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Unirational differential curves and differential rational parametrizations

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ABSTRACT

In this paper, we study unirational differential curves and the corresponding differential rational parametrizations. We first investigate basic properties of proper differential rational parametrizations for unirational differential curves. Then we show that the implicitization problem of proper linear differential rational parametric equations can be solved by means of differential resultants. Furthermore, for linear differential curves, we give an algorithm to determine whether an implicitly given linear differential curve is unirational, and in the affirmative case, to compute a proper differential rational parametrization for the differential curve.

1. Introduction

The study of unirational varieties and the corresponding rational parametrizations is a basic topic in computational algebraic geometry. The central problem in this field is to determine whether an algebraic variety is rationally parametrizable, and, for unirational varieties, to give efficient algorithms to transform between the implicit representations and parametric representations. These problems have been fully explored for algebraic curves by Sendra et al. (2007) using symbolic computation methods. They are also well-understood for algebraic surfaces (Hartshorne, 1977; Schicho, 1998).

The differential implicitization and rational parametrization problems for differential varieties are basic problems in differential algebraic geometry. Differential varieties with differential rational parametrizations are called unirational. The differential parametrization problem is to decide whether

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an implicitly given differential variety is unirational, and to find a differential rational parametrization in the affirmative case. The study of this problem plays an important role in the classification problem of differential varieties up to differentially birational equivalence and also has potential applications in other fields. For instance, assessing differential flatness of control systems is closely related to the parametrization problem (Fliess and Glad, 1993; Fliess et al., 1995). The differential implicitization of differential rational parametric equations was first studied via the differential characteristic set method by Gao (2003). In the special case when given linear differential polynomial parametric equations, it was treated via linear complete differential resultants (Rueda and Sendra, 2007; Rueda, 2011).

However, as far as we know, there are still no general results on the differential parametrization problem. The work of Feng and Gao (2006), on finding rational general solutions for a univariate algebraic ODE $f(y) = 0$, is the first step in solving the rational parametrization problem for differential varieties. Their work gives necessary and sufficient conditions for an ODE to have a rational general solution, and a polynomial algorithm to compute the rational general solution of a first order autonomous ODE if it exists. Then the subsequent work by Winkler and his coauthors extended the method to study rational general solutions for non-autonomous parametrizable first-order ODEs (Ngô and Winkler, 2010, 2011), higher order ODEs (Huang et al., 2013) and even partial differential equations (Grasegger et al., 2018). While these are important contributions on the parametrization of zero-dimensional differential varieties in the one-dimensional space $\mathbb{A}^1$ (Winkler, 2019), it seems that the rational parametrization problem for differential varieties of positive differential dimension has not been studied.

In this paper, we study unirational ordinary differential curves and the corresponding differential rational parametrizations. A (plane) irreducible differential curve $\mathcal{C}$ is a one-dimensional irreducible differential variety in $\mathbb{A}^2$. The differential characteristic set method guarantees the unique existence of an irreducible differential polynomial $A(x, y) \in \mathcal{F}\{x, y\}$ such that $\mathcal{C}$ is the general component of $A(x, y) = 0$, thus this differential curve is often represented by $(\mathcal{C}, A)$. If $(\mathcal{C}, A)$ has a generic point of the form $\mathcal{P}(u) \in \mathcal{F}\langle u\rangle^2$ with u a differential parameter, it is called a unirational differential curve and $\mathcal{P}(u)$ is called a differential rational parametrization of $\mathcal{C}$. $\mathcal{P}(u)$ is called proper if it defines a differential birational map between $\mathbb{A}^1$ and $\mathcal{C}$. The differential Lüroth theorem guarantees that unirational differential curves always have proper differential rational parametrizations (Gao, 2003).

For unirational differential curves, we first explore basic properties of proper differential rational parametrizations. In particular, Theorem 15 gives the order property of properness and Theorem 18 shows that proper parametrizations are unique up to Möbius transformations. These results extend similar properties of proper parametrizations of algebraic curves to their differential counterparts. For proper linear differential rational parametrizations, we give further properties and show differential resultants can be used to compute the corresponding implicit equations of proper linear differential rational parametric equations. This could be considered a generalization of Rueda-Sendra's work on implicitization of linear differential polynomial parametric equations via linear complete differential resultants (Rueda and Sendra, 2007).

Concerning the rational parametrizability problem, it is well known that an algebraic curve is unirational if and only if its genus is equal to 0 (Sendra et al., 2007, Theorem 4.63), so the determination of unirationality of algebraic curves can be reduced to the computation of the genus of the curve. Compared with the algebraic case, the rational parametrizability problem of differential curves is much more complicated to deal with. More precisely, the problem can be stated as follows: given an irreducible differential polynomial $A(x, y)$, decide whether the differential curve $(\mathcal{C}, A)$ is unirational, and if it is unirational, give efficient algorithms to compute a parametrization. In this paper, we start from the simplest nontrivial case by considering linear differential curves. And for linear differential curves, we give an algorithm to determine whether the implicitly given differential curve is unirational, and, in the affirmative case, to compute a proper linear differential polynomial parametrization for the unirational linear differential curve.

This paper is organized as follows. In Section 2, we introduce some notions and preliminary results in differential algebra. In Section 3, we explore the basic properties of proper differential rational parametrizations for unirational differential curves. In Section 4, further properties of proper linear

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differential rational parametrizations are given, and in particular, the corresponding implicit equations can be computed via the method of differential resultants. In Section 5, we deal with the problem of algorithmically deciding whether an implicitly given linear differential curve is unirational and computing a proper rational parametrization in the affirmative case. In Section 6, we propose several problems for further study.

2. Preliminaries

2.1. Differential polynomial algebra and characteristic sets

In this section, we will introduce the basic notions and notation to be used in this paper. For more details about differential algebra, please refer to (Ritt, 1950; Kolchin, 1973; Buium and Cassidy, 1998; Sit, 2002).

Let $\mathcal{F}$ be an ordinary differential field of characteristic 0 with derivation δ. For example, $\mathcal{F}=\mathbb{Q}(t)$ with $\delta=\frac{\mathrm{d}}{\mathrm{d}t}$. An element $c\in\mathcal{F}$ such that $\delta(c)=0$ is called a constant of $\mathcal{F}$. The set of all constants of $\mathcal{F}$ constitutes a differential subfield of $\mathcal{F}$, called the field of constants of $\mathcal{F}$ and denoted by $C_{\mathcal{F}}$. For an element a in $\mathcal{F}$, we use $a', a'', a^{(k)}$ to indicate the derivatives $\delta(a), \delta^2(a), \delta^k(a)\ (k\geq 3)$.

Let $\mathcal{G}$ be a differential extension field of $\mathcal{F}$. A family $\Sigma\subset\mathcal{G}$ is said to be *differentially algebraically dependent over* $\mathcal{F}$ if the family $\Theta(\Sigma):=\{\delta^k(a): a\in\Sigma, k\in\mathbb{N}\}$ is algebraically dependent over $\mathcal{F}$. Otherwise, Σ is said to be *differentially algebraically independent over* $\mathcal{F}$, or a family of *differential indeterminates over* $\mathcal{F}$. An element $\alpha\in\mathcal{G}$ that is differentially algebraically dependent over $\mathcal{F}$ is called *differentially algebraic* over $\mathcal{F}$ (otherwise, *differentially transcendental* over $\mathcal{F}$). A maximal subset Ω of $\mathcal{G}$ that is differentially algebraically independent over $\mathcal{F}$ is said to be a *differential transcendence basis* of $\mathcal{G}$ over $\mathcal{F}$. The cardinality of Ω is the *differential transcendence degree* of $\mathcal{G}$ over $\mathcal{F}$, denoted by $\mathrm{d.tr.deg}\,\mathcal{G}/\mathcal{F}$. The transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ is denoted by $\mathrm{tr.deg}\,\mathcal{G}/\mathcal{F}$. Given $S\subset\mathcal{G}$, we denote respectively by $\mathcal{F}\{S\}=\mathcal{F}[\Theta(S)]$ and $\mathcal{F}\langle S\rangle=\mathcal{F}\big(\Theta(S)\big)$ the smallest differential subring and differential subfield of $\mathcal{G}$ containing $\mathcal{F}$ and S.

Let $\mathcal{E}$ be a fixed universal differential extension field of $\mathcal{F}$ (Kolchin, 1973, p. 134). Let $x, y, y_1, \ldots, y_n$ be a set of differential indeterminates over $\mathcal{E}$. Consider the differential polynomial ring $\mathcal{F}\{y_1,\ldots,y_n\}=\mathcal{F}[y_j^{(k)}: j=1,\ldots,n; k\in\mathbb{N}]$. If $n=2$, we usually use the notation $\mathcal{F}\{x,y\}$ instead. A differential ideal in $\mathcal{F}\{y_1,\ldots,y_n\}$ is an ordinary algebraic ideal closed under δ. A prime differential ideal is a differential ideal which is prime as an ordinary ideal. For $\Sigma\subset\mathcal{F}\{y_1,\ldots,y_n\}$, the differential ideal in $\mathcal{F}\{y_1,\ldots,y_n\}$ generated by Σ is denoted by $[\Sigma]$. Let $f\in\mathcal{F}\{y_1,\ldots,y_n\}$. For each y_j, the order of f w.r.t. y_j is defined to be the largest number k such that $y_j^{(k)}$ appears effectively in f, denoted by $\mathrm{ord}_{y_j}f$, and in case y_j and its derivatives do not appear in f, we set $\mathrm{ord}_{y_j}f=-\infty$. The *order* of f is defined to be $\max_{j=1}^n\{\mathrm{ord}_{y_j}f\}$, denoted by $\mathrm{ord}(f)$.

Let $\mathbb{A}^n(\mathcal{E})$ denote the n-dimensional *differential affine space* over $\mathcal{E}$. Let Σ be a subset of differential polynomials in $\mathcal{F}\{y_1,\ldots,y_n\}$. A point $\eta=(\eta_1,\ldots,\eta_n)\in\mathbb{A}^n(\mathcal{E})$ is called a differential zero of Σ if $f(\eta)=0$ for any $f\in\Sigma$. The set of differential zeros of Σ is denoted by $\mathbb{V}(\Sigma)$, which is called a *differential variety* defined over $\mathcal{F}$. For a differential variety V, we denote $\mathbb{I}(V)$ to be the set of all differential polynomials in $\mathcal{F}\{y_1,\ldots,y_n\}$ that vanish at every point of V. Clearly, $\mathbb{I}(V)$ is a radical differential ideal in $\mathcal{F}\{y_1,\ldots,y_n\}$. Similarly as in algebraic geometry, $\mathbb{V}(\mathbb{I}(V))=V$. A differential variety V is said to be irreducible if it is not the union of two proper differential subvarieties, or equivalently, $\mathbb{I}(V)$ is a prime differential ideal. A point $\eta\in\mathbb{A}^n(\mathcal{E})$ is called a *generic point* of a prime differential ideal P (or $\mathbb{V}(P)$) if $\mathbb{I}(\eta)=P$. It is well-known that a non-unit differential ideal is prime if and only if it has a generic point (Ritt, 1950, p. 27).

A ranking $\mathscr{R}$ of $\mathcal{F}\{y_1,\ldots,y_n\}$ is a total ordering $<$ on the set of derivatives $\Theta(y)\triangleq\{y_j^{(k)}: j=1,\ldots,n; k\in\mathbb{N}\}$ that is compatible with the derivation: 1) $w<\delta w$ and 2) $w<v\Rightarrow\delta w<\delta v$ for all $w, v\in\Theta(y)$. A ranking $\mathscr{R}$ is called an *elimination ranking* if $y_i<y_j$ implies that $\delta^k y_i<\delta^l y_j$ for all $k,l\geq 0$. Let g be a differential polynomial in $\mathcal{F}\{y_1,\ldots,y_n\}$ not in $\mathcal{F}$ and $\mathscr{R}$ be a ranking endowed on it. The highest derivative w.r.t. $\mathscr{R}$ which appears effectively in g is called the *leader* of g and is denoted by $\mathrm{ld}(g)$. Let d be the degree of g in $\mathrm{ld}(g)$. We may rewrite g as a univariate polynomial in $\mathrm{ld}(g)$. Then

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$$g = I_d \mathrm{ld}(g)^d + I_{d-1}\mathrm{ld}(g)^{d-1} + \cdots + I_0. \tag{1}$$

The leading coefficient I_d is called the *initial* of g and is denoted by I_g. The partial derivative $\frac{\partial g}{\partial \,\mathrm{ld}(g)}$ is called the *separant* of g and is denoted by S_g. The pair $(\mathrm{ld}(g), d)$ is called the *rank* of g and is denoted by $\mathrm{rk}(g)$. Let f and g be two differential polynomials and $\mathrm{rk}(g) = (\mathrm{ld}(g), d)$. We say f is *partially reduced* w.r.t. g if no proper derivative of $\mathrm{ld}(g)$ appears in f, and f is *reduced* w.r.t. g if f is partially reduced w.r.t. g and $\deg_{\mathrm{ld}(g)} f < d$. Let $\mathcal{A}$ be a set of differential polynomials not intersecting $\mathcal{F}$. $\mathcal{A}$ is said to be an *autoreduced set* if each element of $\mathcal{A}$ is reduced w.r.t. every other one. Every autoreduced set is finite.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ be an autoreduced set, and f be a differential polynomial. There exists a reduction algorithm (Kolchin, 1973, p. 79, Proposition 1), called Ritt-Kolchin's Remainder Algorithm, which reduces f to a differential polynomial r such that r is reduced w.r.t. $\mathcal{A}$. More precisely, there exist $d_i, e_i \in \mathbb{N}$ such that

$$\prod_{i=1}^{t} \mathrm{S}_{A_i}^{d_i} \mathrm{I}_{A_i}^{e_i} \cdot f \equiv r, \bmod [\mathcal{A}]. \tag{2}$$

This r is called the *Ritt-Kolchin remainder* of f w.r.t. $\mathcal{A}$. Denote $\mathrm{H}_{\mathcal{A}} = \prod_{i=1}^{t} \mathrm{S}_{A_i}\mathrm{I}_{A_i}$. The *saturation ideal* of $\mathcal{A}$ is defined as

$$\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} = \{f \in \mathcal{F}\{x, y\} \mid \exists m \in \mathbb{N}, \text{such that } \mathrm{H}_{\mathcal{A}}^{m} f \in [\mathcal{A}]\}. \tag{3}$$

Let $S \subseteq \mathcal{F}\{y_1, \ldots, y_n\} \setminus \mathcal{F}$. An autoreduced set $\mathcal{A}$ contained in S is said to be a *characteristic set* of S if S does not contain any nonzero element reduced w.r.t. $\mathcal{A}$. A characteristic set $\mathcal{A}$ of a proper differential ideal $\mathcal{I}$ reduces to zero all elements of $\mathcal{I}$. If additionally $\mathcal{I}$ is prime, $\mathcal{A}$ reduces to zero only the elements of $\mathcal{I}$ and $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ (Kolchin, 1973, p. 167, Lemma 2).

Suppose $\mathcal{A} = \{A_1, \ldots, A_t\}$ is an autoreduced set with $\mathrm{ld}(A_1) < \cdots < \mathrm{ld}(A_t)$. We call $\mathcal{A}$ *irreducible* if for each i, there cannot exist any relation of the form $T_i A_i \equiv B_i C_i \mod (A_1, \ldots, A_{i-1})$, where B_i, C_i and T_i are differential polynomials reduced w.r.t. $A_1, \ldots, A_{i-1}$ with $\mathrm{ld}(B_i) = \mathrm{ld}(C_i) = \mathrm{ld}(A_i)$ and $\mathrm{ld}(T_i) < \mathrm{ld}(A_i)$. Obviously, if $\mathrm{rk}(A_i) = (\mathrm{ld}(A_i), 1)$ for each i, then $\mathcal{A}$ is an irreducible autoreduced set. Irreducible autoreduced sets can characterize prime differential ideals.

Lemma 1. *((Ritt, 1950, p. 89 and p. 107), (Wu, 1989)) Let $\mathcal{A}$ be an autoreduced set. Then a necessary and sufficient condition for $\mathcal{A}$ to be a characteristic set of a prime differential ideal is that $\mathcal{A}$ is irreducible. In the case $\mathcal{A}$ is irreducible,* $\mathrm{sat}(\mathcal{A})$ *is prime with a characteristic set $\mathcal{A}$.*

Definition 2. ((Kolchin, 1947), (Gao et al., 2013, Section 2.3)) Let P be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$ with a generic point $\eta = (\eta_1, \ldots, \eta_n)$.

- The *differential dimension* of P or $\mathbb{V}(P)$ is defined as the differential transcendence degree of $\mathcal{F}\langle\eta\rangle$ over $\mathcal{F}$.
- A *parametric set* of P is a subset $\{y_i \mid i \in I\} \subset \{y_1, \ldots, y_n\}$ such that $\{\eta_i \mid i \in I\}$ is a differential transcendence basis of $\mathcal{F}\langle\eta\rangle$ over $\mathcal{F}$.
- The *relative order* of P with respect to a parametric set $U = \{y_i \mid i \in I\}$, denoted by $\mathrm{ord}_U P$, is defined as

  $$\mathrm{ord}_U P = \mathrm{tr.deg}\,\mathcal{F}\langle\eta\rangle / \mathcal{F}\langle\eta_i : i \in I\rangle.$$

  If $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ is a characteristic set of P under an elimination ranking with $\mathrm{ld}(A_i) = y_{c_i}^{(o_i)}$, then $U = \{y_1, \ldots, y_n\} \setminus \{y_{c_1}, \ldots, y_{c_\ell}\}$ is a parametric set of P and $\mathrm{ord}_U P = \sum_{i=1}^{\ell} o_i$.

Prime differential ideals whose characteristic sets consist of a single differential polynomial are of particular interest to us. The following result on Ritt's general component theorem will often be used in this paper.

Lemma 3. *(Ritt, 1950, p. 30 and p. 45) Let $A \in \mathcal{F}\{y_1, \dots, y_n\} \setminus \mathcal{F}$ be an irreducible differential polynomial and S_A be the separant of A under some ranking. Then we have*

(1) $\mathrm{sat}(A) = [A] : \mathrm{S}_A^\infty$ *is a prime differential ideal of differential dimension $n-1$ and $\{A\}$ is a characteristic set of* $\mathrm{sat}(A)$ *under any ranking. In particular, if $B \in \mathrm{sat}(A)$ and $\mathrm{ord}(B) \le \mathrm{ord}(A)$, then B is divisible by A. We call* $\mathrm{sat}(A)$ *the general component of A.*
(2) *Conversely, any prime differential ideal in $\mathcal{F}\{y_1, \dots, y_n\}$ of differential dimension $n-1$ is the general component of an irreducible differential polynomial.*

2.2. Pure differential transcendental extension and differential Lüroth's theorem

In this section, we suppose $u \in \mathcal{E}$ is differentially transcendental over $\mathcal{F}$ and consider the pure differential transcendental extension field $\mathcal{F}\langle u\rangle$. Let $P, Q \in \mathcal{F}\{u\}$ be nonzero. The fraction P/Q is called in reduced form if $\gcd(P(u), Q(u)) = 1$. Given $R(u) \in \mathcal{F}\langle u\rangle$ with $R(u) = \frac{P(u)}{Q(u)}$ in reduced form, the *order* of $R(u)$ is defined to be $\max\{\mathrm{ord}_u P, \mathrm{ord}_u Q\}$, denoted by $\mathrm{ord}_u(R(u))$ or simply by $\mathrm{ord}(R(u))$. Clearly, it is well-defined. Throughout this paper, $\frac{P(u)}{Q(u)} \in \mathcal{F}\langle u\rangle$ is always assumed to be in reduced form.

Concerning the differential field extension $\mathcal{F} \subsetneq \mathcal{G} \subseteq \mathcal{F}\langle u\rangle$, it is well-known that the algebraic theorem of Lüroth has a differential analog, called Differential Lüroth's theorem, which states that $\mathcal{G}$ is always a simple extension of $\mathcal{F}$ (Kolchin, 1944).

Theorem 4 *(Differential Lüroth's theorem). Let $\mathcal{F}$ be an ordinary differential field of characteristic 0 with u a differential indeterminate over $\mathcal{F}$ and let $\mathcal{G}$ be a differential field such that $\mathcal{F} \subsetneq \mathcal{G} \subseteq \mathcal{F}\langle u\rangle$. Then there exists an element $\omega \in \mathcal{G}$ such that $\mathcal{G} = \mathcal{F}\langle \omega\rangle$.*

Such an element ω in Theorem 4 is called a *Lüroth generator* of $\mathcal{G}/\mathcal{F}$. Following Kolchin's proof in (Kolchin, 1947) (the proof was first given in (Kolchin, 1944) and corrected in (Kolchin, 1947)), if $A(y) \in \mathcal{G}\{y\}$ is the minimal differential polynomial of u over $\mathcal{G}$ (i.e., with lowest order and degree in $\mathcal{G}\{z\}$ such that $A(u) = 0$), then for any pair $(a, b) \in \mathcal{F}^2$ of coefficients of A satisfying that $a/b \notin \mathcal{F}$, this a/b can serve as a Lüroth generator. Based on Kolchin's idea and Wu-Ritt's zero decomposition theorem, Gao and Xu (2002) gave an algorithmic process to compute a Lüroth generator (also see (Gao, 2003)).

Remark 5. We describe the algorithmic process to compute a Lüroth generator in the case $\mathcal{G} = \mathcal{F}\left\langle \frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)} \right\rangle$ for the sake of later use in section 4. Consider $f_1 = Q_1(u)x - P_1(u)$, $f_2 = Q_2(u)y - P_2(u) \in \mathcal{F}\{u, x, y\}$ and the prime differential ideal $\mathcal{P} = [f_1, f_2] : (Q_1Q_2)^\infty \subset \mathcal{F}\{u, x, y\}$. Then $\{f_1, f_2\}$ is a characteristic set of $\mathcal{P}$ w.r.t. the elimination ranking $u < x < y$. Compute a characteristic set $B_1(x, y), B_2(x, y, u)$ of $\mathcal{P}$ w.r.t. the elimination ranking $x < y < u$ with Wu-Ritt's zero decomposition theorem (Wu, 1989). Rewrite $B_2(x, y, u) = \sum_i g_i(x, y)\theta_i(u)$ as a differential polynomial in u, and let $\bar{B}(z) = B_2(P_1(u)/Q_1(u), P_2(u)/Q_2(u), z) \in \mathcal{G}\{z\}$.

1) By the proof of (Gao, 2003, Theorem 6.2), $\bar{B}(z)$ is the minimal differential polynomial of u over $\mathcal{G}$ and if $\zeta = \frac{g_{i_1}(P_1(u)/Q_1(u), P_2(u)/Q_2(u))}{g_{i_2}(P_1(u)/Q_1(u), P_2(u)/Q_2(u))} \notin \mathcal{F}$ for some indices i_1, i_2, then $\mathcal{G} = \mathcal{F}\langle\zeta\rangle$.
2) By (Gao, 2003, Theorem 6.1), $\mathcal{G} = \mathcal{F}\langle u\rangle$ if and only if $B_2 = g(x, y)u + h(x, y)$ for some $g, h \in \mathcal{F}\{x, y\} \setminus \{0\}$.

In the following, we prove some technical results for later use.

Lemma 6. *Let $P(u), Q(u) \in \mathcal{F}\{u\}$ with $\gcd(P, Q) = 1$ and $m = \mathrm{ord}(\frac{P(u)}{Q(u)}) \ge 0$. Then we have*

(1) *For each $s \in \mathbb{N}_{>0}$, $(P/Q)^{(s)} = \frac{P_s}{Q^{s+1}}$, where P_s is a differential polynomial of order $m+s$ and linear in $u^{(m+s)}$;*

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(2) $\text{tr.deg}_{\mathcal{F}\langle \frac{P(u)}{Q(u)}\rangle}\mathcal{F}\langle u\rangle = \text{ord}(\frac{P(u)}{Q(u)})$.

Proof. (1) The proof is by induction on s. For $s=1$, $(P/Q)' = \frac{P'Q-PQ'}{Q^2}$ and $P_1 = P'Q - PQ' = S_P Q \cdot u^{(\text{ord}(P)+1)} - S_Q P \cdot u^{(\text{ord}(Q)+1)} + T$ with $\text{ord}(T) \le m$. If $\text{ord}(P) \ne \text{ord}(Q)$, clearly, $\text{rk}(P_1) = (u^{(m+1)}, 1)$. Otherwise, since $\gcd(P, Q) = 1$, $S_P Q - S_Q P \ne 0$ and $\text{rk}(P_1) = (u^{(m+1)}, 1)$ follows. Suppose it holds for $s-1$. Then $(P/Q)^{(s)} = \left(\frac{P_{s-1}}{Q^s}\right)' = \frac{P'_{s-1}Q - sP_{s-1}Q'}{Q^{s+1}}$. Let $P_s = P'_{s-1}Q - sP_{s-1}Q'$. By the induction hypothesis, $\text{rk}(P_s) = (u^{(m+s)}, 1)$.

(2) It is trivial for the case $m=0$. Consider the case when $m \ge 1$. Since $u, u', \dots, u^{(m)}$ are algebraically dependent over $\mathcal{F}\langle\frac{P(u)}{Q(u)}\rangle$, $\text{tr.deg}_{\mathcal{F}\langle \frac{P(u)}{Q(u)}\rangle}\mathcal{F}\langle u\rangle = \text{tr.deg}_{\mathcal{F}\langle \frac{P(u)}{Q(u)}\rangle}\mathcal{F}\langle\frac{P(u)}{Q(u)}\rangle(u, u', \dots, u^{(m-1)})$. If $u, u', \dots, u^{(m-1)}$ are algebraically dependent over $\mathcal{F}\langle\frac{P(u)}{Q(u)}\rangle$, there exists $s \in \mathbb{N}$ such that $u, u', \dots,$ $u^{(m-1)}$ are algebraically dependent over $\mathcal{F}\left(\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}\right)$. Thus

$$\begin{aligned}
&\text{tr.deg}\,\mathcal{F}\left(u, u', \dots, u^{(m-1)}, \frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}\right)\Big/\mathcal{F} \\
=\ &\text{tr.deg}\,\mathcal{F}\Big(\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}\Big)\Big/\mathcal{F} + \\
&\text{tr.deg}\,\mathcal{F}\Big(\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}\Big)\Big(u, u', \dots, u^{(m-1)}\Big)\Big/\mathcal{F}\Big(\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}\Big) \\
\le\ & s+m.
\end{aligned}$$

So $\text{tr.deg}\,\mathcal{F}(u, u', \dots, u^{(m-1)})(\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)})/\mathcal{F}(u, u', \dots, u^{(m-1)}) \le s$, contradicting the fact that $\frac{P(u)}{Q(u)}, (\frac{P(u)}{Q(u)})', \dots, (\frac{P(u)}{Q(u)})^{(s)}$ are algebraically independent over $\mathcal{F}(u, u', \dots, u^{(m-1)})$. Thus, $\text{tr.deg}_{\mathcal{F}\langle \frac{P(u)}{Q(u)}\rangle}\mathcal{F}\langle u\rangle = m$. □

By the additivity property of transcendence degrees, Lemma 6 (2) implies that the order is additive with respect to the composition of differential rational functions.

Corollary 7. *Let $R_1, R_2 \in \mathcal{F}\langle u\rangle \setminus \mathcal{F}$. Then* $\text{ord}_u\big(R_2(R_1(u))\big) = \text{ord}_u R_1 + \text{ord}_u R_2$.

Proof. This follows by considering $\mathcal{F}\langle R_2(R_1(u))\rangle \subset \mathcal{F}\langle R_1(u)\rangle \subset \mathcal{F}\langle u\rangle$ and Lemma 6 (2). □

The following result is an exercise from (Kolchin, 1973, p. 159, Ex. 9) which will be used in Section 3.

Lemma 8. *Let $t, u \in \mathcal{E}$ be differentially transcendental elements over $\mathcal{F}$ and $\mathcal{F}\langle t\rangle = \mathcal{F}\langle u\rangle$. Then there exist $a, b, c, d \in \mathcal{F}$ with $ad - bc \ne 0$ such that $u = (at+b)/(ct+d)$.*

Proof. Write $u = \frac{P(t)}{Q(t)}$ with $P, Q \in \mathcal{F}\{y\}$ and $\gcd(P, Q) = 1$. Observe that $P(y) - uQ(y)$ is irreducible in $\mathcal{F}\langle u\rangle\{y\}$. Let $\mathcal{J} = \text{sat}(P - uQ) \subset \mathcal{F}\langle u\rangle\{y\}$. Fix a generic zero s of $\mathcal{J}$, then we have $Q(s) \ne 0$. Indeed, if $Q(s) = 0$, then $Q(y)$ is divisible by $P(y) - uQ(y)$ by Lemma 3, a contradiction. Thus, u is differentially algebraic over $\mathcal{F}\langle s\rangle$. Therefore, s is differentially transcendental over $\mathcal{F}$. So there exists a differential isomorphism $\phi: \mathcal{F}\langle s\rangle \cong \mathcal{F}\langle t\rangle$ with $\phi(s) = t$. Since $\phi(u) = u$, ϕ is a differential isomorphism over $\mathcal{F}\langle u\rangle$. Thus, t is a generic zero of $\mathcal{J}$. Since t is also a generic zero of $\text{sat}(y-t) \subset \mathcal{F}\langle u\rangle\{y\}$, $\mathcal{J} = \text{sat}(y-t)$. By Lemma 3, $y-t$ is divisible by $P - uQ$ over $\mathcal{F}\langle u\rangle$. So $P(y), Q(y) \in \mathcal{F}[y]$ are both of degree at most 1. Thus, there exist $a, b, c, d \in \mathcal{F}$ such that $P(y) = ay+b$ and $Q(y) = cy+d$ with $ad - bc \ne 0$. □

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3. Unirational differential curves and proper differential rational parametrizations

In this section, we introduce the notions of unirational differential curves and proper differential rational parametrizations, and investigate the basic properties for proper differential rational parametrizations.

Definition 9. A differential curve (over $\mathcal{F}$) is a differential variety $C \subset \mathbb{A}^n(\mathcal{E})$ (over $\mathcal{F}$) which has differential dimension 1. If additionally C is irreducible, C is called an irreducible differential curve. A differential curve $C \subset \mathbb{A}^2$ is called a plane differential curve.

Throughout the paper, we focus on the study of plane differential curves over the base differential field $\mathcal{F}$, and so we always omit "plane" and "over $\mathcal{F}$" for convenience.

If $C \subset \mathbb{A}^2$ is an irreducible differential curve, then $\mathbb{I}(C)$ is a prime differential ideal in $\mathcal{F}\{x, y\}$ with differential dimension 1. So by Lemma 3, there exists a unique irreducible differential polynomial $A \in \mathcal{F}\{x, y\}$ (up to an element in $\mathcal{F}$) such that C is the general component of A. We call C *the differential curve defined by* A, and denote it by (C, A) for simplicity.

Definition 10. (Unirational differential curves) Let (C, A) be an irreducible differential curve. We call C a unirational differential curve if C has a generic point of the form

$$\mathcal{P}(u) = \Big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\Big), \tag{4}$$

where $u \in \mathcal{E}$ is differentially transcendental over $\mathcal{F}\langle x, y\rangle$, $P_i, Q_i \in \mathcal{F}\{u\}$[1] and $\gcd(P_i, Q_i) = 1$ for $i = 1, 2$. And we call (4) a differential rational parametrization of C or A.

We now give an alternative characterization of unirational differential curves in terms of the theory of differential fields in Proposition 11, which is similar to the algebraic case (Sendra et al., 2007, Theorem 4.9) and also can be seen as the geometric version of the differential Lüroth's theorem.

Proposition 11. *An irreducible differential curve C is unirational if and only if the differential function field of C, $\mathcal{F}\langle C\rangle = \mathrm{Frac}(\mathcal{F}\{x, y\}/\mathbb{I}(C))$, is differentially isomorphic to $\mathcal{F}\langle u\rangle$ over $\mathcal{F}$.*

Proof. Let C be a unirational differential curve with a differential rational parametrization $\mathcal{P}(u)$. By Theorem 4, there exists $R(u) \in \mathcal{F}\langle u\rangle \setminus \mathcal{F}$ s.t. $\mathcal{F}\langle \mathcal{P}(u)\rangle = \mathcal{F}\langle R(u)\rangle$. Then it is easy to verify that the parametrization $\mathcal{P}(u)$ defines a differential isomorphism

$$\begin{aligned} \varphi : \mathcal{F}\langle C\rangle &\longrightarrow \mathcal{F}\langle R(u)\rangle \\ f(x, y) &\longmapsto f(\mathcal{P}(u)). \end{aligned} \tag{5}$$

Since $R(u)$ is differentially transcendental over $\mathcal{F}$, $\mathcal{F}\langle R(u)\rangle$ is differentially isomorphic to $\mathcal{F}\langle u\rangle$. Therefore, $\mathcal{F}\langle C\rangle$ is differentially isomorphic to $\mathcal{F}\langle u\rangle$. Conversely, let $\varphi : \mathcal{F}\langle C\rangle \to \mathcal{F}\langle u\rangle$ be a differential isomorphism. Let $\mathcal{P}(u) = (\varphi(x), \varphi(y))$. Then $\mathcal{P}(u) \notin \mathcal{F}^2$ and $\mathbb{I}(\mathcal{P}(u)) = \mathbb{I}(C)$. Thus, C is unirational with a differential rational parametrization $\mathcal{P}(u)$. □

For ease of notation, in this paper when we speak of a differential parametrization $\mathcal{P}(u) = (\frac{P_1}{Q_1}, \frac{P_2}{Q_2}) \in \mathcal{F}\langle u\rangle^2$, we always assume each P_i/Q_i is in reduced form, that is, $P_i, Q_i \in \mathcal{F}\{u\}$ and $\gcd(P_i, Q_i) = 1$. And we define the order of $\mathcal{P}(u)$ to be $\max\{\mathrm{ord}_u(\frac{P_1}{Q_1}), \mathrm{ord}_u(\frac{P_2}{Q_2})\}$, denoted by $\mathrm{ord}(\mathcal{P})$. The following result shows that differential parametrizations of a unirational differential curve satisfy certain order property.

[1] Here we automatically have at least one $P_i/Q_i \in \mathcal{F}\langle u\rangle \setminus \mathcal{F}$.

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Proposition 12. *Let (C, A) be a unirational differential curve with* $\mathrm{ord}_x A \geq 0$ *and* $\mathrm{ord}_y A \geq 0$. *Suppose* $\mathcal{P}(u) = (P_1/Q_1, P_2/Q_2) \in \mathcal{F}\langle u\rangle^2$ *is a differential rational parametrization of* C. *Then*

$$\mathrm{ord}_x A + \mathrm{ord}_u(P_1/Q_1) = \mathrm{ord}_y A + \mathrm{ord}_u(P_2/Q_2).$$

In particular, $\mathrm{ord}_x A \leq \mathrm{ord}_u(P_2/Q_2)$ *and* $\mathrm{ord}_y A \leq \mathrm{ord}_u(P_1/Q_1)$.

Proof. Denote $m_i = \mathrm{ord}_u(P_i/Q_i)$ for $i = 1, 2$. Then $m_i \geq 0$. Let $s_1 = \mathrm{ord}_x A$ and $s_2 = \mathrm{ord}_y A$. The fact $\mathbb{I}(\mathcal{P}(u)) = \mathrm{sat}(A) \subset \mathcal{F}\{x, y\}$ implies that s_1 and s_2 are respectively the minimal indices ℓ_1, ℓ_2 such that

$$P_1/Q_1, (P_1/Q_1)', \ldots, (P_1/Q_1)^{(\ell_1)}, P_2/Q_2, (P_2/Q_2)', \ldots, (P_2/Q_2)^{(\ell_2)} \tag{6}$$

are algebraically dependent over $\mathcal{F}$, or equivalently, in the module $(\Omega_{\mathcal{F}\langle u\rangle/\mathcal{F}}, d)$ of Kähler differentials, $d(P_1/Q_1), d(P_1/Q_1)', \ldots, d(P_1/Q_1)^{(\ell_1)}, d(P_2/Q_2), d(P_2/Q_2)', \ldots, d(P_2/Q_2)^{(\ell_2)}$ are linearly dependent over $\mathcal{F}\langle u\rangle$ (Johnson, 1969, p. 94). By Lemma 6, each $(P_i/Q_i)^{(k)}$ is of order $m_i + k$, so $d(P_i/Q_i)^{(k)}$ is linear in $d(u^{(m_i+k)})$ for $k \geq 0$. Thus, $m_1 + s_1 = m_2 + s_2$ follows. Note that when $\ell_1 = m_2$ and $\ell_2 = m_1$, the $m_1 + m_2 + 2$ elements in (6) are contained in $\mathcal{F}(u, u', \ldots, u^{(m_1+m_2)})$, and thus are algebraically dependent over $\mathcal{F}$. So $\mathrm{ord}_x A \leq m_2$ and $\mathrm{ord}_y A \leq m_1$. □

Below we give some examples and non-examples for unirational differential curves.

Example 13.

(1) Let $A = x'^2 - 4xy^2 \in \mathbb{Q}(t)\{x, y\}$ with $\delta = \frac{\mathrm{d}}{\mathrm{d}t}$. Then (C, A) is a unirational differential curve with a differential rational parametrization $\mathcal{P}_1 = (u^2, u')$. Note that $\mathcal{P}_2 = \big((u')^2, u''\big)$ is another parametrization of (C, A) and $\mathrm{ord}(\mathcal{P}_1) < \mathrm{ord}(\mathcal{P}_2)$.

(2) Let $A = y' - x' \in \mathbb{Q}(t)\{x, y\}$. Then (C, A) is not unirational. If we suppose the contrary, then (C, A) would have a parametrization $(R_1(u), R_2(u)) \in (\mathbb{Q}(t)\langle u\rangle)^2$. However, $a = R_2(u) - R_1(u) \in C_{\mathbb{Q}(t)\langle u\rangle} = \mathbb{Q}$ and consequently $y - x - a \in \mathrm{sat}(A)$, a contradiction.
Actually, if $A = B^{(k)}$ for some $B \in \mathcal{F}\{x, y\} \setminus \mathcal{F}$ and $k > 0$, then (C, A) is not unirational. Otherwise, there exists $\mathcal{P}(u) \in \mathcal{F}\langle u\rangle^2$ such that $\mathbb{I}(\mathcal{P}(u)) = \mathrm{sat}(A)$, which implies that $b = B(\mathcal{P}(u)) \in \mathcal{F}\langle u\rangle$ is differentially algebraic over $\mathcal{F}$ and consequently, $b \in \mathcal{F}$. Thus, $B(x, y) - b \in \mathrm{sat}(A)$. By Lemma 3, $B(x, y) - b$ is divisible by A, a contradiction to $A = B^{(k)}$.

From the above examples, we learn that not all differential curves are unirational and for a unirational differential curve, its differential rational parametrizations are not unique. In fact, if $\mathcal{P}_1(u)$ is a differential rational parametrization of (C, A), then for any $R(u) \in \mathcal{F}\langle u\rangle \setminus \mathcal{F}$, $\mathcal{P}_2 = \mathcal{P}_1\big(R(u)\big)$ is also a differential rational parametrization of (C, A), and thus C has infinitely many differential rational parametrizations. These facts lead to the following two natural problems:

Problem 1. Given $A \in \mathcal{F}\{x, y\}$, decide whether the differential curve (C, A) is unirational or not.

Problem 2. If (C, A) is unirational, find "optimal" differential rational parametrizations for it w.r.t. some criteria, for instance, having minimal order and degree.

We first study Problem 2 in this section by introducing the notion of proper differential rational parametrizations and giving the main basic properties, while leaving Problem 1 to be considered in Section 5.

Definition 14. Let C be a unirational differential curve with a differential rational parametrization $\mathcal{P}(u) = \Big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\Big)$. The parametrization $\mathcal{P}(u)$ is said to be proper if $\mathcal{F}\langle \frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\rangle = \mathcal{F}\langle u\rangle$.

Equivalently, the notion of properness can be stated by means of differentially birational maps between $\mathcal{C}$ and $\mathbb{A}^1$. More precisely, let $(\mathcal{C}, A)$ be a unirational differential curve with a differential rational parametrization $\mathcal{P}(u) \in \mathcal{F}\langle u\rangle^2$. This $\mathcal{P}(u)$ induces the differentially rational map

$$\begin{array}{rcl} \mathcal{P}: \mathbb{A}^1 & \dashrightarrow & \mathcal{C} \subset \mathbb{A}^2 \\ u & \longmapsto & \mathcal{P}(u). \end{array}$$

Then the differential parametrization $\mathcal{P}(u)$ is proper if and only if the map $\mathcal{P}$ is differentially birational, that is, $\mathcal{P}$ has an inverse differential rational map

$$\begin{array}{rcl} \mathcal{U}: \mathcal{C} & \dashrightarrow & \mathbb{A}^1 \\ (x, y) & \longmapsto & U(x, y), \end{array}$$

where $U(x, y) \in \mathcal{F}\langle x, y\rangle$ and the denominator of U does not vanish identically on $\mathcal{C}$. This $\mathcal{U}$ is called the inversion of the proper differential rational parametrization $\mathcal{P}(u)$.

Gao defined properness for differential rational parametric equations (DRPEs) in (Gao, 2003) and under his definition, $\left(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\right)$ is called proper if for a generic zero (a_1, a_2) of $\mathcal{C}$, there exists a unique $\tau \in \mathcal{E}$ such that $a_i = P_i(\tau)/Q_i(\tau)$. By (Gao, 2003, Theorem 6.1), the equivalence of these definitions can be easily seen.

In (Gao, 2003, Theorem 6.2), Gao gave a method to compute a proper reparametrization for any improper DRPEs based on a constructive proof of Theorem 4. Given an arbitrary rational parametrization $\mathcal{P}(u)$ of a unirational differential curve $\mathcal{C}$, Gao's method produces an algorithm to compute a proper differential rational parametrization for $\mathcal{C}$ from $\mathcal{P}(u)$, which in particular shows that each unirational differential curve has a proper rational parametrization.

In the following we show that proper differential rational parametrizations possess essential properties of the unirational differential curves. We first give the order property of proper differential rational parametrizations. Recall that the order of a reduced differential rational function is equal to the maximum of the orders of its denominator and numerator.

Theorem 15. *Let $(\mathcal{C}, A)$ be a unirational differential curve and $\left(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\right)$ be a proper differential rational parametrization of $\mathcal{C}$. Then we have*

$$\operatorname{ord}\left(\frac{P_1(u)}{Q_1(u)}\right) = \operatorname{ord}_y A, \ \ \operatorname{ord}\left(\frac{P_2(u)}{Q_2(u)}\right) = \operatorname{ord}_x A. \tag{7}$$

Proof. For the special cases that either $\frac{P_1(u)}{Q_1(u)} = a_1 \in \mathcal{F}$ or $\frac{P_2(u)}{Q_2(u)} = a_2 \in \mathcal{F}$, by Lemma 8 we have either 1) $A = x - a_1$ and $\mathcal{P}(u) = (a_1, \frac{\alpha_1 u+\beta_1}{\gamma_1 u+\xi_1})$, or 2) $A = y - a_2$ and $\mathcal{P}(u) = (\frac{\alpha_2 u+\beta_2}{\gamma_2 u+\xi_2}, a_2)$, for some $\alpha_i, \beta_i, \gamma_i, \xi_i \in \mathcal{F}$ with $\alpha_i\xi_i - \beta_i\gamma_i \neq 0$, where (7) holds.

So it suffices to consider the case that $m_i = \operatorname{ord}(\frac{P_i(u)}{Q_i(u)}) \geq 0$ for $i = 1, 2$. In this case, both $\{x\}$ and $\{y\}$ are parametric sets of $\operatorname{sat}(A)$. By Definition 2, the relative order of $\operatorname{sat}(A)$ w.r.t. the parametric set $\{x\}$ is

$$\operatorname{ord}_{\{x\}}\operatorname{sat}(A) = \operatorname{tr.deg}_{\mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}\rangle}\mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\rangle = \operatorname{tr.deg}_{\mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}\rangle}\mathcal{F}\langle u\rangle.$$

By Lemma 6, $\operatorname{tr.deg}_{\mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}\rangle}\mathcal{F}\langle u\rangle = m_1$. Since A is a characteristic set of $\operatorname{sat}(A)$ w.r.t. the elimination ranking: $x < y$, again by Definition 2, $\operatorname{ord}_y A = \operatorname{ord}_{\{x\}}\operatorname{sat}(A) = m_1$. By Proposition 12, $\operatorname{ord}_x A - \operatorname{ord}_y A = m_2 - m_1$, and thus $\operatorname{ord}_x A = m_2, \operatorname{ord}_y A = m_1$. $\square$

Remark 16. In the algebraic case, properness of a rational parametrization can be characterized via the degree of the implicit equation of a unirational curve (Sendra et al., 2007, Theorem 4.21). That is, a parametrization $P(t)$ of a unirational curve $\mathcal{V}(f)$ is proper if and only if $\deg(P(t)) = \max\{\deg_x f, \deg_y f\}$. However, in the differential case, we do not have such a characterization of properness via the orders of differential curves and the converse of Theorem 15 is not valid. For a

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non-example, let $A = y' - xy \in \mathcal{F}\{x, y\}$. Clearly, $\mathcal{P}(u) = (\frac{2u'}{u}, u^2)$ is a parametrization of (C, A) which satisfies (7). But $\mathcal{F}\langle \mathcal{P}(u)\rangle = \mathcal{F}\langle u^2\rangle \neq \mathcal{F}\langle u\rangle$, so $\mathcal{P}(u)$ is not proper.

As a direct consequence of Theorem 15, we can show that an algebraic curve is unirational in the algebraic sense if and only if it is unirational in the differential sense.

Corollary 17. *Let $(\mathcal{F}, \delta)$ be a differential field which is algebraically closed. Let $A \in \mathcal{F}[x, y]$ be an irreducible polynomial. Then the differential curve (C, A) is unirational if and only if the genus of the algebraic curve defined by $A = 0$ is 0.*

Proof. By (Sendra et al., 2007, Theorem 4.63), an algebraic curve is rational if and only if its genus is 0. So it suffices to show that if the differential curve (C, A) is unirational, then the algebraic curve defined by $A = 0$ is rational in the algebraic sense. Indeed, if (C, A) is unirational with a proper differential rational parametrization $(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$, by Theorem 15, $\max\{\mathrm{ord}(\frac{P_1(u)}{Q_1(u)}), \mathrm{ord}(\frac{P_2(u)}{Q_2(u)})\} = \mathrm{ord}(A) = 0$, so the algebraic curve $A = 0$ is rational. □

In the algebraic case, it was shown in (Sendra et al., 2007, Lemma 4.17) that proper rational parametrizations of a rational algebraic curve enjoy some "uniqueness" property up to the Möbius transformations. We now show this property can be extended to the differential case.

Theorem 18. *If $\mathcal{P}_1(u), \mathcal{P}_2(u)$ are two proper differential rational parametrizations of (C, A), then there exist $a, b, c, d \in \mathcal{F}$, s.t. $\mathcal{P}_2(u) = \mathcal{P}_1(\frac{au+b}{cu+d})$. Conversely, given any proper parametrization $\mathcal{P}(u)$ of (C, A), $\mathcal{P}(\frac{au+b}{cu+d})$ is also a proper parametrization of (C, A) for $ad - bc \neq 0$.*

Proof. Assume $\mathcal{P}_1(u) = \left(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\right), \mathcal{P}_2(u) = \left(\frac{P_3(u)}{Q_3(u)}, \frac{P_4(u)}{Q_4(u)}\right)$. Since $\mathcal{F}\langle \frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\rangle = \mathcal{F}\langle u\rangle$, there exist $M(x, y), N(x, y) \in \mathcal{F}\{x, y\}$ s.t. $u = \frac{M(\mathcal{P}_1(u))}{N(\mathcal{P}_1(u))}$. Then,

$$\frac{P_i\left(\frac{M(\mathcal{P}_1(u))}{N(\mathcal{P}_1(u))}\right)}{Q_i\left(\frac{M(\mathcal{P}_1(u))}{N(\mathcal{P}_1(u))}\right)} = \frac{P_i(u)}{Q_i(u)}, \; i = 1, 2.$$

Let $r(u) = \frac{M(\mathcal{P}_2(u))}{N(\mathcal{P}_2(u))}$. Since $\mathbb{I}\left(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\right) = \mathbb{I}\left(\frac{P_3(u)}{Q_3(u)}, \frac{P_4(u)}{Q_4(u)}\right)$, we obtain $\frac{P_3(u)}{Q_3(u)} = \frac{P_1(r(u))}{Q_1(r(u))}$, $\frac{P_4(u)}{Q_4(u)} = \frac{P_2(r(u))}{Q_2(r(u))}$. Then $\mathcal{F}\langle u\rangle = \mathcal{F}\langle \frac{P_3(u)}{Q_3(u)}, \frac{P_4(u)}{Q_4(u)}\rangle \subset \mathcal{F}\langle r(u)\rangle$, which implies $\mathcal{F}\langle u\rangle = \mathcal{F}\langle r(u)\rangle$. By Lemma 8, $r(u) = \frac{au+b}{cu+d}$ for some $a, b, c, d \in \mathcal{F}$ with $ad - bc \neq 0$. The converse part is easy to check. □

Remark 19. Let (C, A) be a unirational differential curve. Theorem 18 and its proof imply the following two facts about proper differential rational parametrizations.

1) Proper differential rational parametrizations of (C, A) are of the smallest order among all its rational parametrizations. Indeed, let $\mathcal{P}(u)$ be any differential rational parametrization of (C, A) and $\mathcal{P}_1(u)$ be a proper one, then by the proof of Theorem 18, there exists $r(u) \in \mathcal{F}\langle u\rangle$ such that $\mathcal{P}(u) = \mathcal{P}_1(r(u))$. Therefore, by Corollary 7, $\mathrm{ord}(\mathcal{P}_1(u)) \leq \mathrm{ord}(\mathcal{P}(u))$. And $\mathrm{ord}(\mathcal{P}(u)) = \mathrm{ord}(\mathcal{P}_1(u))$ if and only if $\mathrm{ord}(r(u)) = 0$.
2) Although the orders of proper differential rational parametrizations of (C, A) are the same, their degrees could be distinct when $\mathrm{ord}(A) > 0$. Take $A = y'' - x$ for a simple example. Clearly, $\mathcal{P}(u) = (u'', u)$ is a proper rational parametrization of (C, A) of degree 1 and $\mathcal{P}(1/u) = (\frac{-uu''+2(u')^2}{u^3}, \frac{1}{u})$ is another proper parametrization of degree 3. It is interesting to estimate the lowest degree of proper parametrizations in terms of the numerical data of A.

We illustrate Theorem 15 and Theorem 18 by giving the following examples.

Example 20.

(1) Let $A = y''x + (y')^2 y - y'x' \in \mathbb{Q}\{x, y\}$. Then $\mathcal{P}(u) = (uu'', u')$ is a proper parametrization of $(\mathcal{C}, A)$. Note that $\mathrm{ord}(uu'') = 2 = \mathrm{ord}_y A$ and $\mathrm{ord}(u') = 1 = \mathrm{ord}_x A$.
(2) Let $A = y' - x' - x \in \mathbb{Q}\{x, y\}$. Then $\mathcal{P}_1 = (u', u + u'), \mathcal{P}_2 = (\frac{-u'}{u^2}, \frac{u-u'}{u^2})$ are two proper parametrizations of $(\mathcal{C}, A)$. Here, $\mathcal{P}_2(u) = \mathcal{P}_1(\frac{1}{u})$. Note that $\mathrm{ord}(\mathcal{P}_i) = \mathrm{ord}(A)$ and $\deg(\mathcal{P}_2) > \deg(\mathcal{P}_1)$.

4. Proper linear differential rational parametrizations and the implicitization problem

In this section, we shall first explore further properties for proper linear differential rational parametrizations and then study the implicitization problem using differential resultants.

Definition 21. We call $\mathcal{P}(u) = \big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\big) \in \mathcal{F}\langle u\rangle^2 \setminus \mathcal{F}^2$ a linear differential rational parametrization (LDRP) if for $i = 1, 2$, $P_i, Q_i \in \mathcal{F}\{u\}$ are of degree at most 1 and $\gcd(P_i, Q_i) = 1$.

Although the converse of Theorem 15 is not valid in general as explained in Remark 16, when restricted to linear differential rational parametrizations, the next theorem shows that properness can be characterized via the orders of implicit equations of unirational differential curves.

Theorem 22. *Let $(\mathcal{C}, A)$ be a unirational differential curve which has a linear differential rational parametrization $\mathcal{P}(u) = \big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\big)$. Then, $\mathcal{P}(u)$ is proper if and only if*

$$\mathrm{ord}\Big(\frac{P_1(u)}{Q_1(u)}\Big) = \mathrm{ord}_y A,\ \mathrm{ord}\Big(\frac{P_2(u)}{Q_2(u)}\Big) = \mathrm{ord}_x A.$$

Proof. Suppose $\mathrm{ord}\big(\frac{P_1(u)}{Q_1(u)}\big) = \mathrm{ord}_y A$, $\mathrm{ord}\big(\frac{P_2(u)}{Q_2(u)}\big) = \mathrm{ord}_x A$. We need to show that $\mathcal{P}(u)$ is proper. Let $J = [Q_1(u)x - P_1(u), Q_2(u)y - P_2(u)] : (Q_1Q_2)^\infty$. Then J is a prime differential ideal in $\mathcal{F}\{x, y, u\}$, and $\{Q_1(u)x - P_1(u), Q_2(u)y - P_2(u)\}$ is its characteristic set w.r.t. the elimination ranking $u < x < y$ (Ritt, 1950, p. 107). Now we compute a characteristic set $B_1(x, y), B_2(x, y, u)$ of J w.r.t. the elimination ranking $x < y < u$. Since P_i, Q_i are of degree at most 1, by the zero-decomposition theorem, $B_2(x, y, u)$ is a linear differential polynomial in u.

Let $s := \mathrm{ord}_u B_2$. Rewrite B_2 in the form $B_2(x, y, u) = I_s(x, y)u^{(s)} + \cdots + I_0(x, y)u + I(x, y)$, where $I_i(x, y), I(x, y) \in \mathcal{F}\{x, y\}$. Let $\bar{B}(z) = \frac{B_2(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}, z)}{I_s(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})} \in \mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\rangle\{z\}$. By Remark 5, $\bar{B}(z)$ is the minimal polynomial of u over $\mathcal{F}\langle\mathcal{P}(u)\rangle$ with $\mathrm{ord}_z(\bar{B}) = s$. Suppose the contrary that $\mathcal{P}(u)$ is not proper. Then by Remark 5 2), $s \geq 1$ and consequently there exists a coefficient v of $\bar{B}(z)$ such that $\mathrm{ord}_u v = s \geq 1$. So $v \notin \mathcal{F}$ and by Remark 5 1), $\mathcal{F}\langle\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\rangle = \mathcal{F}\langle v\rangle$ follows. Thus, there exist $P_3, Q_3, P_4, Q_4 \in \mathcal{F}\{z\}$ such that $\frac{P_1(u)}{Q_1(u)} = \frac{P_3(v)}{Q_3(v)}, \frac{P_2(u)}{Q_2(u)} = \frac{P_4(v)}{Q_4(v)}$. Let $\mathcal{P}_1(u) := (\frac{P_3(u)}{Q_3(u)}, \frac{P_4(u)}{Q_4(u)})$. Since $\mathcal{F}\langle\mathcal{P}_1(u)\rangle = \mathcal{F}\langle u\rangle$, $\mathcal{P}_1(u)$ is a proper parametrization of $(\mathcal{C}, A)$ with $\mathrm{ord}(\frac{P_3(u)}{Q_3(u)}) = \mathrm{ord}(\frac{P_1(u)}{Q_1(u)}) - s < \mathrm{ord}_y A, \mathrm{ord}(\frac{P_4(u)}{Q_4(u)}) = \mathrm{ord}(\frac{P_2(u)}{Q_2(u)}) - s < \mathrm{ord}_x A$ by Corollary 7, which contradicts Theorem 15. Thus, $\mathcal{P}(u)$ should be proper. Combined with Theorem 15, $\mathcal{P}(u)$ is proper if and only if $\mathrm{ord}\big(\frac{P_1(u)}{Q_1(u)}\big) = \mathrm{ord}_y A$, $\mathrm{ord}\big(\frac{P_2(u)}{Q_2(u)}\big) = \mathrm{ord}_x A$. □

Remark 23. By the proof of Theorem 22, if $\mathcal{P}(u) = \big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\big)$ is a non-proper linear differential rational parametrization of $(\mathcal{C}, A)$, then the Lüroth generator of $\mathcal{F}\langle\mathcal{P}(u)\rangle/\mathcal{F}$ is of order $s \geq 1$ and $\mathrm{ord}\big(\frac{P_1(u)}{Q_1(u)}\big) = \mathrm{ord}_y A + s$, $\mathrm{ord}\big(\frac{P_2(u)}{Q_2(u)}\big) = \mathrm{ord}_x A + s$.

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4.1. Implicitization of proper LDRPs by differential resultants

In the algebraic case, it was shown in (Sendra et al., 2007, Theorem 4.41) that resultants can be used to solve the implicitization problem for rational parametrizations. For the differential case, the implicitization problem for linear differential polynomial parametric equations was studied by Rueda and Sendra (2007) via linear complete differential resultants. In the following, we present results on implicitization for linear differential rational parametrizations with the method of differential resultants. Before that, we need a technical result.

Recall that the wronskian determinant of $\xi_1, \ldots, \xi_n$ is

$$\mathrm{wr}(\xi_1, \ldots, \xi_n) = \begin{vmatrix} \xi_1 & \delta(\xi_1) & \cdots & \delta^{n-1}(\xi_1) \\ \xi_2 & \delta(\xi_2) & \cdots & \delta^{n-1}(\xi_2) \\ \cdots & \cdots & \cdots & \cdots \\ \xi_n & \delta(\xi_n) & \cdots & \delta^{n-1}(\xi_n) \end{vmatrix}.$$

It is well-known that $\mathrm{wr}(\xi_1, \ldots, \xi_n) = 0$ gives a necessary and sufficient condition that $\xi_1, \ldots, \xi_n$ are linearly dependent over constants.

Lemma 24. *Let $L = \delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0 \in \mathcal{F}[\delta]$. Suppose $L_1 = \delta^{n_1} + b_{n_1-1}\delta^{n_1-1} + \cdots + b_0 \in \mathcal{G}[\delta]$ is a right divisor of L over some differential extension field $\mathcal{G}$ of $\mathcal{F}$. Then all the b_i belong to a finite differential algebraic extension field of $\mathcal{F}$. In particular,* $\mathrm{tr.deg}\,\mathcal{F}\langle b_0, \ldots, b_{n_1-1}\rangle/\mathcal{F} < \infty$.

Proof. $\mathrm{Sol}(L) = \{y \in \mathcal{E} \mid L(y) = 0\}$ is a linear space of dimension n over $C_{\mathcal{E}}$, the field of constants of $\mathcal{E}$. Let $\xi_1, \ldots, \xi_n$ be a basis of $\mathrm{Sol}(L)$. Then the $a_{n-1}, \ldots, a_1, a_0$ satisfy the following linear equations

$$\delta^n(\xi_i) + a_{n-1}\delta^{n-1}(\xi_i) + \cdots + a_0\xi_i = 0, \ (i = 1, \ldots, n).$$

So we have

$$\begin{pmatrix} \xi_1 & \delta(\xi_1) & \cdots & \delta^{n-1}(\xi_1) \\ \xi_2 & \delta(\xi_2) & \cdots & \delta^{n-1}(\xi_2) \\ \cdots & \cdots & \cdots & \cdots \\ \xi_n & \delta(\xi_n) & \cdots & \delta^{n-1}(\xi_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} -\delta^n(\xi_1) \\ -\delta^n(\xi_2) \\ \vdots \\ -\delta^n(\xi_n) \end{pmatrix}.$$

Since $\xi_1, \ldots, \xi_n$ are linearly independent over $C_{\mathcal{E}}$, the wronskian determinant $\mathrm{wr}(\xi_1, \ldots, \xi_n)$ is nonzero. Thus, $a_i = \frac{\mathrm{wr}_i(\xi_1, \ldots, \xi_n)}{\mathrm{wr}(\xi_1, \ldots, \xi_n)} \in \mathcal{F}$, where $\mathrm{wr}_i(\xi_1, \ldots, \xi_n)$ is obtained from $\mathrm{wr}(\xi_1, \ldots, \xi_n)$ by replacing its $(i+1)$-th column by $(-\delta^n(\xi_1) \ -\delta^n(\xi_2) \ \cdots \ -\delta^n(\xi_n))^{\mathrm{T}}$.

Since L_1 is a right divisor of L, the solution space $\mathrm{Sol}(L_1)$ $(\subset \mathrm{Sol}(L))$ of L_1 in $\mathcal{E}$ is of dimension n_1 and there exist $c_{ij} \in C_{\mathcal{E}}\,(i = 1, \ldots, n_1;\ j = 1, \ldots, n)$ such that

$$\eta_i = c_{i1}\xi_1 + c_{i2}\xi_2 + \cdots + c_{in}\xi_n, \ i = 1, \ldots, n_1$$

is a basis of $\mathrm{Sol}(L_1)$. Similarly, we can recover the coefficients b_j of L_1 from the η_i's following the above steps. Thus, $b_j \in \mathcal{F}(c_{ij})\langle \xi_1, \ldots, \xi_n\rangle$, which is a finitely generated differential algebraic extension field of $\mathcal{F}$. □

Differential resultant for two univariate differential polynomials was first introduced by Ritt (1932). Carrà-Ferro then proposed to use algebraic Macaulay resultants to compute differential resultants for $n+1$ differential polynomials in n differential variables (Carrà-Ferro, 1997a,b), which is incomplete in that under her method, even the differential resultant of two generic nonlinear univariate differential polynomials is identically zero. The first rigorous definition of the differential resultant for $n+1$ differential polynomials in n differential variables was given by Gao et al. (2013). Although Ferro's matrix formulae do not work for the general case, these definitions for the differential resultant of two linear univariate differential polynomials are equivalent. Now we recall the definition of differential resultants for two linear univariate differential polynomials via the matrix formulae.

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Definition 25. Let $f_1, f_2 \in \mathbb{D}\{u\}$ be linear differential polynomials of order $m_1, m_2 \geq 0$ over a differential domain $\mathbb{D}$. Let $PS = \{f_1^{(m_2)}, f_1^{(m_2-1)}, \ldots, f_1, f_2^{(m_1)}, f_2^{(m_1-1)}, \ldots, f_2\}$ and $L = m_1 + m_2 + 2$. Let M be the $L \times L$ matrix whose k-th row is the coefficient vector of the k-th polynomial in PS w.r.t. $u^{(m_1+m_2)} > u^{(m_1+m_2-1)} > \cdots > u > 1$ (i.e., the resultant matrix of PS w.r.t. $u^{(j)}, j \leq m_1 + m_2$). This M is called the differential resultant matrix of f_1 and f_2 w.r.t. u, and $\det(M)$ is defined to be the differential resultant of f_1, f_2, denoted by $\delta\text{-Res}_u(f_1, f_2)$, or simply by $\delta\text{-Res}(f_1, f_2)$.

The following result shows that the differential resultant can be applied to compute the implicit equation for proper linear differential rational parametric equations.

Theorem 26. *Let $\mathcal{P}(u) = (\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$ be a linear differential rational parametrization with $m_i = \text{ord}(\frac{P_i}{Q_i}) \geq 0$ for $i = 1, 2$. If $\mathcal{P}(u)$ is proper, then the differential resultant*

$$R(x, y) := \delta\text{-Res}_u\big(xQ_1(u) - P_1(u), yQ_2(u) - P_2(u)\big) \neq 0.$$

Furthermore, $\text{ord}_x R = m_2$, $\text{ord}_y R = m_1$ *and R is linear in $x^{(m_2)}$ and $y^{(m_1)}$.*

Proof. Let $f_1 = xQ_1(u) - P_1(u)$, $f_2 = yQ_2(u) - P_2(u) \in \mathcal{F}\{u, x, y\}$. Denote $m = m_1 + m_2 + 2$. Let $M \in \mathcal{F}\{x, y\}^{m\times m}$ be the differential resultant matrix of f_1, f_2 w.r.t. u. Then $R := \delta\text{-Res}_u(f_1, f_2) = \det(M) \in \mathcal{F}\{x, y\}$. To show $R \neq 0$, it suffices to prove that $\text{coeff}(\det(M), x^{(m_2)}) \neq 0$.

Note the fact that only $f^{(m_2)}$ effectively involves $x^{(m_2)}$ and $f^{(m_2)}$ is linear in $x^{(m_2)}$ with coefficient $Q_1(u)$. So $\text{coeff}(\det(M), x^{(m_2)}) = \det(M_1)$, where $M_1 \in \mathcal{F}\{x, y\}^{m\times m}$ be the resultant matrix of $Q_1, f_1^{(m_2-1)}, \ldots, f_1', f_1, f_2^{(m_1)}, \ldots, f_2', f_2$ w.r.t. the variables $u^{(m_1+m_2)}, u^{(m_1+m_2-1)}, \ldots, u', u$. For $j = 1, \ldots, m-1$, multiply the j-th column of M_1 by $u^{(m-1-j)}$ and add it to the last column, then compute $\det(M_1)$ by the last column. So there exist $a_i, b_j, a \in \mathcal{F}\{x, y\}$ such that

$$\det(M_1) = a(x, y)Q_1(u) + \sum_{i=0}^{m_2-1} a_i(x, y)f_1^{(i)} + \sum_{j=0}^{m_1} b_j(x, y)f_2^{(j)}. \tag{8}$$

Clearly, $a(x, y) = p(y) \cdot \det(M_2) \cdot (-1)^{m_1+1}$, where $p(y) = \text{coeff}(f_2, u^{(m_2)}) \neq 0$ and M_2 is the submatrix obtained from M_1 by removing the 1-st, the $(m_2 + 2)$-th rows, and the 1-st, the m-th columns. We claim that $\det(M_2) \neq 0$.

If $m_i = 0$ for some i, then $\det(M_2) = (\text{coeff}(f_i, u))^{m-2} \neq 0$. Now suppose $m_1, m_2 > 0$. Assume $\mathcal{P}(u) = (\frac{L_{11}(u)+a_{11}}{L_{12}(u)+a_{12}}, \frac{L_{21}(u)+a_{21}}{L_{22}(u)+a_{22}})$ where $L_{ij} \in \mathcal{F}[\delta]$, and for each i, L_{i1} and L_{i2} are not both equal to zero. Clearly,

$$\det(M_2) = \delta\text{-Res}^h(xL_{12}(u) - L_{11}(u), yL_{22}(u) - L_{21}(u)).$$

By (Chardin, 1991, Theorem 2), $\delta\text{-Res}^h(xL_{12}(u) - L_{11}(u), yL_{22}(u) - L_{21}(u)) \neq 0$ if and only if $\text{gcrd}(xL_{12} - L_{11}, yL_{22} - L_{21}) = 1$ (over $\mathcal{F}\langle x, y\rangle$). We now show $\text{gcrd}(xL_{12} - L_{11}, yL_{22} - L_{21}) = 1$. Suppose the contrary, that is, $\text{gcrd}(xL_{12} - L_{11}, yL_{22} - L_{21}) = D(\delta)$ which is monic of degree greater than 0. By Lemma 24, $D(\delta)$ can not effectively involve x or y. For if not, suppose $D(\delta)$ effectively involves x, which contradicts the fact that the coefficients of a monic right divisor of $yL_{22} - L_{21}$ have finite transcendence degree over $\mathcal{F}\langle y\rangle$ (x is differentially transcendental over $\mathcal{F}\langle y\rangle$). So $D(\delta) \in \mathcal{F}[\delta]$. As a consequence, $\mathcal{F}\langle \mathcal{P}(u)\rangle \subseteq \mathcal{F}\langle D(u)\rangle \subsetneqq \mathcal{F}\langle u\rangle$, contradicting the hypothesis that $\mathcal{P}(u)$ is proper. Thus, $\text{gcrd}(xL_{12} - L_{11}, yL_{22} - L_{21}) = 1$ and $\det(M_2) \neq 0$ follows.

Since $\det(M_2) \neq 0$, $a(x, y) = (-1)^{m_1+1}p(y) \cdot \det(M_2) \neq 0$. Let $A(x, y) \in \mathcal{F}\{x, y\}$ be an irreducible differential polynomial such that $\mathbb{I}\big(\mathcal{P}(u)\big)$ is the general component of A. Since $\mathcal{P}(u)$ is proper, by Theorem 15, $\text{ord}_x A = m_2$ and $\text{ord}_y A = m_1$. The fact $\text{ord}_x \det(M_2) < m_2$ implies that $\text{ord}_x a(x, y) < m_2$. So $a(\mathcal{P}(u)) \neq 0$. By (8), $\det(M_1)\big|_{(x,y)=\mathcal{P}(u)} \neq 0$ and thus $\text{coeff}(R(x, y), x^{(m_2)}) = \det(M_1) \neq 0$. So $R \neq 0$, $\text{ord}_x R = m_2$ and R is linear in $x^{(m_2)}$. Since $R \in [f_1, f_2] \cap \mathcal{F}\{x, y\} \subset \mathbb{I}(\mathcal{P}(u)) = \text{sat}(A)$, $\text{ord}_y R \geq \text{ord}_y A = m_1$. Since $\text{ord}_y R \leq m_1$, $\text{ord}_y R = m_1$ and R is linear in $y^{(m_1)}$. This completes the proof. □

By Theorem 26, the implicitization of a proper linear differential rational parametrization can be reduced to the computation of the corresponding differential resultant.

Corollary 27. *Let $(\mathcal{C}, A)$ be a unirational differential curve with a linear differential rational parametrization $\mathcal{P}(u) = \big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\big)$ with $m_i = \mathrm{ord}(\frac{P_i}{Q_i}) \geq 0$ for $i = 1, 2$. Suppose $\mathcal{P}(u)$ is proper. Then A is the main irreducible factor of $R = \delta\text{-}\mathrm{Res}\big(xQ_1(u) - P_1(u), yQ_2(u) - P_2(u)\big)$. That is, if $R = AB$, then $\mathrm{ord}(B) < \mathrm{ord}(R)$. In particular,*

$$\deg_{x^{(m_2)}} A = 1 \text{ and } \deg_{y^{(m_1)}} A = 1.$$

Another direct consequence of Theorem 26 gives a necessary condition for a unirational differential curve to possess a proper linear differential rational parametrization.

Corollary 28. *Suppose $A \in \mathcal{F}\{x, y\}$ defines a unirational differential curve. A necessary condition such that $(\mathcal{C}, A)$ has a proper linear differential rational parametrization is that A is quasi-linear under any ranking.*

We give the following examples to illustrate Theorem 26 and Corollary 27.

Example 29.

(1) Let $\mathcal{P}(u) = (\frac{u'}{u}, u)$. Then $\mathcal{P}(u)$ is proper. Furthermore, the differential resultant $\delta\text{-}\mathrm{Res}\big(ux - u', y - u\big) = xy - y'$, which is exactly the implicit equation of $\mathcal{P}(u)$.

(2) Let $\mathcal{F} = (\mathbb{Q}(t), \frac{d}{dt})$ and $\mathcal{P}(u) = (\frac{u''}{tu+1}, u')$. Clearly, $\mathcal{F}\langle \frac{u''}{tu+1}, u' \rangle = \mathcal{F}\langle u \rangle$, so $\mathcal{P}(u)$ is proper. The differential resultant of $(tu + 1)x - u''$ and $y - u'$ w.r.t. u is $R(x, y) = -txy'' + tx'y' + y'x + t^2yx^2 - x^2$, which is irreducible. By Corollary 27, R is the implicit equation of $\mathcal{P}(u)$.

Remark 30. In the algebraic case, given a rational parametrization $\mathcal{P}(u) = \big(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)}\big)$, it was shown in (Sendra et al., 2007, Section 4.5) that the resultant of $xQ_1(u) - P_1(u)$ and $yQ_2(u) - P_2(u)$ is equal to a power of the implicit equation of $\mathcal{P}(u)$, and in particular, if $\mathcal{P}(u)$ is proper, then the resultant is irreducible and is exactly the implicit equation. But in the differential case, even for linear differential rational parametrizations, the situation may become more complicated. First, given a proper linear differential rational parametrization, although Theorem 26 shows the corresponding differential resultant is nonzero and quasi-linear, we do not know whether it is irreducible. Second, if the given parametrization is not proper, either case may happen: (C1) The differential resultant is zero. For instance, given a nonproper parametrization $\mathcal{P}_1(u) = (\frac{u'}{u}, \frac{u}{u'})$, we have $\delta\text{-}\mathrm{Res}_u\big(xu - u', yu' - u\big) = 0$; or (C2) The differential resultant is nonzero. For a simple example, let $\mathcal{P}_2(u) = (\frac{u''+1}{u}, \frac{u''+1}{u})$, which is obviously not proper. The differential resultant $\delta\text{-}\mathrm{Res}_u\big(xu - u'' - 1, yu - u'' - 1\big) = (y - x)^3 \neq 0$. Note that $y - x$ is the implicit equation of $\mathcal{P}_2(u)$. In case the differential resultant is nonzero, we expect its square-free part could serve as the implicit equation as in the algebraic case, but it has not been proved yet.

4.2. Properness of LDRPs by differential resultants

Given a linear differential rational parametrization $\mathcal{P}(u)$, Corollary 27 shows the implicitization problem can be solved by computing the corresponding differential resultant provided that $\mathcal{P}(u)$ is proper. To make this idea algorithmic, we first need to give a method to decide whether a given linear differential rational parametrization is proper. Rueda and Sendra proved that a linear differential polynomial parametrization $(P(u), Q(u)) \in \mathcal{F}\{u\}^2$ is proper if and only if the differential resultant $\delta\text{-}\mathrm{Res}(x - P(u), y - Q(u)) \neq 0$ (Rueda and Sendra, 2007, Theorem 30). However, this result is not valid for linear differential rational parametrizations as it is shown in Remark 30.

In this section, we shall give a characterization of properness for linear differential rational parametrizations with the use of differential resultant combined with the order property. Before that, we need some preparations by studying a particular differential remainder sequence.

Given a linear differential rational parametrization $\mathcal{P}(u)=\left(\frac{P_1(u)}{Q_1(u)},\frac{P_2(u)}{Q_2(u)}\right)$ with $\gcd(P_i,Q_i)=1$ and $m_i=\mathrm{ord}(\frac{P_i(u)}{Q_i(u)})\geq 0$ for $i=1,2$. Without loss of generality, suppose $m_1\geq m_2$. Denote

$$f_1(x,y,u)=P_1(u)-xQ_1(u),\ f_2(x,y,u)=P_2(u)-yQ_2(u). \tag{9}$$

Fix the elimination ranking $\mathscr{R}: x<y<u$. Let $f_3(x,y,u)=\delta\text{-prem}(f_1,f_2)$ be the Ritt-Kolchin remainder of f_1 with respect to f_2 under $\mathscr{R}$. Since $f_1\notin \mathrm{sat}(f_2)$, $f_3\neq 0$, $\mathrm{ord}_u f_3<m_2$ and $\mathrm{ord}_y f_3\leq m_1-m_2$. And $Q_1\notin \mathrm{sat}(f_2)$ implies $\mathrm{ord}_x f_3=0$. If $\mathrm{ord}_u f_3\geq 0$, let $f_4(x,y,u)=\delta\text{-prem}(f_2,f_3)$. Then, we have

$$\mathrm{ord}_u f_4<\mathrm{ord}_u f_3,\ \mathrm{ord}_x f_4\leq m_2-\mathrm{ord}_u f_3. \tag{10}$$

If $\mathrm{ord}_u f_4\geq 0$, then let $f_5=\delta\text{-prem}(f_3,f_4)$. Continue the differential reduction process when $\mathrm{ord}_u f_{l-1}\geq 0$ until we get $f_l=\delta\text{-prem}(f_{l-2},f_{l-1})\in\mathcal{F}\{x,y\}$ for some $l\in\mathbb{N}$. Then, we obtain a sequence of differential remainders

$$f_1(x,y,u),\ f_2(x,y,u),\ f_3(x,y,u),\ldots,\ f_{l-1}(x,y,u),\ f_l(x,y). \tag{11}$$

Lemma 31. *The obtained sequence (11) satisfies the following properties:*

1) *For* $2\leq i\leq l-1$, $\mathrm{ord}_u f_i+\mathrm{ord}_x f_{i+1}\leq m_2$ *and* $\mathrm{ord}_u f_i+\mathrm{ord}_y f_{i+1}\leq m_1$.
2) *For each* $i\geq 2$, f_i *has the following representation form*

$$f_{i+1}=\sum_{k=0}^{m_2-\mathrm{ord}_u f_i} A_{i,k}(x,y)f_1^{(k)}+\sum_{j=0}^{m_1-\mathrm{ord}_u f_i} B_{i,j}(x,y)f_2^{(j)} \tag{12}$$

where $A_{i,k},B_{i,j}\in\mathcal{F}\{x,y\}$, *and* $B_{i,m_1-\mathrm{ord}_u f_i}\neq 0$, $A_{i,m_2-\mathrm{ord}_u f_i}\neq 0$ *are products of separants of* f_k $(k\leq i)$.

Proof. We shall show 1) and 2) by induction on i. First note that $\mathrm{ord}_u f_2+\mathrm{ord}_x f_3=m_2$ and $\mathrm{ord}_u f_2+\mathrm{ord}_y f_3\leq m_1$. By the differential reduction formula for $f_3=\delta\text{-prem}(f_1,f_2)$, there exist $a\in\mathbb{N}$ and $C_k\in\mathcal{F}\{x,y\}$ such that $f_3=(\mathrm{S}_{f_2})^a f_1-(\mathrm{S}_{f_2})^{a-1}\mathrm{S}_{f_1}f_2^{(m_1-m_2)}-\sum_{k=0}^{m_1-m_2-1}C_k f_2^{(k)}$. So both 1) and 2) holds for $i=2$.

Now suppose 1) and 2) holds for $i\leq j\ (\geq 2)$. We consider the case for $i=j+1$. Since both f_1 and f_2 are linear differential polynomials in u, all the $f_i\ (i\leq l-1)$ are linear in u and its derivatives and thus $\mathrm{ord}_u f_i<\mathrm{ord}_u f_{i-1}$. By the induction hypothesis, $\mathrm{ord}_u f_j+\mathrm{ord}_x f_{j+1}\leq m_2$, and f_{j+1} has a representation form as (12) with $B_{j,m_1-\mathrm{ord}_u f_j}\neq 0$, $A_{j,m_2-\mathrm{ord}_u f_j}\neq 0$. Since $f_{j+2}=\delta\text{-prem}(f_j,f_{j+1})$, f_{j+2} is a linear combination of f_j and $f_{j+1},f'_{j+1},\ldots,f_{j+1}^{(s)}$ with coefficients in $\mathcal{F}\{x,y\}$ and in particular the nonzero coefficient for $f_{j+1}^{(s)}$ is a product of separants of f_j,f_{j+1} where $s=\mathrm{ord}_u f_j-\mathrm{ord}_u f_{j+1}$. Thus, f_{j+2} has a representation form as (12) with $B_{j+1,m_1-\mathrm{ord}_u f_{j+1}}\neq 0$, $A_{j+1,m_2-\mathrm{ord}_u f_{j+1}}\neq 0$ being products of the separants of $f_k\ (k\leq j+1)$. And

$$\begin{aligned}\mathrm{ord}_x f_{j+2}&\leq \max\{\mathrm{ord}_x f_{j+1}+\mathrm{ord}_u f_j-\mathrm{ord}_u f_{j+1},\mathrm{ord}_x f_j\}\\&\leq \max\{m_2-\mathrm{ord}_u f_j+\mathrm{ord}_u f_j-\mathrm{ord}_u f_{j+1},m_2-\mathrm{ord}_u f_{j-1}\}\\&\leq m_2-\mathrm{ord}_u f_{j+1}.\end{aligned}$$

So $\mathrm{ord}_u f_{j+1}+\mathrm{ord}_x f_{j+2}\leq m_2$. Similarly, $\mathrm{ord}_u f_{j+1}+\mathrm{ord}_y f_{j+2}\leq m_1$ can be shown. Thus, 1) and 2) are proved by induction. □

Now, we are ready to propose the main theorem.

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Theorem 32. *Let* $\mathcal{P}(u)=(\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$ *be a linear differential rational parametrization with* $m_i=\mathrm{ord}(\frac{P_i(u)}{Q_i(u)}) \geq 0\,(i=1,2)$. *Then* $\mathcal{P}(u)$ *is proper if and only if*

$$R := \delta\text{-Res}\big(xQ_1(u) - P_1(u), yQ_2(u) - P_2(u)\big) \neq 0 \ \ \textit{and} \ \ \mathrm{ord}_x R = m_2, \mathrm{ord}_y R = m_1. \tag{13}$$

Proof. "$\Rightarrow$": It follows from Theorem 26.

"$\Leftarrow$": Suppose we have (13). Without loss of generality, assume $m_1 \geq m_2$. If $m_2 = 0$, then $\mathcal{F}\langle \frac{P_2(u)}{Q_2(u)}\rangle = \mathcal{F}\langle u\rangle$ and thus $\mathcal{P}(u)$ is proper. So it suffices to consider the case when $m_2 \geq 1$.

Let $f_1 = xQ_1(u) - P_1(u), f_2 = yQ_2(u) - P_2(u) \in \mathcal{F}\{u, x, y\}$. Fix the elimination ranking $\mathscr{R} : x < y < u$. Do the differential reduction process as in Lemma 31 under $\mathscr{R}$ and we consider the obtained sequence

$$f_1(x, y, u), f_2(x, y, u), f_3(x, y, u), \ldots, f_{l-1}(x, y, u), f_l(x, y).$$

By Lemma 31, there exist $a, a_i, b_j \in \mathcal{F}\{x, y\}$ with $a \neq 0, b_{m_2-\mathrm{ord}_u f_{l-1}} \neq 0$ such that

$$f_l(x, y) = a(x, y) f_2^{(m_1-\mathrm{ord}_u f_{l-1})} + \sum_{i=0}^{m_1-\mathrm{ord}_u f_{l-1}-1} a_i(x, y) f_2^{(i)} + \sum_{j=0}^{m_2-\mathrm{ord}_u f_{l-1}} b_j(x, y) f_1^{(j)}. \tag{14}$$

Claim A. f_l *and* f_{l-1} *satisfy the following properties:*

1) $f_l(x, y) \neq 0$, $\mathrm{ord}_x f_l = m_2$ *and* $\mathrm{ord}_y f_l = m_1$.
2) $f_{l-1}(x, y, u) = g(x, y)u + h(x, y)$, *where* $g, h \in \mathcal{F}\{x, y\} \setminus \{0\}$.

We now proceed to prove **Claim A**. We first show that $f_l \neq 0$ and $\mathrm{ord}_u f_{l-1} = 0$. Suppose the contrary that $f_l = 0$. Then by (14), $f_1, f_1', \ldots, f_1^{(m_2-\mathrm{ord}_u f_{l-1})}, f_2, f_2', \ldots, f_2^{(m_1-\mathrm{ord}_u f_{l-1})}$ are linearly dependent over $\mathcal{F}\langle x, y\rangle$. As a consequence, $f_1, f_1', \ldots, f_1^{(m_2)}, f_2, f_2', \ldots, f_2^{(m_1)}$ are linearly dependent over $\mathcal{F}\langle x, y\rangle$, which contradicts the fact that δ-Res$(f_1, f_2) \neq 0$. Thus $f_l(x, y) \neq 0$. And by (14), 1 can be written as a linear combination of $f_1, f_1', \ldots, f_1^{(m_2-\mathrm{ord}_u f_{l-1})}, f_2, f_2', \ldots, f_2^{(m_1-\mathrm{ord}_u f_{l-1})}$ over $\mathcal{F}\langle x, y\rangle$ with a nonzero coefficient for $f_2^{(m_1-\mathrm{ord}_u f_{l-1})}$, that is,

$$1 = \frac{a(x, y)}{f_l(x, y)} f_2^{(m_1-\mathrm{ord}_u f_{l-1})} + \sum_{i=0}^{m_1-\mathrm{ord}_u f_{l-1}-1} \frac{a_i(x, y)}{f_l(x, y)} f_2^{(i)} + \sum_{j=0}^{m_2-\mathrm{ord}_u f_{l-1}} \frac{b_j(x, y)}{f_l(x, y} f_1^{(j)}. \tag{15}$$

If $\mathrm{ord}_u f_{l-1} > 0$, then by differentiating both sides of (15), we obtain that $f_1, f_1', \ldots, f_1^{(m_2)}, f_2, f_2', \ldots, f_2^{(m_1)}$ are linearly dependent over $\mathcal{F}\langle x, y\rangle$, which leads to a contradiction. So $\mathrm{ord}_u f_{l-1} = 0$ and $f_{l-1} = g(x, y)u + h(x, y)$ for some $g, h \in \mathcal{F}\{x, y\}$ with $g \neq 0$.

It remains to show that $\mathrm{ord}_x f_l = m_2$, $\mathrm{ord}_y f_l = m_1$ and $h \neq 0$. Denote $m = m_1 + m_2 + 2$. Let $M \in \mathcal{F}\langle x, y\rangle^{m\times m}$ be the resultant matrix of $f_1^{(m_2)}, \ldots, f_1', f_1, f_2^{(m_1)}, \ldots, f_2', f_2$ w.r.t. the variables $u^{(m_1+m_2)}, u^{(m_1+m_2-1)}, \ldots, u', u$. We perform row operations on M using the a_i and b_j in (14) as follows. For all the $i \neq m_2 + 2$, add a multiple c_i of the i-th row to the $(m_2 + 2)$-th row of M successively and denote the obtained matrix by M_1, where for $i = 1, \ldots, m_2 + 1$, $c_i = b_{m_2+1-i}/a$; and for $i \geq m_2 + 3$, $c_i = a_{m_1+m_2+2-i}/a$. By (14), the $(m_2 + 2)$-th row of M_1 becomes $(0\ 0\ \cdots\ 0\ f_l/a)$. Thus,

$$\delta\text{-Res}(f_1, f_2) = \det(M) = \det(M_1) = (-1)^{m_1} c \cdot f_l/a \cdot \det(M_2), \tag{16}$$

where $c \neq 0$ is the coefficient of $f_1^{(m_2)}$ in $u^{(m_1+m_2)}$, and M_2 is the matrix obtained by deleting the 1-st row, the (m_2+2)-th row, the 1-st column and the last column. Obviously, $\mathrm{ord}_x c \leq 0 < m_2$ and $\mathrm{ord}_y c = -\infty$. Since M_2 is free of coefficients of $f_1^{(m_2)}$ and $f_2^{(m_1)}$, $\mathrm{ord}_x\det(M_2) < m_2$ and $\mathrm{ord}_y\det(M_2) < m_1$. By Lemma 31, $a(x, y) = B_{l-1,m_1-\mathrm{ord}_u(f_{l-1})}$ is a product of separants of $f_k\,(k \leq l - 1)$, and for such

f_k, $\text{ord}_x f_k < m_2$ and $\text{ord}_y f_k < m_1$. So $\text{ord}_x a(x, y) < m_2$ and $\text{ord}_y a(x, y) < m_1$. Meanwhile, (13) assumes that $\text{ord}_x \delta\text{-Res}(f_1, f_2) = m_2$ and $\text{ord}_y \delta\text{-Res}(f_1, f_2) = m_1$. So by (16), we have $\text{ord}_x f_l = m_2$ and $\text{ord}_y f_l = m_1$. We finish the proof of **Claim A** by showing $h \neq 0$. If $h = 0$, then $f_l = g^k f_{l-2}(x, y, 0)$ for some $k \in \mathbb{N}$ and thus $\text{ord}_x f_l \leq \max\{\text{ord}_x g, \text{ord}_x f_{l-2}\} < m_2$ by Lemma 31, a contradiction. Thus $h \neq 0$ and **Claim A** is proved.

Since $\det(M)$ is linear in $x^{(m_2)}$ and $\text{ord}_x f_l = m_2$, by (16), f_l is linear in $x^{(m_2)}$. Recall that $\text{ord}_x f_{l-1} < m_2$ and $f_{l-1} = g(x, y)u + h(x, y)$ is linear in u. Thus, $\mathcal{A} = \{f_l(x, y), f_{l-1}(x, y, u)\}$ is an autoreduced set w.r.t. the elimination ranking $\mathscr{R}_1 : y < x < u$. Since $\text{rk}(f_l) = (x^{(m_2)}, 1)$ and $\text{rk}(f_l) = (u, 1)$, by Lemma 1 and the paragraph above it, $\mathcal{A}$ is irreducible and $\mathcal{P} = \text{sat}(\mathcal{A})$ is a prime differential ideal with $\mathcal{A}$ being a characteristic set of it under $\mathscr{R}_1$. We shall show $[f_1, f_2] : (Q_1 Q_2)^\infty \subset \mathcal{P}$ by proving i) $f_1, f_2 \in \mathcal{P}$ and ii) $Q_1, Q_2 \notin \mathcal{P}$.

To show i), by Lemma 31, for each $1 \leq i \leq l-1$, $\text{ord}_x f_i < m_2$ and thus the separant $S_{f_i} \notin \mathcal{P}$. By the reduction process, for each $1 \leq i \leq l-2$, there exists $k_i \in \mathbb{N}$ such that $S_{f_{i+1}}^{k_i} f_i \equiv f_{i+2} \bmod [f_{i+1}]$. Therefore, $f_{l-2} \in \mathcal{P}$ and consequently $f_2, f_1 \in \mathcal{P}$.

To show ii), let $M_3 \in \mathcal{F}\{x, y\}^{m \times m}$ be the resultant matrix of $Q_1, f_1^{(m_2-1)}, \ldots, f_1', f_1, f_2^{(m_1)}, \ldots, f_2', f_2$ w.r.t. $u^{(m_1+m_2)}, \ldots, u', u$. Then $\det(M_3) = \text{coeff}(R, x^{(m_2)}) \neq 0$, for R is linear in $x^{(m_2)}$. Therefore, $Q_1, f_1^{(m_2-1)}, \ldots, f_1', f_1, f_2^{(m_1)}, \ldots, f_2', f_2$ are linearly independent over $\mathcal{F}(x^{[m_2-1]}, y^{[m_1]})$, and

$$\begin{aligned}&\text{Span}_{\mathcal{F}(x^{[m_2-1]}, y^{[m_1]})}(Q_1, f_1^{(m_2-1)}, \ldots, f_1', f_1, f_2^{(m_1)}, \ldots, f_2', f_2)\\ &\quad = \text{Span}_{\mathcal{F}(x^{[m_2-1]}, y^{[m_1]})}(1, u, \ldots, u^{(m_1+m_2)}).\end{aligned}$$

The representation of 1 in terms of $Q_1, f_1^{(m_2-1)}, \ldots, f_2$ yields a nonzero differential polynomial $G \in [Q_1, f_1, f_2] \cap \mathcal{F}\{x, y\}$ with $\text{ord}_x G < m_2$. If $Q_1 \in \mathcal{P}$, then $G \in \mathcal{P} \cap \mathcal{F}\{x, y\} = \text{sat}(f_l)$, which is impossible. Thus, $Q_1 \notin \mathcal{P}$. Note that $\frac{\partial^2 R}{\partial x^{(m_2)} \partial y^{(m_1)}} = \det(M_4) = 0$, where M_4 is the resultant matrix of $Q_1, f_1^{(m_2-1)}, \ldots, f_1', f_1, Q_2, f_2^{(m_1-1)} \ldots, f_2', f_2$ w.r.t. $u^{(m_1+m_2)}, \ldots, u', u$. So by (16), if B is the irreducible factor of f_l effectively involving $x^{(m_2)}$, then $\text{ord}_y B = m_1$ and B is linear in $y^{(m_1)}$. Thus, $\mathcal{P} \cap \mathcal{F}\{x, y\} = \text{sat}(f_l) = \text{sat}(B)$ and B is a characteristic set of $\text{sat}(f_l)$ w.r.t. any ranking. Since $\text{coeff}(R, y^{(m_1)}) \neq 0$, repeating the above procedures, we obtain some $H \in [Q_2, f_1, f_2] \cap \mathcal{F}\{x, y\}$ with $\text{ord}_y H < m_1$, and consequently $Q_2 \notin \mathcal{P}$. Thus, $\big([f_1, f_2] : (Q_1 Q_2)^\infty\big) \subset \mathcal{P}$.

Suppose $\mathcal{P}(u)$ is a differential rational parametrization of $A(x, y) \in \mathcal{F}\{x, y\}$. Then $A \in \big([f_1, f_2] : (Q_1 Q_2)^\infty\big) \cap \mathcal{F}\{x, y\} \subset \mathcal{P} \cap \mathcal{F}\{x, y\} = \text{sat}(f_l)$. Since $\text{ord}_x A \leq m_2$ by Proposition 12, $\text{ord}_x A = m_2$ and $\text{ord}_y A = m_1 - m_2 + \text{ord}_x A = m_1$ follows. By Theorem 22, $\mathcal{P}(u)$ is proper. □

We conclude this section by giving two algorithmic consequences of Theorem 32 and its proof. First, we get back to the implicitization problem, and devise an algorithm based on Theorem 32 and Corollary 27 to decide whether a given linear differential rational parametrization is proper, and in the affirmative case, to compute its implicit equation.

Algorithm Proper-Implicitization:
Input: A linear differential rational parametrization $\mathcal{P}(u) = (\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$ with $\text{ord}(\frac{P_i}{Q_i}) \geq 0$.
Output: The implicit equation of $\mathcal{P}(u)$, if it is proper, otherwise, return "$\mathcal{P}(u)$ is not proper".

1. Compute the differential resultant $R = \delta\text{-Res}(Q_1(u)x - P_1(u), Q_2(u)y - P_2(u))$ w.r.t. u.
2. If $R = 0$, then return "$\mathcal{P}(u)$ is not proper".
3. If $\text{ord}_x R = \text{ord}(\frac{P_2(u)}{Q_2(u)})$ and $\text{ord}_y R = \text{ord}(\frac{P_1(u)}{Q_1(u)})$, return the main factor of R as the implicit equation; else, return "$\mathcal{P}(u)$ is not proper".

Remark 33. Concerning the complexity of the algorithm, note that the main step is the computation of the differential resultant R, for which we just need to compute the determinant of a matrix of size $m_1 + m_2 + 2$ with $m_i = \text{ord}(\frac{P_i}{Q_i})$. So the complexity of the computation of the differential resultant is $(m_1 + m_2 + 2)^\omega$, where ω is the exponent of matrix multiplication. Recall that the implicit equation is

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irreducible. If the given parametrization is proper, we need to factor R to obtain the implicit equation A, so the complexity of the multivariate polynomial factorization problem will be involved in order to analyze the computing complexity of the whole algorithm. There are polynomial-time algorithms for multivariate polynomial factorization and irreducibility testing over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and algebraic number fields (Kaltofen, 2003), but as $\mathcal{F}$ is an arbitrary field, we can not give a general complexity analysis for this step. Since the implicitization algorithm based on the method of characteristic sets is not factorization-free and $\mathrm{sat}(R) = \mathrm{sat}(A)$, this algorithm in general is more efficient. However, when the given parametrization is not proper, the current method via differential resultants may fail to compute the implicit equation (see Remark 30), while the characteristic set method in (Gao, 2003) and the elimination method given in (Ovchinnikov et al., 2018) can provide general techniques to solve the implicitization problem.

The second application of Theorem 32 is to compute inversion maps of proper linear differential rational parametrizations. Recall that in the algebraic case, Sendra et al. proposed a method to compute the inversion maps of a proper rational parametrization via gcd computations (Sendra et al., 2007, Section 4.4). For differential rational parametric equations (DRPEs), Gao gave a general method based on the Wu-Ritt characteristic set method to decide whether a set of DRPEs is proper and in the affirmative case, to compute the inversion maps (Gao, 2003, Theorem 6.1). For proper linear differential rational parametrizations, we show the sequence of differential remainders constructed in the proof of Theorem 32 could be used to compute the inversion maps.

Corollary 34. *Let (C, A) be a unirational differential curve with a proper linear differential rational parametrization $\mathcal{P}(u) = (\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$. Suppose $m_i = \mathrm{ord}(\frac{P_i}{Q_i}) \geq 1$ for $i = 1, 2$. Perform the differential reduction process under the elimination ranking $x < y < u$ for $f_1 = xQ_1 - P_1$, $f_2 = yQ_2 - P_2$ and suppose the obtained sequence is $f_1(x, y, u), f_2(x, y, u), \ldots, f_{l-1}(x, y, u), f_l(x, y)$. Then $f_{l-1} = g(x, y)u + h(x, y)$ for some $g, h \in \mathcal{F}\{x, y\} \setminus \{0\}$ and*

$$\begin{array}{rcl} \mathcal{U} : C & \dashrightarrow & \mathbb{A}^1 \\ (x, y) & \longmapsto & U(x, y), \end{array}$$

given by $U(x, y) = -\frac{h(x,y)}{g(x,y)}$ is the inversion map of $\mathcal{P}(u)$. Moreover, $\mathrm{ord}_x U < m_2, \mathrm{ord}_y U < m_1$.

Proof. By the proof of Theorem 32, f_{l-1} is linear in u with $\mathrm{ord}_x f_{l-1} < m_2$ and $\mathrm{ord}_y f_{l-1} < m_1$. Thus, $u \in \mathcal{F}\big(\frac{P_1}{Q_1}, (\frac{P_1}{Q_1})', \ldots, (\frac{P_1}{Q_1})^{(m_2-1)}, \frac{P_2}{Q_2}, (\frac{P_2}{Q_2})', \ldots, (\frac{P_2}{Q_2})^{(m_1-1)}\big)$. □

Remark 35. We are interested in proper linear differential rational parametrizations in this section. However, it may happen that even if a unirational differential curve has a linear differential rational parametrization, it does not have a proper linear differential rational parametrization. For example, let $A = (x'x + x^3 - 2x^2 - x)y' + (xy + y^2)x'' + 3y^2x'x + y^2x^3 - 2yx'^2 + 6yx'x + 3yx' + 4yx^3 + 3yx^2 - 2xy - y + 5x^3 + 2x^2$. Then $\mathcal{P}(u) = (\frac{u''}{u' + u'''}, \frac{u' + 2u''}{u^{(4)}})$ is a linear differential rational parametrization of (C, A) and $\mathcal{Q}(u) = \big(\frac{u}{u' + u^2 + 1}, \frac{2u+1}{u'' + 3uu' + u^3}\big)$ is a proper differential rational parametrization of (C, A). By Theorem 18, any proper differential rational parametrization of (C, A) is of degree greater than 1.

5. Rational parametrization for linear differential curves

We now deal with the parametrization problem for differential curves, which asks for criteria to decide algorithmically whether an implicitly given differential curve is unirational or not, and in the affirmative case, to return a differential rational parametrization. In general, it is a very difficult problem. In this section, we start from the simplest nontrivial case by considering the unirationality problem of linear differential curves.

Definition 36. Let $(C, A(x, y))$ be an irreducible differential curve. We call C a linear differential curve if A is a linear differential polynomial in $\mathcal{F}\{x, y\}$.

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In the algebraic case, by Gaussian elimination, we know each linear variety is unirational and has a polynomial parametric representation. However, in the differential case, even a linear differential curve might not be unirational as shown in Example 13 (2).

Every linear differential polynomial $A(x, y) \in \mathcal{F}\{x, y\}$ is of the form

$$A = L_1(x) + L_2(y) + a$$

for some linear differential operators $L_1, L_2 \in \mathcal{F}[\delta]$ where at least one L_i is nonzero and $a \in \mathcal{F}$. We shall show in Theorem 39 that a necessary and sufficient condition for $A(x, y)$ to be unirational is that the greatest common left divisor of L_1, L_2 is 1. Before that, we recall the extended left Euclidean algorithm given by Bronstein and Petkovšek (1996).

In general, $\mathcal{F}[\delta]$ is a non-commutative domain and there exist the left and the right Euclidean divisions in $\mathcal{F}[\delta]$. Given $L_1, L_2 \in \mathcal{F}[\delta]$ with $L_2 \neq 0$, by the left Euclidean division, we obtain $Q, R \in \mathcal{F}[\delta]$ with $\deg(R) < \deg(L_2)$ satisfying $L_1 = L_2 Q + R$, where Q and R are called respectively the *left-quotient* and the *left-remainder* of L_1 w.r.t. L_2, denoted by $\mathrm{lquo}(L_1, L_2)$ and $\mathrm{lrem}(L_1, L_2)$. If $R = 0$, then L_2 is called a left divisor of L_1, and correspondingly, L_1 is called a right multiple of L_2. A common left divisor of L_1 and L_2 with the highest degree is called a *greatest common left divisor* of L_1 and L_2. There exists a unique, monic (i.e., reduced in Ore's sense), greatest common left divisor, denoted by $\mathrm{gcld}(L_1, L_2)$. A common right multiple of L_1, L_2 of minimal degree is called a *least common right multiple*. And we have analogous notions for right Euclidean divisions. Below, we restate the extended left Euclidean algorithm $\mathbf{ELE}(L_1, L_2)$ for later use and list its basic properties in Proposition 37, which were given in (Bronstein and Petkovšek, 1996, pp. 14-15).

Left Euclidean Algorithm: $\mathbf{ELE}(L_1, L_2)$
Input: $L_1, L_2 \in \mathcal{F}[\delta]$.
Output: The tuple $(R_{n-1}, A_n, B_n, A_{n-1}, B_{n-1}) \in \mathcal{F}[\delta]^5$.

1. $R_0 := L_1, A_0 := 1, B_0 := 0$;
   $R_1 := L_2, A_1 := 0, B_1 := 1$;
   $i := 1$.
2. While $R_i \neq 0$ do
   $i := i + 1$;
   $Q_{i-1} := \mathrm{lquo}(R_{i-2}, R_{i-1})$;
   $R_i := \mathrm{lrem}(R_{i-2}, R_{i-1})$;
   $A_i := A_{i-2} - A_{i-1} Q_{i-1}$;
   $B_i := B_{i-2} - B_{i-1} Q_{i-1}$.
3. $n := i$, and **return** $(R_{n-1}, A_n, B_n, A_{n-1}, B_{n-1})$.

Proposition 37. *Let $L_1, L_2 \in \mathcal{F}[\delta]$. The algorithm* $\mathbf{ELE}(L_1, L_2)$ *can be used to compute a greatest common left divisor and a least common right multiple of L_1, L_2 and the obtained sequences A_i, B_i, R_i satisfy the following properties:*

(1). *R_{n-1} is a greatest common left divisor of L_1, L_2;*
(2). *$L_1 A_n = -L_2 B_n$ is a least common right multiple of L_1, L_2;*
(3). *$R_i = L_1 A_i + L_2 B_i$ for $0 \le i \le n$;*
(4). $\deg(A_i) = \deg(L_2) - \deg(R_{i-1})$, $\deg(B_i) = \deg(L_1) - \deg(R_{i-1})$ *for $2 \le i \le n$.*

The following result given in (Rueda and Sendra, 2007) will be used to derive Theorem 39.

Lemma 38. (Rueda and Sendra, 2007, Theorem 30) *Let $L_1, L_2 \in \mathcal{F}[\delta]$. Then $\mathcal{F}\langle L_1(u), L_2(u)\rangle = \mathcal{F}\langle u\rangle$ if and only if* $\mathrm{gcrd}(L_1, L_2) = 1$ *(i.e., a greatest common right divisor of L_1, L_2 belongs to $\mathcal{F} \setminus \{0\}$).*

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Theorem 39. *Let $F = L_1(x) + L_2(y) + a \in \mathcal{F}\{x, y\} \setminus \mathcal{F}$ with $L_1, L_2 \in \mathcal{F}[\delta]$. Then,*

$$(\mathcal{C}, F) \text{ is unirational if and only if } \operatorname{gcld}(L_1, L_2) = 1.$$

Furthermore, each unirational linear differential curve has a proper linear differential polynomial parametrization.

Proof. For the necessity, suppose $(\mathcal{C}, F)$ is unirational. If $\operatorname{gcld}(L_1, L_2) \neq 1$, then there exist $L \in \mathcal{F}[\delta] \setminus \mathcal{F}$ and $L_3, L_4 \in \mathcal{F}[\delta]$ such that $L_1 = LL_3, L_2 = LL_4$. Then $F = L(L_3(x) + L_4(y)) + a$ with $\operatorname{ord}(L_3(x) + L_4(y)) < \operatorname{ord}(F)$. Since there exists $\mathcal{P}(u) \in \mathcal{F}\langle u\rangle^2$ such that $\operatorname{sat}(F) = \mathbb{I}(\mathcal{P}(u))$, we have $F(\mathcal{P}(u)) = 0$, which implies that $L_3(\mathcal{P}(u)) + L_4(\mathcal{P}(u)) \in \mathcal{F}$. Thus, $A := L_3(x) + L_4(y) - b \in \operatorname{sat}(F)$ for some $b \in \mathcal{F}$. Since $\operatorname{ord}(A) < \operatorname{ord}(F)$, this leads to a contradiction. So $\operatorname{gcld}(L_1, L_2) = 1$.

To show the sufficiency, suppose $\operatorname{gcld}(L_1, L_2) = 1$. By performing the algorithm **ELE(**L_1, L_2**)**, we obtain $A_i, B_i, R_i \in \mathcal{F}[\delta]$ satisfying the properties given in Proposition 37. In particular, $L_1 A_n = -L_2 B_n$ is a least common right multiple of L_1 and L_2. The fact that $\operatorname{gcld}(L_1, L_2) = 1$ yields $c := R_{n-1} \in \mathcal{F} \setminus \{0\}$, and consequently $\deg(A_n) = \deg(L_2), \deg(B_n) = \deg(L_1)$. Let

$$\mathcal{P}(u) = \Big(A_n(u) + A_{n-1}(-a/c), B_n(u) + B_{n-1}(-a/c)\Big) \in \mathcal{F}\{u\}^2.$$

We shall show that $\mathcal{P}(u)$ is a proper linear differential polynomial parametrization of $(\mathcal{C}, F)$.

We first prove $\mathcal{F}\langle\mathcal{P}(u)\rangle = \mathcal{F}\langle u\rangle$. Since $\mathcal{F}\langle\mathcal{P}(u)\rangle = \mathcal{F}\langle A_n(u), B_n(u)\rangle$, by Lemma 38, it suffices to prove that $\operatorname{gcrd}(A_n, B_n) = 1$. If $\operatorname{gcrd}(A_n, B_n) \neq 1$, there exists $C(\delta) \in \mathcal{F}[\delta] \setminus \mathcal{F}$ such that $A_n = C_1(\delta)C(\delta), B_n = C_2(\delta)C(\delta)$ for some $C_1, C_2 \in \mathcal{F}[\delta]$. Since $L_1 A_n = -L_2 B_n$, we obtain $L_1 C_1 = -L_2 C_2$ is also a common right multiple of L_1, L_2, which contradicts the fact that $L_1 A_n$ is a least common right multiple of A, B. Therefore, $\mathcal{F}\langle\mathcal{P}(u)\rangle = \mathcal{F}\langle u\rangle$.

By Theorem 15, there exists an irreducible differential polynomial $G(x, y) \in \mathcal{F}\{x, y\}$ with $\operatorname{ord}_x G = \deg(L_1), \operatorname{ord}_y G = \deg(L_2)$ such that $\mathbb{I}(\mathcal{P}(u)) = \operatorname{sat}(G)$. Since $L_1 A_{n-1} + L_2 B_{n-1} = c$, by acting this operator on $-a/c$, we have $L_1(A_{n-1}(\frac{-a}{c})) + L_2(B_{n-1}(\frac{-a}{c})) = -a$. So $F(\mathcal{P}(u)) = L_1(A_n(u)) + L_2(B_n(u)) = (L_1 A_n + L_2 B_n)(u) = 0$ and $F \in \operatorname{sat}(G)$ follows. Since $\operatorname{ord}(G) = \operatorname{ord}(F)$ and F is linear, $F = eG$ for some $e \in \mathcal{F}$ and $\mathbb{I}(\mathcal{P}(u)) = [F]$. Thus, $\mathcal{P}(u)$ is a proper linear differential polynomial parametrization of F and $(\mathcal{C}, F)$ is unirational. The later assertion follows directly from the above proof. □

By the proof of Theorem 39, we can devise an algorithm to determine whether an implicitly given linear differential curve is unirational or not, and in the affirmative case, to construct a proper linear differential polynomial parametrization for it.

Algorithm Linear-Differential-Curve-Parametrization: LDCP(F)
Input: $F = L_1(x) + L_2(y) + a \in \mathcal{F}\{x, y\} \setminus \mathcal{F}$ with $L_1, L_2 \in \mathcal{F}[\delta]$.
Output: A proper linear differential polynomial parametrization $\mathcal{P}(u)$ of $(\mathcal{C}, F)$, if it is unirational; No, in the contrary case.

1. Perform **ELE**$(L_1, L_2) = (R_{n-1}, A_n, B_n, A_{n-1}, B_{n-1})$;
2. If $R_{n-1} \notin \mathcal{F}$, then **return** No;
3. **Return** $\mathcal{P}(u) = \Big(A_n(u) + A_{n-1}(-a/R_{n-1}), B_n(u) + B_{n-1}(-a/R_{n-1})\Big)$.

Below, we give examples to illustrate Theorem 39.

Example 40.

(1) Let $A = x'' - y' \in \mathbb{Q}\{x, y\}$, then $L_1 = \delta^2, L_2 = \delta$ and $(\mathcal{C}, A)$ is not unirational.

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(2) Let $A = y' - x' - x \in \mathbb{Q}\{x, y\}$, then $L_1 = -\delta - 1, L_2 = \delta$, $(\mathcal{C}, A)$ is unirational with a proper parametrization $(u', u + u')$.
(3) Let $A = x' + x + ty' + (t+1)y \in \mathbb{Q}(t)\{x, y\}$ with $\delta = \frac{d}{dt}$. For this example, $L_1 = \delta + 1, L_2 = t\delta + t + 1$. Since $\mathrm{gcld}(L_1, L_2) = \delta + 1$, $(\mathcal{C}, A)$ is not unirational.
(4) Let $A = tx' + tx + y' + y \in \mathbb{Q}(t)\{x, y\}$ with $\delta = \frac{d}{dt}$. Then $(\mathcal{C}, A)$ is unirational with a proper parametrization $(u' + u, -tu' + (1-t)u)$. Here $L_1 = t\delta + t, L_2 = \delta + 1$. Note that $\mathrm{gcld}(L_1, L_2) = 1$ but $\mathrm{gcrd}(L_1, L_2) = \delta + 1$.

Given two differential polynomials $A, B \in \mathcal{F}\{x, y\}$, we have shown in Example 13 (2) that if A is a proper derivative of B, then $(\mathcal{C}, A)$ is not differentially unirational. If additionally both A and B are linear, we have a stronger result as follows.

Proposition 41. *Let A, B be two linear differential polynomials in $\mathcal{F}\{x, y\} \setminus \mathcal{F}$.*

1). *If $A \in [B]$ with $\mathrm{ord}(A) > \mathrm{ord}(B)$, then $(\mathcal{C}, A)$ is not differentially unirational.*
2). *Suppose A is differentially unirational. If $[A] \subseteq [B]$, then $[A] = [B]$.*

Proof. Without loss of generality, suppose $\mathrm{ord}(A) = \mathrm{ord}_x A$ and set $s = \mathrm{ord}_x A - \mathrm{ord}_x B$. Suppose $A \in [B]$. Since A, B are linear, there exists $a_i \in \mathcal{F}$ with $a_s \neq 0$ such that

$$A = a_0 \cdot B + a_1 \cdot B' + \cdots + a_s \cdot B^{(s)}.$$

Clearly, the result 2) is a direct consequence of 1), so it suffices to show 1). If $\mathrm{ord}(A) > \mathrm{ord}(B)$, then $s > 0$ and $a_0 + a_1\delta + \cdots + a_s\delta^s$ is a common left divisor of L_1, L_2, where $A = L_1(x) + L_2(y) + b \in \mathcal{F}\{x, y\}$. By Theorem 39, A is not differentially unirational. □

Remark 42. Proposition 41 shows that given two linear differential curves $\mathcal{C}_2 \subseteq \mathcal{C}_1$, if $\mathcal{C}_1$ is unirational, then $\mathcal{C}_1 = \mathcal{C}_2$. However, in general, the inclusion of unirational differential curves does not imply the equality. For example, let $A = y'x - x'y + xy^2$ and $B = y$, then $(\mathcal{C}_2, B) \subsetneqq (\mathcal{C}_1, A)$. Note that $(\mathcal{C}_1, A)$ and $(\mathcal{C}_2, B)$ are unirational with parametrizations $(u', \frac{u'}{u})$ and $(u, 0)$ respectively.

Now consider the implicitization of linear differential curves with given linear differential polynomial parametrization equations. In (Rueda and Sendra, 2007, Sec. 8.1., Algorithm 2), an algorithm was devised to compute the implicit equation of the linear differential curve using differential resultants. We give an alternative method based on the differential remainder sequence introduced in (11).

Proposition 43. *Let $\mathcal{P}(u) = (P_1(u), P_2(u)) \in (\mathcal{F}\{u\} \setminus \mathcal{F})^2$ be a linear differential polynomial parametrization. Let $f_1 = x - P_1(u)$, $f_2 = y - P_2(u) \in \mathcal{F}\{u, x, y\}$. Suppose*

$$f_1(x, y, u),\ f_2(x, y, u),\ \ldots,\ f_{l-1}(x, y, u),\ f_l(x, y)$$

is the differential remainder sequence under the elimination ranking $\mathscr{R} : x < y < u$ obtained by the differential reduction process as in Lemma 31. Then $\mathcal{P}(u)$ parametrizes $(\mathcal{C}, f_l(x, y))$

Proof. Let $A(x, y) = 0$ be the implicit equation of $x = P_1(u), y = P_2(u)$. That is, A is linear with $\mathbb{I}(\mathcal{P}(u)) = [A(x, y)]$. Since $f_{i-2} \equiv f_i \bmod [f_{i-1}]$ for $i = 3, \ldots, l$, each $f_i \in [f_1, f_2]$ and $f_1, f_2 \in [f_l, f_{l-1}]$. We first show that $f_l(x, y) \neq 0$. Suppose the contrary, then $f_2, f_1 \in [f_{l-1}]$. Since $f_{l-1} \in [f_1, f_2]$, we have $\mathrm{ord}_x f_{l-1} \geq 0$ or $\mathrm{ord}_y f_{l-1} \geq 0$, which contradicts the fact that $f_2, f_1 \in [f_{l-1}]$. So $f_l(x, y) \neq 0$. Without loss of generality, suppose $\mathrm{ord}(P_1) \geq \mathrm{ord}(P_2)$. Two cases are considered:

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- Case 1) $l = 3$. Here, $f_3(x, y) \in \mathbb{I}(\mathcal{P}(u)) = [A]$. Since $\text{ord}_x f_3 = 0$ and $\text{ord}_x A \geq 0$, $f_3 = cA$ for some $c \in \mathcal{F}$ and $\mathbb{I}(\mathcal{P}(u)) = [f_3]$ follows.
- Case 2) $l \geq 4$. Since $\text{ord}_y f_i = \text{ord}_u f_1 - \text{ord}_u f_{i-1}$ for $i \geq 3$, $\text{ord}_y f_{i-1} < \text{ord}_y f_i$ for $4 \leq i \leq l$. Thus, $\mathcal{A}$: $f_l(x, y), f_{l-1}(x, y, u)$ is a characteristic set of the prime differential ideal $[f_l, f_{l-1}]$ under $\mathscr{R}$. Since $f_2, f_1 \in [f_l, f_{l-1}]$, $A(x, y) \in [f_1, f_2] \subseteq [f_l, f_{l-1}]$. Therefore, $\text{ord}_y f_l \leq \text{ord}_y A$. Since $f_l \in \mathbb{I}(\mathcal{P}(u)) = [A]$, $f_l = cA$ for some $c \in \mathcal{F}$ and $\mathbb{I}(\mathcal{P}(u)) = [f_l]$.

Thus, $\mathcal{P}(u)$ is a differential parametrization of the differential curve $(\mathcal{C}, f_l)$. □

Remark 44. Example 40 (1) shows that the linear differential curve $\mathcal{C} = \mathbb{V}(x'' - y') \subset \mathbb{A}^2$ is not unirational. However, if we allow differential rational parametrizations involving arbitrary constants as in (Gao, 2003, Example 1.2.), then $\mathcal{C}$ has a parametrization of the form $x = u$ and $y = u' + c$ where c is an arbitrary constant. It is also an interesting topic to study generalized "unirational" differential curves with rational parametrizations involving arbitrary constants.

Remark 45. Theorem 39 shows that a unirational linear differential curve always admits proper linear differential polynomial parametrizations. For a general problem, one may ask which kinds of unirational differential curves can possess polynomial parametrizations or proper polynomial parametrizations. For algebraic curves, it was shown in (Sendra et al., 2007, Theorem 6.11) that a curve has polynomial parametrizations if and only if it has proper polynomial parametrizations, and such a curve is called a polynomial curve. In (Sendra et al., 2007, Section 6.2), Sendra et al. solved the polynomiality problem positively by proposing an algorithm to decide whether a plane curve represented by a rational parametrization is polynomial, and in the affirmative case, to compute a proper polynomial parametrization. The key result leading to the algorithm is a simple criterion for polynomiality in (Sendra et al., 2007, Corollary 6.14) by just comparing the square-free parts of the denominators of the two components of a given proper rational parametrization.

In the differential case, the polynomiality problem is more complicated to solve. First, it may happen that a unirational differential curve admits polynomial parametrizations but does not have proper polynomial parametrizations. For example, $(\mathcal{C}, x'^2 - 4xy)$ is a unirational differential curve with a nonproper polynomial parametrization $\mathcal{P}_1(u) = (u^2, u'^2)$ and a proper rational parametrization $\mathcal{P}_2(u) = (u, \frac{u'^2}{4u})$. Since $\mathcal{P}_2\left(\frac{au+b}{cu+d}\right)$ is not a polynomial parametrization for $ad - bc \neq 0$, it follows by Theorem 18 that $\mathcal{C}$ does not admit a proper polynomial parametrization. Second, given a differential rational parametrization, even if we can use the reparametrization algorithm given in (Gao, 2003) to compute a proper rational parametrization, it is still hard to derive a criterion for polynomiality from a proper differential rational parametrization.

6. Problems for further study

There are several problems for further study. Given a linear differential rational parametrization $\mathcal{P}(u) = (\frac{P_1(u)}{Q_1(u)}, \frac{P_2(u)}{Q_2(u)})$, Theorem 32 provides an algorithm to decide whether a given linear differential rational parametrization is proper or not, and in the affirmative case, to compute the implicit equation. It is interesting to see whether in general the differential resultant can be used to compute the implicit equations of proper differential rational parametric equations.

In the algebraic case, Sendra and Winkler (2001) introduced the notion of tracing index for rational parametrizations and gave an algorithmic approach based on greatest common divisors for computing the tracing index of a given rational parametrization and deciding whether it is proper. It is interesting to define a similar notion of tracing index for differential rational parametrizations and devise an efficient algorithm to decide properness of differential rational parametrizations.

The most important and unsolved problem is to give general methods to determine the rational parametrizability of nonlinear differential curves and if so, to develop efficient algorithms to compute proper differential rational parametrizations. Motivated by the results for algebraic curves, the

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determination problem may amount to define new differential invariants such as differential genus for differential curves as proposed in (Feng and Gao, 2006) and (Gao, 2003).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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