b(s,u)a(t,v)}{sv-tu}h(t,v),\\ L(s,u;t,v) &= \frac{\mathbf{L}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu}=\frac{c(s,u)a(t,v)h(t,v)-c(t,v)a(s,u)h(s,u)}{sv-tu}.\end{aligned} \tag{30}$$

Note that $\mathbf{M}(s,u)$ and $\mathbf{L}(s,u)$ are linearly independent for any parameter (s,u) corresponding to the point $\mathbf{Q}$, since, by assumption, $a(s,u) \neq 0$. Hence, we next examine $I_{\mathbf{Q}}(M,L)$ instead of $I_{\mathbf{Q}}(F,G)$.

Lemma 4.7. *Let $m \triangleq \deg(h)$ be the multiplicity of the singularity $\mathbf{Q}$. Then*

$$I_{\mathbf{Q}}\big(h(t,v), L(s,u;t,v)\big) = m(m-1). \tag{31}$$

Proof. Let $\{s_i,u_i\}_{i=1}^{k}$ be all the distinct parameters corresponding to the point $\mathbf{Q}$ and let $\mathbf{S}_{ij} = (s_i,u_i;s_j,u_j)$. If m_i is the multiplicity of (s_i,u_i) as a root of $h(s,u)$, then

$$\begin{aligned}
I_{\mathbf{S}_{ij}}(h,L) &= I_{\mathbf{S}_{ij}}\left(h(t,v), \frac{c(s,u)a(t,v)-a(s,u)c(t,v)}{sv-tu}h(t,v) + a(s,u)c(t,v)\frac{h(t,v)-h(s,u)}{sv-tu}\right)\\
&= I_{\mathbf{S}_{ij}}\left(h(t,v), a(s,u)c(t,v)\frac{h(t,v)-h(s,u)}{sv-tu}\right)\\
&= I_{\mathbf{S}_{ij}}\left(h(t,v), \frac{h(t,v)-h(s,u)}{sv-tu}\right)\\
&= \begin{cases} m_i(m_i-1), & i=j,\\ m_i m_j, & i\neq j.\end{cases}
\end{aligned} \tag{32}$$

Therefore

$$\begin{aligned}
I_{\mathbf{Q}}(h,L) &= \sum_{i,j} I_{\mathbf{S}_{ij}}(h,L) = \sum_{i=1}^{k} m_i(m_i-1) + \sum_{i\neq j} m_i m_j\\
&= \sum_{i,j=1}^{k} m_i m_j - \sum_{i=1}^{k} m_i = m^2 - m = m(m-1). \qquad \square
\end{aligned} \tag{33}$$

Theorem 4.8. *Let $\nu_{\mathbf{Q}^*}$ denote the multiplicity of an infinitely near point $\mathbf{Q}^*$ of the point $\mathbf{Q} = (0:0:1)$ on the parametrization* (28)*. Then*

$$I_{\mathbf{Q}}(M,L) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*}-1), \tag{34}$$

where the sum is taken over all the infinitely near points $\mathbf{Q}^*$ of the singularity $\mathbf{Q}$ including $\mathbf{Q}$ itself.

Proof. Set

$$\mathbf{P}^0 = \mathbf{P} = (ah, bh, c),$$

where $\gcd(a,b) = \gcd(h,c) = 1$. We can also make a coordinate transformation so that $\gcd(a,h) = 1$. Set $a_0 = a, b_0 = b, c_0 = c, h_0 = h$. We assume that the curve $\mathbf{P}^0(s,u)$ has no infinitely near singularities after k blow ups. We shall prove the theorem by induction on k.

Blowing up the curve $\mathbf{P}^0(s,u)$ at the singularity $(0:0:1)$, we get

$$\mathbf{P}^1(s,u) = \big(a_0^2 h_0, b_0 c_0, a_0 c_0\big). \tag{35}$$

Assume that $\mathbf{Q}^*$ is a singular point on $\mathbf{P}^1(s,u)$ related to $\mathbf{Q}$. To blow up the curve $\mathbf{P}^1(s,u)$ at the singular point $\mathbf{Q}^*$, we translate the coordinates so that the point $\mathbf{Q}^*$ is moved to $(0:0:1)$. Then we have another parametrization for the curve $\mathbf{P}^1(s,u)$:

$$\mathbf{P}^1(s,u) = (a_1 h_1, b_1 h_1, c_1), \tag{36}$$

where $\gcd(a_1,b_1) = \gcd(h_1,c_1) = 1$, and the roots of h_1 are parameters of the point $\mathbf{Q}^*$ with the correct multiplicity. (Strictly speaking the coordinate transformation leads to a different curve, but we are examining the parameters corresponding to the points on the curve, which never change under coordinate transformations.)

We have a pair of syzygies for the parametrization (35) of the curve $\mathbf{P}^1$:

$$\begin{aligned}
\mathbf{S}_1(s,u) &\triangleq (0, a_0, -b_0),\\
\mathbf{T}_1(s,u) &\triangleq (c_0, 0, -a_0 h_0).
\end{aligned} \tag{37}$$

Construct two polynomials from $\mathbf{S}_1(s,u)$ and $\mathbf{T}_1(s,u)$:

$$S_1(s,u;t,v) \triangleq \frac{\mathbf{S}_1(s,u)\cdot\mathbf{P}^1(t,v)}{sv-tu} = \frac{a_0(s,u)b_0(t,v)-b_0(s,u)a_0(t,v)}{sv-tu}c_0(t,v),$$
$$T_1(s,u;t,v) \triangleq \frac{\mathbf{T}_1(s,u)\cdot\mathbf{P}^1(t,v)}{sv-tu} = \frac{c_0(s,u)a_0(t,v)h_0(t,v)-c_0(t,v)a_0(s,u)h_0(s,u)}{sv-tu}a_0(t,v). \tag{38}$$

We also have a pair of syzygies for the parametrization (36) of the curve $\mathbf{P}^1$:

$$\mathbf{M}_1(s,u) = \big(-b_1(s,u), a_1(s,u), 0\big),$$
$$\mathbf{L}_1(s,u) = \big(c_1(s,u), 0, -a_1(s,u)h_1(s,u)\big). \tag{39}$$

Define

$$\begin{aligned} M_1(s,u;t,v) &\triangleq \frac{\mathbf{M}_1(s,u)\cdot\mathbf{P}^1(t,v)}{sv-tu} \\ &= \frac{a_1(s,u)b_1(t,v)-b_1(s,u)a_1(t,v)}{sv-tu}h_1(t,v) \\ &= \overline{M}_1(s,u;t,v)h_1(t,v), \\ L_1(s,u;t,v) &\triangleq \frac{\mathbf{L}_1(s,u)\cdot\mathbf{P}^1(t,v)}{sv-tu} \\ &= \frac{c_1(s,u)a_1(t,v)h_1(t,v)-c_1(t,v)a_1(s,u)h_1(s,u)}{sv-tu}. \end{aligned} \tag{40}$$

Note that $\mathbf{S}_1(s,u), \mathbf{T}_1(s,u)$ and $\mathbf{M}_1(s,u), \mathbf{L}_1(s,u)$ are both independent syzygy pairs of the same curve $\mathbf{P}^1(s,u)$. By Lemma 4.6, for each infinitely near point $\mathbf{Q}^*$ in the first neighborhood of the point $\mathbf{Q}$, we have

$$I_{\mathbf{Q}^*}(S_1,T_1) = I_{\mathbf{Q}^*}(M_1,L_1) = I_{\mathbf{Q}^*}(\overline{M}_1,L_1) + I_{\mathbf{Q}^*}\big(h_1(t,v),L_1\big). \tag{41}$$

On the other hand, let

$$\begin{aligned} M_0(s,u;t,v) &\triangleq M(s,u;t,v) \\ &= \frac{a_0(s,u)b_0(t,v)-b_0(s,u)a_0(t,v)}{sv-tu}h_0(t,v) \\ &\triangleq \overline{M}_0(s,u;t,v)h_0(t,v), \\ L_0(s,u;t,v) &\triangleq L(s,u;t,v) \\ &= \frac{c_0(s,u)a_0(t,v)h_0(t,v)-c_0(t,v)a_0(s,u)h_0(s,u)}{sv-tu}. \end{aligned} \tag{42}$$

By Remark 2.2 and Lemma 2.3, a parameter pair $(s,u;t,v)$ corresponds to an infinitely near singularity in the first neighborhood of $\mathbf{Q}$ if and only if $\overline{M}_0(s,u;t,v) = L_0(s,u;t,v) = 0$. Hence the intersection of $\overline{M}_0(s,u;t,v)$ and $L_0(s,u;t,v)$ provides all the parameters of the infinitely near singularities in the first neighborhood of $\mathbf{Q}$ whereas the intersection of $h(t,v)$ and $L_0(s,u;t,v)$ provides all the parameters corresponding to the singularity $\mathbf{Q}$. Comparing the expressions of $\overline{M}_0, L_0$ and S_1, T_1 in Eqs. (42) and (38) and recalling that $a(s,u), c(s,u) \neq 0$ at parameters corresponding to $\mathbf{Q}$, we get

$$I_{\mathbf{Q}^*}(\overline{M}_0,L_0) = I_{\mathbf{Q}^*}(S_1,T_1). \tag{43}$$

Thus

$$I_{\mathbf{Q}}(\overline{M}_0,L_0) = \sum_{\mathbf{Q}^*} I_{\mathbf{Q}^*}(S_1,T_1), \tag{44}$$

where the sum is taken over all the infinitely near singularities $\mathbf{Q}^*$ in the first neighborhood of $\mathbf{Q}$. Substituting Eqs. (41) into (44), we get

$$I_{\mathbf{Q}}(\overline{M}_0,L_0) = \sum_{\mathbf{Q}^*}\big(I_{\mathbf{Q}^*}(\overline{M}_1,L_1) + I_{\mathbf{Q}^*}\big(h_1(t,v),L_1\big)\big). \tag{45}$$

But by Lemma 4.7

$$I_{\mathbf{Q}^*}\big(h_1(t,v),L_1\big) = \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*}-1). \tag{46}$$

Therefore

$$I_{\mathbf{Q}}(\overline{M}_0,L_0) = \sum_{\mathbf{Q}^*}\big(I_{\mathbf{Q}^*}(\overline{M}_1,L_1) + \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*}-1)\big), \tag{47}$$

where the sum is taken over all the infinitely near singularities $\mathbf{Q}^*$ in the first neighborhood of $\mathbf{Q}$.

For each infinitely near singularity $\mathbf{Q}^*$ in the first neighborhood of $\mathbf{Q}$, continue to examine $I_{\mathbf{Q}^*}(\overline{M}_1, L_1)$ by blowing up the curve $\mathbf{P}^1(s,u)$ at the point $\mathbf{Q}^*$. After k blow ups we will finally get

$$I_{\mathbf{Q}}(\overline{M}_0, L_0) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1), \tag{48}$$

where the sum is taken over the infinitely near singularities in all the neighborhoods of the point $\mathbf{Q}$ (not including $\mathbf{Q}$ itself). Invoking Lemma 4.7 again, we finally have

$$I_{\mathbf{Q}}(M, L) = I_{\mathbf{Q}}(\overline{M}_0, L_0) + I_{\mathbf{Q}}(h_0, L_0) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1), \tag{49}$$

where the sum is taken over all the infinitely near singularities of the point $\mathbf{Q}$ including $\mathbf{Q}$ itself. □

Remark 4.9. The procedure in the proof of Theorem 4.8 must stop after a finite number of blow ups; otherwise $I_{\mathbf{Q}}(\overline{M}, L)$ would be infinite, which is impossible. Therefore we have proved the following result.

Corollary 4.2. *Every singularity on a rational planar curve reduces to an ordinary singularity after a finite number of blow ups.*

Now by Lemma 4.6 and Theorem 4.8 we arrive directly at the result in Theorem 4.5.

Remark 4.10. The above proof also implies that the intersection of $M(s,u;t,v)$ and $L(s,u;t,v)$ (as well as the intersection of $F(s,u;t,v)$ and $G(s,u;t,v)$) gives the correct multiplicities for parameter pairs $(s_i,u_i;s_j,u_j)$ such that $h(s_i,u_i) = h(s_j,u_j) = 0$. That is,

$$I_{\mathbf{S}_{ij}}(M, L) = \begin{cases} \sum_{\mathbf{Q}^*} m_i^{\mathbf{Q}^*} m_j^{\mathbf{Q}^*}, & \text{if } (s_i,u_i) \neq (s_j,u_j), \\ \sum_{\mathbf{Q}^*} m_i^{\mathbf{Q}^*}(m_i^{\mathbf{Q}^*} - 1), & \text{if } (s_i,u_i) = (s_j,u_j), \end{cases} \tag{50}$$

where $m_i^{\mathbf{Q}^*}$ is the multiplicity of (s_i,u_i) as a parameter for the infinitely near singularity $\mathbf{Q}^*$, and the sum is taken over all the infinitely near singularities $\mathbf{Q}^*$ of $\mathbf{Q}$ including $\mathbf{Q}$ itself.

Corollary 4.3. *Let (s_0,u_0) and (s_1,u_1) be two parameters of multiplicity m_0 and m_1 corresponding to the same singularity $\mathbf{Q}$ on the curve $\mathbf{P}(s,u)$. Then*

- *If $(s_0,u_0) \neq (s_1,u_1)$, and if $I_{(s_0,u_0;s_1,u_1)}(F,G) > m_0 m_1$, then there are infinitely near singularities corresponding to the parameters (s_0,u_0) and (s_1,u_1);*
- *If $(s_0,u_0) = (s_1,u_1)$, and if $I_{(s_0,u_0;s_1,u_1)}(F,G) > m_0(m_0-1)$, then there are infinitely near singularities corresponding to the parameter (s_0,u_0).*

Proof. This result follows directly from Remark 4.10. Moreover, the first situation implies that the curve $\mathbf{P}(s,u)$ has parallel tangents at the two distinct parameters (s_0,u_0) and (t_0,v_0); while the second situation implies that the curve $\mathbf{P}(s,u)$ has extra multiple tangents at the parameter (s_0,u_0). □

Remark 4.11. Theorem 4.5 shows that $I_{\mathbf{Q}}(F,G) = 2\delta$, where in algebraic geometry $\delta = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1)/2$ is usually called the δ-invariant of a singularity $\mathbf{Q}$. The multiplicity m and the *δ-invariant* of a singularity $\mathbf{Q}$ can both be computed by the MAPLE command 'singularities'.

Corollary 4.4. *Let $H(s,u) \triangleq \mathrm{Res}_{t,v}(F(s,u;t,v), G(s,u;t,v))$. Then the roots of $H(s,u)$ are the parameters corresponding to the singularities on the rational planar curve $\mathbf{P}(s,u)$. Moreover, let (s_i,u_i) be all the distinct roots of $H(s,u)$ corresponding to the same singularity $\mathbf{Q}$, and let the multiplicity of (s_i,u_i) as a root of $H(s,u)$ be k_i. Then the δ-invariant of the singularity $\mathbf{Q}$ is $\delta = \sum_i k_i/2$.*

Proof. A parameter (s_0,u_0) is a root of $H(s,u)$ if and only if there exists a parameter (t_0,v_0) such that $F(s_0,u_0;t_0,v_0) = G(s_0,u_0;t_0,v_0) = 0$. Thus by Theorem 4.2, a root (s_0,u_0) of $H(s,u)$ must correspond to a singularity on the curve $\mathbf{P}(s,u)$.

Let $(s_i,u_i;s_j,u_j)$ be all the distinct common roots of $F(s,u;t,v)$ and $G(s,u;t,v)$ that correspond to the same singularity $\mathbf{Q}$ on the curve $\mathbf{P}(s,u)$. Then by Remark 4.11 and Definition 4.2,

$$\delta = I_{\mathbf{Q}}(F,G)/2 = \sum_{i,j} I_{\mathbf{S}_{ij}}(F,G)/2.$$

Since $H(s,u) \triangleq \mathrm{Res}_{t,v}(F(s,u;t,v), G(s,u;t,v))$,

$$\sum_j I_{\mathbf{S}_{ij}}(F,G) = k_i.$$

Therefore $\delta = \sum_i k_i/2$. □

Remark 4.12. Abhyankar showed in Abhyankar (1990, p. 153) that to find all the singularities of a planar polynomial curve $\mathbf{P}(s) = (a(s), b(s), 1)$, one need only study the Taylor t-resultant

$$\mathrm{Res}_\tau\left(a'(s) + \frac{a''(s)}{2!}\tau + \frac{a'''(s)}{3!}\tau^2 + \cdots,\ b'(s) + \frac{b''(s)}{2!}\tau + \frac{b'''(s)}{3!}\tau^2 + \cdots\right). \tag{51}$$

In our approach, since $\mathbf{L}_1(s) = (-1, 0, a(s))$ and $\mathbf{L}_2(s) = (0, -1, b(s))$ are a pair of syzygies for the curve $\mathbf{P}(s)$, we study the resultant $H(s)$ of the two polynomials

$$\begin{aligned} L_1(s,t) &\triangleq \frac{\mathbf{L}_1(s)\cdot\mathbf{P}(t)}{s-t} = \frac{a(s)-a(t)}{s-t} \\ &= a'(s) + \frac{a''(s)}{2!}(s-t) + \frac{a'''(s)}{3!}(s-t)^2 + \cdots, \\ L_2(s,t) &\triangleq \frac{\mathbf{L}_2(s)\cdot\mathbf{P}(t)}{s-t} = \frac{b(s)-b(t)}{s-t} \\ &= b'(s) + \frac{b''(s)}{2!}(s-t) + \frac{b'''(s)}{3!}(s-t)^2 + \cdots. \end{aligned} \tag{52}$$

Let

$$\begin{aligned} \tilde{L}_1(s,\tau) &\triangleq a'(s) + \frac{a''(s)}{2!}\tau + \frac{a'''(s)}{3!}\tau^2 + \cdots, \\ \tilde{L}_2(s,\tau) &\triangleq b'(s) + \frac{b''(s)}{2!}\tau + \frac{b'''(s)}{3!}\tau^2 + \cdots. \end{aligned}$$

Comparing the matrices $\mathrm{Syl}_t(L_1(s,t), L_2(s,t))$ and $\mathrm{Syl}_\tau(\tilde{L}_1(s,\tau), \tilde{L}_2(s,\tau))$ we can conclude that $\mathrm{Res}_t(L_1(s,t), L_2(s,t)) = \mathrm{Res}_\tau(\tilde{L}_1(s,\tau), \tilde{L}_2(s,\tau))$. Therefore, the polynomial $H(s)$ in our approach agrees with the t-resultant of Abhyankar when rational planar curves are replaced by the simpler case of polynomial planar curves. Thus our approach extends Abhyankar's method from polynomial curves to rational curves. For an alternative approach, see Buse (2009).

Theorem 4.5 shows that the intersection of the two algebraic curves $F(s,u;t,v) = 0$ and $G(s,u;t,v) = 0$ constructed from a parametrization and a μ-basis not only provides all the parameters of the singularities on the rational planar curve, but also provides the correct multiplicities of the singularities including all the infinitely near singularities. Next we shall apply Theorem 4.5 to prove a classical result in algebraic geometry.

Theorem 4.13. *Let $\nu_{\mathbf{Q}^*}$ denote the multiplicity of the point $\mathbf{Q}^*$ on a degree n rational planar curve. Then*

$$(n-1)(n-2) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1), \tag{53}$$

where the sum is taken over all the singularities on the curve including the infinitely near singularities.

Proof. By Theorem 4.2, a parameter pair $(s,u;t,v)$ corresponds to a singularity on the curve $\mathbf{P}(s,u)$ if and only if $F(s,u;t,v) = G(s,u;t,v) = 0$. By Theorems 4.2 and 4.5, the total number of intersections of $F(s,u;t,v)$ and $G(s,u;t,v)$ is

$$\sum_{\mathbf{Q}} I_{\mathbf{Q}}(F,G) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1), \tag{54}$$

where the sum is taken over all the singularities on the curve $\mathbf{P}(s,u)$ including the infinitely near singularities. On the other hand, let μ and $n-\mu$ be the degrees of a μ-basis for the curve $\mathbf{P}(s,u)$. By Bezout's Theorem on a product of two projective lines, the total number of intersections of $F(s,u;t,v) = 0$ and $G(s,u;t,v) = 0$ is $(n-1)(\mu-1) + (n-1)(n-\mu-1) = (n-1)(n-2)$. Therefore we conclude that

$$(n-1)(n-2) = \sum_{\mathbf{Q}^*} \nu_{\mathbf{Q}^*}(\nu_{\mathbf{Q}^*} - 1). \qquad \square \tag{55}$$

In classical algebraic geometry both sides of Eq. (53) are divided by 2. Our extra factor of 2 arises because by Remark 4.3 each parameter pair $(s, u; t, v)$ corresponding to a singular point $\mathbf{Q}$ is counted twice: once as $(s, u; t, v)$ and once as $(t, v; s, u)$.

5. Examples

In this section we shall illustrate our theorems in the previous section with two examples.

Example 5.1. Consider the planar rational quartic curve

$$\mathbf{P}(s,u) = \left(s^4 + s^3u - 5s^2u^2 - 3su^3 - 3u^4, s^4 - 2s^3u - 5s^2u^2 + 6su^3 + 6u^4, u^4\right).$$

A μ-basis for $\mathbf{P}(s,u)$ is given by:

$$\mathbf{p}(s,u) = \left(2u^2 - su, u^2 + su, 6s^2 - 9su\right),$$
$$\mathbf{q}(s,u) = \left(s^2 - 3su, -s^2 + 3u^2, 15s^2 - 27su - 18u^2\right).$$

Computing the two polynomials $F(s,u;t,v)$ and $G(s,u;t,v)$, we get

$$F(s,u;t,v) = \frac{\mathbf{p}(s,u) \cdot \mathbf{P}(t,v)}{sv - tu} = 6sv^3 - 3t^3u + 15tv^2u,$$
$$G(s,u;t,v) = \frac{\mathbf{q}(s,u) \cdot \mathbf{P}(t,v)}{sv - tu} = 3st^3 - 9stv^2 + 6sv^3 - 3t^3u + 6t^2vu + 15tv^2u - 18v^3u.$$

Now compute

$$H(s,u) \triangleq \mathrm{Res}_{(t,v)}\left(F(s,u;t,v), G(s,u;t,v)\right)$$
$$= -5832(s-u)(s-2u)(s+2u)(s+u)\left(s - \sqrt{3}u\right)\left(s + \sqrt{3}u\right).$$

Hence by Corollary 4.4 there are 6 parameters on the curve $\mathbf{P}(s,u)$ corresponding to singularities:

$$(s_1,u_1) = (1,1), \qquad (s_2,u_2) = (-2,1), \qquad (s_3,u_3) = (2,1),$$
$$(s_4,u_4) = (-1,1), \qquad (s_5,u_5) = \left(\sqrt{3},1\right), \qquad (s_6,u_6) = \left(-\sqrt{3},1\right).$$

Grouping these 6 parameters, we get three singular points:

$$\mathbf{Q}_1 = \mathbf{P}(s_1,u_1) = \mathbf{P}(s_2,u_2) = (-9:6:1),$$
$$\mathbf{Q}_2 = \mathbf{P}(s_3,u_3) = \mathbf{P}(s_4,u_4) = (-5:-2:1),$$
$$\mathbf{Q}_3 = \mathbf{P}(s_5,u_5) = \mathbf{P}(s_6,u_6) = (-9:0:1).$$

Next compute the inversion formula for $\mathbf{Q}_i$, $i = 1,2,3$:

$$h_1 = (su_1 - s_1u)\gcd\left(F(s,u;s_1,u_1), G(s,u;s_1,u_1)\right) = 6(s-1)(s+2),$$
$$h_2 = (su_3 - s_3u)\gcd\left(F(s,u;s_3,u_3), G(s,u;s_3,u_3)\right) = 6(s-2)(s+1),$$
$$h_3 = (su_5 - s_5u)\gcd\left(F(s,u;s_5,u_5), G(s,u;s_5,u_5)\right) = \left(s - \sqrt{3}\right)\left(s + \sqrt{3}\right).$$

Hence $\mathbf{Q}_i$, $i = 1,2,3$, are all double points with multiplicity $m_i = 2$. Since $\deg(\mathbf{P}) = 4$ and $(4-1)\times(4-2) = \sum_{i=1}^{3} m_i(m_i - 1) = 3 \times 2 = 6$, by Theorem 4.13 there are no infinitely near singularities arising from $\mathbf{Q}_i$, $i = 1,2,3$.

Fig. 5 shows the intersection of the two algebraic curves $F(s,u;t,u) = 0$ and $G(s,u;t,u) = 0$. Note that by Remark 4.3, the intersection points are symmetric with respect to the dashed line $s = t$ because if (s,t) corresponds to a singularity of $\mathbf{P}(s,u)$, then so too does (t,s). Fig. 6 shows the quartic rational planar curve $\mathbf{P}(s,u)$ with three double points.

Example 5.2. Consider the planar rational septic curve

$$\mathbf{P}(s,u) = \left((s-u)^2(s-2u)^2(s-3u)u^2, (s-u)^2(s-2u)^4(s-3u), u^7\right).$$

A μ-basis for $\mathbf{P}(s,u)$ is given by:

$$\mathbf{p}(s,u) = \left(-u^5, 0, s^5 - 9s^4u + 31s^3u^2 - 51s^2u^3 + 40su^4 - 12u^5\right),$$
$$\mathbf{q}(s,u) = \left(s^2 - 4su + 4u^2, -u^2, 0\right).$$

Computing the two polynomials $F(s,u;t,v)$ and $G(s,u;t,v)$, we get:

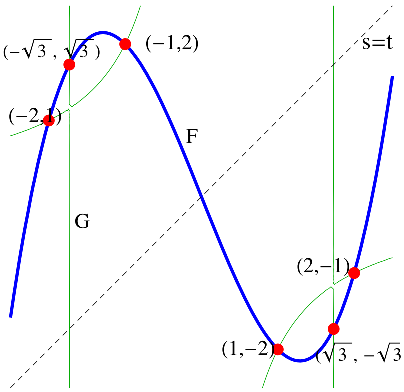


Fig. 5. Intersection of the two algebraic curves $F(s,u;t,u)$ and $G(s,u;t,u)$.

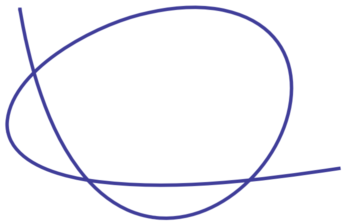


Fig. 6. Rational quartic planar curve with three double points.

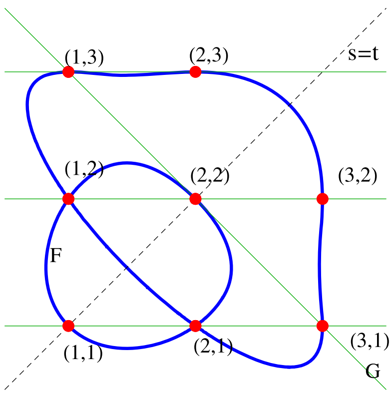


Fig. 7. Intersection of the two algebraic curves $F(s,1;t,1)$ and $G(s,1;t,1)$.

$$\begin{aligned}
F(s,u;t,v) &= \frac{\mathbf{p}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu} \\
&= \big(v^4s^4 - 9s^3uv^4 + s^3tuv^3 + 31s^2u^2v^4 - 9s^2tu^2v^3 + s^2t^2u^2v^2 \\
&\quad - 51su^3v^4 + 31stu^3v^3 - 9st^2u^3v^2 + st^3u^3v + 40u^4v^4 - 51tu^4v^3 \\
&\quad + 31t^2u^4v^2 - 9t^3u^4v + t^4u^4\big)v^2, \\
G(s,u;t,v) &= \frac{\mathbf{q}(s,u)\cdot\mathbf{P}(t,v)}{sv-tu} \\
&= svt^5 - 9st^4v^2 - 12sv^6 + 31st^3v^3 - 51st^2v^4 + 40stv^5 - 172v^5tu \\
&\quad + 48v^6u + 67v^2ut^4 - 13vut^5 + 244v^4ut^2 - 175v^3ut^3 + t^6u.
\end{aligned}$$

Fig. 7 shows the intersection of the two algebraic curves $F(s,1;t,1)$ and $G(s,1;t,1)$. Note that $G(s,1;t,1)$ can be factored into three double lines $t-1=0$, $t-2=0$, $t-3=0$ and a line $s+t-4=0$. Again notice that the intersection points (s,t) are symmetric with respect to the dashed line $s=t$.

Table 1
Parameters corresponding to infinitely near singularities.

Singularity	Intersection parameter	Actual multiplicity	Expected multiplicity	Infinitely near singularities?
	(1, 1; 1, 1)	2	2 × 1	
	(2, 1; 2, 1)	4	2 × 1	Yes
	(1, 1; 2, 1)	4	2 × 2	
$\mathbf{Q}_1$	(2, 1; 1, 1)	4	2 × 2	
	(1, 1; 3, 1)	3	2 × 1	Yes
	(3, 1; 1, 1)	3	1 × 2	Yes
	(2, 1; 3, 1)	2	2 × 1	
	(3, 1; 2, 1)	2	1 × 2	
$\mathbf{Q}_2$	(1, 0; 1, 0)	6	2 × 1	Yes

Computing

$$\begin{aligned} H(s,u) &\triangleq \mathrm{Res}_{(t,v)}\big(F(s,u;t,v), G(s,u;t,v)\big) \\ &= (s-u)^9(s-2u)^{10}(s-3u)^5u^6, \end{aligned}$$

we find that there are 4 different parameters on the curve $\mathbf{P}(s,u)$ corresponding to singularities:

$$(s_1,u_1) = (1,1), \qquad (s_2,u_2) = (2,1), \qquad (s_3,u_3) = (3,1), \qquad (s_4,u_4) = (1,0).$$

Grouping these four parameters, we get two singularities

$$\begin{aligned} &\mathbf{Q}_1 = \mathbf{P}(s_1,u_1) = \mathbf{P}(s_2,u_2) = \mathbf{P}(s_3,u_3) = (0:0:1), \\ &\mathbf{Q}_2 = \mathbf{P}(s_4,u_4) = (1:1:0). \end{aligned}$$

The inversion formulas for $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are:

$$\begin{aligned} &h_1 = (su_1 - s_1u)\gcd\big(F(s,u;s_1,u_1), G(s,u;s_1,u_1)\big) = (s-u)^2(s-2u)^2(s-3u), \\ &h_2 = (su_4 - s_4u)\gcd\big(F(s,u;s_4,u_4), G(s,u;s_4,u_4)\big) = u^2. \end{aligned}$$

Hence the point $\mathbf{Q}_1$ has multiplicity $m_1 = 5$ and the point $\mathbf{Q}_2$ has multiplicity $m_2 = 2$.

Next we check whether there are infinitely near singularities arising from $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Let k_i be the highest power of $(su_i - s_iu)$ that divides $H(s,u)$. Then $k_1 = 9$, $k_2 = 10$, $k_3 = 5$, $k_4 = 6$. Therefore, the δ-invariants of the point $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are $\delta_1 = (k_1+k_2+k_3)/2 = 12$ and $\delta_2 = k_4/2 = 3$. Note that $12 = \delta_1 > m_1(m_1-1)/2 = 10$ and $3 = \delta_2 > m_2(m_2-1)/2 = 1$. Therefore, there are infinitely near singularities arising form both $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

Now we need to find the parameters corresponding to the infinitely near singularities. To find these parameters, we must compute the intersection multiplicity k_{ij} of the two algebraic curves $F = 0$ and $G = 0$ at each parameter pair $(s_i,u_i;s_j,u_j)$. Observing Fig. 7, we can see that there are several different intersections of $F = 0$ and $G = 0$ along the vertical lines $(s,u) = (1,1)$, $(s,u) = (2,1)$ and $(s,u) = (3,1)$. So we rotate the coordinates by setting

$$s = \frac{1}{2}\bar{s} + \frac{\sqrt{3}}{2}\bar{t}, \qquad t = \frac{\sqrt{3}}{2}\bar{s} - \frac{1}{2}\bar{t}.$$

Then the two polynomials $F(s,1;t,1)$ and $G(s,1;t,1)$ become $\widetilde{F}(\bar{s},1;\bar{t},1)$ and $\widetilde{G}(\bar{s},1;\bar{t},1)$, and

$$\begin{aligned} H(\bar{s}) &\triangleq \mathrm{Res}_{\bar{t}}\left(\widetilde{F}(\bar{s},1;\bar{t},1), \widetilde{G}(\bar{s},1;\bar{t},1)\right) \\ &= \frac{1}{1048576}\big(-2\bar{s}+3+2\sqrt{3}\big)^2\big(-2\bar{s}+1+\sqrt{3}\big)^2\big(-2\bar{s}+2+3\sqrt{3}\big)^2 \\ &\quad\times\big(-2\bar{s}+1+3\sqrt{3}\big)^3\big(-2\bar{s}+3+\sqrt{3}\big)^3\big(-2\bar{s}+2+\sqrt{3}\big)^4\big(-2\bar{s}+1+2\sqrt{3}\big)^4\big(-\bar{s}+1+\sqrt{3}\big)^4. \end{aligned}$$

The parameter pairs $(s_i,u_i;s_j,u_j)$ are also changed under rotation. For example, the parameter pair $(s_1,u_1;s_2,u_2) = (1,1;2,1)$ becomes $(\frac{1}{2}+\sqrt{3},1;\frac{\sqrt{3}}{2}-1,1)$. Now the intersection multiplicity for the parameter pair $(s_1,u_1;s_2,u_2)$ is the highest power of $2\bar{s}-1-2\sqrt{3}$ that divides $H(\bar{s})$, i.e., $k_{12} = 4$. Similarly, we can get the other values of k_{ij} directly from the expression for $H(\bar{s})$.

Table 1 compares the actual intersection multiplicity k_{ij} and the expected multiplicity of each parameter pair $(s_i,u_i;s_j,u_j)$ as an intersection of the two algebraic curves $F(s,u;t,v) = 0$ and $G(s,u;t,v) = 0$. Note that if there is no corresponding infinitely near point, then the expected multiplicity is m_im_j if $(s_i,u_i) \neq (s_j,u_j)$ and $m_i(m_i-1)$ if $(s_i,u_i) = (s_j,u_j)$.

From Table 1 we can see that for $\mathbf{Q}_1$ there are infinitely near singular points arising from the parameters $(2,1;2,1)$, $(1,1;3,1)$ and $(3,1;1,1)$, whose expected multiplicities are less than the actual multiplicities as intersections of

(a) Rational planar curve with a singularity of multiplicity 5.

(b) Blow up curve from the point $\mathbf{Q}_1$ with two infinitely near double points.

Fig. 8. Rational planar curve and its blow up curve for the point $\mathbf{Q}_1 = (0:0:1)$.

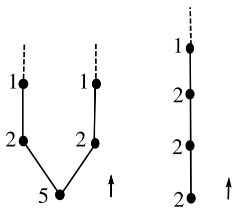


Fig. 9. Singularity trees of the singular points $(0:0:1)$ (left) and $(0:1:0)$ (right) of the rational planar curve $\mathbf{P}(s,u) = ((s-u)^2(s-2u)^2(s-3u)u^2, (s-u)^2(s-2u)^4(s-3u), u^7)$.

$F(s,u;t,v) = 0$ and $G(s,u;t,v) = 0$. This observation suggests that the curve $\mathbf{P}(s,u)$ has a multiple tangent at $(s,u) = (2,1)$, and has parallel tangents at the two distinct parameters $(s,u) = (1,1)$ and $(s,u) = (3,1)$. Thus $\mathbf{Q}_1$ has two infinitely near double points. Fig. 8(a) shows the singularity $\mathbf{Q}_1$ of multiplicity 5 on the rational planar curve $\mathbf{P}(s,u)$. Fig. 8(b) shows the two infinitely near double points of the blow up curve $\mathbf{P}^1(s,u)$.

For the other double point $\mathbf{Q}_2$, from the expression for $H(s,u)$ we get $k_{44} = 6$, but $m_4(m_4 - 1) = 2 \times 1 = 2$. Therefore there are also two infinitely near double points to $\mathbf{Q}_2$.

Fig. 9 shows the singularity trees of the singular points $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

Finally note that the theorems described in this paper for degree n rational planar curves $\mathbf{P}(s,u) = (a(s,u), b(s,u), c(s,u))$ remain valid if we replace the μ-basis $\mathbf{p}(s,u), \mathbf{q}(s,u)$ by the two obvious syzygies $(-c(s,u), 0, a(s,u))$ and $(0, -c(s,u), b(s,u))$. The proofs are essentially the same. The main computational advantages of using these two moving lines instead of a μ-basis is that we do not need to calculate a μ-basis. But we pay a price in this approach because the algebraic curves

$$
\begin{aligned}
F(s,u;t,v) &\triangleq \frac{-a(t,v)c(s,u) + c(t,v)a(s,u)}{sv - tu}, \\
G(s,u;t,v) &\triangleq \frac{-b(t,v)c(s,u) + c(t,v)b(s,u)}{sv - tu}
\end{aligned}
\tag{56}
$$

have $n(n-1)$ extraneous roots where $c(s,u) = c(t,v) = 0$, and these roots must be discarded in order to correctly compute the singularities of $\mathbf{P}(s,u)$. Thus in this approach the degrees of F and G are unnecessarily high. Hence the main computational advantage of the μ-basis approach is that there are no extraneous roots and the associated algebraic curves $F(s,u;t,v)$ and $G(s,u;t,v)$ have minimal degree.

6. Conclusions and future research

We have shown how to apply μ-bases to construct two algebraic curves whose intersection points are the parameters with proper multiplicity of the singularities of a given rational planar curve. This approach extends Abhyankar's method of t-resultants from planar polynomial curves to rational planar curves. We have also derived the classical result that for a rational planar curve of degree n the sum of all the singularities including infinitely near singularities with proper multiplicity is $(n-1)(n-2)/2$. Based on these results we have developed algorithms to find all the singularities for rational planar curves. Notice, however, that in these algorithms, we first need to solve for the roots of $H(s,u) = \mathrm{Res}_{t,v}(F(s,u;t,v), G(s,u;t,v))$, for which we can typically get only approximate solutions. Therefore our approach sometimes suffers from numerical problems. We hope to address these numerical issues in our subsequent research.

One way to address these numerical issues would be to avoid altogether the computation of the roots of $H(s,u)$ and instead to prove the conjecture of Chen, Wang and Liu, using the Smith form of the Bezout matrix of the μ-basis to compute the parameters both for the singular points and for the infinitely near singular points. Since our current methods allow us to use resultants to compute the parameters corresponding to infinitely near singular points, we plan to provide such a proof in a sequel to this paper – see Jia and Goldman (2009).

In the future we also hope to extend our results both to rational spaces curves and to rational surfaces. For rational space curves, we would like to develop similar techniques using μ-bases to find all the singularities with their proper multiplicities. We also hope to find some tight bounds on the number of these singularities with their proper multiplicities. For rational surfaces, we plan first to extend our techniques to rational ruled surfaces, where the theory of computing the singular locus of rational ruled surfaces via μ-bases is already well developed (Jia et al., 2009). Here we expect to use the method of μ-bases to find the singular locus and to characterize the multiplicity of each point on this locus. Eventually, when the theory of μ-bases is fully developed, we hope to extend these results to arbitrary rational surfaces.

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