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Chow form for projective differential variety ☆

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In this paper, a generic intersection theorem in projective differential algebraic geometry is given. Then, the Chow form for an irreducible projective differential variety is defined and basic properties of the differential Chow form in affine differential case are shown to be valid for its projective differential counterpart. Finally, we apply the differential Chow form to a result of linear dependence over projective varieties given by Kolchin.

1. Introduction

Differential algebra or differential algebraic geometry founded by Ritt and Kolchin aims to study algebraic differential equations in a similar way that polynomial equations are studied in commutative algebra or algebraic geometry [18,9]. An excellent survey on this subject can be found in [2].

The Chow form, also known as the Cayley form or the Cayley–Chow form, is a basic concept in algebraic geometry [5,6] and has many important applications in transcendental number theory [15,17], elimination theory [1,3], and algebraic computational complexity [7].

Recently, the theory of Chow forms for affine differential algebraic varieties was developed and basic properties of the algebraic Chow form were extended to its differential counterpart in a nontrivial

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way [4]. Furthermore, a theory of differential resultants and sparse differential resultants was also given [4,13,14]. In this paper, we will study the Chow form for projective differential varieties.

It is known that most results in projective algebraic geometry are more complete than their affine analogs. But in differential algebraic geometry, this is not the case. Due to the complicated structure, projective differential varieties are not studied thoroughly. The basis of projective differential algebraic geometry was established by Kolchin in his paper [10]. There, he cited a remark by Ritt:

> Consider an irreducible algebraic variety V in complex projective space $P_n(\mathbb{C})$ and $n+1$ meromorphic functions $f_0, f_1, \ldots, f_n$ on some region of $\mathbb{C}$. J.F. Ritt once remarked to me that there exists an irreducible ordinary differential polynomial $h \in \mathbb{C}\{y_0, \ldots, y_n\}$, dependent only on V and having order equal to the dimension of V, that enjoys the following property: A necessary and sufficient condition that there exist $c_0, \ldots, c_n \in \mathbb{C}$ not all zero such that $(c_0 : \cdots : c_n)$ is a point of V and $\sum_j c_j f_j = 0$ is that $(f_0, \ldots, f_n)$ be in the general solution of h.

Kolchin commented that "This provides an occasion to describe the beginning of a theory of algebraic differential equations in a projective space $P_n(\mathcal{E})$". And he devoted two papers on differential projective spaces: [10] and [11], published posthumously. In the former, Kolchin developed the foundation for a theory of differentially homogenous differential ideals and their differential zero sets in $P_n(\mathcal{E})$. Also, Kolchin proved Ritt's result mentioned above. In the following, we will use "the result on linear dependence over projective varieties" to refer to this result.

In this paper, we will establish the theory for projective differential Chow forms. We first consider the dimension and order for the intersection of an irreducible projective differential variety by a generic projective differential hyperplane. Precisely, the intersection of an irreducible projective differential variety of dimension $d > 0$ and order o with a generic projective differential hyperplane is shown to be an irreducible projective differential variety of dimension $d - 1$ and order o. Then we define Chow forms for irreducible projective differential varieties. As an application, we will show that the differential polynomial h in Ritt's remark mentioned above is the Chow form of a projective differential variety associated with V.

The rest of the paper is organized as follows. In Section 2, we will present basic notations and preliminary results in projective differential algebraic geometry. In Section 3, we will prove the generic intersection theorem of a projective differential variety by a generic projective differential hyperplane. In Section 4, the Chow form for an irreducible projective differential variety is defined and its basic properties are given. In Section 5, we will apply the differential Chow form theory to the result on linear dependence over projective varieties given by Kolchin. Finally, we present the conclusion and propose a conjecture on the Chow form for the projective differential variety.

2. Preliminaries

In this section, some basic notations and preliminary results in projective differential algebraic geometry will be given. For more details, please refer to [18,9,10,4].

2.1. Basic notions in differential algebra

Let $\mathcal{F}$ be a fixed ordinary differential field of characteristic zero, with a derivation operator δ. An element $c \in \mathcal{F}$ such that $\delta(c) = 0$ is called a constant of $\mathcal{F}$. The set $\mathcal{C}$ of constants of $\mathcal{F}$ is a field, called the field of constants of $\mathcal{F}$. In this paper, unless otherwise indicated, δ is kept fixed during any discussion and we use exponents (i) and exponents $[t]$ to indicate derivatives under δ and the set of derivatives up to the order t respectively.

Let S be a subset of a differential field $\mathcal{G}$ which contains $\mathcal{F}$. We will denote respectively by $\mathcal{F}[S]$, $\mathcal{F}(S)$, $\mathcal{F}\{S\}$, and $\mathcal{F}\langle S\rangle$ the smallest subring, the smallest subfield, the smallest differential subring, and the smallest differential subfield of $\mathcal{G}$ containing $\mathcal{F}$ and S. Let Θ denote the free commutative semigroup with unit (written multiplicatively) generated by δ. Then $\mathcal{F}\{S\} = \mathcal{F}[\Theta(S)]$ and $\mathcal{F}\langle S\rangle = \mathcal{F}(\Theta(S))$. A differential extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be finitely generated if $\mathcal{G}$ has a finite subset S such that $\mathcal{G} = \mathcal{F}\langle S\rangle$.

A subset Σ of a differential extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be *differentially dependent* over $\mathcal{F}$ if the set $(\theta\alpha)_{\theta\in\Theta,\,\alpha\in\Sigma}$ is algebraically dependent over $\mathcal{F}$, and is said to be *differentially independent* over $\mathcal{F}$, or to be a family of *differential indeterminates* over $\mathcal{F}$ in the contrary case. In the case Σ consists of only one element α, we say that α is differentially algebraic or differentially transcendental over $\mathcal{F}$ respectively. The maximal subset Ω of $\mathcal{G}$ which is differentially independent over $\mathcal{F}$ is said to be a differential transcendence basis of $\mathcal{G}$ over $\mathcal{F}$. We use $\mathrm{d.tr.deg}\,\mathcal{G}/\mathcal{F}$ (see [9, p. 105–109]) to denote the *differential transcendence degree* of $\mathcal{G}$ over $\mathcal{F}$, which is the cardinal number of Ω. Considering $\mathcal{F}$ and $\mathcal{G}$ as ordinary algebraic fields, we denote the algebraic transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ by $\mathrm{tr.deg}\,\mathcal{G}/\mathcal{F}$.

A differential extension field $\mathcal{E}$ of $\mathcal{F}$ is called a *universal differential extension field*, if for any finitely generated differential extension field $\mathcal{F}_1$ of $\mathcal{F}$ in $\mathcal{E}$ and any finitely generated differential extension field $\mathcal{F}_2$ of $\mathcal{F}_1$ not necessarily in $\mathcal{E}$, $\mathcal{F}_2$ can be embedded in $\mathcal{E}$ over $\mathcal{F}_1$, i.e. there exists a differential extension field $\mathcal{F}_3$ in $\mathcal{E}$ that is differentially isomorphic to $\mathcal{F}_2$ over $\mathcal{F}_1$. Such a differential universal extension field of $\mathcal{F}$ always exists [9, Theorem 2, p. 134]. By definition, for any natural number n, we can find in $\mathcal{E}$ a subset of cardinality n whose elements are differentially independent over $\mathcal{F}$.

Now suppose $\mathbb{Y}=\{y_1,y_2,\ldots,y_n\}$ is a set of differential indeterminates over $\mathcal{E}$. For any $y\in\mathbb{Y}$, denote $\delta^k y$ by $y^{(k)}$. The elements of $\mathcal{F}\{\mathbb{Y}\}=\mathcal{F}[y_j^{(k)}\colon\ j=1,\ldots,n;\ k\in\mathbb{N}]$ are called *differential polynomials* over $\mathcal{F}$ in $\mathbb{Y}$, and $\mathcal{F}\{\mathbb{Y}\}$ itself is called the *differential polynomial ring* over $\mathcal{F}$ in $\mathbb{Y}$. A differential polynomial ideal $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ is an ordinary algebraic ideal which is closed under derivation, i.e. $\delta(\mathcal{I})\subset\mathcal{I}$. And a prime (resp. radical) differential ideal is a differential ideal which is prime (resp. radical) as an ordinary algebraic polynomial ideal. For convenience, a prime differential ideal is assumed not to be the unit ideal in this paper.

Let f be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$. We define the order of f w.r.t. y_i to be the greatest number k such that $y_i^{(k)}$ appears effectively in f, which is denoted by $\mathrm{ord}(f,y_i)$. And if y_i does not appear in f, then we set $\mathrm{ord}(f,y_i)=-\infty$. The *order* of f is defined to be $\max_i \mathrm{ord}(f,y_i)$, that is, $\mathrm{ord}(f)=\max_i \mathrm{ord}(f,y_i)$.

A *ranking* $\mathscr{R}$ is a total order over $\Theta(\mathbb{Y})$, which is compatible with the derivations over the alphabet:

(1) $\delta\theta y_j>\theta y_j$ for $\delta\in\Theta$ and all derivatives $\theta y_j\in\Theta(\mathbb{Y})$.
(2) $\theta_1 y_i>\theta_2 y_j\Longrightarrow\delta\theta_1 y_i>\delta\theta_2 y_j$ for $\delta\in\Theta$ and $\theta_1 y_i,\theta_2 y_j\in\Theta(\mathbb{Y})$.

By convention, $1<\theta y_j$ for all $\theta y_j\in\Theta(\mathbb{Y})$.

Two important kinds of rankings are the following:

(1) *Elimination ranking*: $y_i>y_j\Longrightarrow\delta^k y_i>\delta^l y_j$ for any $k,l\geqslant 0$.
(2) *Orderly ranking*: $k>l\Longrightarrow\delta^k y_i>\delta^l y_j$ for any $i,j\in\{1,2,\ldots,n\}$.

Let p be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$ and $\mathscr{R}$ a ranking endowed on it. The greatest derivative w.r.t. $\mathscr{R}$ which appears effectively in p is called the *leader* of p, which will be denoted by u_p or $\mathrm{ld}(p)$. The two conditions mentioned above imply that the leader of θp is θu_p for $\theta\in\Theta$. Let the degree of p in u_p be d. As a univariate polynomial in u_p, p can be rewritten as

$$p=I_d u_p^d+I_{d-1}u_p^{d-1}+\cdots+I_0.$$

I_d is called the *initial* of p and is denoted by I_p. The partial derivative of p w.r.t. u_p is called the *separant* of p, which will be denoted by S_p. Clearly, S_p is the initial of any proper derivative of p. The *rank* of p is u_p^d, and is denoted by $\mathrm{rk}(p)$.

Let p and q be two differential polynomials and u_p^d the rank of p. q is said to be *partially reduced* w.r.t. p if no proper derivatives of u_p appear in q. q is said to be *reduced* w.r.t. p if q is partially reduced w.r.t. p and $\deg(q,u_p)<d$. Let $\mathcal{A}$ be a set of differential polynomials. $\mathcal{A}$ is said to be an *auto-reduced set* if each polynomial of $\mathcal{A}$ is reduced w.r.t. any other element of $\mathcal{A}$. Every auto-reduced set is finite.

Let $\mathcal{A} = A_1, A_2, \ldots, A_t$ be an auto-reduced set with S_i and I_i as the separant and initial of A_i, and f be any differential polynomial. Then there exists an algorithm, called Ritt's algorithm of reduction, which reduces f w.r.t. $\mathcal{A}$ to a polynomial r that is reduced w.r.t. $\mathcal{A}$, satisfying the relation

$$\prod_{i=1}^{t} \mathrm{S}_i^{d_i} \mathrm{I}_i^{e_i} \cdot f \equiv r, \ \mathrm{mod}\ [\mathcal{A}],$$

where d_i, e_i $(i = 1, 2, \ldots, t)$ are nonnegative integers. The differential polynomial r is called the *differential remainder* of f w.r.t. $\mathcal{A}$.

Let $\mathcal{A}$ be an auto-reduced set. Denote $\mathrm{H}_{\mathcal{A}}$ to be the set of all the initials and separants of $\mathcal{A}$ and $\mathrm{H}_{\mathcal{A}}^{\infty}$ to be the minimal multiplicative set containing $\mathrm{H}_{\mathcal{A}}$. The *saturation ideal* of $\mathcal{A}$ is defined to be

$$\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} = \{p \colon \exists h \in \mathrm{H}_{\mathcal{A}}^{\infty}, \text{ s.t. } hp \in [A]\}.$$

An auto-reduced set $\mathcal{C}$ contained in a differential polynomial set $\mathcal{S}$ is said to be a *characteristic set* of $\mathcal{S}$, if $\mathcal{S}$ does not contain any nonzero element reduced w.r.t. $\mathcal{C}$. A characteristic set $\mathcal{C}$ of a differential ideal $\mathcal{J}$ reduces to zero all elements of $\mathcal{J}$. If the differential ideal is prime, $\mathcal{C}$ reduces to zero only the elements of $\mathcal{J}$ and $\mathcal{J} = \mathrm{sat}(\mathcal{C})$ [9, Lemma 2, p. 167] is valid.

In this paper, we need the following theorem on a property of differential specialization. A set of elements $E \subset \mathcal{E}$ is said to be *differentially free* from $\mathcal{F}\langle\mathbb{U}\rangle$, if $\mathbb{U}$ is a set of differential indeterminates over $\mathcal{F}\langle E\rangle$.

Theorem 2.1. *(See [4, Theorem 2.16].) Let $\{u_1, \ldots, u_r\} \subset \mathcal{E}$ be a set of differential indeterminates over $\mathcal{F}$, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$ $(i = 1, \ldots, m)$ differential polynomials in $\mathbb{U} = (u_1, \ldots, u_r)$ and $\mathbb{Y} = (y_1, \ldots, y_n)$. Let $\bar{\mathbb{Y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$, where $\bar{y}_i \in \mathcal{E}$ are differentially free from $\mathcal{F}\langle\mathbb{U}\rangle$. If $P_i(\mathbb{U}, \bar{\mathbb{Y}})$ $(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\bar{\mathbb{U}}$ in $\mathcal{F}$, $P_i(\bar{\mathbb{U}}, \bar{\mathbb{Y}})$ $(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}$.*

2.2. Differentially homogenous differential ideals

In this paper, we fix $\mathcal{F}$ to be an ordinary differential field with the derivation operator δ, the field of constants $\mathcal{C}$, and $\mathcal{E}$ to be a universal differential field of $\mathcal{F}$ with the field of constants $\mathcal{K}$.

Let $\mathbb{Y} = (y_0, \ldots, y_n)$ and consider the differential polynomial ring $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}\{y_0, \ldots, y_n\}$ over $\mathcal{F}$. For any element λ of any differential overring of $\mathcal{F}\{\mathbb{Y}\}$, set $\lambda\mathbb{Y} = (\lambda y_0, \lambda y_1, \ldots, \lambda y_n)$. Following Kolchin [10], we have the following definition.

Definition 2.2. A differential polynomial $f \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is called differentially homogenous of degree m if for a new differential indeterminate λ over $\mathcal{F}\{\mathbb{Y}\}$, we have $f(\lambda y_0, \lambda y_1, \ldots, \lambda y_n) = \lambda^m f(y_0, y_1, \ldots, y_n)$.

The following result will be used in this paper.

Lemma 2.3. *Let f be a nonzero differentially homogenous polynomial of degree m and $\mathscr{R}$ any ranking of $\mathbb{Y}$. Then both of its initial and separant w.r.t. $\mathscr{R}$ are differentially homogenous.*

Proof. We first consider the separant of f. Let $y_i^{(o)}$ be the leader of f w.r.t. $\mathscr{R}$. Then we have $\lambda \frac{\partial f}{\partial y_i^{(o)}}(\lambda y_0, \ldots, \lambda y_n) = \frac{(\lambda y_i)^{(o)}}{y_i^{(o)}} \frac{\partial f}{\partial y_i^{(o)}}(\lambda y_0, \ldots, \lambda y_n) = \frac{\partial}{\partial y_i^{(o)}} f(\lambda y_0, \ldots, \lambda y_n) = \frac{\partial}{\partial y_i^{(o)}} \lambda^m f(y_0, \ldots, y_n) = \lambda^m \frac{\partial f}{\partial y_i^{(o)}}(y_0, \ldots, y_n)$. Thus, $\frac{\partial f}{\partial y_i^{(o)}}(\lambda\mathbb{Y}) = \lambda^{m-1} \frac{\partial f}{\partial y_i^{(o)}}(\mathbb{Y})$. It follows that the separant of f is differentially homogenous of degree $m - 1$.

Now we consider the initial of f. Rewriting f as a univariate polynomial in $y_i^{(0)}$, we have $f = I_f \cdot (y_i^{(0)})^l + I_1 \cdot (y_i^{(0)})^{l-1} + \cdots + I_{l-1} \cdot (y_i^{(0)}) + I_l$, where I_f is the *initial* of f. In the above, we have proved that $\partial f/\partial y_i^{(0)} = l I_f \cdot (y_i^{(0)})^{l-1} + (l-1) I_1 \cdot (y_i^{(0)})^{l-2} + \cdots + I_{l-1}$ is differentially homogenous. If $l = 1$, it follows that $I_f = \partial f/\partial y_i^{(0)}$ is differentially homogenous. While $l > 1$, the leader of $\partial f/\partial y_i^{(0)}$ is $y_i^{(0)}$ too. Continuing in this way, $\frac{\partial^l f}{\partial (y_i^{(0)})^l} = l! I_f$ is differentially homogenous. Thus, I_f is differentially homogenous. □

More generally, let $(y_{ij})_{1\leqslant i\leqslant p,\,0\leqslant j\leqslant n_i}$ be a family of differential indeterminates over $\mathcal{E}$, set $\mathbb{Y}_i = (y_{i0}, \ldots, y_{in_i})$ $(1 \leqslant i \leqslant p)$, and consider the differential polynomial ring $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\} = \mathcal{F}\{(y_{ij})_{1\leqslant i\leqslant p, 0\leqslant j\leqslant n_i}\}$ over $\mathcal{F}$. Let $f \in \mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ and $(d_1, \ldots, d_p) \in \mathbb{N}^p$. If for any index i, f is differentially homogenous in $\mathbb{Y}_i$ of degree d_i, f is said to be *differentially p-homogenous* in $(\mathbb{Y}_1, \ldots, \mathbb{Y}_p)$ of degree $(d_1, \ldots, d_p)$.

Let $\mathcal{I}$ be a differential ideal of $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$. Denote $\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p)^\infty = \{f \in \mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\} \mid (y_{1j_1} \cdots y_{pj_p})^e f \in \mathcal{I},\ 0 \leqslant j_1 \leqslant n_1, \ldots, 0 \leqslant j_p \leqslant n_p \text{ and for some } e\}$ and $\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p) = \{f \in \mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\} \mid (y_{1j_1} \cdots y_{pj_p}) f \in \mathcal{I},\ 0 \leqslant j_1 \leqslant n_1, \ldots, 0 \leqslant j_p \leqslant n_p\}$. Clearly, $\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p)^\infty$ is a differential ideal, and is a radical differential ideal coinciding with $\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p)$ when $\mathcal{I}$ is a radical differential ideal.

Definition 2.4. Let $\mathcal{I}$ be a differential ideal of $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$. $\mathcal{I}$ is called a *differentially p-homogenous differential ideal* if $\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p) = \mathcal{I}$ and for every $P \in \mathcal{I}$ and each index i and a differential indeterminate λ over $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$, $P(\mathbb{Y}_1, \ldots, \lambda\mathbb{Y}_i, \ldots, \mathbb{Y}_p) \in \mathcal{F}\{\lambda\}\mathcal{I}$ in the differential ring $\mathcal{F}\{\lambda, \mathbb{Y}_1 \cdots \mathbb{Y}_p\}$. If $p = 1$ and $\mathbb{Y}_1 = \mathbb{Y}$, $\mathcal{I}$ is called a *differentially homogenous differential ideal* of $\mathcal{F}\{\mathbb{Y}\}$.

For $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ and any ranking $\mathscr{R}$ of $((y_{ij})_{1\leqslant i\leqslant p,\,0\leqslant j\leqslant n_i})$. $\mathcal{I}$ has a *characteristic set* w.r.t. $\mathscr{R}$ which is not unique. To impose the uniqueness condition on characteristic set, Kolchin gave the definition of canonical characteristic set which is unique for a differential ideal and a fixed ranking $\mathscr{R}$ [10]. Kolchin gave the following theorem to test whether $\mathcal{I}$ is differentially p-homogenous [10].

Theorem 2.5. *Let $\mathcal{I}$ be a prime differential ideal of $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ and let $\mathcal{A}$ denote the canonical characteristic set of $\mathcal{I}$ w.r.t. some ranking of $(y_{ij})_{1\leqslant i\leqslant p,\,0\leqslant j\leqslant n_i}$. Then the followings are equivalent.*

- *$\mathcal{I}$ is differentially p-homogenous.*
- *$\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p) = \mathcal{I}$ and for every zero $(\eta_1, \ldots, \eta_p)$ of $\mathcal{I}$ in $\mathcal{E}^{n_1+1} \times \cdots \times \mathcal{E}^{n_p+1}$ and each $s \in \mathcal{E}\backslash\{0\}$ and each index i, $(\eta_1, \ldots, s\eta_i, \ldots, \eta_p)$ is a zero of $\mathcal{I}$.*
- *$\mathcal{I} : (\mathbb{Y}_1 \cdots \mathbb{Y}_p) = \mathcal{I}$ and each element of $\mathcal{A}$ is differentially p-homogenous in $(\mathbb{Y}_1, \ldots, \mathbb{Y}_p)$.*

Let $n \in \mathbb{N}$ and consider the projective space $\mathbf{P}(n)$ over $\mathcal{E}$. Any element $(a_0, a_1, \ldots, a_n)$ of $\mathcal{E}^{n+1}$ different from the origin is a representative of a unique point α of $\mathbf{P}(n)$, denoted by $(a_0 : a_1 : \cdots : a_n)$. Given $n_1, \ldots, n_p \in \mathbb{N}$, we consider the p-projective space $\mathbf{P}(n_1, \ldots, n_p) = \mathbf{P}(n_1) \times \cdots \times \mathbf{P}(n_p)$. For any point $\alpha = (\alpha_1, \ldots, \alpha_p)$ of $\mathbf{P}(n_1, \ldots, n_p)$, if $\mathbf{a}_i = (a_{i0}, a_{i1}, \ldots, a_{in_i})$ is a representative of α_i $(1 \leqslant i \leqslant p)$, then the element $(a_{ij})_{1\leqslant i\leqslant p,\,0\leqslant j\leqslant n_i} = (\mathbf{a}_1, \ldots, \mathbf{a}_p)$ of $\mathcal{E}^{n_1+1} \times \cdots \times \mathcal{E}^{n_p+1}$ is called a representative of α.

Consider a differential polynomial $P \in \mathcal{E}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ and a point $\alpha \in \mathbf{P}(n_1, \ldots, n_p)$. Say that P vanishes at α, and that α is a zero of P, if P vanishes at every representative of α. For a subset $\mathscr{M}$ of $\mathbf{P}(n_1, \ldots, n_p)$, let $\mathbb{I}_{\mathcal{F}}(\mathscr{M})$ denote the set of differential polynomials in $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ that vanish on $\mathscr{M}$ and write $\mathbb{I}(\mathscr{M}) = \mathbb{I}_{\mathcal{F}}(\mathscr{M})$. Let $\mathbb{V}(S)$ denote the set of points of $\mathbf{P}(n_1, \ldots, n_p)$ that are zeros of the subset S of $\mathcal{E}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$. And a subset V of $\mathbf{P}(n_1, \ldots, n_p)$ is called a *projective differential $\mathcal{F}$-variety* if there exists an $S \subset \mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ such that $V = \mathbb{V}(S)$.

As in the affine differential case, we have a one-to-one correspondence between projective differential varieties and radical differentially homogenous differential ideals.

Theorem 2.6. *(See [10].) The mapping from the set of projective differential $\mathcal{F}$-varieties of $\mathbf{P}(n_1,\ldots,n_p)$ into the set of differentially p-homogenous radical differential ideals of $\mathcal{F}\{\mathbb{Y}_1,\ldots,\mathbb{Y}_p\}$, given by the formula $V \longrightarrow \mathbb{I}_{\mathcal{F}}(V)$, and the mapping in the opposite direction, given by the formula $\mathcal{I} \longrightarrow \mathbb{V}(\mathcal{I})$, are bijective and inverse to each other. And a projective differential $\mathcal{F}$-variety V is $\mathcal{F}$-irreducible if and only if $\mathbb{I}(V)$ is prime.*

3. Generic intersection for projective differential varieties

In this section, we will consider the order and dimension of the intersection of a projective differential variety by a generic projective differential hyperplane. Before doing this, we first give a rigorous definition of dimension and order for differentially homogenous differential ideals.

3.1. Order and dimension in projective differential algebraic geometry

In the whole paper, when talking about prime differential ideals, we always imply that they are distinct from the unit differential ideal.

For a differentially homogenous differential ideal $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$, we now define the concepts of differentially independent set modulo $\mathcal{I}$, parametric set, and differential dimension polynomial similar to the affine case. More precisely, a variable set $\mathbb{U} \subset \mathbb{Y}$ is said to be an independent set modulo $\mathcal{I}$ if $\mathcal{I} \cap \mathcal{F}\{\mathbb{U}\} = \{0\}$. And a *parametric set* of $\mathcal{I}$ is a maximal differentially independent set modulo $\mathcal{I}$.

Lemma 3.1. *Let $\mathcal{I}$ be a differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$. Then its parametric set is not empty.*

Proof. Suppose the contrary. That is, for each y_i, $\mathcal{I} \cap \mathcal{F}\{y_i\} \neq [0]$ $(i = 0,\ldots,n)$. Let $\mathcal{A}_i$ be the canonical characteristic set of $\mathcal{I}$ w.r.t. the elimination ranking $y_i \prec y_0 \prec \cdots \prec y_n$. Then by Theorem 2.5, each element of $\mathcal{A}_i$ is differentially homogenous. Since $\mathcal{I} \cap \mathcal{F}\{y_i\} \neq [0]$, there exists $A_{i0} \in \mathcal{A}_i$ such that $A_{i0} \in \mathcal{F}\{y_i\}$. If $\mathrm{ord}(A_{i0}) > 0$, it is easy to see that A_{i0} cannot be differentially homogenous. So $A_{i0} \in \mathcal{F}[y_i]$. Using the fact that $\mathcal{I}$ is prime, $A_{i0} = y_i$ follows. Thus, for each i, $y_i \in \mathcal{I}$. It follows that $1 \in \mathcal{I} : \mathbb{Y}$. By Theorem 2.5, we have $\mathcal{I} = \mathcal{I} : \mathbb{Y}$, so $\mathcal{I} = \mathcal{F}\{\mathbb{Y}\}$, which is a contradiction. □

In [8], Kolchin introduced differential dimension polynomials for prime differential polynomial ideals. Also see [12] for more detailed discussion. Following Kolchin, we give the definition of differential dimension polynomials for differentially homogenous prime differential ideals.

Definition 3.2. Let $\mathcal{I}$ be a differentially homogenous prime differential ideal of $\mathcal{F}\{\mathbb{Y}\}$. Then there exists a unique numerical polynomial $\omega_{\mathcal{I}}(t)$ such that $\omega_{\mathcal{I}}(t) = \dim(\mathcal{I} \cap \mathcal{F}[(y_i^{(k)})_{0 \leqslant i \leqslant n,\, 0 \leqslant k \leqslant t}])$ for all sufficiently big $t \in \mathbb{N}$. $\omega_{\mathcal{I}}(t)$ is called the *differential dimension polynomial* of $\mathcal{I}$.

We can use the differential dimension polynomial to define the differential dimension and order for a prime differential ideal.

Lemma 3.3. *Let $\mathcal{I}$ be a differentially homogenous prime differential ideal of $\mathcal{F}\{\mathbb{Y}\}$. Then $\omega_{\mathcal{I}}(t)$ can be written in the form $\omega_{\mathcal{I}}(t) = (d+1)(t+1) + o$ for $d \geqslant 0$. We define d to be the differential dimension of $\mathcal{I}$ and o is called the order of $\mathcal{I}$.*

Proof. By Lemma 3.1, the cardinality of a parametric set of $\mathcal{I}$ is not less than 1. Then by [8, Theorem 2], it follows. □

In the affine case, we can read the information of a differential ideal, such as dimension and order, from its generic point. However, in the projective case, it is a bit more complicated to do this. Firstly, we give the definition of generic points for a differentially homogenous prime differential ideal following Kolchin's notation [10].

Consider a point $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbf{P}(n_1, \ldots, n_p)$. Choose a representative $\mathbf{a}_i = (a_{i0}, \ldots, a_{in_i}) \in \mathcal{E}^{n_i+1}$ $(1 \leqslant i \leqslant p)$, and for each i choose one index j_i such that $a_{ij_i} \neq 0$. Denote $\mathbf{b}_i = (a_{ij_i}^{-1} a_{i0}, \ldots, 1, \ldots, a_{ij_i}^{-1} a_{in_i}) \in \mathcal{E}^{n_i+1}$ $(1 \leqslant i \leqslant p)$ and $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_p)$. For any subfield K of $\mathcal{E}$, the field extension $K(\alpha) \triangleq K(\mathbf{b}) = K((a_{ij_i}^{-1} a_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i})$ is independent of the choice of the representative $(a_0, \ldots, a_n)$ and indices j_i. Denote the set of all points α such that $K(\alpha) = K$ by $\mathbf{P}_K(n_1, \ldots, n_p)$.

Consider again the point $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbf{P}(n_1, \ldots, n_p)$. Denote the differential field $\mathcal{F}\langle \mathcal{F}(\alpha) \rangle$ by $\mathcal{F}\langle \alpha \rangle$. Clearly, the differential transcendence polynomial of $(a_{ij_i}^{-1} a_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i}$ is independent of the choices made above, and may therefore be called the *differential transcendence polynomial* of α over $\mathcal{F}$, denoted by $\omega_{\alpha/\mathcal{F}}$. It can be written in the form $\omega_{\alpha/\mathcal{F}}(t) = a_1(t+1) + a_0$ where $a_i \in \mathbb{Z}$. Then a_1 is the differential transcendence degree of $\mathcal{F}\langle \alpha \rangle$ over $\mathcal{F}$.

Consider a second point $\alpha' = (\alpha'_1, \ldots, \alpha'_p) \in \mathbf{P}(n_1, \ldots, n_p)$ and a representative $(\mathbf{a}'_1, \ldots, \mathbf{a}'_p)$ of α'; for each i write $\mathbf{a}'_i = (a'_{i0}, \ldots, a'_{in_i}) \in \mathcal{E}^{n_i+1}$ and fix j'_i such that $a'_{ij'_i} \neq 0$. If $\mathbb{I}_{\mathcal{F}}(\alpha) \subset \mathbb{I}_{\mathcal{F}}(\alpha')$, then $a_{ij'_i} \neq 0$ $(1 \leqslant i \leqslant p)$, that is, the indices j_i can be chosen equal to the indices j'_i, and evidently $(a'^{-1}_{ij'_i} a'_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i}$ is a differential specialization of $(a_{ij'_i}^{-1} a_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i}$ over $\mathcal{F}$. Conversely, if there exist indices $j_1, \ldots, j_p$ such that $a_{ij_i} \neq 0$, $a'_{ij_i} \neq 0$ $(1 \leqslant i \leqslant p)$ and $(a'^{-1}_{ij_i} a'_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i}$ is a differential specialization of $(a_{ij_i}^{-1} a_{ij})_{1 \leqslant i \leqslant p, 0 \leqslant j \leqslant n_i}$ over $\mathcal{F}$, then $\mathbb{I}_{\mathcal{F}}(\alpha) \subset \mathbb{I}_{\mathcal{F}}(\alpha')$. Under these conditions call α' a *differential specialization of* α *over* $\mathcal{F}$.

Let $\mathcal{I}$ be a differentially p-homogenous prime differential ideal of $\mathcal{F}\{\mathbb{Y}_1, \ldots, \mathbb{Y}_p\}$ and V the corresponding projective differential $\mathcal{F}$-variety of $\mathbf{P}(n_1, \ldots, n_p)$. Thus V is $\mathcal{F}$-irreducible, $V = \mathbb{V}(\mathcal{I})$, and $\mathcal{I} = \mathbb{I}(\mathbb{V})$. And for a point $\alpha \in \mathbf{P}(n_1, \ldots, n_p)$, $\mathbb{I}_{\mathcal{F}}(\alpha) = \mathcal{I}$ if and only if the set of all differential specializations of α over $\mathcal{F}$ is V.

Call such a point α a *generic point of* $\mathcal{I}$ *in* $\mathbf{P}(n_1, \ldots, n_p)$ or a *generic point of* V *over* $\mathcal{F}$. And we call $\omega_{\alpha/\mathcal{F}}(t)$ the *differential dimension polynomial* of V and denoted by ω_V and d.tr.deg $\mathcal{F}\langle \alpha \rangle / \mathcal{F}$ is called the *differential dimension* of V. Then ω_V and $\omega_{\mathcal{I}}$ have the following relation.

Theorem 3.4. *Let* $\mathcal{I}$ *be a differentially homogenous prime differential ideal of* $\mathcal{F}\{\mathbb{Y}\}$ *and* $V = \mathbb{V}(\mathcal{I})$. *Then* $\omega_{\mathcal{I}}(t) = (t+1) + \omega_V(t)$.

Proof. Without loss of generality, assume that $V \not\subseteq \mathbb{V}(y_0)$. Let $(1, \xi_1, \ldots, \xi_n)$ be a generic point of V and $u \in \mathcal{E}$ a differential indeterminate over $\mathcal{F}\langle \xi_1, \ldots, \xi_n \rangle$. Firstly, we claim that $\omega_{\mathcal{I}}(t) = \text{tr.deg}\, \mathcal{F}(u^{[t]}, (u\xi_1)^{[t]}, \ldots, (u\xi_n)^{[t]})/\mathcal{F}$ for all sufficiently big $t \in \mathbb{N}$. Denote $\mathcal{I}^a$ to be the affine differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ consisting of all elements of $\mathcal{I}$. By the definition of $\omega_{\mathcal{I}}(t)$, it only needs to show that $(u, u\xi_1, \ldots, u\xi_n)$ is a generic point of $\mathcal{I}^a$. Firstly, each polynomial in $\mathcal{I}^a$ vanishes at $(u, u\xi_1, \ldots, u\xi_n)$. Suppose $f^a \in \mathcal{F}\{\mathbb{Y}\}$ with $f^a(u, u\xi_1, \ldots, u\xi_n) = 0$. Since u is a differential indeterminate over $\mathcal{F}\langle \xi_1, \ldots, \xi_n \rangle$, we can regard $f^a(u, u\xi_1, \ldots, u\xi_n)$ as a differential polynomial in u, which is identically zero. Thus, for each $\lambda \in \mathcal{E}$, $f^a(\lambda, \lambda\xi_1, \ldots, \lambda\xi_n) = 0$. That is, f^a vanishes at every representative of $(1, \xi_1, \ldots, \xi_n)$ and $f^a \in \mathcal{I}$ follows. So $f^a \in \mathcal{I}^a$, and we have shown that $(u, u\xi_1, \ldots, u\xi_n)$ is a generic point of $\mathcal{I}^a$.

Thus, we have

$$\begin{aligned}
\omega_{\mathcal{I}}(t) &= \text{tr.deg}\, \mathcal{F}\big(u^{[t]}, (u\xi_1)^{[t]}, \ldots, (u\xi_n)^{[t]}\big)/\mathcal{F} \\
&= \text{tr.deg}\, \mathcal{F}\big(u^{[t]}, (\xi_1)^{[t]}, \ldots, (\xi_n)^{[t]}\big)/\mathcal{F} \\
&= \text{tr.deg}\, \mathcal{F}\big(u^{[t]}\big)/\mathcal{F} + \text{tr.deg}\, \mathcal{F}\big(u^{[t]}\big)\big((\xi_1)^{[t]}, \ldots, (\xi_n)^{[t]}\big)/\mathcal{F}\big(u^{[t]}\big) \\
&= (t+1) + \text{tr.deg}\, \mathcal{F}\big((\xi_1)^{[t]}, \ldots, (\xi_n)^{[t]}\big)/\mathcal{F} \\
&= (t+1) + \omega_V(t). \qquad \square
\end{aligned}$$

By the above theorem, we know that the differentially homogenous prime differential ideal $\mathcal{I}$ has the same dimension and order as its corresponding projective differential variety V.

Remark 3.5. The differential dimension polynomial of an irreducible projective differential variety is a birational invariant but not a differential birational invariant. By this we mean that if V_1 and V_2 are two irreducible projective differential varieties with generic points α and β respectively, then the condition $\mathcal{F}(\alpha)=\mathcal{F}(\beta)$ implies that $\omega_{V_1}(t)=\omega_{V_2}(t)$ but the weaker condition $\mathcal{F}\langle\alpha\rangle=\mathcal{F}\langle\beta\rangle$ does not. Nevertheless, ω_V carries with it two fundamental differential birational invariants: the degree and the leading coefficient. And if the degree is 1, the leading coefficient is just equal to its differential dimension.

By Lemma 3.1, for every nontrivial differentially homogenous prime differential ideal $\mathcal{I}$, there exists at least one index i such that $\mathcal{I}\cap\mathcal{F}\{y_i\}=[0]$. Since we can permute variables when necessary, it only needs to consider the case $\mathcal{I}\cap\mathcal{F}\{y_0\}=[0]$. Denote the set of all nontrivial differentially homogenous prime differential ideals $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ with $\mathcal{I}\cap\mathcal{F}\{y_0\}=[0]$ by $\mathcal{S}$. Denote the set of all prime differential ideals in $\mathcal{F}\{y_1,\dots,y_n\}$ by $\mathcal{T}$. Now we give a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{T}$.

Define the maps

$$\phi:\mathcal{S}\subset\mathcal{F}\{\mathbb{Y}\}\longrightarrow\mathcal{T}\subset\mathcal{F}\{y_1,\dots,y_n\}\quad\text{and}$$
$$\psi:\mathcal{T}\subset\mathcal{F}\{y_1,\dots,y_n\}\longrightarrow\mathcal{S}\subset\mathcal{F}\{\mathbb{Y}\}\tag{3.1}$$

as follows: For each $\mathcal{I}\subset\mathcal{S}$, suppose $(\eta_0,\dots,\eta_n)$ is a generic point of $\mathcal{I}$. Clearly, $\eta_0\neq 0$. Let $\phi(\mathcal{I})$ be the prime differential ideal in $\mathcal{F}\{y_1,\dots,y_n\}$ with $(\eta_1/\eta_0,\dots,\eta_n/\eta_0)$ as its generic point. Conversely, for a prime differential ideal $\mathcal{J}^a$ in $\mathcal{F}\{y_1,\dots,y_n\}$ with a generic point $(\xi_1,\dots,\xi_n)$, let $\psi(\mathcal{J}^a)$ be a differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with $(v,v\xi_1,\dots,v\xi_n)$ as a generic point, where $v\in\mathcal{E}$ is a differential indeterminate over $\mathcal{F}\langle\xi_1,\dots,\xi_n\rangle$. Clearly, $\phi\circ\psi=\mathrm{id}_{\mathcal{T}}$ and $\psi\circ\phi=\mathrm{id}_{\mathcal{S}}$. By Theorem 3.4, $\omega_{\mathcal{I}}(t)=(t+1)+\omega_{\phi(\mathcal{I})}(t)$. That is, $\mathcal{I}$ has the same dimension and order as $\phi(\mathcal{I})$ which is called the canonical affine representative of $\mathcal{I}$.

Remark 3.6. The set of affine zeros of $\mathcal{I}$ as a differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ is called the cone over V, whose differential dimension and that of V differ by 1 by Theorem 3.4.

As above, we give the definition of ϕ and ψ in the language of generic points. Now we will give an interpretation from the perspective of characteristic sets.

Lemma 3.7. *Let $\mathcal{A}:=A_1,\dots,A_{n-d}$ be a canonical characteristic set of $\mathcal{I}\in\mathcal{S}$ w.r.t. any elimination ranking $\mathscr{R}$ with $y_0\prec y_i$ for each i. Denote $B_i=A_i(1,y_1,\dots,y_n)$. Then $\mathcal{B}:=B_1,\dots,B_{n-d}$ is a characteristic set of $\phi(\mathcal{I})$ w.r.t. the elimination ranking induced by $\mathscr{R}$.*

Proof. Since $\mathcal{I}$ is a differentially homogenous prime differential ideal, each A_i is differentially homogenous by Theorem 2.5. Denote the separant and initial of A_i by S_i and I_i. By Lemma 2.3, S_i and I_i are differentially homogenous. It follows that $\mathrm{S}_i(1,y_1,\dots,y_n)$ and $\mathrm{I}_i(1,y_1,\dots,y_n)$ are not zero. Consequently, $\mathrm{ld}(B_i)=\mathrm{ld}(A_i)$, and the separant and initial of B_i are $\mathrm{S}_i(1,y_1,\dots,y_n)$ and $\mathrm{I}_i(1,y_1,\dots,y_n)$ respectively. So $\mathcal{B}$ is an auto-reduced set w.r.t. the elimination ranking induced by $\mathscr{R}$.

Let $(\eta_0,\eta_1,\dots,\eta_n)$ be a generic point of $\mathcal{I}$. Then $(\eta_1/\eta_0,\dots,\eta_n/\eta_0)$ is a generic point of $\phi(\mathcal{I})$. Clearly, $B_i(\eta_1/\eta_0,\dots,\eta_n/\eta_0)=A_i(1,\eta_1/\eta_0,\dots,\eta_n/\eta_0)=0$. That is, $B_i\in\phi(\mathcal{I})$, so $\mathcal{B}$ is an auto-reduced set in $\phi(\mathcal{I})$. Let f be any polynomial in $\phi(\mathcal{I})$ and r the remainder of f w.r.t. $\mathcal{B}$. Then $r\in\phi(\mathcal{I})$ and $r(\eta_1/\eta_0,\dots,\eta_n/\eta_0)=0$. Let $R=y_0^l r(y_1/y_0,\dots,y_n/y_0)\in\mathcal{F}\{\mathbb{Y}\}$. Then $R(\eta_0,\eta_1,\dots,\eta_n)=0$ and R is reduced w.r.t. $\mathcal{A}$. Thus, $R\equiv 0$ and $r\equiv 0$ follows. As a consequence, $\mathcal{B}$ reduces every differential polynomial in $\phi(\mathcal{I})$ to zero. So, $\mathcal{B}$ is a characteristic set of $\phi(\mathcal{I})$ w.r.t. the elimination ranking induced by $\mathscr{R}$. And the lemma is proved. □

Remark 3.8. Since the set of irreducible projective differential varieties and the set of differentially homogenous prime differential ideals have a one-to-one correspondence, ϕ also gives a one-to-one map between the set of irreducible projective differential varieties not contained in $y_0 = 0$ and the set of irreducible affine varieties with the inverse map ψ.

3.2. A generic intersection theorem

In affine differential algebraic geometry, we have proved the following intersection theorem.

Theorem 3.9. *(See [4, Theorem 3.14].) Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$ with dimension $d > 0$ and order h. Let $\{u_0, u_1, \ldots, u_n\} \subset \mathcal{E}$ be a set of differential indeterminates over $\mathcal{F}$. Then $\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1 y_1 + \cdots + u_n y_n]$ is a prime differential ideal in $\mathcal{F}\langle u_0, u_1, \ldots, u_n\rangle\{y_1, \ldots, y_n\}$ with dimension $d - 1$ and order h.*

A *generic projective differential hyperplane* is the differential zero set of $u_0 y_0 + u_1 y_1 + \cdots + u_n y_n = 0$ in $\mathbf{P}(n)$ where $u_i \in \mathcal{E}$ are differential indeterminates over $\mathcal{F}$. Now we try to give the projective version of the above theorem. Before doing this, we give the following lemma.

Lemma 3.10. *Let $\mathcal{I}$ be a differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension $d > 0$ and $u_0 y_0 + u_1 y_1 + \cdots + u_n y_n = 0$ be a generic projective differential hyperplane. Denote $\mathbf{u}_0 = (u_0, \ldots, u_n)$. Then the differential ideal $\mathcal{I}_0 = [\mathcal{I}, u_0 y_0 + u_1 y_1 + \cdots + u_n y_n] : (\mathbb{Y})^\infty \subset \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ is a differentially 2-homogenous prime differential ideal and $\mathcal{I}_0 \cap \mathcal{F}\{\mathbf{u}_0\} = [0]$.*

Proof. Let $(\xi_0, \ldots, \xi_n)$ be a generic point of $\mathcal{I}$ that is free from $\mathcal{F}\langle u_0, \ldots, u_n\rangle$. Without loss of generality, suppose $\xi_0 \neq 0$. Denote $\mathcal{J} = [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n] : (\mathbb{Y}\mathbf{u}_0)^\infty \subset \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$. We now prove that $\mathcal{I}_0$ is a differentially 2-homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ by showing that $\mathcal{J}$ is a differentially 2-homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}, \mathbf{u}_0\}$ and $\mathcal{I}_0 = \mathcal{J}$.

Firstly, we show that for any point $\mathbf{a} \in \mathbf{P}(n) \times \mathbf{P}(n)$, if $\mathcal{J}$ vanishes at $\mathbf{a}$, then $\mathcal{J}$ vanishes at every representative of $\mathbf{a}$. Now, suppose $\mathcal{J}$ vanishes at $\mathbf{a}$. For any differential polynomial $H \in \mathcal{J}$, there exists some $e \in \mathbb{N}$ such that $(y_j u_k)^e H \in [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n]$ for any $0 \leqslant j, k \leqslant n$. Since $\mathcal{I}$ and $u_0 y_0 + \cdots + u_n y_n$ vanish at every representative of $\mathbf{a}$, H vanishes at it. It follows that $\mathcal{J}$ vanishes at every representative of $\mathbf{a}$. In this way, we say $\mathbf{a}$ is a zero of $\mathcal{J}$. Since $\mathcal{I}_0 \subset \mathcal{J}$, $\mathcal{I}_0$ has the same property.

Now let $\zeta = (\xi_0, \ldots, \xi_n; -(u_1\xi_1 + \cdots + u_n\xi_n)/\xi_0, u_1, \ldots, u_n)$. We now show that ζ is a generic point of $\mathcal{J}$. For any $f \in \mathcal{J}$, there exists $e \in \mathbb{N}$ such that $(y_j u_k)^e H \in [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n]$. Take $j = 0$ and $k = 1$. Since $\xi_0 \neq 0$, it follows that H vanishes at ζ. Conversely, suppose G is any differential polynomial in $\mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ such that G vanishes at ζ. Let G_1 be the differential remainder of G w.r.t. $u_0 y_0 + \cdots + u_n y_n$ with u_0 as its leader, then we have $y_0^{a_0} G \equiv G_1$, mod $[u_0 y_0 + \cdots + u_n y_n]$ for some $a_0 \in \mathbb{N}$. Then G_1 is free from u_0 and its derivatives with $G_1(\zeta) = 0$. Regard G_1 as a polynomial in $\mathcal{F}\langle u_1, \ldots, u_n\rangle\{\mathbb{Y}\}$. Then it vanishes at $(\xi_0, \ldots, \xi_n)$. Since $[\mathcal{I}] \subset \mathcal{F}\langle u_1, \ldots, u_n\rangle\{\mathbb{Y}\}$ is a prime differential ideal with a generic point $(\xi_0, \ldots, \xi_n)$, $G_1 \in [\mathcal{I}] \cap \mathcal{F}\{\mathbb{Y}; u_1, \ldots, u_n\}$. Thus, $y_0^{a_0} G \in [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n]$. And for any index j_0 such that $\xi_{j_0} \neq 0$, similarly in this way, we can show that there exists $a_{j_0} \in \mathbb{N}$ such that $y_{j_0}^{a_{j_0}} G \in [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n]$. And if $\xi_{j_0} = 0$, then $y_{j_0} \in \mathcal{I}$, and $y_{j_0} G \in [\mathcal{I}, u_0 y_0 + \cdots + u_n y_n]$. Thus, it follows that $G \in \mathcal{J}$ and ζ is a generic point of $\mathcal{J}$. Similarly, we can show that ζ is also a generic point of $\mathcal{I}_0$. By Theorem 2.5, $\mathcal{I}_0 = \mathcal{J}$ is a differentially 2-homogenous prime differential ideal.

Suppose $\dim(\mathcal{I}) > 0$. Then there exists an $i \in \{1, \ldots, n\}$ such that $\mathcal{I} \cap \mathcal{F}\{y_0, y_i\} = [0]$. It follows that ξ_i/ξ_0 is differentially independent over $\mathcal{F}$. By Theorem 2.1, $u_0, \ldots, u_n$ are differentially independent modulo $\mathcal{I}_0$. □

Now, we give the following generic intersection theorem.

Theorem 3.11. *Let $\mathcal{I}$ be a differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension $d > 0$ and order h. Let $\mathbf{u}_0 = \{u_0, u_1, \ldots, u_n\} \subset \mathcal{E}$ be a set of differential indeterminates over $\mathcal{F}$. Then*

$\mathcal{I}_1 = [\mathcal{I}, u_0y_0 + u_1y_1 + \cdots + u_ny_n] : \mathbb{Y}^\infty$ is a differentially homogenous prime differential ideal in $\mathcal{F}\langle\mathbf{u}_0\rangle\{\mathbb{Y}\}$ with dimension $d-1$ and order h.

Proof. By Theorem 3.9, $[\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$ is a prime differential ideal of dimension $d-1$ and order h in $\mathcal{F}\langle\mathbf{u}_0\rangle\{y_1, \ldots, y_n\}$, where ϕ is defined in (3.1). Notice the fact that if we can show $\phi(\mathcal{I}_1) = [\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$, then it follows that $\mathcal{I}_1$ is a differentially homogenous prime differential ideal in $\mathcal{F}\langle\mathbf{u}_0\rangle\{\mathbb{Y}\}$ with dimension $d-1$ and order h. So it remains to show that $\phi(\mathcal{I}_1) = [\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$.

Firstly, we show that $\mathcal{I}_1$ is a differentially homogenous prime differential ideal. Let $\mathcal{I}_0$ be the differential ideal $[\mathcal{I}, u_0y_0 + u_1y_1 + \cdots + u_ny_n] : \mathbb{Y}^\infty$ in $\mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$. By Lemma 3.10, $\mathcal{I}_0$ is a differentially 2-homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ and $\mathcal{I}_0 \cap \mathcal{F}\{\mathbf{u}_0\} = [0]$. Now we show that $\mathcal{I}_1 = [\mathcal{I}_0] \subset \mathcal{F}\langle u_0, u_1, \ldots, u_n\rangle\{\mathbb{Y}\}$ is a nontrivial prime differential ideal. If $1 \in \mathcal{I}_1$, then we have $\mathcal{I}_0 \cap \mathcal{F}\{u_0, u_1, \ldots, u_n\} \neq [0]$, a contradiction. So $\mathcal{I}_1 \neq [1]$. Suppose $f_1, f_2 \in \mathcal{F}\langle u_0, u_1, \ldots, u_n\rangle\{\mathbb{Y}\}$ with $f_1f_2 \in \mathcal{I}_1$. Then there exist $h_i(\mathbf{u}_0)$ $(i = 1, 2)$ such that $h_if_i \in \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$. So $(h_1f_1)(h_2f_2) \in \mathcal{I}_0$. Then $h_1f_1 \in \mathcal{I}_0$ or $h_2f_2 \in \mathcal{I}_0$, and it follows that $f_1 \in \mathcal{I}_1$ or $f_2 \in \mathcal{I}_1$. Thus, $\mathcal{I}_1$ is a differentially homogenous prime differential ideal. Moreover, $\mathcal{I}_1$ and $\mathcal{I}_0$ have the following relations:

(i) $\mathcal{I}_1 \cap \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\} = \mathcal{I}_0$.
(ii) Any characteristic set of $\mathcal{I}_1$ in $\mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ is a characteristic set of $\mathcal{I}_0$.
(iii) Any characteristic set of $\mathcal{I}_0$ with $\mathbf{u}_0$ contained in its parametric set is a characteristic set of $\mathcal{I}_1$.

Since $\mathcal{I}_1 \neq [1]$, by Lemma 3.1, there exists i such that $\mathcal{I}_1 \cap \mathcal{F}\langle\mathbf{u}_0\rangle\{y_i\} = [0]$. Without loss of generality, we suppose $\mathcal{I}_1 \cap \mathcal{F}\langle\mathbf{u}_0\rangle\{y_0\} = [0]$. Let $\mathcal{A} \subset \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$ be a characteristic set of $\mathcal{I}_1$ w.r.t. the elimination ranking $y_0 \prec y_1 \prec \cdots \prec y_n$. Then $\mathcal{A}$ is also a characteristic set of $\mathcal{I}_0$ w.r.t. the elimination ranking $u_0 \prec \cdots \prec u_n \prec y_0 \prec y_1 \prec \cdots \prec y_n$. Let $\mathcal{B}$ be the auto-reduced set obtained from $\mathcal{A}$ by setting $y_0 = 1$ in each element of $\mathcal{A}$. By Lemma 3.7, $\mathcal{B}$ is a characteristic set of $\phi(\mathcal{I}_1)$. Both $\phi(\mathcal{I}_1)$ and $[\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$ are prime differential ideals. If we can show that $\mathcal{B}$ is a characteristic set of $[\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$, then $\phi(\mathcal{I}_1) = [\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$ follows.

We claim that $\mathcal{B}$ is a characteristic set of $\mathcal{I}_0^a = [\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n] \subset \mathcal{F}\{\mathbb{Y}; \mathbf{u}_0\}$. We already know that if $(\xi_0, \ldots, \xi_n)$ is a generic point of $\mathcal{I}$, then $(\xi_0, \ldots, \xi_n; -(u_1\xi_1 + \cdots + u_n\xi_n)/\xi_0, u_1, \ldots, u_n)$ is a generic point of $\mathcal{I}_0$ and $(\xi_1/\xi_0, \ldots, \xi_n/\xi_0; -(u_1\xi_1/\xi_0 + \cdots + u_n\xi_n/\xi_0), u_1, \ldots, u_n)$ is a generic point of $\mathcal{I}_0^a$. Recall that $\mathcal{B}$ has the same leaders as $\mathcal{A}$. It is easy to see that $B_i \in \mathcal{I}_0^a$. For any differential polynomial $f \in \mathcal{I}_0^a$, let r be the remainder of f w.r.t. $\mathcal{B}$. Then $r(\xi_1/\xi_0, \ldots, \xi_n/\xi_0; -(u_1\xi_1/\xi_0 + \cdots + u_n\xi_n/\xi_0), u_1, \ldots, u_n) = 0$. Let $r^{\mathrm{h}} = y_0^l r(y_1/y_0, \ldots, y_n/y_0; u_0, \ldots, u_n)$ where l is the denomination of r [11]. Then $r^{\mathrm{h}}(\xi_0, \ldots, \xi_n; -(u_1\xi_1 + \cdots + u_n\xi_n)/\xi_0, u_1, \ldots, u_n) = 0$. So $r^{\mathrm{h}} \in \mathcal{I}_0$. But $\mathcal{A}$ is also a characteristic set of $\mathcal{I}_0$, thus, $r^{\mathrm{h}} = 0$ and $r = 0$ follows. Thus, $\mathcal{B}$ is a characteristic set of $\mathcal{I}_0^a$. It follows that $\mathcal{B}$ is a characteristic set of $[\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$. Thus, $\phi(\mathcal{I}_1) = [\phi(\mathcal{I}), u_0 + u_1y_1 + \cdots + u_ny_n]$ and the theorem is proved. □

4. Chow forms for projective differential varieties

In this section, we will define the Chow form for an irreducible projective differential variety and prove its basic properties.

Let V be an irreducible projective differential $\mathcal{F}$-variety in $\mathbf{P}(n)$ of differential dimension d and order h. Suppose V does not lie in the hyperplane $y_0 = 0$. Intuitively, the Chow form of V is obtained by intersecting V with $d+1$ generic projective differential hyperplanes.

Let $\{u_{ij}\colon i = 0, \ldots, d;\ j = 0, \ldots, n\} \subset \mathcal{E}$ be a set of differential indeterminates over $\mathcal{F}$ and set

$$\mathbf{u} = \{u_{ij}\colon i = 0, \ldots, d;\ j = 1, \ldots, n\}.$$

Let $\xi = (1, \xi_1, \ldots, \xi_n)$ be a generic point of V, which is free from u_{ij}. Denote $\zeta_i = -\sum_{j=1}^n u_{ij}\xi_j$ $(i = 0, \ldots, d)$. Since $\dim(V) = d$, d.tr.deg$\,\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle/\mathcal{F} = d$. Then as in the affine differential case

[4, Lemma 4.1], we can prove that $\text{d.tr.deg}\,\mathcal{F}\langle\mathbf{u}\rangle\langle\zeta_0,\ldots,\zeta_d\rangle/\mathcal{F}\langle\mathbf{u}\rangle = d$. Since $\zeta_0,\ldots,\zeta_d$ are differentially dependent over $\mathcal{F}\langle\mathbf{u}\rangle$, there exists a relation

$$f(\mathbf{u};\zeta_0,\ldots,\zeta_d) = 0$$

where f is a differential polynomial in $\mathcal{F}\langle\mathbf{u}\rangle\{u_{00},\ldots,u_{d0}\}$ with minimal order. We choose f to be an irreducible differential polynomial in $\mathcal{F}\{\mathbf{u};u_{00},\ldots,u_{d0}\}$. Denote $\mathbf{u}_i = (u_{i0},\ldots,u_{in})$ $(i = 0,\ldots,d)$.

Definition 4.1. The above irreducible differential polynomial $f(\mathbf{u};u_{00},\ldots,u_{d0}) \in \mathcal{F}\{\mathbf{u};u_{00},\ldots,u_{d0}\}$ is defined to be *the differential Chow form*, or simply the Chow form, of V, denoted by $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$.

Let $\mathcal{I}$ be the differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ associated to V, where $\mathbb{Y} = (y_0,\ldots,y_n)$. Let $\mathbb{P}_i = u_{i0}y_0 + u_{i1}y_1 + \cdots + u_{in}y_n$ $(i = 0,\ldots,d)$. Then we have the following theorem which can be used to compute the Chow form.

Theorem 4.2. *The ideal $\mathcal{J} = [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d] : (\mathbb{Y}\mathbf{u}_0\cdots\mathbf{u}_d)^\infty$ in $\mathcal{F}\{\mathbb{Y},\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is a differentially $(d+2)$-homogenous prime differential ideal in $\mathbb{Y},\mathbf{u}_0,\ldots,\mathbf{u}_d$ with $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\} = \mathrm{sat}(F)$, where F is differentially $(d+1)$-homogenous.*

Proof. Firstly, we show that for any point $\mathbf{a} \in \mathbf{P}(n)\times\mathbf{P}(n)\times\cdots\times\mathbf{P}(n) = (\mathbf{P}(n))^{d+2}$, if $\mathcal{J}$ vanishes at $\mathbf{a}$, then $\mathcal{J}$ vanishes at every representative of $\mathbf{a}$. Now, suppose $\mathcal{J}$ vanishes at $\mathbf{a}$. For any differential polynomial $H \in \mathcal{J}$, there exists some $e \in \mathbb{N}$ such that $(y_j u_{0j_0}u_{1j_1}\cdots u_{dj_d})^e H \in [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d]$ for any $0 \leqslant j, j_0,\ldots,j_d \leqslant n$. Since $\mathcal{I}$ and $\mathbb{P}_0,\ldots,\mathbb{P}_d$ vanish at every representative of $\mathbf{a}$, H vanishes at it. It follows that $\mathcal{J}$ vanishes at every representative of $\mathbf{a}$. In this way, we say $\mathbf{a}$ is a zero of $\mathcal{J}$.

To prove $\mathcal{J}$ is a prime differential ideal, it suffices to show that $\mathbf{c} = (1,\xi_1,\ldots,\xi_n;\zeta_0,u_{01},\ldots,u_{0n};\ldots;\zeta_d,u_{d1},\ldots,u_{dn})$ is a generic zero of $\mathcal{J}$. Firstly, it is easy to see that $\mathbf{c}$ is a zero of $\mathcal{J}$. Now, suppose that G is any differential polynomial in $\mathcal{F}\{\mathbb{Y},\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ such that $G(\mathbf{c}) = 0$. Now, $\mathbb{P}_0,\ldots,\mathbb{P}_d$ form an auto-reduced set w.r.t. any elimination ranking of $\bigcup_{i=0}^d \mathbf{u}_i \cup \mathbb{Y}$ such that $z \prec u_{00} \prec \cdots \prec u_{d0}$ for any $z \in \mathbb{Y}\cup\mathbf{u}$. Let G_1 be the differential remainder of G w.r.t. $\mathbb{P}_0,\ldots,\mathbb{P}_d$. Then G_1 is free of u_{i0} $(i = 0,\ldots,d)$ and

$$y_0^a G \equiv G_1, \text{ mod } [\mathbb{P}_0,\ldots,\mathbb{P}_d] \quad (a \in \mathbb{N}).$$

So G_1 vanishes at $\mathbf{c}$. Since $G_1 \in \mathcal{F}\{\mathbb{Y},\mathbf{u}\}$, now rewritten G_1 as a differential polynomial in $\mathbf{u}$ with coefficients in $\mathbb{Y}$, i.e., $G_1 = \sum_\phi \phi(\mathbf{u})g_\phi$ where $\phi(\mathbf{u})$ are different differential monomials in $\mathbf{u}$ and g_ϕ are differential polynomials in $\mathcal{F}\{\mathbb{Y}\}$. Thus, $G_1(\mathbf{c}) = \sum_\phi \phi(\mathbf{u})g_\phi(\xi) = 0$. Since $\mathbf{u}$ are differential indeterminates over $\mathcal{F}\langle\xi\rangle$, $g_\phi(\xi) = 0$ for any ϕ. So, $g_\phi \in \mathcal{I}$ and $y_0^a G \in [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d]$. And for any index j_0 such that $\xi_{j_0} \neq 0$, it is easy to show that $\zeta_0,\ldots,\zeta_d$ and $\mathbf{u}\backslash\{(u_{ij_0})_{0\leqslant i\leqslant d}\}$ are differentially independent over $\mathcal{F}\langle\xi\rangle$. Similarly in this way, we can show that there exists $a_{j_0} \in \mathbb{N}$ such that $y_{j_0}^{a_{j_0}} G \in [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d]$. And if $\xi_{j_0} = 0$, then $y_{j_0} \in \mathcal{I}$, and $y_{j_0}G \in [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d]$. Thus, it follows that $G \in \mathcal{J}$.

By Theorem 2.5, $\mathcal{J}$ is a differentially $(d+2)$-homogenous prime differential ideal. Clearly, $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is a differentially $(d+1)$-homogenous prime differential ideal with a generic zero $(\zeta_0,u_{01},\ldots,u_{0n};\ldots;\zeta_d,u_{d1},\ldots,u_{dn})$. Since $\mathbf{u},\zeta_1,\ldots,\zeta_d$ are differentially independent over $\mathcal{F}$, the canonical characteristic set of $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ consists of only one differential polynomial, which is differentially $(d+1)$-homogenous by Theorem 2.5. By the definition of differential Chow form above, this polynomial differs from F by only one factor in $\mathcal{F}$. It follows that $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\} = \mathrm{sat}(F)$, and F is differentially $(d+1)$-homogenous. □

Similar to the differential affine case [4, Lemma 4.9], we have the following result.

Lemma 4.3. *Let $F(\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_d)$ be the Chow form of an irreducible projective differential variety V and $F^*(\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_d)$ obtained from F by interchanging $\mathbf{u}_\rho$ and $\mathbf{u}_\tau$. Then F^* and F differ at most by a sign.*

Furthermore, $\mathrm{ord}(F, u_{ij})$ $(i = 0, \ldots, d;\ j = 0, 1, \ldots, n)$ *are the same for all* u_{ij} *occurring in* F. *In particular,* u_{i0} $(i = 0, \ldots, d)$ *appear effectively in* F. *And a necessary and sufficient condition for some* u_{ij} $(j > 0)$ *not occurring effectively in* F *is that* $y_j \in \mathbb{I}(V)$.

Based on the above lemma, we define $\mathrm{ord}(F) = \mathrm{ord}(F, u_{00})$. By Definition 4.1, we know that F is also the differential Chow form of $\phi(V)$. Since V is of dimension d and order h, $\phi(V)$ is of dimension d and order h too. Thus, by [4, Theorem 4.11], we have the following theorem.

Theorem 4.4. *Let* V *be an irreducible projective differential variety of dimension* d *and order* h, *and* $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ *the differential Chow form of* V. *Then* $\mathrm{ord}(F) = h$.

We can also prove the above theorem directly using Theorem 3.11. Similarly to the affine case [4, Theorem 4.27], the projective differential Chow form has the following Poisson-type product formula [16].

Theorem 4.5. *Let* $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ *be the Chow form of an irreducible projective differential* $\mathcal{F}$*-variety which is of dimension* d *and order* h, *and does not lie on the hyperplane* $y_0 = 0$. *Then, there exist* $\xi_{\tau 1}, \ldots, \xi_{\tau n}$ *in a differential extension field* $(\mathcal{F}_\tau, \delta_\tau)$ $(\tau = 1, \ldots, g)$ *of* $(\mathcal{F}\langle\tilde{\mathbf{u}}\rangle, \delta)$ *such that*

$$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} \left(u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho} \right)^{(h)} \tag{4.1}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ *is in* $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$, $\tilde{\mathbf{u}} = \bigcup_{i=0}^{d} \mathbf{u}_i \backslash \{u_{00}\}$ *and* $g = \deg(f, u_{00}^{(h)})$. *Note that Eq.* (4.1) *is formal and should be understood in the following precise meaning*: $(u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho})^{(h)} \triangleq \delta^h u_{00} + \delta_\tau^h (\sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho})$.

For an element $\eta = (\eta_0, \eta_1, \ldots, \eta_n)$, denote its truncation up to order k as $\eta^{[k]} = (\eta_0, \eta_1, \ldots, \eta_n, \ldots, \eta_0^{(k)}, \eta_1^{(k)}, \ldots, \eta_n^{(k)})$.

Now we introduce the following notations:

$$\begin{aligned}
{}^a\mathbb{P}_0^{(0)} &= {}^a\mathbb{P}_0 := u_{00}y_0 + u_{01}y_1 + \cdots + u_{0n}y_n, \\
{}^a\mathbb{P}_0^{(1)} &= {}^a\mathbb{P}_0' := u_{00}'y_0 + u_{00}y_0' + u_{01}'y_1 + u_{01}y_1' + \cdots + u_{0n}'y_n + u_{0n}y_n', \\
&\cdots \\
{}^a\mathbb{P}_0^{(s)} &:= \sum_{j=0}^{n}\sum_{k=0}^{s} \binom{s}{k} u_{0j}^{(k)} y_j^{(s-k)}
\end{aligned} \tag{4.2}$$

which are considered to be algebraic polynomials in $\mathcal{F}(\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_n^{[s]})[\mathbb{Y}^{[s]}]$, and $u_{ij}^{(k)}$, $y_i^{(j)}$ are treated as algebraic indeterminates. A point $\eta = (\eta_0, \eta_1, \ldots, \eta_n)$ is said to be lying on ${}^a\mathbb{P}_0^{(k)}$ if regarded as an algebraic point, $\eta^{[k]}$ is a zero of ${}^a\mathbb{P}_0^{(k)}$. Then similar to the affine case [4, Theorem 4.36], the following theorem holds.

Theorem 4.6. *The points* $(1, \xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, g)$ *in* (4.1) *are generic points of the projective differential* $\mathcal{F}$*-variety* V. *If* $d > 0$, *they also satisfy the equations* $\sum_{\rho=0}^{n} u_{\sigma\rho} y_\rho = 0$ $(\sigma = 1, \ldots, d)$. *Moreover, they are the only elements of* V *which also lie on* $\mathbb{P}_i$ $(i = 1, \ldots, d)$[1] *as well as on* ${}^a\mathbb{P}_0^{(j)}$ $(j = 0, \ldots, h - 1)$.

[1] If $d = 0$, $\mathbb{P}_i$ $(i = 1, \ldots, d)$ is empty.

For a differential polynomial F, when we say η is in the *general solution* of $F=0$, we mean η is a zero of $\mathrm{sat}(F)$. As to the relations between the differential Chow form and the projective differential variety, we have the following theorem.

Theorem 4.7. *Let $F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)$ be the differential Chow form of V and $S_F=\frac{\partial F}{\partial u_{00}^{(h)}}$. Suppose that $\mathbf{u}_i$ are differentially specialized over $\mathcal{F}$ to sets $\mathbf{v}_i\subset\mathcal{E}$ and $\overline{\mathbb{P}}_i$ are obtained by substituting $\mathbf{u}_i$ by $\mathbf{v}_i$ in $\mathbb{P}_i$ $(i=0,\dots,d)$. If $\overline{\mathbb{P}}_i=0$ $(i=0,\dots,d)$ meet V, then $(\mathbf{v}_0,\dots,\mathbf{v}_d)$ is in the general solution of the differential equation $F=0$. Furthermore, if $F(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$ and $S_F(\mathbf{v}_0,\dots,\mathbf{v}_d)\neq 0$, then the $d+1$ differential hyperplanes $\overline{\mathbb{P}}_i=0$ $(i=0,\dots,d)$ meet V.*

Proof. Let $\mathcal{I}$ be the differentially homogenous prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ associated to V. If $\overline{\mathbb{P}}_i=0$ $(i=0,\dots,d)$ meet V, there exists $\mathbf{a}=(a_0,\dots,a_n)\in\mathbf{P}(n)$ with $a_{i_0}\neq 0$ such that $\overline{\mathbb{P}}_i$ and $\mathcal{I}$ vanish at $\mathbf{a}$. Since $[\mathcal{I},\mathbb{P}_0,\mathbb{P}_1,\dots,\mathbb{P}_d]:(\mathbb{Y})^\infty\cap\mathcal{F}\{\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d\}=\mathrm{sat}(F)$, for any differential polynomial $G\in\mathrm{sat}(F)$, there exists $e\in\mathbb{N}$ such that $y_j^eG\in[\mathcal{I},\mathbb{P}_0,\mathbb{P}_1,\dots,\mathbb{P}_d]$ for every $j=0,1,\dots,n$. So $(a_{i_0})^eG(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$, and $G(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$ follows. Thus, $(\mathbf{v}_0,\dots,\mathbf{v}_d)$ is in the general solution of $F=0$.

Conversely, suppose $F(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$ and $S_F(\mathbf{v}_0,\dots,\mathbf{v}_d)\neq 0$. Let $(1,\xi_1,\dots,\xi_n)$ be a generic point of V. Denote $\zeta_i=-\sum_{j=1}^n u_{ij}\xi_j$ $(i=0,\dots,d)$. By Definition 4.1, $F(\mathbf{u};\zeta_0,\dots,\zeta_d)=0$. Differentiating $F(\mathbf{u};\zeta_0,\dots,\zeta_d)=0$ w.r.t. $u_{0\rho}^{(h)}$, we have

$$\overline{\frac{\partial F}{\partial u_{0\rho}^{(s)}}}-\xi_\rho\overline{S_F}=0, \tag{4.3}$$

where $\overline{\frac{\partial F}{\partial u_{0\rho}^{(s)}}}$ and $\overline{S_F}$ are obtained by replacing $(u_{00},\dots,u_{d0})$ with $(\zeta_0,\dots,\zeta_d)$ in $\frac{\partial F}{\partial u_{0\rho}^{(s)}}$ and S_F respectively. Since $(1,\xi_1,\dots,\xi_n;\zeta_0,u_{01},\dots,u_{0n};\dots;\zeta_d,u_{d1},\dots,u_{dn})$ is a generic point of $[\mathcal{I},\mathbb{P}_0,\mathbb{P}_1,\dots,\mathbb{P}_d]:(\mathbb{Y})^\infty$, by Eq. (4.3), $A_\rho=S_Fy_\rho-\frac{\partial F}{\partial u_{0\rho}^{(s)}}y_0\in[\mathcal{I},\mathbb{P}_0,\mathbb{P}_1,\dots,\mathbb{P}_d]:(\mathbb{Y})^\infty$ $(\rho=1,\dots,n)$. It is easy to see that $F,A_1,\dots,A_n$ is a characteristic set of $[\mathcal{I},\mathbb{P}_0,\mathbb{P}_1,\dots,\mathbb{P}_d]:(\mathbb{Y})^\infty$ w.r.t. the elimination ranking $u_{01}\prec\dots\prec u_{dn}\prec u_{10}\prec\dots\prec u_{d0}\prec u_{00}\prec y_0\prec y_1\prec\dots\prec y_n$. Thus, for any $H\in\mathcal{I}$ and each $\mathbb{P}_i$, there exist e and e_i such that $S_F^eH\in[F,A_1,\dots,A_n]$ and $S_F^{e_i}\mathbb{P}_i\in[F,A_1,\dots,A_n]$. Let $\bar{y}_i=\frac{\partial F}{\partial u_{0\rho}^{(s)}}(\mathbf{v}_0,\dots,\mathbf{v}_d)/S_F(\mathbf{v}_0,\dots,\mathbf{v}_d)$ for $i=1,\dots,n$. Clearly, $(1,\bar{y}_1,\dots,\bar{y}_n;\mathbf{v}_0,\dots,\mathbf{v}_d)$ is a common zero of $F,A_1,\dots,A_n$. Thus, $H(1,\bar{y}_1,\dots,\bar{y}_n)=0$ and $\overline{\mathbb{P}}_i(1,\bar{y}_1,\dots,\bar{y}_n)=0$. That is to say, $(1,\bar{y}_1,\dots,\bar{y}_n)$ is a common point of V and $\overline{\mathbb{P}}_i=0$. $\square$

5. Application to linear dependence over projective variety

Let V be an irreducible algebraic variety in $\mathbf{P}_{\mathcal{K}}(n)$ that is defined over $\mathcal{C}$. Note that $\mathcal{K}$ is the constant field of $\mathcal{E}$. Call an element $\mathbf{v}=(v_0,\dots,v_n)\in\mathcal{E}^{n+1}$ *linearly dependent over* V if there exists a $\gamma\in V$ such that for a representative $(c_0,\dots,c_n)$ of γ, $\sum_{j=0}^n c_jv_j=0$. In this section, V is fixed to be an algebraic variety in $\mathbf{P}_{\mathcal{K}}(n)$ that is defined and irreducible over $\mathcal{C}$. In [10], Kolchin gave the following theorem.

Theorem 5.1. *Let $\mathfrak{R}$ denote the set of points of $\mathbf{P}(n)$ that are linearly dependent over V. Then there exists a differential polynomial $R_V\in\mathcal{C}\{\mathbb{Y}\}$, irreducible over $\mathcal{C}$, such that an element belongs to $\mathfrak{R}$ if and only if it is in the general solution of the differential equation $R_V=0$. R_V is unique up to a nonzero factor in $\mathcal{C}$ and is differentially homogenous with its order equal to the dimension of V.*

In a footnote of his paper [10], Kolchin mentioned that H. Morikawa pointed out to him that the differential polynomial R_V is the algebraic Chow form of V computed at the signed minors of the matrix $(y_j^{(i)})_{0\leqslant i\leqslant d,\,0\leqslant j\leqslant n}$. That is, if $G((u_{ij})_{0\leqslant i\leqslant d,\,0\leqslant j\leqslant n})$ is the Chow form of V, then $R_V=G((y_j^{(i)})_{0\leqslant i\leqslant d,\,0\leqslant j\leqslant n})$.

Clearly, V has a natural structure of projective differential $\mathcal{C}$-variety, defined by its defining polynomial equations $P_i=0$ $(1\leqslant i\leqslant s)$, together with the differential equations $y_iy_j'-y_jy_i'=0$ $(0\leqslant i<j\leqslant n)$. Let V^δ be the projective differential variety defined by $P_i=0$ $(1\leqslant i\leqslant s)$ and $y_iy_j'-y_jy_i'=0$ $(0\leqslant i<j\leqslant n)$.

In this section, we will explore the relationship between R_V and the differential Chow form of V^δ. Before giving the main result, we first prove two lemmas.

Lemma 5.2. *If* $\dim(V)=d$*, then* V^δ *defined as above is an irreducible projective differential variety of differential dimension zero and order* d*.*

Proof. Let $\mathcal{I}_0=\mathbb{I}(V)\subset\mathcal{C}[\mathbb{Y}]$ and $(c_0,\dots,c_n)\in\mathcal{K}^{n+1}$ a generic point of $\mathcal{I}_0$. Without loss of generality, suppose $c_0\neq0$. Consider the differential ideal $\mathcal{J}=[\mathcal{I}_0,(y_iy_j'-y_jy_i')_{0\leqslant i<j\leqslant n}]:(\mathbb{Y})^\infty$ of $\mathcal{C}\{\mathbb{Y}\}$. Then we have $V^\delta=\mathbb{V}(\mathcal{J})$. We claim that $\mathcal{J}$ is a differentially homogenous prime differential ideal.

Let $u\in\mathcal{E}$ be a differential indeterminate over $\mathcal{K}$. To prove $\mathcal{J}$ is a prime differential ideal, it suffices to prove that $(uc_0,\dots,uc_n)$ is a generic point of $\mathcal{J}$. Firstly, it is easy to show that $(uc_0,\dots,uc_n)$ is a zero of $\mathcal{J}$. Now, let $G\in\mathcal{C}\{\mathbb{Y}\}$, which vanishes at $(uc_0,\dots,uc_n)$. Let $\mathscr{R}$ be any ranking of $\mathbb{Y}$ such that $y_0\prec y_j$ $(j=1,\dots,n)$. Then $\mathcal{A}:=y_0y_j'-y_jy_0'$ $(j=1,\dots,n)$ is an auto-reduced set w.r.t. $\mathscr{R}$. Suppose the remainder of G w.r.t. $\mathcal{A}$ is G_1. Then $G_1\in\mathcal{C}\{y_0\}[y_1,\dots,y_n]$ and there exists some $a\in\mathbb{N}$ such that

$$y_0^aG\equiv G_1,\ \bmod[\mathcal{A}].$$

Since $G(uc_0,\dots,uc_n)=0$, $G_1(uc_0,\dots,uc_n)=0$. Rewrite G_1 as an algebraic polynomial in the proper derivatives of y_0 with coefficients in $\mathcal{C}[\mathbb{Y}]$, then we have $G_1=\sum_\phi\phi(y_0',y_0'',\dots)G_\phi(\mathbb{Y})$ where $\phi(y_0',y_0'',\dots)$ are distinct monomials in $y_0',y_0'',\dots$. Since $u',u'',\dots$ are algebraic indeterminates over $\mathcal{K}(u)$, $G_\phi(uc_0,\dots,uc_n)=0$, so, $G_\phi\in\mathcal{I}_0$ for each ϕ. Thus, $y_0^aG\in[\mathcal{I}_0,(y_iy_j'-y_jy_i')_{0\leqslant i<j\leqslant n}]$. Similarly, for any j_0 such that $c_{j_0}\neq0$, we can show that $y_{j_0}^{a_{j_0}}G\in[\mathcal{I}_0,(y_iy_j'-y_jy_i')_{0\leqslant i<j\leqslant n}]$ for some $a_{j_0}\in\mathbb{N}$. And for any j_0 such that $c_{j_0}=0$, $y_{j_0}\in\mathcal{I}_0$. So, $G\in\mathcal{J}$ and $(uc_0,\dots,uc_n)$ is a generic point of $\mathcal{J}$. Thus, $\mathcal{J}$ is a prime differential ideal. Clearly, $\mathcal{J}:(\mathbb{Y})=\mathcal{J}$ and for any zero η of $\mathcal{J}$ and every $s\in\mathcal{E}^\star$, $s\eta$ is a zero of $\mathcal{J}$. By Theorem 2.5, $\mathcal{J}$ is a differentially homogenous prime differential ideal. So V^δ is an irreducible projective differential variety and $(uc_0,\dots,uc_n)$ is a generic point of it.

The differential dimension polynomial of V^δ is

$$\begin{aligned}\omega_{V^\delta}(t)&=\mathrm{tr.deg}\,\mathcal{C}\left(\left(\frac{uc_j}{uc_0}\right)^{(k)}:\ 1\leqslant j\leqslant n,k\leqslant t\right)\Big/\mathcal{C}\\&=\mathrm{tr.deg}\,\mathcal{C}\big((c_j/c_0)^{(k)}:\ 1\leqslant j\leqslant n,k\leqslant t\big)/\mathcal{C}\\&=\mathrm{tr.deg}\,\mathcal{C}(c_j/c_0:\ 1\leqslant j\leqslant n)/\mathcal{C}=\dim(V)=d.\end{aligned}$$

Thus, V^δ is of dimension zero and order d and the lemma follows. □

Lemma 5.3. *Let* $G_{ij}=y_iy_j'-y_jy_i'$ $(0\leqslant i,j\leqslant n)$*. Then for all* l*,* j_0 *and* m*,* $y_{j_0}^my_l^{(m)}\equiv h_m(y_{j_0})\cdot y_l,\ \bmod[G_{j_0l}]$ *where* $h(y_{j_0})\in\mathcal{C}\{y_{j_0}\}$*.*

Proof. Fix l and j_0. For $m=1$, $y_{j_0}y_l'=y_{j_0}'y_l+G_{j_0l}$. Suppose it holds for $1,\dots,m-1$. Since $G_{j_0l}^{(m-1)}=(y_{j_0}y_l'-y_{j_0}'y_l)^{(m-1)}=\sum_{s=0}^{m-1}\binom{m-1}{s}y_{j_0}^{(m-1-s)}y_l^{(s+1)}-\sum_{s=0}^{m-1}\binom{m-1}{s}y_{j_0}^{(m-s)}y_l^{(s)}$, $y_{j_0}y_l^{(m)}=G_{j_0l}^{(m-1)}+\sum_{s=1}^{m-1}(\binom{m-1}{s}-\binom{m-1}{s-1})y_{j_0}^{(m-s)}y_l^{(s)}+y_{j_0}^{(m)}y_l$. By the hypothesis, $y_{j_0}^sy_l^{(s)}\equiv h_s(y_{j_0})\cdot y_l,\ \bmod[G_{j_0l}]$ holds for $s \leqslant m-1$. Thus, $y_{j_0}^m y_l^{(m)} = y_{j_0}^{m-1} G_{j_0 l}^{(m-1)} + \sum_{s=1}^{m-1} (\binom{m-1}{s} - \binom{m-1}{s-1}) y_{j_0}^{(m-s)} y_{j_0}^{m-1} y_l^{(s)} + y_{j_0}^{(m)} y_{j_0}^{m-1} y_l \equiv h_m(y_{j_0}) \cdot y_l$, mod $[G_{j_0 l}]$, where $h_m = \sum_{s=1}^{m-1} (\binom{m-1}{s} - \binom{m-1}{s-1}) y_{j_0}^{(m-s)} y_{j_0}^{m-s-1} h_s + y_{j_0}^{(m)} y_{j_0}^{m-1}$. □

Now we give the main theorem as follows.

Theorem 5.4. *Let $\mathbf{u}_0 = (u_{00}, \ldots, u_{0n})$. Then the differential polynomial $R_V(\mathbf{u}_0)$ defined in Theorem 5.1 is equal to the differential Chow form of V^δ in the sense of multiplied by a nonzero constant in $\mathcal{C}$.*

Proof. By Lemma 5.2, V^δ is an irreducible projective differential variety of dimension zero and order d. Let $F(\mathbf{u}_0)$ be the differential Chow form of V^δ. Then by Theorem 4.4, $\operatorname{ord}(F, \mathbf{u}_0) = d$. By Theorem 5.1, $\operatorname{ord}(R_V, \mathbf{u}_0) = d$.

Let $\mathbb{P}_0 = \sum_{i=0}^n u_{0i} y_i$. Since $R_V(\mathbf{u}_0)$ and $F(\mathbf{u}_0)$ are irreducible differential polynomials in $\mathcal{C}\{\mathbf{u}_0\}$ with the same order, if we can prove $R_V(\mathbf{u}_0) \in [\mathbb{I}(V^\delta), \mathbb{P}_0] : (\mathbb{Y}\mathbf{u}_0)^\infty \cap \mathcal{F}\{\mathbf{u}_0\} = \operatorname{sat}(F(\mathbf{u}_0))$, R_V and F differ at most by a nonzero constant in $\mathcal{C}$. Now we are going to show that $R_V(\mathbf{u}_0) \in [\mathbb{I}(V^\delta), \mathbb{P}_0] : (\mathbb{Y}\mathbf{u}_0)^\infty$.

Let $\mathcal{I}_0 = \mathbb{I}(V) \subset \mathcal{C}\{\mathbb{Y}\}$ and $(c_0, \ldots, c_n)$ a generic point of V. Let $F_0(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ be the algebraic Chow form of V, where $\mathbf{u}_1, \ldots, \mathbf{u}_d$ are the vectors of coefficients of the generic algebraic hyperplanes $\mathbb{L}_1, \ldots, \mathbb{L}_d$ respectively, and $\mathbb{L}_0 = \mathbb{P}_0$ is regarded as an algebraic hyperplane at the very moment. For the algebraic Chow form, we have $(\mathcal{I}_0, \mathbb{L}_0, \mathbb{L}_1, \ldots, \mathbb{L}_d) : \mathbb{Y}^\infty \cap \mathcal{F}\{\mathbf{u}_0\} = (F_0)$. Suppose $c_{j_0} \neq 0$. Then there exists an a_{j_0} such that

$$y_{j_0}^{a_{j_0}} F_0(\mathbf{u}_0, \ldots, \mathbf{u}_d) = \sum_{k=0}^d h_k \mathbb{L}_k + \sum_i g_i f_i \tag{5.1}$$

where $f_i \in \mathcal{I}_0 \subset \mathcal{C}[\mathbb{Y}]$ and $h_k, g_i \in \mathcal{C}[\mathbb{Y}, \mathbf{u}_0, \ldots, \mathbf{u}_d]$. Denote $\mathbf{u}_0^{(k)} = (u_{00}^{(k)}, \ldots, u_{0n}^{(k)})$. Replace $\mathbf{u}_k$ by $\mathbf{u}_0^{(k)}$ in (5.1) for $k = 1, \ldots, d$, we obtain

$$y_{j_0}^{a_{j_0}} F_0\bigl(\mathbf{u}_0, \mathbf{u}_0', \ldots, \mathbf{u}_0^{(d)}\bigr) = \sum_{k=0}^d \widehat{h_k} \widehat{\mathbb{L}_k} + \sum_i \widehat{g_i} f_i, \tag{5.2}$$

where $\widehat{\mathbb{L}_k} = u_{00}^{(k)} y_0 + u_{01}^{(k)} y_1 + \cdots + u_{0n}^{(k)} y_n$. Denote $G_{ij} = y_i y_j' - y_j y_i'$ $(0 \leqslant i, j \leqslant n)$. Now we claim that there exists $b_k \in \mathbb{N}$ such that

$$y_{j_0}^{b_k} \widehat{\mathbb{L}_k} \in \bigl[\mathbb{P}_0, (G_{ij})_{0 \leqslant i, j \leqslant n}\bigr] \quad (k = 0, \ldots, d). \tag{5.3}$$

Assuming the claim holds. Denote $e_{j_0} = \max_k \{a_{j_0} + b_k\}$. Then by (5.2), for each j_0 such that $c_{j_0} \neq 0$, we have $y_{j_0}^{e_{j_0}} F_0(\mathbf{u}_0, \mathbf{u}_0', \ldots, \mathbf{u}_0^{(d)}) \in [\mathbb{I}(V^\delta), \mathbb{P}_0]$. So $R_V(\mathbf{u}_0) = F_0(\mathbf{u}_0, \mathbf{u}_0', \ldots, \mathbf{u}_0^{(d)}) \in [\mathbb{I}(V^\delta), \mathbb{P}_0] : (\mathbb{Y}\mathbf{u}_0)^\infty$. Thus, it suffices to prove the claim.

Now we are going to prove the claim (5.3) by induction on k. Firstly, $\mathbb{L}_0 = \mathbb{P}_0$. And for $k = 1$, $\widehat{\mathbb{L}_1} = \sum_{l=0}^n u_{0l}' y_l = (\sum_{l=0}^n u_{0l} y_l)' - \sum_{l=0}^n u_{0l} y_l' = \mathbb{P}_0' - \sum_{l=0}^n u_{0l} y_l'$. By Lemma 5.3, $y_{j_0} \widehat{\mathbb{L}_1} = y_{j_0} \mathbb{P}_0' - \sum_{l=0}^n u_{0l} y_{j_0} y_l' = y_{j_0} \mathbb{P}_0' - \sum_{l=0}^n u_{0l} (G_{j_0 l} - y_{j_0}' y_l) = y_{j_0} \mathbb{P}_0' - \sum_{l=0}^n u_{0l} G_{j_0 l} + y_{j_0}' \mathbb{P}_0$. So it holds for $k = 1$. Now suppose the claim holds for integers less than k and we now deal with k. Since $\widehat{\mathbb{L}_k} = \sum_{l=0}^n u_{0l}^{(k)} y_l = \mathbb{P}_0^{(k)} - \sum_{l=0}^n \sum_{m=1}^k \binom{k}{m} u_{0l}^{(k-m)} y_l^{(m)}$, it follows that $y_{j_0}^k \widehat{\mathbb{L}_k} = y_{j_0}^k \mathbb{P}_0^{(k)} - \sum_{m=1}^k \binom{k}{m} \sum_{l=0}^n u_{0l}^{(k-m)} y_{j_0}^k y_l^{(m)} \equiv y_{j_0}^k \mathbb{P}_0^{(k)} - \sum_{m=1}^k \binom{k}{m} h_m(y_{j_0}) y_{j_0}^{k-m} \widehat{\mathbb{L}_{k-m}}$, mod $[(G_{ij})_{0 \leqslant i, j \leqslant n}]$. By the hypothesis, for $s < k$, $y_{j_0}^{b_s} \widehat{\mathbb{L}_s} \in [\mathbb{P}_0, (G_{ij})_{0 \leqslant i, j \leqslant n}]$. Thus, the claim follows. □

Combining Theorem 5.1 with Theorem 5.4, we have the following corollary.

Corollary 5.5. *Let V^δ be defined as above and $F(u_{00}, u_{01}, \ldots, u_{0n})$ its differential Chow form. If $v_{0j} \in \mathcal{E}$ is any differential specialization of u_{0j} $(j = 0, 1, \ldots, n)$, then V^δ meets the differential hyperplane $v_{00}y_0 + v_{01}y_1 + \cdots + v_{0n}y_n = 0$ if and only if $(v_{00}, v_{01}, \ldots, v_{0n})$ is in the general solution of the differential equation $F = 0$.*

6. Conclusion

In this paper, we first prove a theorem for the intersection of an irreducible projective differential variety with generic projective differential hyperplanes. Then we define the Chow form for projective differential varieties and give its basic properties. Finally, we show that the formula on linear dependence over an algebraic projective variety given by Kolchin is actually the differential Chow form of the projective variety teated as a differential projective variety in certain sense.

For an algebraic projective variety V of dimension d, its projective Chow form gives a necessary and sufficient condition for V having common points with the $d + 1$ hyperplanes $\sum_{j=0}^{n} u_{ij}x_j = 0$ [6, p. 50]. For a particular kind of projective differential varieties, Corollary 5.5 tells us that $\mathrm{sat}(F)$ gives a necessary and sufficient condition that V and the differential hyperplane $\sum_{j=0}^{n} u_{0j}y_j = 0$ intersect, where F is the projective differential Chow form of V. However, up to now, for general projective differential varieties, Theorem 4.7 only gives a necessary condition. Due to Corollary 5.5, we conjecture that the Chow form also gives a sufficient condition. That is,

Conjecture. *Let V be an irreducible projective differential variety over $\mathcal{F}$ with $\dim(V) = d$. Suppose $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is the projective differential Chow form of V. If $v_{ij} \in \mathcal{E}$ is a differential specialization of u_{ij} $(i = 0, \ldots, d;\ j = 0, 1, \ldots, n)$, then a necessary and sufficient condition that V and the $d + 1$ differential hyperplanes*

$$v_{i0}y_0 + v_{i1}y_1 + \cdots + v_{in}y_n = 0 \quad (i = 0, \ldots, d)$$

have points in common is that $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d)$ is in the general solution of the differential equation $F = 0$, where $\mathbf{v}_i = (v_{i0}, \ldots, v_{in})$.

The above conjecture cannot be proved by showing that the projective differential space is differentially complete which is invalid [10]. One possible way is to extend the proof for the affine counterpart given in [6] to the differential case, where the corresponding problem is to study differential resultant for two differentially homogenous differential polynomials in two differential variables.

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