Advances in Applied Mathematics 72 (2016) 77–112

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Computation of differential Chow forms for ordinary prime differential ideals ☆

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ARTICLE INFO

Article history:
Received 7 December 2014
Received in revised form 3 September 2015
Accepted 4 September 2015
Available online 16 September 2015

MSC:
primary 12H05, 68W30
secondary 14Q99

Keywords:
Differential Chow form
Jacobi bound
Characteristic set
Single exponential algorithm

ABSTRACT

In this paper, we propose algorithms for computing differential Chow forms for ordinary prime differential ideals which are given by characteristic sets. The algorithms are based on an optimal bound for the order of a prime differential ideal in terms of a characteristic set under an arbitrary ranking, which shows the Jacobi bound conjecture holds in this case. Apart from the order bound, we also give a degree bound for the differential Chow form. In addition, for a prime differential ideal given by a characteristic set under an orderly ranking, a much simpler algorithm is given to compute its differential Chow form. The computational complexity of the algorithms is single exponential in terms of the Jacobi number, the maximal degree of the differential polynomials in a characteristic set, and the number of variables.

☆ Partially supported by a National Key Basic Research Project of China (2011CB302400) and by grants from NSFC (60821002, 11301519).
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http://dx.doi.org/10.1016/j.aam.2015.09.004
0196-8858/

1. Introduction

Differential algebra, founded by Ritt and Kolchin, aims to study differential polynomial equations in a way similar to how polynomial equations are studied in algebraic geometry [39,27]. There are many interesting open problems in differential algebra; see [39, p. 177] and [3]. One classical problem is the Jacobi bound conjecture, which is one of the five problems formulated by Kolchin at the ICM, Moscow in 1966 [26].

In this paper, we first prove the Jacobi bound conjecture for prime differential ideals given by characteristic sets. Then, based on this result, we propose algorithms to compute differential Chow forms for prime differential ideals represented by characteristic sets.

1.1. Around the Jacobi bound conjecture

The first main problem we consider in this paper is related to the Jacobi bound conjecture.

Let $\mathcal{S} = \{f_1, \ldots, f_m\} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ ($m \leq n$) be a system of differential polynomials. Let e_{ij} be the greatest natural number k such that $y_j^{(k)}$ effectively appears in f_i. If y_j and its derivatives do not appear in f, set $e_{ij} = -\infty$. Set $e_{ij}^* = \max\{e_{ij}, 0\}$. The *(strong) Jacobi number* $\mathrm{Jac}(\mathcal{S})$ (see Definition 11), the *weak Jacobi number*, of $\mathcal{S}$, are defined to be the maximal diagonal sums of matrices (e_{ij}), (e_{ij}^*) respectively. The *Ritt number* $\mathrm{R}(\mathcal{S})$ of $\mathcal{S}$ is defined to be $\mathrm{R}(\mathcal{S}) = \sum_{j=1}^{n} \max_i\{e_{ij}^*\}$.

The order of an irreducible differential variety V (see Definition 5), or the prime differential ideal $\mathbb{I}(V)$, is the sum of orders of the elements of any characteristic set of this ideal with respect to an orderly ranking. In the case when the differential dimension of V is equal to zero, the order of V is just what in the classical literature is called the number of arbitrary constants on which the solution of the system $\mathcal{S} = 0$ depends. So it measures the size of the zero set of $\mathcal{S}$.

The Jacobi bound conjecture is the following:

Jacobi bound conjecture. *(See [38].) Suppose $m = n$. Let V be any component of the radical differential ideal generated by $\mathcal{S}$ with differential dimension* 0*. Then the order of V does not exceed* $\mathrm{Jac}(\mathcal{S})$.

If we replace $\mathrm{Jac}(\mathcal{S})$ by $\mathrm{Jac}^*(\mathcal{S})$, we get the so-called *weak Jacobi bound conjecture.*

The Jacobi bound was proposed heuristically by Jacobi [20]. Ritt proved the conjecture in the case when $\mathcal{S}$ is a linear system and also in the case when $n \leq 2$ [38]. Ritt's results have been extended to partial differential systems by Tomasovic [44]. Lando showed that the weak Jacobi bound conjecture holds for the case $e_{ij} \leq 1$ [31]. A system $\mathcal{S}$ is called *independent* (see Definition 13) if it satisfies the regularity hypothesis defined by Johnson [23]. Kondratieva et al. proved that the Jacobi bound conjecture holds for independent (partial) differential systems [29].

Without the assumption $m = n$, Ritt proved that the order of any component of $\mathcal{S}$ of differential dimension 0 is bounded by the Ritt number of $\mathcal{S}$ [39, p. 135]. Combining Ritt's result on relative orders [39, p. 135] and [12, Theorem 2.11], we obtain that the Ritt number is also an upper bound for the order of each component of $\mathcal{S}$. The work of Greenspan and of Lando, summarized in [7], has provided improvements on the bound, but as remarked by Cohn, their work seems unlikely to have yielded bounds best possible for the information in E or E^* [8].

It was shown by Cohn [8, p. 3] that the Jacobi bound conjecture implies the famous *differential dimension conjecture* [39, p. 178].

Differential dimension conjecture. *Suppose $m \leq n$ and the system $\mathcal{S} = 0$ has solutions. Then the differential dimension of each component of the radical differential ideal generated by $\mathcal{S}$ is at least $n - m$.*

In addition to radical differential ideals given by generators, Golubitsky et al. also worked on the order bound problem for prime differential ideals given by characteristic sets under arbitrary rankings [15]. Let $\mathcal{I} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ be a prime differential ideal and $\mathcal{A}$ a characteristic set of $\mathcal{I}$ under an arbitrary ranking. Golubitsky et al. showed that the order of $\mathcal{I}$ does not exceed the number $|\mathcal{A}| \cdot \max\{\mathrm{ord}(C) : C \in \mathcal{A}\}$ [15, p. 337], where $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. Moreover, since this bound is likely to be non-optimal, they also conjectured the following stronger upper bound in terms of the Ritt number.

Conjecture. *(See [15, p. 337].) For each $j = 1, \ldots, n$, let o_j be the greatest number k such that $y_j^{(k)}$ appears in $\mathcal{A}$. Suppose $o_{k_1} \geq o_{k_2} \geq \cdots \geq o_{k_n}$. Then $\mathrm{ord}(\mathcal{I}) \leq \sum_{i=1}^{|\mathcal{A}|} o_{k_i}$.*

Clearly, the Jacobi number is in general smaller than this conjectured bound.

In this paper, we prove the Jacobi bound conjecture holds for prime differential ideals given by characteristic sets with respect to arbitrary rankings, which implies the above conjecture proposed by Golubitsky et al. is true.

Theorem 1. *Let $\mathcal{I} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ be a prime differential ideal of differential dimension d, and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathcal{I}$ under any fixed ranking $\mathscr{R}$. Then $\mathrm{ord}(\mathcal{I}) \leq \mathrm{Jac}(\mathcal{A})$.*

1.2. The computation of differential Chow form

The Chow form, also known as the Cayley form, is a basic concept in algebraic geometry [5,19] and also a powerful tool in elimination theory. Suppose $V \subset \mathbb{C}^n$ is an unmixed variety of dimension d. Given a matrix $(a_{ij}) \in \mathbb{C}^{(d+1)\times(n+1)}$, let

$$\mathbb{L}_i := (a_{i0} + a_{i1}y_1 + \cdots + a_{in}y_n = 0),\ \ i = 0, \ldots, d$$

be $d+1$ hyperplanes. The Zariski closure of the set $\{(a_{ij}) : V \cap \mathbb{L}_0 \cap \cdots \cap \mathbb{L}_d \neq \emptyset\}$ is an irreducible hypersurface. The minimal defining polynomial of this hypersurface is called the *algebraic Chow form* of V.

The Chow form has important applications in many fields, for example, [2,35,37,45]. Recent studies also show that the Chow form has closely related to sparse elimination theory [14,42,36,24]. There are efficient algorithms to compute the Chow form [30,4,21].

Recently, the algebraic Chow form was generalized to the differential algebraic setting and the theory of differential Chow forms in both affine and projective differential algebraic geometry was developed [12,32]. The theory of sparse differential resultants and efficient algorithms to compute sparse differential resultants were then developed in [33,34]. A natural next problem is to develop efficient algorithms to compute the differential Chow form. In general, there is no algorithm to test whether a given differential ideal is prime or not, because this primality testing problem is equivalent to the Ritt problem; see [16, Theorem 1] and [17, Conjecture 1.1]. However, for most applications, prime differential ideals are given by characteristic sets.

Thus, the second main problem we consider in this paper is the following:

Given a prime differential ideal $\mathcal{I} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ represented by a characteristic set $\mathcal{A}$ under an arbitrary ranking $\mathscr{R}$, devise an algorithm to compute its differential Chow form, and estimate the computational complexity.

Although, as mentioned in [12, Remark 4.4], the differential Chow form could be computed by means of algorithms for algebraic transformation of differential characteristic decompositions from one ranking to another [1,40,15], on the whole, either there is no computational complexity analysis, or the algorithms are so general that they are not efficient. In this paper, taking advantage of properties of the differential Chow form, we will propose single-exponential algorithms for computing differential Chow forms for prime differential ideals represented by characteristic sets under arbitrary rankings. Our algorithms require only linear algebraic computations over the base field of the ideals.

The differential dimension of $\mathcal{I}$ is equal to $d = n - |\mathcal{A}|$ (see Lemma 4). Let

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d)$$

be generic differential hyperplanes (see Definition 7). The differential Chow form is the unique (up to a factor in $\mathcal{F}$) differential polynomial $F(u_{ij})$ with minimal order and of minimal degree under this order contained in $[\mathcal{I}, \mathbb{P}_0, \cdots, \mathbb{P}_d] \cap \mathcal{F}\{u_{ij}\}$ (Definition 9). By [12, Theorem 4.11], the order of the differential Chow form is equal to the order of the corresponding prime differential ideal. So by Theorem 1, the order of the differential Chow form is bounded by $\mathrm{Jac}(\mathcal{A})$. Apart from giving an upper bound for the order in Theorem 1, we also give a Beźout-type degree bound for the differential Chow form of $\mathcal{I}$ in terms of the degrees of elements in $\mathcal{A}$.

Based on the order and degree bounds, we are able to use the method of undetermined coefficients to give algorithms to compute the differential Chow form. In the special case when $\mathscr{R}$ is an orderly ranking (defined in Section 2.2), the order of $\mathcal{I}$ is equal to the sum of orders of elements in $\mathcal{A}$, and we give a simpler algorithm to compute the differential Chow form. In the general case when $\mathscr{R}$ is an arbitrary ranking, we give two different algorithms to compute the differential Chow form. One uses the searching strategy prioritizing order over degree and the other uses the searching strategy prioritizing degree over order. All the algorithms have single exponential complexity in terms of the Jacobi number, the number of variables, the degrees and the orders of the polynomials in $\mathcal{A}$.

The paper is organized as follows. In Section 2, we give some basic notation and preliminary results about differential algebra. In Section 3, we prove the Jacobi bound conjecture for prime differential ideals given by characteristic sets. In Section 4, we give an algorithm to compute differential Chow forms for prime differential ideals represented by characteristic sets under orderly rankings. In Section 5, we give two different algorithms to compute differential Chow forms for prime differential ideals given by characteristic sets under arbitrary rankings.

2. Preliminaries

In this section, some basic notation and preliminary results in differential algebra will be given. For more details about differential algebra, please refer to [3,28,27,39,41].

2.1. Differential polynomial algebra

Let $\mathcal{F}$ be a fixed ordinary differential field of characteristic zero with a derivation operator δ. For ease of notation, we use primes and exponents (i) to denote derivatives under δ, and for each $a \in \mathcal{F}$, denote $a^{[n]} = \{a, a^{(1)}, \ldots, a^{(n)}\}$ and $a^{[\infty]} = \{a^{(i)} : i \geq 0\}$. Throughout this paper, unless otherwise indicated, δ is kept fixed during any discussion. A typical example of differential field is $\mathbb{Q}(t)$ which is the field of rational functions in the variable t with $\delta = \frac{d}{dt}$.

Let $\mathcal{G}$ be a differential extension field of $\mathcal{F}$ and S a subset of $\mathcal{G}$. We denote respectively by $\mathcal{F}[S]$, $\mathcal{F}(S)$, $\mathcal{F}\{S\}$, and $\mathcal{F}\langle S\rangle$ the smallest subring, the smallest subfield, the smallest differential subring, and the smallest differential subfield of $\mathcal{G}$ containing $\mathcal{F}$ and S. And $\mathcal{G}$ is said to be *finitely differentially generated* over $\mathcal{F}$ if there exists a finite subset $S \subset \mathcal{G}$ such that $\mathcal{G} = \mathcal{F}\langle S\rangle$.

Let Θ be the free commutative semigroup with unit (written multiplicatively) generated by δ. A subset Σ of a differential extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be *differentially dependent* over $\mathcal{F}$ if the set $(\theta\alpha)_{\theta\in\Theta,\alpha\in\Sigma}$ is algebraically dependent over $\mathcal{F}$, and otherwise, it is said to be *differentially independent* over $\mathcal{F}$, or to be a family of *differential indeterminates* over $\mathcal{F}$ (abbr. differential $\mathcal{F}$-indeterminates). If Σ consists of only one element α, we simply say that α is *differentially algebraic* or *differentially transcendental* over $\mathcal{F}$ respectively. A maximal subset Ω of $\mathcal{G}$ which is differentially independent over $\mathcal{F}$ is said to be a *differential transcendence basis* of $\mathcal{G}$ over $\mathcal{F}$. The cardinality of Ω is called the *differential transcendence degree* of $\mathcal{G}$ over $\mathcal{F}$, denoted by $\text{d.tr.deg}\,\mathcal{G}/\mathcal{F}$.

Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two differential extension fields of $\mathcal{F}$. A homomorphism (resp. isomorphism) ϕ from $\mathcal{G}_1$ to $\mathcal{G}_2$ is called *a differential homomorphism (resp. isomorphism) over* $\mathcal{F}$ if ϕ commutes with δ and leaves each element of $\mathcal{F}$ invariant.

A differential extension field $\mathcal{E}$ of $\mathcal{F}$ is called a *universal differential extension field*, if for any finitely differentially generated extension field $\mathcal{F}_1 \subset \mathcal{E}$ of $\mathcal{F}$ and any finitely differentially generated extension field $\mathcal{F}_2$ of $\mathcal{F}_1$ not necessarily contained in $\mathcal{E}$, there exists a differential extension field $\mathcal{F}_3 \subset \mathcal{E}$ of $\mathcal{F}_1$ such that $\mathcal{F}_3$ is differentially isomorphic to $\mathcal{F}_2$ over $\mathcal{F}_1$. Such a universal differential extension field of $\mathcal{F}$ always exists [27, p. 134, Theorem 2].

Now suppose $\mathcal{E}$ is a universal differential extension field of $\mathcal{F}$, and $\mathbb{Y} = \{y_1, \ldots, y_n\}$ is a set of differential indeterminates over $\mathcal{E}$. For any $y \in \mathbb{Y}$, denote $\delta^k y$ by $y^{(k)}$. The elements of $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[y_j^{(k)} : j = 1, \ldots, n; k \in \mathbb{N}]$ are called *differential polynomials* over $\mathcal{F}$ in $\mathbb{Y}$, and $\mathcal{F}\{\mathbb{Y}\}$ itself is called the *differential polynomial ring* over $\mathcal{F}$ in $\mathbb{Y}$. A differential polynomial ideal $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ is an algebraic ideal which is closed under derivation, i.e. $\delta(\mathcal{I}) \subseteq \mathcal{I}$. A *prime differential ideal* is a differential ideal which is also a prime ideal. For convenience, a prime differential ideal is assumed not to be the unit ideal in this paper.

By a differential affine space, we mean any one of the sets $\mathcal{E}^n (n \in \mathbb{N})$. Let Σ be a subset of differential polynomials in $\mathcal{F}\{\mathbb{Y}\}$. A point $\eta \in \mathcal{E}^n$ is called a *zero* of Σ if $f(\eta) = 0$ for any $f \in \Sigma$. The set of all zeros of Σ is denoted by $\mathbb{V}(\Sigma)$, which is called a *differential variety* defined over $\mathcal{F}$. A point $\eta \in \mathbb{V}(\mathcal{I})$ is called a *generic point* of a prime differential ideal $\mathcal{I} \subseteq \mathcal{F}\{\mathbb{Y}\}$ if for any $f \in \mathcal{F}\{\mathbb{Y}\}$ we have $f(\eta) = 0 \Leftrightarrow f \in \mathcal{I}$. It is well known that:

Lemma 2. *A non-unit differential ideal is prime if and only if it has a generic point.*

2.2. Characteristic sets of a differential polynomial ideal

Let f be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$. The *order of f with respect to y_i* is the greatest number k such that $y_i^{(k)}$ appears effectively in f, denoted by $\operatorname{ord}(f, y_i)$. If y_i does not appear in f, set $\operatorname{ord}(f, y_i) = -\infty$. The *order* of f is defined to be $\max_i\{\operatorname{ord}(f, y_i)\}$, denoted by $\operatorname{ord}(f)$.

A *ranking* $\mathscr{R}$ is a total order over $\Theta(\mathbb{Y})$ if satisfying 1) $\delta\alpha > \alpha$ for all $\alpha \in \Theta(\mathbb{Y})$ and 2) $\alpha_1 > \alpha_2 \Rightarrow \delta\alpha_1 > \delta\alpha_2$ for all $\alpha_1, \alpha_2 \in \Theta(\mathbb{Y})$. Below are two important kinds of rankings:

1. *Elimination ranking*: $y_i > y_j \Rightarrow \delta^k y_i > \delta^l y_j$ for any $k, l \geq 0$.
2. *Orderly ranking*: $k > l \Rightarrow \delta^k y_i > \delta^l y_j$ for any $i, j \in \{1, \ldots, n\}$.

Let f be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$ endowed with a ranking $\mathscr{R}$. The *leader* of f is the greatest derivative with respect to $\mathscr{R}$ which appears effectively in f, denoted by $\operatorname{ld}(f)$. Regarding f as a univariate polynomial in $\operatorname{ld}(f)$, its leading coefficient is called the *initial* of f, denoted by I_f, and the partial derivative of f with respect to $\operatorname{ld}(f)$ is called the *separant* of f, denoted by S_f. For any two differential polynomials f, g in $\mathcal{F}\{\mathbb{Y}\}$, f is said to be of *lower rank* than g, denoted by $f < g$, if 1) $\mathrm{ld}(f) < \mathrm{ld}(g)$, or 2) $\mathrm{ld}(f) = \mathrm{ld}(g)$ and $\deg(f, \mathrm{ld}(f)) < \deg(g, \mathrm{ld}(f))$. And f is said to be *reduced* with respect to g if no proper derivatives of $\mathrm{ld}(g)$ appear in f and $\deg(f, \mathrm{ld}(g)) < \deg(g, \mathrm{ld}(g))$. Let $\mathcal{A}$ be a set of differential polynomials. Then $\mathcal{A}$ is said to be an *auto-reduced set* if each element of $\mathcal{A}$ is reduced with respect to any other element of $\mathcal{A}$. Every auto-reduced set is finite [27, p. 77].

Let $\mathcal{A}$ be an auto-reduced set. We denote $\mathrm{H}_{\mathcal{A}}$ to be the set of all initials and separants of $\mathcal{A}$ and $\mathrm{H}_{\mathcal{A}}^{\infty}$ the minimal multiplicative set containing $\mathrm{H}_{\mathcal{A}}$. The *saturation differential ideal* of $\mathcal{A}$ is defined by

$$\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} = \{f \in \mathcal{F}\{\mathbb{Y}\} \mid \exists h \in \mathrm{H}_{\mathcal{A}}^{\infty}, \mathrm{s.t.} hf \in [\mathcal{A}]\}.$$

The *algebraic saturation ideal* of $\mathcal{A}$ is defined by $\mathrm{asat}(\mathcal{A}) = (\mathcal{A}) : \mathrm{I}_{\mathcal{A}}^{\infty}$, where $\mathrm{I}_{\mathcal{A}}^{\infty}$ is the multiplicative set generated by the initials of polynomials in $\mathcal{A}$. We use capital calligraphic letters such as $\mathcal{A}, \mathcal{B}, \ldots$ to denote auto-reduced sets and use notation $\mathcal{A} = A_1, \ldots, A_t$ to specify the list of the elements of $\mathcal{A}$ arranged by increasing rank.

An auto-reduced set $\mathcal{C}$ contained in a differential polynomial set $\mathcal{S}$ is said to be a *characteristic set* of $\mathcal{S}$ if $\mathcal{S}$ does not contain any nonzero element reduced with respect to $\mathcal{C}$. A characteristic set $\mathcal{C}$ of a differential ideal $\mathcal{J}$ reduces all elements of $\mathcal{J}$ to zero. Furthermore, if $\mathcal{J}$ is prime, then $\mathcal{J} = \mathrm{sat}(\mathcal{C})$.

Definition 3. For an auto-reduced set $\mathcal{A} = A_1, \ldots, A_t$ with $\mathrm{ld}(A_i) = y_{c_i}^{(o_i)}$, the set $\mathbb{Y} \setminus \{y_{c_1}, \ldots, y_{c_t}\}$ is called the *parametric set* of $\mathcal{A}$ and the *order of* $\mathcal{A}$ is defined by $\mathrm{ord}(\mathcal{A}) = \sum_{i=1}^{t} o_i$.

We conclude this section by recalling the definition of differential dimension and order for a prime differential ideal $\mathcal{I}$, which are closely related to characteristic sets of $\mathcal{I}$.

Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ and $\xi = (\xi_1, \ldots, \xi_n)$ a generic point of $\mathcal{I}$. The *differential dimension* of $\mathcal{I}$ or $\mathbb{V}(\mathcal{I})$ is defined as the differential transcendence degree of the differential extension field $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle$ over $\mathcal{F}$, that is, $\dim(\mathcal{I}) = \mathrm{d.tr.deg}\, \mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle/\mathcal{F}$. Suppose $\dim(\mathcal{I}) = d$ and $\{\xi_{i_1}, \ldots, \xi_{i_d}\}$ is a differential transcendence basis. Then $U = \{y_{i_1}, \ldots, y_{i_d}\}$ is called *a parametric set* of $\mathcal{I}$. The *relative order* of $\mathcal{I}$ with respect to U is defined as the transcendence degree of $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle$ over $\mathcal{F}\langle\xi_{i_1}, \ldots, \xi_{i_d}\rangle$, denoted by

$$\mathrm{ord}_U(\mathcal{I}) = \mathrm{tr.deg}\langle\xi_1, \ldots, \xi_n\rangle / \langle\xi_{i_1}, \ldots, \xi_{i_d}\rangle.$$

The differential dimension and relative orders can be read off from characteristic sets.

Lemma 4. *(See [6, Theorem 4.11].) Let $\mathcal{A}$ be a characteristic set of a prime differential ideal $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ endowed with some ranking. The cardinality of the parametric set U of*

$\mathcal{A}$ gives the differential dimension of $\mathcal{I}$. The relative order of $\mathcal{I}$ with respect to U is equal to the order of $\mathcal{A}$.

In [25, Theorem 1], Kolchin proved that there exists a unique numerical polynomial $\omega_{\mathcal{I}}(t)$ such that for all sufficiently large $t \in \mathbb{N}$,

$$\omega_{\mathcal{I}}(t) = \text{tr.deg}\, \mathcal{F}(\eta_i^{(j)} : i = 1, \ldots, n; j \leq t)/\mathcal{F}.$$

This $\omega_{\mathcal{I}}(t)$ is called the *differential dimension polynomial* of $\mathcal{I}$ [25, p. 572].

Definition 5. (See [40, Theorem 13].) The differential dimension polynomial of $\mathcal{I}$ is of the form

$$\omega_{\mathcal{I}}(t) = \dim(\mathcal{I}) \cdot (t + 1) + h.$$

The nonnegative integer h is defined to be the *order of* $\mathcal{I}$, denoted by $\text{ord}(\mathcal{I})$. Or equivalently, if $\mathcal{A}$ is a characteristic set of $\mathcal{I}$ under any orderly ranking, then $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{A})$.

By the relation between the order and relative orders of a prime differential ideal [12, Theorem 2.11] and Lemma 4, we have the following result.

Lemma 6. *Let $\mathcal{I}$ be a prime differential ideal of differential dimension d. Then*

$$\text{ord}(\mathcal{I}) = \max_{\mathcal{A}} \text{ord}(\mathcal{A}),$$

where $\mathcal{A}$ runs over all characteristic sets of $\mathcal{I}$ under arbitrary rankings.

2.3. Differential Chow form for a prime differential ideal

In this section, we recall the definition of the differential Chow form and some of its basic properties. For more details about the differential Chow form, please refer to [12].

Definition 7. A *generic differential hyperplane* is the solution set of a linear differential polynomial equation

$$u_0 + u_1 y_1 + \cdots + u_n y_n = 0$$

contained in $\mathcal{E}^n$ where the coefficients $u_i \in \mathcal{E}$ are differentially independent over $\mathcal{F}$. We also call $\mathbb{P} = u_0 + u_1 y_1 + \cdots + u_n y_n$ a generic differential hyperplane.

Let $\mathcal{I} \subseteq \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal of differential dimension d and

$$\mathbb{P}_i = u_{i0} + u_{i1} y_1 + \cdots + u_{in} y_n \ \ (i = 0, \ldots, d)$$

be $d+1$ generic differential hyperplanes. For each i, denote

$$\mathbf{u}_i = \{u_{i0}, u_{i1}, \dots, u_{in}\} \text{ and } \mathbf{u} = \{u_{ij} : 0 \le i \le d; 1 \le j \le n\}.$$

Let

$$\mathcal{I}_{\mathbb{Y},\mathbf{u}} = [\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d]_{\mathcal{F}\{\mathbb{Y},\mathbf{u}_0,\dots,\mathbf{u}_d\}}$$

be the differential ideal generated by $\mathcal{I}$ and the $\mathbb{P}_i$ in $\mathcal{F}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_d\}$. Then by [12, Lemma 4.3], we have the following result.

Lemma 8. $\mathcal{I}_{\mathbb{Y},\mathbf{u}} \cap \mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$ *is a prime differential ideal of differential codimension one.*

By [12, Lemma 3.10], there exists a unique (up to a factor in $\mathcal{F}$) irreducible differential polynomial $F(\mathbf{u}_0, \dots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$ such that $\{F\}$ is a characteristic set of $\mathcal{I}_{\mathbb{Y},\mathbf{u}} \cap \mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$ under any ranking endowed on $\mathbf{u}_0 \cup \dots \cup \mathbf{u}_d$. That is,

$$[\mathcal{I}, \mathbb{P}_0, \dots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \dots, \mathbf{u}_d\} = \text{sat}(F) \tag{1}$$

is valid for whichever ranking we choose.

Definition 9. The unique irreducible differential polynomial in (1) is defined to be the *differential Chow form* of $\mathcal{I}$.

The following theorem gives some basic properties of differential Chow forms.

Theorem 10. *(See [12, Theorem 1.1].) Let $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal of differential dimension d and order h with $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$ its differential Chow form. Then the following assertions hold:*

1) *The order of the F is equal to the order of $\mathcal{I}$. That is,* $\text{ord}(F) = h$.
2) *$F(\mathbf{u}_0, \dots \mathbf{u}_d)$ is differentially homogeneous of the same degree in each $\mathbf{u}_i$ ($i = 0, \dots, d$). Namely, there exists $r \in \mathbb{N}$ such that for each i and a differential indeterminate λ,*

$$F(\mathbf{u}_0, \dots, \lambda\mathbf{u}_i, \dots, \mathbf{u}_d) = \lambda^r \cdot F(\mathbf{u}_0, \dots, \mathbf{u}_i, \dots, \mathbf{u}_d).$$

3) *Let*

$$\mathcal{C} = F, \frac{\partial F}{\partial u_{00}^{(h)}} y_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \dots, \frac{\partial F}{\partial u_{00}^{(h)}} y_n - \frac{\partial F}{\partial u_{0n}^{(h)}}.$$

Then $\mathcal{C}$ is a characteristic set of $\mathcal{I}_{\mathbb{Y},\mathbf{u}}$ with respect to the elimination ranking $\mathbf{u} < u_{d0} < \dots < u_{00} < y_1 < \dots < y_n$ [12, Lemma 4.10].

4) *Suppose* $F_{\rho\tau}$ *is obtained from* F *by interchanging* $\mathbf{u}_\rho$ *and* $\mathbf{u}_\tau$ *in* F*. Then* $F_{\rho\tau}$ *and* F *differ at most by a sign [12, Lemma 4.9].*
5) *Suppose* ζ *is a generic point of* $\mathrm{sat}(F) \subset \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$*. Let*

$$\eta_i = \frac{\partial F}{\partial u_{0i}^{(h)}} \Big/ \frac{\partial F}{\partial u_{00}^{(h)}} \Big|_{(\mathbf{u}_0,\ldots,\mathbf{u}_d)=\zeta}, \ i = 1, \ldots, n.$$

Then $(\eta_1, \ldots, \eta_n)$ *is a generic point of* $\mathcal{I}$ *[12, Theorem 4.13].*

From property 5) of Theorem 10 and the definition of differential Chow form, we can see that differential Chow forms can uniquely characterize their corresponding differential ideals.

When $d = n$, $\mathcal{I} = [0]$. The differential Chow form of $\mathcal{I}$ is the determinant of the $(n+1) \times (n+1)$ matrix whose $(i-1)$-th row vector is $(u_{i0}, u_{i1}, \ldots, u_{in})$. This is the simplest case to compute the differential Chow form. In the remaining part of the paper, all prime differential ideals in question are assumed to have differential dimension $d < n$.

3. Jacobi bound for the order of a prime differential ideal in terms of characteristic sets

In this section, we prove that the order of a prime differential ideal $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ is bounded by the Jacobi number (defined below) of $\mathcal{A}$, where $\mathcal{A}$ is a characteristic set of $\mathcal{I}$ under an arbitrary ranking, not necessarily an orderly ranking.

Definition 11. (See [8, p. 2, Section 4].) Let $A = (a_{ij})$ be an $r \times n$ matrix. A *diagonal* of A is a sequence of the a_{ij} with no two entries in the same row or in the same column of A, whose cardinality is equal to $\min\{r, n\}$. Denote $\mathrm{Jac}(A) = \max_D \sum_{a_{ij} \in D} a_{ij}$ where D runs over all the diagonals of A.

Let $\mathcal{S} = \{f_1, \ldots, f_r\} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ be a system of differential polynomials. Let $e_{ij} = \mathrm{ord}(f_i, y_j)$ if y_j occurs effectively in f_i and we set $e_{ij} = -\infty$ otherwise. Let E denote the $r \times n$ matrices (e_{ij}). The *Jacobi number* $\mathrm{Jac}(\mathcal{S})$ of $\mathcal{S}$ is defined by $\mathrm{Jac}(\mathcal{S}) = \mathrm{Jac}(E)$.

It is conjectured that the order of each zero-dimensional component of $\mathcal{S}$ is bounded by the Jacobi number of $\mathcal{S}$. This is the famous Jacobi bound conjecture as introduced in Section 1, which is so far open in general. In this section, given a characteristic set $\mathcal{A}$ of a prime differential ideal $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$, we will show that the Jacobi bound conjecture is valid in this case. That is, we will prove the order of $\mathcal{I}$ is bounded by the Jacobi number of $\mathcal{A}$. We should point out that if $\mathscr{R}$ is an orderly ranking, it is trivial because $\mathrm{ord}(\mathcal{I}) = \mathrm{ord}(\mathcal{A}) = \mathrm{Jac}(\mathcal{A})$. However, for an arbitrary ranking $\mathscr{R}$, as explained in Section 1, it is much more complicated. The main tool here is using the result about the Jacobi bound for ordinary differential polynomials independent over a prime differential ideal proved by Kondratieva et al. [29].

Before proving the main result, we first recall Kähler differentials and some results from [29] for later use. For more details on Kähler differentials, please refer to [10] and [22].

Let $\mathcal{F}$ be a field and A an $\mathcal{F}$-algebra. The module of *Kähler differentials* of A over $\mathcal{F}$, written $\Omega_{A/\mathcal{F}}$, is the A-module generated by the set $\{d(a) : a \in A\}$ subject to the relations

$$d(aa') = ad(a') + a'd(a)$$
$$d(r_1a_1 + r_2a_2) = r_1d(a_1) + r_2d(a_2)$$

for all $a_1, a_2 \in A$, and $r_1, r_2 \in \mathcal{F}$.

Theorem 12. *(See [22, p. 94].) Let k be a field of characteristic zero and K a field extension of k. Then the elements $\eta_1, \ldots, \eta_r$ of K are algebraically independent over k if and only if $d(\eta_1), \ldots, d(\eta_r)$ are linearly independent over K.*

Furthermore, if $\mathcal{F}$ is a differential field and $\mathcal{F}\{\mathbb{Y}\}$ is the differential polynomial ring over $\mathcal{F}$ in $\mathbb{Y} = \{y_1, \ldots, y_n\}$ with a derivation operator δ, then the module of Kähler differentials of $\mathcal{F}\{\mathbb{Y}\}$ over $\mathcal{F}$, $\Omega_{\mathcal{F}\{\mathbb{Y}\}/\mathcal{F}}$, has a uniquely canonical structure of differential module over $\mathcal{F}\{\mathbb{Y}\}$ such that for each $f \in \mathcal{F}\{\mathbb{Y}\}$,

$$\delta d(f) = d\delta(f),$$

which was first introduced by Johnson [22, Proposition].

Suppose that $\mathcal{I}$ is a prime ideal of a commutative ring A and M is an A-module. A set $H \subseteq M$ is called *independent* over $\mathcal{I}$ if $\{h + \mathcal{I}M \mid h \in H\}$ is a system of elements of $M/M\mathcal{I}$ linearly independent over the quotient ring $A/\mathcal{I}$.

Definition 13. (See [29, Definition 2].) Let $\mathcal{I}$ be a prime differential ideal of the differential polynomial ring $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}\{y_1, \ldots, y_n\}$. The set $\{f_1, \ldots, f_r\} \subset \mathcal{F}\{\mathbb{Y}\}$ is called *independent* over $\mathcal{I}$ if the set $\{df_i^{(k)} : 1 \leq i \leq r; k \geq 0\} \subset \Omega_{\mathcal{F}\{\mathbb{Y}\}/\mathcal{F}}$ is independent over $\mathcal{I}$.

In [23, Theorem 2], Johnson proved that if a system of n differential polynomials in n differential variables contained in a prime differential ideal $\mathcal{I}$ is independent over $\mathcal{I}$, then the differential dimension of $\mathcal{I}$ is equal to zero. Further, Kondratieva et al. proved that the Jacobi bound conjecture is valid in this case, which will be needed in our proof for the main result.

Lemma 14. *(See [29, Theorem 3].) Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$. Suppose $f_1, \ldots, f_n \in \mathcal{I}$. If $f_1, \ldots, f_n$ are independent over $\mathcal{I}$, then $\operatorname{ord}(\mathcal{I}) \leq \operatorname{Jac}(f_1, \ldots, f_n)$.*

We now proceed to prove the main result. The following lemmas are crucial to prove it.

Lemma 15. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$. Then the set*

$$\{y_i^{(k)} : 1 \leq i \leq n; k \geq 0\}$$

is independent over $\mathcal{I}$.

Proof. Denote $\mathcal{R} = \mathcal{F}\{y_1, \ldots, y_n\}$. Note that $\{y_i^{(k)} : 1 \leq i \leq n; k \geq 0\}$ is algebraically independent over $\mathcal{F}$, so by Theorem 12, $\{d(y_i^{(k)}) : 1 \leq i \leq n; k \geq 0\} \subset \Omega_{\mathcal{R}/\mathcal{F}}$ is linearly independent over $\mathcal{R}$, which is a linear basis of $\Omega_{\mathcal{R}/\mathcal{F}}$. Suppose $\sum_{i=1}^n \sum_{k\geq 0} \bar{a}_{ik}\big(d(y_i^{(k)}) + \mathcal{I}\Omega_{\mathcal{F}\{y_1,\ldots,y_n\}/\mathcal{F}}\big) = 0$ for some $a_{ik} \in \mathcal{R}$ and $\bar{a}_{ik} \in \mathcal{R}/\mathcal{I}$. So there exists $b_{ik} \in \mathcal{I}$ such that $\sum_{i=1}^n \sum_{k\geq 0} a_{ik} d(y_i^{(k)}) = \sum_{i=1}^n \sum_{k\geq 0} b_{ik} d(y_i^{(k)})$ in $\Omega_{\mathcal{R}/\mathcal{F}}$. Thus, $a_{ik} = b_{ik} \in \mathcal{I}$, which implies that $d(y_i^{(k)}) + \mathcal{I}\Omega_{\mathcal{F}\{y_1,\ldots,y_n\}/\mathcal{F}}\, (i = 1, \ldots, n; k \geq 0)$ are linearly independent over $\mathcal{I}$. By Definition 13, $\{y_i^{(k)} : 1 \leq i \leq n; k \geq 0\}$ is independent over $\mathcal{I}$. □

Lemma 16. *Let $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal of differential dimension d, and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set under any fixed ranking $\mathscr{R}$. Let $L_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n$ $(i = 1, \ldots, d)$ be d independent generic differential hyperplanes with coefficient vector $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, and $\mathcal{J} = [\mathcal{I}, L_1, \ldots, L_d]_{\mathcal{F}\langle \mathbf{u}_1,\ldots,\mathbf{u}_d\rangle\{\mathbb{Y}\}}$. Then*

$$A_1, \ldots, A_{n-d}, L_1, \ldots, L_d$$

are independent over $\mathcal{J}$.

Proof. For the sake of convenience, suppose $\mathrm{ld}(A_i) = y_{d+i}^{(o_i)}$ $(i = 1, \ldots, n-d)$ with $A_i < A_j$ $(i < j)$ and the parametric set of $\mathcal{A}$ is $\{y_1, \ldots, y_d\}$. Let $\mathcal{F}_d = \mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d\rangle$. By [12, Theorem 3.6], $\mathcal{J} = [\mathcal{I}, L_1, \ldots, L_d]_{\mathcal{F}_d\{\mathbb{Y}\}}$ is a prime differential ideal of differential dimension 0. By Definition 13, we need to show that the set

$$\Big\{d(A_i^{(l)}) + \mathcal{J}\Omega_{\mathcal{F}_d\{\mathbb{Y}\}/\mathcal{F}_d}, d(L_j^{(l)}) + \mathcal{J}\Omega_{\mathcal{F}_d\{\mathbb{Y}\}/\mathcal{F}_d} : 1 \leq i \leq n-d; 1 \leq j \leq d; l \geq 0\Big\}$$
$$\subset \Omega_{\mathcal{F}_d\{\mathbb{Y}\}/\mathcal{F}_d}/\mathcal{J}\Omega_{\mathcal{F}_d\{\mathbb{Y}\}/\mathcal{F}_d}$$

is linearly independent over $\mathcal{J}$.

Let $o = \max_i\{\mathrm{ord}(A_i)\}$. Then $d(A_i^{(l)}) = \sum_{j=1}^n \sum_{t=0}^{o+l} \frac{\partial A_i^{(l)}}{\partial y_j^{(t)}} d(y_j^{(t)})$ and each $d(L_j^{(l)})$ has a similar expression. By Lemma 15, the $d(y_j^{(t)})$ are linearly independent over $\mathcal{J}$. Thus, it suffices to prove that for each $k \geq 0$, the Jacobi matrix N_k of $\mathcal{S}_k = \{A_1^{[k]}, \ldots, A_{n-d}^{[k]}, L_1^{[k]}, \ldots, L_d^{[k]}\}$ with respect to the variables $\{y_i^{(l)} : i \leq i \leq n; 0 \leq l \leq o+k\}$ has full row rank module $\mathcal{J}$. Note N_k is of size $(n(k+1)) \times (n(o+k+1))$. Let T be the $(n(k+1)) \times (n(k+1))$ submatrix of N_k with columns indexed by variables $y_{d+1}^{(o_1)}, \ldots, y_n^{(o_{n-d})}; \ldots; y_{d+1}^{(o_1+k)}, \ldots, y_n^{(o_{n-d}+k)}; y_1, \ldots, y_d; \ldots; y_1^{(k)}, \ldots, y_d^{(k)}$. Then after interchanging rows or columns when necessary, T can be written in the following block form:

$$T = \begin{pmatrix} M_1 & \mathbf{0} & \cdots & \mathbf{0} & * & * & \cdots & * \\ * & M_1 & \cdots & \mathbf{0} & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & M_1 & * & * & \cdots & * \\ * & * & * & * & M_2 & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & * & * & * & M_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \cdots & M_2 \end{pmatrix},$$

where $M_1 = \begin{pmatrix} \mathrm{S}_{A_1} & 0 & \cdots & 0 \\ * & \mathrm{S}_{A_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \mathrm{S}_{A_{n-d}} \end{pmatrix}$ and $M_2 = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1d} \\ u_{21} & u_{22} & \cdots & u_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ u_{d1} & u_{d2} & \cdots & u_{dd} \end{pmatrix}$. Here, for each $0 \leq l \leq k$, M_1 is the Jacobian matrix of $A_1^{(l)}, \ldots, A_{n-d}^{(l)}$ with respect to $y_{d+1}^{(o_1+l)}, \ldots, y_n^{(o_{n-d}+l)}$, while M_2 is the Jacobian matrix of $L_1^{(l)}, \ldots, L_d^{(l)}$ with respect to $y_1^{(l)}, \ldots, y_d^{(l)}$.

To complete the proof, it is enough to show that T has full rank module $\mathcal{J}$, or equivalently, $\det(T) \notin \mathcal{J}$. Let $\widetilde{\mathbf{u}} = \{u_{ij} : 1 \leq i \leq d; 1 \leq j \leq n\}$. Note that $\det(T) \in \mathcal{F}\{\mathbb{Y}, \widetilde{\mathbf{u}}\}$. Now, we claim that for each $f \in \mathcal{J} \cap \mathcal{F}\{\mathbb{Y}, \widetilde{\mathbf{u}}\}$, if we rewrite f as a differential polynomial in $\widetilde{\mathbf{u}}$ with coefficients in $\mathcal{F}\{\mathbb{Y}\}$, that is, $f = \sum_\phi \phi(\widetilde{\mathbf{u}})g_\phi(\mathbb{Y})$, then for each ϕ, $g_\phi(\mathbb{Y}) \in \mathcal{I}$.

Indeed, let $\mathcal{J}_0 = [\mathcal{I}, L_1, \ldots, L_d]_{\mathcal{F}\{\mathbb{Y},\mathbf{u}_1,\ldots,\mathbf{u}_d\}}$ and $\xi = (\xi_1, \ldots, \xi_n)$ be a generic point of $\mathcal{I}$ free[1] from $\mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d\rangle$. Let $\zeta = (\xi, -\sum_{i=1}^n u_{1i}\xi_i, u_{11}, \ldots, u_{1n}, \ldots, -\sum_{i=1}^n u_{di}\xi_i, u_{d1}, \ldots, u_{dn})$. It is easy to show that ζ is a generic point of $\mathcal{J}_0$ and $\mathcal{J} \bigcap \mathcal{F}\{\widetilde{\mathbf{u}}, \mathbb{Y}\} \subset \mathcal{J}_0$. So $f(\zeta) = 0$, and consequently, for each ϕ, $g_\phi(\xi) = 0$, which implies that $g_\phi(\mathbb{Y}) \in \mathcal{I}$.

Rewrite $\det(T)$ as a differential polynomial in $\widetilde{\mathbf{u}}$ and suppose $\det(T) = \sum_\phi \phi(\widetilde{\mathbf{u}})g_\phi(\mathbb{Y})$, where ϕ runs through all distinct differential monomials in $\widetilde{\mathbf{u}}$. Take $\phi^*(\widetilde{\mathbf{u}}) = (\prod_{i=1}^d u_{ii})^k$. Then its coefficient is $g_{\phi^*}(\mathbb{Y}) = (\prod_{i=1}^{n-d} s_{A_i})^{k+1}$, which is not in $\mathcal{I}$. Thus, by the above claim, $\det(T) \notin \mathcal{J}$. □

Corollary 17. *Let $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal and $\mathcal{A}$ a characteristic set of $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$. Then $\mathcal{A}$ is independent over $\mathcal{I}$.*

Proof. By Lemma 16, $\{d(y_i^{(k)}) : i = 1, \ldots, n; k \geq 0\}$ is a linear basis of the free $\mathcal{F}\{\mathbb{Y}\}/\mathcal{I}$-module $\Omega_{\mathcal{F}\{\mathbb{Y}\}/\mathcal{F}}/\mathcal{I}\Omega_{\mathcal{F}\{\mathbb{Y}\}/\mathcal{F}}$. So $\mathcal{A}$ is independent over $\mathcal{I}$ if and only if for each $s \in \mathbb{N}$, the Jacobian matrix M_s of $\mathcal{A}^{[s]} = \{A^{(k)} : A \in \mathcal{A}, k \leq s\}$ with respect to the variables $\{y_j^{(k)} : 1 \leq j \leq n; k \leq \max_A \mathrm{ord}(A^{(s)})\}$ is of full row rank modulo $\mathcal{I}$. Let $r(s) = (s+1) \cdot |\mathcal{A}|$, the number of rows of M_s. By the proof of Theorem 16, M_s has an

[1] By saying that ξ is free from $\mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d\rangle$, we mean the $\mathbf{u}_i$ are differentially independent over $\mathcal{F}\langle\xi\rangle$. Note that such a point ξ always exists.

$r(s) \times r(s)$ minor equal to $(\prod_{i=1}^{n-d} s_{A_i})^{s+1}$, which does not belong to $\mathcal{I}$. So M_s is of full row rank modulo $\mathcal{I}$, which implies that $\mathcal{A}$ is independent over $\mathcal{I}$. □

Now, we prove Theorem 1, the main result in this section, which shows that the Jacobi bound conjecture holds for prime differential ideals represented by characteristic sets with respect to arbitrary rankings. For convenience, we restate the theorem here.

Theorem 18. *Let $\mathcal{I}$ be a prime differential ideal of differential dimension d in $\mathcal{F}\{\mathbb{Y}\}$, and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$. Then* $\mathrm{ord}(\mathcal{I}) \leq \mathrm{Jac}(\mathcal{A})$.

Proof. By [12, Theorem 3.13], $\mathcal{J} = [\mathcal{I}, L_1, \ldots, L_d]_{\mathcal{F}\langle \mathbf{u}_1,\ldots,\mathbf{u}_d\rangle\{\mathbb{Y}\}}$ is a prime differential ideal with $\mathrm{ord}(\mathcal{J}) = \mathrm{ord}(\mathcal{I})$. By Lemma 16, $A_1, \ldots, A_{n-d}, L_1, \ldots, L_d$ are independent over $\mathcal{J}$. So by Lemma 14, $\mathrm{ord}(\mathcal{J}) \leq \mathrm{Jac}(\mathcal{A}, L_1, \ldots, L_d) = \mathrm{Jac}(\mathcal{A})$. Thus, $\mathrm{ord}(\mathcal{I}) \leq \mathrm{Jac}(\mathcal{A})$ follows. □

As a corollary, we show the conjecture proposed by Golubitsky et al. in [15, p. 337] holds.

Corollary 19. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ and $\mathcal{A}$ a characteristic set of $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$. Let $o_j = \max\{0, \mathrm{ord}(A, y_j) : A \in \mathcal{A}\}$ for $j = 1, \ldots, n$. Suppose $o_{i_1} \geq o_{i_2} \geq \ldots \geq o_{i_n}$. Then* $\mathrm{ord}(\mathcal{I}) \leq \sum_{k=1}^{|\mathcal{A}|} o_{i_k}$.

Proof. Since $\mathrm{Jac}(\mathcal{A}) \leq \sum_{k=1}^{|\mathcal{A}|} o_{i_k}$, it is an immediate consequence of Theorem 18. □

We use the following simple example to show that the Jacobi bound is indeed much better than the bound in Corollary 19.

Example 20. Let $\mathcal{I} = \mathrm{sat}(y_1y_2 + 1, y_1^{(n)}y_3^{(n)} + y_1) \subset \mathcal{F}\{y_1, y_2, y_3\}$ be a prime differential ideal with $\mathcal{A} = \{y_1y_2 + 1, y_1^{(n)}y_3^{(n)} + y_1\}$ a characteristic set of $\mathcal{I}$ with respect to the elimination ranking $y_1 < y_2 < y_3$. By Theorem 18, we get an upper bound for the order of $\mathcal{I}$, that is, $\mathrm{ord}(\mathcal{I}) \leq n = \mathrm{Jac}(\mathcal{A})$. While the order bound in Corollary 19 for this example is $2n$. For $n >> 0$, the Jacobi bound n is much smaller than $2n$.

Remark 1. Theorem 18 shows the Jacobi bound conjecture is true for all prime differential ideals in terms of characteristic sets. But in general, we still cannot prove the Jacobi bound conjecture for prime differential ideals given by generators. More precisely, given a prime differential ideal $\mathcal{I} = \{f_1, \ldots, f_m\}$, we do not know whether $\mathrm{ord}(\mathcal{I}) \leq \mathrm{Jac}(f_1, \ldots, f_m)$. One possible idea to prove it from Theorem 18 is to show that a characteristic set $\mathcal{A}$ of $\mathcal{I}$ can be computed from $f_1, \ldots, f_m$ such that $\mathrm{Jac}(\mathcal{A}) = \mathrm{Jac}(f_1, \ldots, f_m)$.

4. Computation of differential Chow forms for prime differential ideals represented by characteristic sets under orderly rankings

In this section, we give an algorithm to compute the differential Chow form for a prime differential ideal represented by a characteristic set with respect to an orderly ranking. This algorithm is based on linear algebraic techniques and has single-exponential computational complexity.

Given a prime differential ideal $\mathrm{sat}(\mathcal{A})$ with $\mathcal{A}$ a characteristic set under an orderly ranking, by Definition 5 and Theorem 10, the order of its differential Chow form is equal to $\mathrm{ord}(\mathcal{A})$. To give an algorithm and estimate the computational complexity of this algorithm, degree bounds are also needed. So before giving the algorithm, we first give a degree bound for the differential Chow form.

4.1. Degree bound of the differential Chow form in terms of a characteristic set under an orderly ranking

In this section, we will give a degree bound for the differential Chow form of a prime differential ideal $\mathcal{I}$ in terms of the orders and degrees of the polynomials in a characteristic set of $\mathcal{I}$. We first recall several properties about the degrees of ideals in the algebraic case.

Let k be a field and $\bar{k}$ its algebraic closure. Let $\mathcal{I}$ be a prime ideal in $k[x_1,\ldots,x_n]$ with $\dim(\mathcal{I}) = d$ and $V \subset \bar{k}^n$ be the irreducible variety defined by $\mathcal{I}$. The *degree* of $\mathcal{I}$ (resp. V), denoted by $\deg(\mathcal{I})$ (resp. $\deg(V)$), is defined to be the number of solutions of the zero dimensional prime ideal $(\mathcal{I}, L_1,\ldots,L_d)_{k_1[x_1,\ldots,x_n]}$ in the algebraic closure of k_1, where $L_i = u_{i0} + \sum_{j=1}^{n} u_{ij}x_j$ for $i = 1,\ldots,d$ are d generic hyperplanes and $k_1 = k((u_{ij})_{1\le i\le n;0\le j\le n})$ [18,19]. That is,

$$\deg(\mathcal{I}) = |\mathbb{V}(\mathcal{I}, L_1,\ldots,L_d)|.$$

The following result gives a relation between the degree of an ideal and that of its elimination ideal, which has been proved in [33, Theorem 2.1] and is also a consequence of [18, Lemma 2].

Lemma 21. *Let $\mathcal{I}$ be a prime ideal in $k[x_1,\ldots,x_n]$ and $\mathcal{I}_r = \mathcal{I} \cap k[x_1,\ldots,x_r]$ for any $1 \le r \le n$. Then $\deg(\mathcal{I}_r) \le \deg(\mathcal{I})$.*

The notion of degree can be defined for general varieties of $\bar{k}^n$ other than irreducible varieties. Let $V \subset \bar{k}^n$ be a variety and $\{V_1,\ldots,V_l\}$ the set of all irreducible components of V. The degree of V is defined to be the sum of the degrees of V_i, that is, $\deg(V) = \sum_{i=1}^{l} \deg(V_i)$. The following lemma shows how degree behaves under intersections.

Lemma 22. *(See [18, Theorem 1].) Let $V_1, \ldots, V_r$ $(r \geq 2)$ be a finite number of varieties in $\bar{k}^n$. Then $\deg(V_1 \cap \cdots \cap V_r) \leq \prod_{i=1}^r \deg(V_i)$. In particular, if $V = \mathbb{V}(f_1, \ldots, f_m)$ for some $f_i \in k[x_1, \ldots, x_n]$, then $\deg(V) \leq \prod_i \deg(f_i)$.*

We now go back to our differential case and proceed to prove an upper bound for the degree of the differential Chow form.

Lemma 23. *Let $\mathcal{I} \subseteq \mathcal{F}\{\mathbb{Y}\} = \mathcal{F}\{y_1, \ldots, y_n\}$ be a prime differential ideal of differential dimension d with $\{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathcal{I}$ with respect to an orderly ranking and $e_i = \mathrm{ord}(A_i)$, $h = \sum_{i=1}^{n-d} e_i$. Suppose F is the differential Chow form of $\mathcal{I}$. Then*

$$(F) = (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}],$$

where $\mathrm{H} = \prod_{i=1}^{n-d} \mathrm{I}_{A_i} \mathrm{S}_{A_i}$, x_0 is a new indeterminant and $\mathbf{u}_i^{[h]} = \{u_{ij}^{(k)} : 0 \leq j \leq n; k \leq h\}$.

Proof. First, we claim that $\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[h]}] = (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbb{Y}^{[h]}]$. Indeed, let

$$\begin{aligned} J &= (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathrm{H}x_0 - 1) \bigcap \mathcal{F}[\mathbb{Y}^{[h]}] \\ &= \mathrm{asat}(A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}) \bigcap \mathcal{F}[\mathbb{Y}^{[h]}]. \end{aligned}$$

For any $f \in \mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[h]}]$, there exists $l \in \mathbb{N}$ such that $H^l f = \sum_{i=1}^{n-d} \sum_{k_i=0}^{h-e_i} g_{k_i} A_i^{(k_i)} = [(\mathrm{H}x_0 - 1 + 1)/x_0]^l f$, where $g_{k_i} \in \mathcal{F}[\mathbb{Y}^{[h]}]$. So $f \in (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathrm{H}x_0 - 1)$, and consequently $f \in J$. On the other hand, for any $f \in J$, we have $f = \sum_{i=1}^{n-d} \sum_{k_i=0}^{h-e_i} g_{k_i} A_i^{(k_i)} + g(\mathrm{H}x_0 - 1)$, here $g_{k_i}, g \in \mathcal{F}[\mathbb{Y}^{[h]}, x_0]$. Thus if we substitute $x_0 = 1/H$ into this equality, we get $f \in \mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[h]}]$. Hence $\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[h]}] = J$.

Thus, we have

$$\begin{aligned} (F) &= \big(\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[h]}], \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}\big) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] \\ &= \Big((A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbb{Y}^{[h]}], \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}\Big) \bigcap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] \\ &\subseteq (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] \\ &\subseteq [A_1, \ldots, A_{n-d}, \mathbb{P}_0, \ldots, \mathbb{P}_d, \mathrm{H}x_0 - 1] \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] \\ &= [\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] \\ &= (F) \end{aligned}$$

Thus, $(F) = (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]$. □

Now, we give a degree bound for the differential Chow form.

Theorem 24. *Let $\mathcal{I} \subseteq \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal of differential dimension d and $\{A_1, \ldots, A_{n-d}\}$ characteristic set with respect to an orderly ranking. Suppose F is the differential Chow form of $\mathcal{I}$. Let $e_i = \mathrm{ord}(A_i)$, $h = \sum_{i=1}^{n-d} e_i$ and $\deg(A_i) = m_i$, then*

$$\deg(F) \le 2^{(h+1)(d+1)} \prod_{i=1}^{n-d} m_i^{h-e_i+1} \Big(2 \sum_{i=1}^{n-d} (m_i - 1) + 1\Big).$$

In particular, let $m = \max\{m_i\}$, then $\deg(F) \le (n-d)2^{(dh+d+h+2)} m^{(h+1)(n-d)-h}$.

Proof. Set $\mathcal{J} = (A_1^{[h-e_1]}, \ldots, A_{n-d}^{[h-e_{n-d}]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1)$, where $\mathrm{H} = \prod_{i=1}^{n-d} \mathrm{I}_{A_i}\mathrm{S}_{A_i}$. By Lemma 22, we have $\deg(\mathcal{J}) \le (\prod_{i=1}^{n-d} m_i^{h-e_i+1}) 2^{(h+1)(d+1)} \big(2\sum_{i=1}^{n-d}(m_i - 1) + 1\big)$. And by Lemmas 21 and 23, we have $\deg(F) = \deg\big(\mathcal{J} \bigcap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]\big) \le \deg(\mathcal{J})$. Thus,

$$\begin{aligned}\deg(F) &\le 2^{(h+1)(d+1)} \prod_{i=1}^{n-d} m_i^{h-e_i+1} \Big(2 \sum_{i=1}^{n-d} (m_i - 1) + 1)\Big) \\ &\le (n-d)2^{(dh+d+h+2)} m^{(h+1)(n-d)-h}. \qquad \square\end{aligned}$$

4.2. Complexity of differential reductions

In this section, we estimate the complexity of performing differential reductions, which will be used when analyzing the computational complexity of differential Chow forms. Before doing so, we first give the computational complexity of algebraic reductions.

Here, for the algebraic reduction, we use the method of solving linear equations as described in [43, p. 72], which was first introduced in [13]. For convenience, we restate the method here to give a formal definition of algebraic reductions and remainders. Here, we fix an ordering $\mathscr{R}$ among variables: $x_1 < \cdots < x_n < \cdots < x_l$ $(l \ge n)$ and consider the polynomial ring $\mathcal{F}[x_1, \ldots, x_l]$. The *leading variable* of a polynomial $f \in \mathcal{F}[x_1, \ldots, x_l]$ is the greatest variable appearing in f with respect to $\mathscr{R}$, denoted by $\mathrm{lvar}(f)$.

Let

$$f \in \mathcal{F}[x_1, \ldots, x_l], \;\; g \in \mathcal{F}[x_1, \ldots, x_n] \;(m = \deg(g, x_n) > 0).$$

The leading variable of g is x_n. We now define the algebraic remainder of f with respect to g. Suppose $\deg(f, x_n) = t$ and $k = t - n$. Write $f = \sum_{i=0}^{t} f_i x_n^i$ and $g = \sum_{i=0}^{m} g_i x_n^i$ as univariate polynomials in x_n with coefficients $f_i \in \mathcal{F}[x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_l]$ and $g_i \in \mathcal{F}[x_1, \ldots, x_{n-1}]$. Using the method of undetermined coefficients to solve the equation

$$f = q'g + r', \;\; \deg(r', x_n) < m$$

over the field $\mathcal{F}(x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_l)$. Let $q' = \sum_{i=0}^{k} q_i x_n^i$. Then, equating the corresponding coefficients of x_n^l $(l = m, \ldots, t)$ in the both sides of $f = q'g + r'$, we get a system of $k+1$ linear equations in $k+1$ variables $q_0, \ldots, q_k$:

$$\begin{aligned} g_m q_k &= f_t \\ g_{m-1} q_k + g_m q_{k-1} &= f_{t-1} \\ \vdots \quad & \quad \vdots \\ g_{m-k} q_k + g_{m-k+1} q_{k-1} + \cdots + g_m q_0 &= f_m. \end{aligned}$$

Clearly, the coefficient matrix M of the above linear equation system is lower triangular and its determinant is equal to g_m^{k+1}. Using the Cramer's rule, we can compute its unique solution $(q_k, \dots, q_0)$ with $g_m^{k+1} q_i \in \mathcal{F}[x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_l]$. Thus, we obtain a polynomial $q = g_m^{k+1}(\sum_{i=0}^k q_i x_n^i) \in \mathcal{F}[x_1, \dots, x_l]$ such that

$$g_m^{k+1} f = qg + r,$$

where $r = g_m^{k+1} f - qg \in \mathcal{F}[x_1, \dots, \dots, x_l]$ and $\deg(r, x_n) < m$.

The above r is called the *algebraic remainder* of f with respect to g under $\mathscr{R}$, denoted by $r = \text{rem}(f, g)$. For given f and g, this method to compute r is called the *algebraic reduction* of f with respect to g under $\mathscr{R}$. If $l < n$ or $t < m$, set $f = \text{rem}(f, g)$ by convention.

Lemma 25. *(See [43, Lemma 3.3.3].) Let f, g and $\mathscr{R}$ as above. Suppose we have computed $r = \text{rem}(f, g) \in \mathcal{F}[x_1, \dots, x_l]$ and $q \in \mathcal{F}[x_1, \dots, x_l]$ satisfying*

$$(\text{lc}(g, x_n))^{k+1} f = qg + r,$$

where $k = \deg_{x_n}(f) - \deg_{x_n}(g)$ and $\text{lc}(g, x_n)$ is the leading coefficient of g as a univariate polynomial in x_n. Then for $j < n$,

$$\begin{aligned} \deg_{x_j}(q) &\le (k+1)\deg_{x_j}(g) + \deg_{x_j}(f), \\ \deg_{x_j}(r) &\le (k+2)\deg_{x_j}(g) + \deg_{x_j}(f), \end{aligned}$$

and for $j > n$,

$$\deg_{x_j}(q), \deg_{x_j}(r) \le \deg_{x_j}(f).$$

For further discussion, we fix an ordering on $x_1, \dots, x_n$. A sequence of polynomials $A_1, \dots, A_t$ in $\mathcal{F}[x_1, \dots, x_n]$ is said to be a *triangular set*, if 1) $r = 1$ and $A_1 \neq 0$, or 2) $A_1 \notin \mathcal{F}$, $\text{lvar}(A_i) < \text{lvar}(A_j)$ for $1 \le i < j$. The *initial* of A_i is the leading coefficient of A_i as a univariate polynomial in $\text{lvar}(A_i)$, denoted by I_{A_i}. Given a triangular set $\mathcal{A} = A_1, \dots, A_r$ and $f \in \mathcal{F}[x_1, \dots, x_n]$, the *remainder sequence* of f with respect to $\mathcal{A}$ is

$$f_t = f, f_{t-1} = \text{rem}(f_t, A_t), \dots, f_1 = \text{rem}(f_2, A_2), f_0 = \text{rem}(f_1, A_1).$$

And f_0 is called the *the remainder of f with respect to $\mathcal{A}$*, denoted by $\text{rem}(f, \mathcal{A})$.

Based on Lemma 25, we now analyze the computational complexity of reducing a polynomial with respect to a triangular set. The similar result of reduction with respect to an ascending chain can be found in [11, Lemma 5.2].

Lemma 26. *Let $\mathcal{A} = \{A_1, \ldots, A_p\}$ be an triangular set in $\mathcal{F}[x_1, \ldots, x_n]$ with respect to any fixed ranking $\mathcal{R}$. Set $m = \max_i\{\deg(A_i)\}$. Then for any $f \in \mathcal{F}[x_1, \ldots, x_n]$, $\deg(f) = D$, the remainder r of f with respect to $\mathcal{A}$ can be computed with at most $[2(D+1)^n(m+1)^{n(p+1)}]^{2.376}$ $\mathcal{F}$-arithmetic operations and the degree of r is bounded by $(m+1)^p(D+1)$.*

Proof. Suppose $\mathrm{lvar}(A_i) = z_i$ $(i = 1, \ldots, p)$. Let $f_p = f, f_{p-1}, \ldots, f_0 = r$ be the remainder sequence of f with respect to $\mathcal{A}$ satisfying the following equations:

$$(\mathrm{I}_{A_i})^{l_i} f_i = q_i A_i + f_{i-1},$$

where

$$l_i = \deg_{z_i}(f_i) - \deg_{z_i}(A_i) + 1.$$

Then, by Lemma 25, $\deg(f_i)$ and $\deg(q_i)$ satisfy the following relations:

$$\deg(f_p) = \deg(f) = D,$$
$$\deg(f_{i-1}) \leq (m+1)\deg(f_i) + m,$$
$$\deg(q_i) \leq (m+1)\deg(f_i).$$

So for all $0 \leq i \leq p$, $\deg(f_i) \leq (m+1)^{p-i}(D+1) - 1$, and for all $1 \leq j \leq p$, $\deg(q_j) \leq (m+1)^{p-j+1}(D+1)-m-1$. It follows that $\deg(r) \leq (m+1)^p(D+1)-1 < (m+1)^p(D+1)$.

We now analyze the computational complexity of the above process. For i, suppose f_i has been computed (we start from $i = p$). Since we know the degree bounds for q_i and f_{i-1}, we can use the method of undetermined coefficients to compute f_{i-1} from $(\mathrm{I}_{A_i})^{l_i} f_i = q_i A_i + f_{i-1}$. Denote $D_i = (m+1)^{p-i}(D+1)-1$ and $Q_j = (m+1)^{p-j+1}(D+1)-m-1$. More precisely, suppose

$$f_{i-1} = \sum_{\phi} c_{0\phi}\phi, \; q_i = \sum_{\varphi} c_{1\varphi}\varphi,$$

where $c_{0\phi}$ and $c_{1\varphi}$ are coefficients to be determined in $\mathcal{F}$, ϕ runs through all the monomials in $x_1, \ldots, x_n$ with total degree bounded by D_{i-1} and the degree in z_k $(k = i, \ldots, p)$ less than $\deg(A_k, z_k)$, and φ runs through the monomials with total degree bounded by Q_i.

Equating the corresponding coefficients of the polynomials in both sides of the equality $(\mathrm{I}_{A_i})^{l_i} f_i = q_i A_i + f_{i-1}$, we get a system of linear equations over $\mathcal{F}$ in variables $c_{0\phi}, c_{1\varphi}$. Thus, f_{i-1} can be computed by solving this linear equation system consisting of at most

$w_{i1} \leq \binom{(m+1)(D_i+1)+n}{n}$ equations in $w_{i2} < \binom{Q_i+n}{n} + \binom{D_{i-1}+n}{n}$ variables. To solve it, we need at most

$$(\max\{w_{i1}, w_{i2}\})^{\omega} \leq [2(m+1)^n(D_i+1)^n]^{\omega} \leq \{2(m+1)^{n(p-i+1)}(D+1)]^n\}^{\omega}$$

$\mathcal{F}$-arithmetic operations, where ω is the matrix multiplication exponent and the currently best known ω is 2.376 [9].

Thus, $f_0 = \text{rem}(f, \mathcal{A})$ can be computed with at most

$$\sum_{i=1}^{p}(\max\{w_{i1}, w_{i2}\})^{\omega} \leq [2(D+1)^n(m+1)^{n(p+1)}]^{2.376}$$

$\mathcal{F}$-arithmetic operations. $\square$

In the following, we get back to the differential case and discuss the differential reduction of a polynomial with respect to an auto-reduced set. For the sake of convenience, we fix a differential ranking $\mathscr{R}$ endowed on the differential polynomial ring $\mathcal{F}\{y_1, \ldots, y_n\}$.

Let $\mathcal{A} = A_1, \ldots, A_t$ be an auto-reduced set with I_i and S_i the initial and separant of A_i respectively. Let f be an arbitrary differential polynomial. Then there exists an algorithm, called Ritt–Kolchin algorithm of differential reduction [41, Section 6], which reduces f with respect to $\mathcal{A}$ to a differential polynomial r that is reduced with respect to $\mathcal{A}$, satisfying

$$\prod_{i=1}^{t} \mathrm{S}_i^{d_i}\mathrm{I}_i^{e_i} \cdot f \equiv r, \mathrm{mod}\,[\mathcal{A}],$$

where d_i and $e_i\,(i = 1, \ldots, t)$ are nonnegative integers. We call this r the *differential remainder* of f with respect to $\mathcal{A}$, denoted by $\delta\text{-rem}(f, \mathcal{A})$.

It is worth pointing out that the differential remainder of f with respect to $\mathcal{A}$ coincides with the algebraic reminder of f with respect to some algebraic triangular set. More precisely, let $s = \text{ord}(f)$ and $\text{ld}(A_i) = y_i^{(o_i)}$ $(i = 1, \ldots, t)$. We call the polynomial sequence

$$\begin{cases} A_1, A_1^{(1)}, \ldots, A_1^{(s)} \\ \cdots \\ A_p, A_p^{(1)}, \ldots, A_p^{(s)} \end{cases}$$

the *prolongation sequence* of $\mathcal{A}$ with respect to f. Let $\mathscr{R}^{\mathrm{a}}$ be the total ordering of algebraic variables $\{y_i^{(k)} : 1 \leq i \leq n; k \geq 0\}$ induced by the differential ranking $\mathscr{R}$. Note that $\text{lvar}(A_i^{(k)}) = y_i^{(o_i+k)}$ and for $k > 0$, $A_i^{(k)}$ is linear in $y_i^{(o_i+k)}$. Arrange polynomials in the prolongation sequence as

$$\mathcal{A}^{[s]} : B_1 < B_2 < \cdots < B_{p(s+1)}$$

with respect to $\mathscr{R}^{\mathrm{a}}$, that is, $\mathrm{lvar}(B_i) < \mathrm{lvar}(B_j)\,(i < j)$. We can perform algebraic reductions for f with respect to the triangular set $\mathcal{A}^{[s]}$. It is easily seen that $\mathrm{rem}(f, \mathcal{A}^{[s]})$ is differentially reduced with respect to $\mathcal{A}$. Actually, from the procedures described by Sit in [41, p. 28–30], by some tedious manipulations, one can show that the differential remainder of f with respect to $\mathcal{A}$ is simply the same as the algebraic remainder of f with respect to $\mathcal{A}^{[s]}$.[2] So differential remainders can be computed via algebraic reductions and throughout this paper, we always use this method to compute the differential remainders. We now give the complexity of differential reductions.

Theorem 27. *Let $\mathcal{A} = \{A_1, \ldots, A_p\}$ be a differential auto-reduced set in $\mathcal{F}\{y_1, \ldots, y_n\}$ under some fixed ranking $\mathscr{R}$ and $f \in \mathcal{F}\{y_1, \ldots, y_n\}$. Set $h = \mathrm{ord}(f)$, $D = \deg(f)$, $e = \max_i\{\mathrm{ord}(A_i)\}$ and $m = \max_i\{\deg(A_i)\}$. Then the differential remainder of f with respect to $\mathcal{A}$ can be computed with at most*

$$2^{2.376}[(D+1)(m+1)^{p(h+1)+1}]^{2.376n(e+h+1)}$$

$\mathcal{F}$-arithmetic operations and its degree is bounded by $(m+1)^{p(h+1)}(D+1)$.

Proof. Since $\delta\text{-rem}(f, \mathcal{A}) = \mathrm{rem}(f, \mathcal{A}^{[h]})$, by Lemma 26, the differential remainder of f with respect to $\mathcal{A}$ is of degree bounded by $(m+1)^{p(h+1)}(D+1)$. And it can be computed with at most

$$2^{2.376}[(D+1)(m+1)^{p(h+1)+1}]^{2.376n(e+h+1)}$$

$\mathcal{F}$-arithmetic operations. □

4.3. An algorithm to compute the differential Chow form

Let $\mathcal{I} = \mathrm{sat}(\mathcal{A}) \subset \mathcal{F}\{\mathbb{Y}\}$ be a prime differential ideal of differential dimension d and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathcal{I}$ with respect to an orderly ranking $\mathscr{R}$. So $\mathrm{ord}(\mathcal{I}) = \mathrm{ord}(\mathcal{A})$. Let

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \ (i = 0, \ldots, d)$$

be generic differential hyperplanes. Let $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$ be the coefficient vector of $\mathbb{P}_i$ and $\mathbf{u} = \{u_{ij} : 0 \le i \le d; 1 \le j \le n\}$. Let $\mathscr{R}_1$ be the elimination ranking with $\mathbf{u} < \mathbb{Y} < u_{00} < \cdots < u_{d0}$ and $\mathscr{R}_1|_{\mathbb{Y}} = \mathscr{R}$. Note, $\mathrm{ld}(\mathbb{P}_i) = u_{i0}$. By [12, Remark 4.4], $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ is a characteristic set of the prime differential ideal $\mathcal{J} = [\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d]_{\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d, \mathbb{Y}\}}$ with respect to the ranking $\mathscr{R}_1$.

[2] One should note that for particular cases, $\mathcal{A}$ does not need to differentiate s times. For example, if $\mathscr{R}$ is an orderly ranking, we just need to perform algebraic reduction of f with respect to $A_1^{[o_1]}, \ldots, A_p^{[o_p]}$, where $o_i = \max\{0, s - o_i\}$.

Algorithm 1 DChowForm-1($\mathcal{A}$).

Input: A characteristic set $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ of a nonzero prime differential ideal $\mathcal{I}$ under an orderly ranking $\mathscr{R}$

Output: The differential Chow form $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ of $\mathcal{I}$.

1. For $i = 0, \ldots, d$, let $\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n$ and $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$.
2. Set $h = \text{ord}(\mathcal{A})$ and $U = \cup_{i=0}^{d} \mathbf{u}_i^{[h]}$.
3. Set $F = 0$ and $t = 1$.
4. While $F = 0$ do
   4.1. Set F_0 to be a homogeneous GPol of degree t in $\mathbf{v}$.
   4.2. Set $\mathbf{c} = \text{coeff}(F_0, \mathbf{v})$.
   4.3. Compute $F_1 = \delta\text{-rem}(F_0, \{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\})$ under the elimination ranking $\mathscr{R}_1 : \mathbf{u} < \mathbb{Y} < u_{00} < \cdots < u_{d0}$ and $\mathscr{R}_1|_{\mathbb{Y}} = \mathscr{R}$.
   4.4. Set $\mathcal{P} = \text{coeff}(F_1, \Theta(\mathbb{Y}) \cup U)$. Note $\mathcal{P}$ is a system of homogeneous linear equations in $\mathbf{c}$.
   4.5. Solve the linear equation system $\mathcal{P} = 0$.
   4.6. If $\mathbf{c}$ has a non-zero solution, then substitute it into F_0 to get F and return F; else $F = 0$.
   4.7. $t := t + 1$.

/*/ GPol stands for algebraic polynomial and generic algebraic polynomial.
/*/ coeff(F, V) returns the set of coefficients of F as an algebraic polynomial in V.

Suppose F is the differential Chow form of $\mathcal{I}$. Then by the definition of the differential Chow form, F is the unique (up to a factor in $\mathcal{F}$) polynomial of minimal order and minimal degree under this order contained in $\mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$. And by Theorem 10, $\text{ord}(F) = \text{ord}(\mathcal{A})$. Therefore, if F_0 is a homogeneous differential polynomial of the smallest degree among all polynomials in $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ with $\text{ord}(F_0) = \text{ord}(\mathcal{A})$, whose differential remainder with respect to $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ under $\mathscr{R}_1$ is zero, then F_0 must be the differential Chow form of $\mathcal{I}$.

With the above idea, we now give Algorithm DChowForm-1 to compute the differential Chow form of $\text{sat}(\mathcal{A})$.

With the fixed order $h = \text{ord}(\mathcal{A})$, the algorithm works adaptively by searching a nonzero $F \in \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] = \mathcal{F}[u_{ij}^{(k)} : k \leq h]$ with a minimal degree t. We start from $t = 1$. If we cannot find a nonzero F with such a degree, then we repeat the procedure with degree $t + 1$. Theorem 24 guarantees the termination of the algorithm. In this way, we need only to handle problems with the real size and need not go to the upper bound in most cases.

Theorem 28. *Let $\mathcal{I} = \text{sat}(\mathcal{A})$ be a nonzero prime differential ideal of differential dimension d and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathcal{I}$ under some orderly ranking.*

Set $h = \sum_i \operatorname{ord}(A_i)$, $m = \max_i\{\deg(A_i)\}$. *Algorithm* DChowForm-1 *computes the differential Chow form* $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ *of* $\mathcal{I}$ *with at most*

$$\left[(n-d)(m+1)^{2(n+1)(h+1)+3}\right]^{2.376(e+h+1)[n+(d+1)(n+1)]+1}$$

$\mathcal{F}$*-arithmetic operations.*

Proof. The algorithm finds a nonzero differential polynomial $F \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ of the smallest degree satisfying that $\operatorname{ord}(F) = h = \operatorname{ord}(\mathcal{A})$ and the differential remainder of f with respect to $\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d$ under $\mathscr{R}_1$ is zero. The existence of such an F is obvious since

$$[\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] = (\operatorname{Chow}(\mathcal{I})),$$

where $\operatorname{Chow}(\mathcal{I})$ is the differential Chow form of $\mathcal{I}$. So this F is the differential Chow form of $\mathcal{I}$.

We estimate the computational complexity of the algorithm below. In each loop of step 4, the complexity is determined by the number of arithmetic operations needed to perform in step 4.3 and step 4.5. In step 4.3, we need to compute the differential remainder F_1 of F_0 with respect to the characteristic set $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$. By Theorem 27, the degree of F_1 is bounded by $(m+1)^{(n+1)(h+1)}(t+1)$ and F_2 can be computed with at most

$$C_{1t} = 2^{2.376}\left[(t+1)(m+1)^{(n+1)(h+1)+1}\right]^{2.376(e+h+1)[n+(d+1)(n+1)]}$$

$\mathcal{F}$-arithmetic operations.

In step 4.6, we need to solve the linear equation system $\mathcal{P} = 0$ in $\mathbf{c}$. It is easy to see that $\mid \mathbf{c} \mid = \binom{(d+1)(h+1)(n+1)+t-1}{t}$, then $\mathcal{P} = 0$ is a linear equation system with $W_{1,t} = \mid \mathbf{c} \mid$ variables and $W_{2,t} = \binom{\deg(F_1)+n(e+h+1)+(d+1)(h+1)(n+1)}{\deg(F_1)}$ equations. To solve it, we need at most

$$C_{2t} = \max\{W_{1,t}, W_{2,t}\}^{\omega}$$

$\mathcal{F}$-arithmetic operations, where ω is the matrix multiplication exponent and currently, the best known ω is 2.376 [9].

Suppose T is the degree bound of the differential Chow form. The iteration in step 4 may loop from 1 to T in the worst case. Thus, in terms of T, the differential Chow form can be computed with

$$\begin{aligned}
&\sum_{t=1}^{T}(C_{1t} + C_{2t}) \\
&\leq T\Big\{\left[(T+1)(m+1)^{(n+1)(h+1)+1}\right]^{2.376(e+h+1)(nd+2n+d+1)} \\
&\quad + \left(\left[(t+1)(m+1)^{(n+1)(h+1)}\right]^{2.376[n(e+h+1)+(d+1)(h+1)(n+1)]}\right)\Big\}
\end{aligned}$$

$\mathcal{F}$-arithmetic operations in the worst case. Here, to derive the above inequalities, we always assume that $(m+1)^{(n+1)(h+1)}(T+1) > n(e+h+1)+(d+1)(h+1)(n+1)$. Hence, the theorem follows by simply replacing T by the degree bound for F given in Theorem 24. $\square$

We use the following example to illustrate the above algorithm.

Example 29. Let $n=1$ and $\mathcal{A}=\{y'-4y\}$. Clearly, $d=\dim(\mathrm{sat}(\mathcal{A}))=0$. We use this simple example to illustrate Algorithm 1.

Let $\mathbb{P}_0=u_{00}+u_{01}y$, and $\mathbf{u}_0=(u_{00},u_{01})$. In step 2, $h=1$ and $U=(u_{00},u_{01},u_{00}',u_{01}')$.

First, for $t=1$, we execute steps 4.1 to 4.7. Set

$$F_0=c_{01}u_{00}+c_{02}u_{01}+c_{03}u_{00}'+c_{04}u_{01}' \text{ and } \mathbf{c}=(c_{01},c_{02},c_{03},c_{04}).$$

In step 4.3, we get

$$F_1=-(c_{01}+4c_{03})u_{01}y+c_{02}u_{02}-c_{03}u_{01}'y+c_{04}u_{01}'.$$

Then in step 4.4, $\mathcal{P}=\{c_{01}+4c_{03},c_{02},c_{03},c_{04}\}$ and $\mathcal{P}=0$ has a unique solution $\mathbf{c}=(0,0,0,0)$. So $F=0$ and in step 4.8, we get $t=2$.

Next, we execute steps 4.1 to 4.7 for $t=2$. Set

$$\begin{aligned}F_0&=c_{01}u_{00}^2+c_{02}u_{00}u_{01}+c_{03}u_{00}u_{00}'+c_{04}u_{00}u_{01}'+c_{05}u_{01}^2+c_{06}u_{01}u_{00}'+c_{07}u_{01}u_{01}'\\&\quad+c_{08}u_{00}'^2+c_{09}u_{00}'u_{01}'+c_{10}u_{01}'^2,\end{aligned}$$

and $\mathbf{c}=(c_{01},\ldots,c_{10})$. In step 4.3, we obtain

$$\begin{aligned}F_1&=(c_{01}+4c_{03}+16c_{08})u_{01}^2y^2+(c_{03}+8c_{08})u_{01}u_{01}'y^2+c_{08}u_{01}'2y^2-(c_{02}+4c_{06})u_{01}^2y\\&\quad-(c_{04}+c_{06}+4c_{09})u_{01}u_{01}'y-c_{09}u_{01}'2y+c_{05}u_{01}^2+c_{07}u_{01}u_{01}'+c_{10}u_{01}'^2.\end{aligned}$$

So in step 4.4, $\mathcal{P}=0$ consists of equations

$$\begin{cases} c_{01}+4c_{03}+16c_{08} &= 0\\ c_{04}+c_{06}+4c_{09} &= 0\\ c_{03}+8c_{08} &= 0\\ c_{02}+4c_{06} &= 0\\ c_{05}=c_{07}=c_{08}=c_{09}=c_{10} &= 0\end{cases}$$

Hence $\mathbf{c}=(0,4q,0,q,0,-q,0,0,0,0)$ where $q\in\mathbb{Q}\backslash\{0\}$. Substitute $\mathbf{c}$ into F_0, then we get

$$F=4u_{00}u_{01}+u_{00}u_{01}'-u_{01}u_{00}'.$$

Therefore, this algorithm returns $F=4u_{00}u_{01}+u_{00}u_{01}'-u_{01}u_{00}'$, which is exactly the differential Chow form of $\mathcal{I}=\mathrm{sat}(\mathcal{A})$.

5. Computation of differential Chow forms for differential ideals represented by characteristic sets under arbitrary differential rankings

In Section 4, we give an algorithm to compute the differential Chow form of a prime differential ideal given by a characteristic set under some orderly ranking $\mathscr{R}$. In this section, we will consider the more general case when $\mathscr{R}$ is an arbitrary ranking. More precisely, for a prime differential ideal $\mathcal{I} = \mathrm{sat}(\mathcal{A})$, where $\mathcal{A}$ is a characteristic set of $\mathcal{I}$ under an arbitrary ranking, we give algorithms to compute the differential Chow form based on the order bound given in Section 3 and the degree bound to be given later. Here, we give two different algorithms according to different searching strategies by giving order and degree distinct priorities.

5.1. An algorithm for computing the differential Chow form: order priority

Let $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ be a prime differential ideal of differential dimension d and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a given characteristic set of $\mathcal{I}$ under an arbitrary fixed ranking $\mathscr{R}$. In this section, we will give an algorithm to compute the differential Chow form F of $\mathcal{I}$ based on linear algebraic techniques.

To give the algorithm, we first need to give a degree bound for the differential Chow form of a prime differential ideal in terms a characteristic set under an arbitrary ranking. The method used here is similar to that in Section 4.3.

Theorem 30. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ of differential dimension d. Let $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ be a characteristic set of $\mathcal{I}$ under an arbitrary ranking. Suppose F is the differential Chow form of $\mathcal{I}$. Set $\deg(A_i) = m_i$. Then*

$$\deg(F) \le 2^{(\mathrm{Jac}(\mathcal{A})+1)(d+1)}\big(2\sum_{i=1}^{n-d}(m_i-1)+1\big)\prod_{i=1}^{n-d} m_i^{\mathrm{Jac}(\mathcal{A})+1}.$$

Proof. Suppose $\mathrm{ord}(\mathcal{I}) = h$. Let $H = \prod_{i=1}^{n-d} \mathrm{I}_{A_i}\mathrm{S}_{A_i}$. We first claim that

$$(F) = (A_1^{[h]}, \ldots, A_{n-d}^{[h]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}],$$

where x_0 is a new indeterminant. Indeed, by the discussion above Theorem 27, for each polynomial $f \in \mathcal{F}[\mathbb{Y}^{[h]}]$, the differential remainder of f with respect to $\mathcal{A}$ is equal to the algebraic remainder of f with respect to $\{A_1^{[h]}, \ldots, A_{n-d}^{[h]}\}$. So similarly to the proof of Lemma 22, it is easy to show that $\mathcal{I}\cap\mathcal{F}[\mathbb{Y}^{[h]}] = (A_1^{[h]}, \ldots, A_{n-d}^{[h]}, \mathrm{H}x_0 - 1)\cap\mathcal{F}[\mathbb{Y}^{[h]}]$. Then

$$\begin{aligned}(F) &= \big(\mathcal{I}\cap\mathcal{F}[\mathbb{Y}^{[h]}], \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}\big)\cap\mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]\\ &= \big((A_1^{[h]}, \ldots, A_{n-d}^{[h]}, \mathrm{H}x_0 - 1)\cap\mathcal{F}[\mathbb{Y}^{[h]}], \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}\big)\cap\mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]\end{aligned}$$

$$\subseteq (A_1^{[h]}, \ldots, A_{n-d}^{[h]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]$$
$$\subseteq [A_1, \ldots, A_{n-d}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] = (F),$$

which proves the claim.

Let

$$\mathcal{J} = (A_1^{[h]}, \ldots, A_{n-d}^{[h]}, \mathbb{P}_0^{[h]}, \ldots, \mathbb{P}_d^{[h]}, \mathrm{H}x_0 - 1) \subset \mathcal{F}[\mathbb{Y}^{[h]}, \mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}, x_0].$$

Then by Lemma 22, we have $\deg(\mathcal{J}) \leq \prod_{i=1}^{n-d} m_i^{h+1} 2^{(h+1)(d+1)}\big(2\sum_{i=1}^{n-d}(m_i - 1) + 1\big)$. From Lemma 21, we get $\deg(F) = \deg\big(\mathcal{J} \bigcap \mathcal{F}(\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]})\big) \leq \deg(\mathcal{J})$. Thus

$$\deg(F) \leq 2^{(h+1)(d+1)}\big(2\sum_{i=1}^{n-d}(m_i - 1) + 1\big)\prod_{i=1}^{n-d} m_i^{h+1}.$$

By Theorem 18, $h \leq \mathrm{Jac}(\mathcal{A})$. Thus, the result follows. □

Now we give Algorithm DChowform-2 to compute the differential Chow form F of $\mathcal{I}$ where the algorithm works adaptively by searching F with order h from $\mathrm{ord}(\mathcal{A})$ to $\mathrm{Jac}(\mathcal{A})$. Indeed, by Lemma 6, $\mathrm{ord}(\mathcal{I}) \geq \mathrm{ord}(\mathcal{A})$, and thus $\mathrm{ord}(F) \geq \mathrm{ord}(\mathcal{A})$, that is why we start from $h = \mathrm{ord}(\mathcal{A})$. For a fixed order h, we search F from $t = 1$. If we cannot find F with such a degree, then we repeat the procedure with $t + 1$ until $t > \prod_{i=1}^{n-d} \deg(A_i)^{h+1} 2^{(h+1)(d+1)}\big(2\sum_{i=1}^{n-d}(\deg(A_i) - 1) + 1\big)$. If for this h, a nonzero F cannot be found, then we repeat the procedure with $h + 1$. In this way, we need only to handle problems with the real size and need not go to the upper bound in most cases. Note that the order bound given in Theorem 18 and the degree bound given in Theorem 30 guarantee the termination of Algorithm 2.

Theorem 31. *Let $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ be a prime differential ideal of differential dimension d and $\mathcal{A} = \{A_1, \ldots, \mathcal{A}_{n-d}\}$ a differential characteristic set under an arbitrary ranking. Set $m_i = \deg(A_i)$, $m = \max\{m_i\}$, $e_i = \mathrm{ord}(A_i)$, and $e = \max\{e_i\}$. Algorithm 2 computes the differential Chow form F of $\mathcal{I}$ with*

$$(\mathrm{Jac}(\mathcal{A}) + 1)\big[(n - d)(m + 1)^{(2n-d+1)(\mathrm{Jac}(\mathcal{A})+1)+3}\big]^{2.376n_1+1}$$

$\mathcal{F}$-arithmetic operations in the worst cases, where $n_1 = \big(e+\mathrm{Jac}(\mathcal{A})+1\big)\big(n+(d+1)(n+1)\big)$.

Proof. Algorithm 2 computes a nonzero differential polynomial with minimal order and minimal degree under this order contained in the differential ideal $[\mathrm{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$, which is exactly the differential Chow form of $\mathrm{sat}(\mathcal{A})$. Theorems 18 and 30 guarantees the termination of the algorithm. So it remains to estimate its computational complexity.

Algorithm 2 DChowform-2($\mathcal{A}$).

Input: A characteristic set $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ of a nonzero prime differential ideal $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$

Output: The differential Chow form $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ of $\mathcal{I}$.

1. For $i = 0, \ldots, d$, let $\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n$ and $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$.
2. Set $h = \mathrm{ord}(\mathcal{A})$.
3. Set $F = 0$.
4. While $F = 0$ do
   4.1. Set $t = 1$, $\mathbf{v} = \cup_{i=0}^{d} \mathbf{u}_i^{[h]}$.
   4.2. While $t \le 2^{(h+1)(d+1)}\big(2\sum_{i=1}^{n-d}(\deg(A_i) - 1) + 1\big)\prod_{i=1}^{n-d}\deg(A_i)^{h+1}$ do
      4.2.1. Set F_0 to be a homogeneous GPol of degree t in $\mathbf{v}$.
      4.2.2. Set $\mathbf{c} = \mathrm{coeff}(F_0, \mathbf{v})$.
      4.2.3. Substitute $u_{i0} = -u_{i1}y_1 - \cdots - u_{in}y_n$ $(i = 0, \ldots, d)$ into F_0 to get F_1.
      4.2.4. Compute $F_2 = \delta\text{-rem}(F_1, \mathcal{A})$ under ranking $\mathscr{R}$.
      4.2.5. Set $\mathcal{P} = \mathrm{coeff}(F_2, \Theta(\mathbb{Y}) \cup \mathbf{v})$. Note $\mathcal{P}$ is a set of linear homogeneous polynomials in $\mathbf{c}$.
      4.2.6. Solve the linear equation system $\mathcal{P} = 0$.
      4.2.7. If $\mathbf{c}$ has a non-zero solution, then substitute it into F_0 to get F and return F; else $F = 0$.
      4.2.8. $t := t + 1$.
   4.3. $h := h + 1$.

/*/ Pol and GPol stand for algebraic polynomial and generic algebraic polynomial.
/*/ coeff(F, V) returns the set of coefficients of F as an algebraic polynomial in V.

Clearly, it is enough to estimate the complexity of steps 4.2.4 and 4.2.6. Similarly as in the proof of Theorem 28, for fixed h and t, step 4.2.4 can be done with at most

$$T_1^{(h,t)} = 2^{2.376}\big[(m+1)^{(n-d)(h+1)+1}(2t+1)\big]^{2.376(e+h+1)[n+(d+1)(n+1)]}$$

$\mathcal{F}$-arithmetic operations, while in step 4.2.6, we needs at most

$$T_2^{(h,t)} = [(m+1)^{(n-d)(h+1)}(2t+1)]^{2.376(e+h+1)[n+(d+1)(n+1)]}$$

$\mathcal{F}$-arithmetic operations.

From Theorem 18, step 4 may loop from $\mathrm{ord}(\mathcal{A})$ to $\mathrm{Jac}(\mathcal{A})$, and for each fixed h, step 4.2 may loop from 1 to $D(h) = 2^{(h+1)(d+1)}\big(2\sum_{i=1}^{n-d}(m_i - 1) + 1\big)\prod_{i=1}^{n-d} m_i^{h+1}$. Thus, the differential Chow form can be computed with at most

$$\sum_{h=\mathrm{ord}(\mathcal{A})}^{\mathrm{Jac}(\mathcal{A})}\sum_{t=1}^{D(h)}(T_1^{(h,t)} + T_2^{(h,t)})$$
$$\le \ (\mathrm{Jac}(\mathcal{A}) + 1)\big[(n-d)(m+1)^{(2n-d+1)(\mathrm{Jac}(\mathcal{A})+1)+3}\big]^{2.376n_1+1}$$

$\mathcal{F}$-arithmetic operations, where $n_1 = \big(e+\mathrm{Jac}(\mathcal{A})+1\big)\big(n+(d+1)(n+1)\big)$ is the cardinality of the set $\{y_i^{(k)}, u_{jl}^{(k)} : 1 \le i \le n; k \le e + \mathrm{Jac}(\mathcal{A}); 0 \le j \le d; 0 \le l \le n\}$. Here, to derive the above inequalities, $D(\mathrm{Jac}(\mathcal{A})) > n_1$ is assumed. □

We use the following example to illustrate the above algorithm.

Example 32. Let $n = 2$, $\mathcal{A} = \{y_2 - y_1'\}$ and $\mathscr{R}$ is the elimination ranking $y_1 < y_2$. Clearly, $d = \dim(\mathrm{sat}(\mathcal{A})) = 1$. We use this simple example to illustrate Algorithm 2.

In step 1, we set $\mathbb{P}_0 = u_{00}+u_{01}y_1+u_{02}y_2$, $\mathbb{P}_1 = u_{10}+u_{11}y_1+u_{12}y_2$, $\mathbf{u}_0 = (u_{00}, u_{01}, u_{02})$ and $\mathbf{u}_1 = (u_{10}, u_{11}, u_{12})$. In step 2, $h = \mathrm{ord}(\mathcal{A}) = 0$.

For $h = 0$, we execute steps 4.1 and 4.2. In step 4.1, $\mathbf{v} = (u_{00}, u_{01}, u_{02}, u_{10}, u_{11}, u_{12})$. The degree bound in step 4.2 is $t \le 4$. We first execute steps 4.2.1 to 4.2.6 for $t = 1$. Set

$$F_0 = c_{01}u_{00} + c_{02}u_{01} + c_{03}u_{02} + c_{04}u_{10} + c_{05}u_{11} + c_{06}u_{12}, \text{and } \mathbf{c} = (c_{01}, \dots, c_{06}).$$

In steps 4.2.3 and 4.2.4, we get the following polynomials:

$$F_1 = -c_{01}u_{11}y_1 - c_{01}u_{12}y_2 + c_{02}u_{01} + c_{03}u_{02} - c_{04}u_{11}y_1 - c_{04}u_{12}y_2 + c_{05}u_{11} + c_{06}u_{12},$$
$$F_2 = -c_{01}u_{11}y_1 - c_{01}u_{12}y_1' + c_{02}u_{01} + c_{03}u_{02} - c_{04}u_{11}y_1 - c_{04}u_{11}y_1' + c_{05}u_{11} + c_{06}u_{12}.$$

So $\mathcal{P} = 0$ consists of equations $\{c_{01} = c_{02} = c_{03} = c_{04} = c_{05} = c_{06} = 0\}$ and $\mathcal{P} = 0$ has a unique solution $\mathbf{c} = (0, 0, 0, 0, 0, 0)$. So $F = 0$ and in step 4.2.8, $t = 2$.

Next we execute steps 4.2.1 to 4.2.6 for $t = 2$. In the following, to save space, we will just list the number of equations and the solutions of the linear equation system $\mathcal{P} = 0$, which are easily computed by Maple due to the strong sparsity of the system. For $t = 2$, $\mathcal{P} = 0$ is a system of 34 homogeneous linear equations which has a unique solution $\mathbf{c} = (c_{01}, \dots, c_{21}) = (0, \dots, 0)$. And for $t = 3$, we get 104 linear homogeneous polynomials in $\mathcal{P}$ and $\mathcal{P} = 0$ has a unique solution $\mathbf{c} = (c_{01}, \dots, c_{56}) = (0, \dots, 0)$. For $t = 4$, $\mathcal{P} = 0$ is a system of 259 linear homogeneous equations in 126 variables, which only has a zero solution $\mathbf{c} = (0, \dots, 0)$. Thus, $F = 0$. Now, in step 4.2, $t = 5 > 4$. So we go on to step 4.3 and obtain $h = 1$.

Since $F = 0$, we execute steps 4.1 and 4.2 for $h = 1$. In step 4.1, set $t = 1$ and $\mathbf{v} = (u_{00}, u_{01}, u_{02}, u_{10}, u_{11}, u_{12}, u_{00}', u_{01}', u_{02}', u_{10}', u_{11}', u_{12}')$. Now, we execute step 4.2 until $t > 8$ or $F \ne 0$. For $t = 1$, $\mathcal{P} = 0$ is the system

$$\{c_{01} = c_{02} = c_{03} = c_{04} = c_{05} = c_{06} = c_{07} = c_{08} = c_{09} = c_{10} = c_{11} = c_{12} = 0\},$$

which has a unique solution $\mathbf{c} = (c_{01}, \dots, c_{12}) = (0, \dots, 0)$. For $t = 2$, $\mathcal{P} = 0$ is a system of 186 linear equations which has a unique solution $\mathbf{c} = (c_{01}, \dots, c_{78}) = (0, \dots, 0)$. For $t = 3$, we get 1122 linear homogeneous polynomials in $\mathcal{P}$ and $\mathcal{P} = 0$ has a unique solution $\mathbf{c} = (c_{01}, \dots, c_{364}) = (0, \dots, 0)$. And for $t = 4$, in step 4.2.5, we get 5082 linear homogeneous polynomials in $\mathcal{P}$, and in step 4.2.6, $\mathcal{P} = 0$ has a nonzero solution $\mathbf{c} = (c_{01}, \dots, c_{1365})$ with

$$c_{164} = c_{171} = c_{283} = c_{388} = c_{462} = c_{506} = c_{668} = c_{760} = q,$$
$$c_{110} = c_{177} = c_{256} = c_{442} = c_{449} = c_{568} = c_{675} = c_{725} = -q,$$

and all the remaining c equal to 0, where $q \in \mathbb{Q}\backslash\{0\}$. Therefore, this algorithm returns

$$\begin{aligned}F = {} & u_{00}u_{01}u_{11}u_{12} - u_{00}u_{02}u_{11}^2 + u_{01}u_{02}u_{10}u_{11} - u_{01}^2u_{10}u_{12} + u'_{00}u_{02}u_{11}u_{12} - u'_{00}u_{01}u_{12}^2 \\ & + u_{00}u_{02}u_{11}u'_{12} - u_{01}u_{02}u_{10}u'_{12} + u_{01}u_{02}u'_{10}u_{12} - u_{02}^2u'_{10}u_{11} + u_{01}u'_{02}u_{10}u_{12} \\ & - u_{00}u_{02'}u_{11}u_{12} - u_{00}u_{02}u'_{11}u_{12} + u_{02}^2u_{10}u'_{11} - u'_{01}u_{02}u_{10}u_{12} + u_{00}u'_{01}u_{12}^2,\end{aligned}$$

which is the differential Chow form of $\mathcal{I} = \text{sat}(\mathcal{A})$.

5.2. An alternative algorithm for computing the differential Chow form: degree priority

Algorithm 2 searches the differential Chow form with the order prior to the degree. In other words, the output of Algorithm 2 is a nonzero polynomial in $[\text{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ with minimal order and minimal degree under this order. Thus, by the definition of differential Chow form, it must be the differential Chow form.

In this section, we give an alternative algorithm for computing the differential Chow form of $\text{sat}(\mathcal{A})$ which uses the searching strategy prioritizing degree over order. To be more precise, this algorithm works adaptively by searching F from degree $t = 1$ and for this fixed t searching it with order h from $\text{ord}(\mathcal{A})$ to the order bound $\text{Jac}(\mathcal{A})$. If a nonzero F with degree t is not found, then we repeat the procedures with degree $t+1$. If we find such a nonzero F, it requires to check whether F is the differential Chow form. Here, we using the following criteria.

Theorem 33. *Let $\mathcal{I} = \text{sat}(\mathcal{A})$ be a prime differential ideal of differential dimension d and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set under an arbitrary ranking $\mathscr{R}$. Suppose $f \in \mathcal{F}\{\mathbf{u}_0, \cdots, \mathbf{u}_d\}$ is an irreducible differentially homogeneous polynomial and $h = \text{ord}(f)$. Let $\mathscr{R}_1$ be the elimination ranking $\mathbf{u} < \mathbb{Y} < u_{00} < \cdots < u_{d0}$ with $\mathscr{R}_1|_{\mathbb{Y}} = \mathscr{R}$. Let $\mathscr{R}_2$ be the elimination ranking $\mathbf{u} < u_{d0} < \cdots < u_{00} < y_1 < \cdots < y_n$. Let $\mathcal{C}_f = \{f, \frac{\partial f}{\partial u_{00}^{(h)}}y_1 - \frac{\partial f}{\partial u_{01}^{(h)}}, \frac{\partial f}{\partial u_{00}^{(h)}}y_2 - \frac{\partial f}{\partial u_{01}^{(h)}}, \ldots, \frac{\partial f}{\partial u_{00}^{(h)}}y_n - \frac{\partial f}{\partial u_{0n}^{(h)}}\}$.*

Then f is the differential Chow form of $\mathcal{I}$ if and only if f satisfies the following conditions:

1) *The differential remainder of f with respect to $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ under the ranking $\mathscr{R}_1$ is zero.*
2) *The differential remainder of each element in the set $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ with respect to $\mathcal{C}_f$ under the ranking $\mathscr{R}_2$ is zero; while the differential remainder of $\text{I}_\mathcal{A}\text{S}_\mathcal{A}$ with respect to $\mathcal{C}_f$ is nonzero. Here $\text{I}_\mathcal{A}\text{S}_\mathcal{A}$ is the product of the initials and separants of elements in $\mathcal{A}$ under $\mathscr{R}$.*

Proof. If f is the differential Chow form of $\mathcal{I}$, then by Theorem 10, 1) and 2) are valid.

For the other direction, assume f satisfies 1) and 2). We will show that f is the differential Chow form of $\mathcal{I}$. Let

$$\mathcal{J} = [\mathrm{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d] \subset \mathcal{F}\{\mathbb{Y}, \mathbf{u}, u_{00}, \ldots, u_{d0}\}.$$

We first claim that $\mathcal{C}_f \subseteq \mathcal{J}$. Since $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ is a characteristic set of $\mathcal{J}$ under the ranking $\mathscr{R}_1$ [12, Remark 4.4], by 1), $f \in \mathcal{J}$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a generic point of $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ over $\mathcal{F}$ that is free from $\mathcal{F}\langle\mathbf{u}\rangle$. Set $\eta_j = -\sum_{i=1}^{n} u_{ji}\xi_i$, then $(\xi_1, \ldots, \xi_n, \eta_0, \ldots, \eta_d)$ is a generic point of $\mathcal{J}$. So $f(\mathbf{u}, \eta_0, \ldots, \eta_d) = 0$. Take the partial derivatives of the both sides of $f(\mathbf{u}, \eta_0, \ldots, \eta_d) = 0$ with respect to $u_{0\rho}^{(h)} (\rho = 1, \ldots, n)$, then we have

$$\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} - \xi_\rho \overline{\frac{\partial f}{\partial u_{00}^{(h)}}} = 0,$$

where $\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}$ and $\overline{\frac{\partial f}{\partial u_{00}^{(h)}}}$ are obtained by replacing $u_{00}, \ldots, u_{d0}$ with $\eta_0, \ldots, \eta_d$ in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ and $\frac{\partial f}{\partial u_{00}^{(h)}}$ respectively. So $\frac{\partial f}{\partial u_{00}^{(h)}} y_\rho - \frac{\partial f}{\partial u_{0\rho}^{(h)}} \in \mathcal{J}$. Thus, we have proved that $\mathcal{C}_f \subseteq \mathcal{J}$.

Obviously, $\mathcal{C}_f$ is an irreducible auto-reduced set with respect to the elimination ranking $\mathscr{R}_2$. Thus, $\mathrm{sat}(\mathcal{C}_f)$ is a prime differential ideal with $\mathcal{C}_f$ a characteristic set [39, p. 107]. We now show that $\mathrm{sat}(\mathcal{C}_f) = \mathcal{J}$. For any $g \in \mathcal{J} = [\mathrm{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d)]$, we have $(\mathrm{I}_\mathcal{A}\mathrm{S}_\mathcal{A})^t g \in [\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d]$ for some $t \in \mathbb{N}$. Since the differential remainder of each $\mathbb{P}_i$ and each element in $\mathcal{A}$ with respect to $\mathcal{C}_f$ is zero, $\mathbb{P}_i \in \mathrm{sat}(\mathcal{C}_f)$ and $\mathcal{A} \subset \mathrm{sat}(\mathcal{C}_f)$. Thus, $(\mathrm{I}_\mathcal{A}\mathrm{S}_\mathcal{A})^t g \in \mathrm{sat}(\mathcal{C}_f)$. Since $\mathrm{sat}(\mathcal{C}_f)$ is a prime differential ideal, and the differential remainder of $\mathrm{I}_\mathcal{A}\mathrm{S}_\mathcal{A}$ with respect to $\mathcal{C}_f$ is nonzero, we have $g \in \mathrm{sat}(\mathcal{C}_f)$ and it follows that $\mathcal{J} \subseteq \mathrm{sat}(\mathcal{C}_f)$. On the other hand, since f is irreducible, $\frac{\partial f}{\partial u_{00}^{(h)}} \notin \mathcal{J}$. For, if not then we have the relation $\frac{\partial f}{\partial u_{00}^{(h)}} \in \mathcal{J} \subseteq \mathrm{sat}(\mathcal{C}_f)$, and therefore $\frac{\partial f}{\partial u_{00}^{(h)}}$ would be divisible by f, a contradiction. So $\mathrm{sat}(\mathcal{C}) = \big([\mathcal{C}_f] : (\frac{\partial f}{\partial u_{00}^{(h)}})^\infty\big) \subseteq \mathcal{J}$. Thus, $\mathrm{sat}(\mathcal{C}_f) = \mathcal{J}$ and $\mathcal{C}_f$ is a characteristic set of $\mathcal{J}$, and consequently f is a characteristic set of $\mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\} = \mathrm{sat}(F)$ under the elimination ranking $\mathbf{u} < u_{d0} < \cdots < u_{00}$, where F is the differential Chow form of $\mathcal{I}$. Since both f and F are irreducible, f and F differ at most by a factor in $\mathcal{F}$. Thus, f is the differential Chow form of $\mathcal{I}$. □

With the above preparations, we now give Algorithm 3.

Theorem 34. *Let $\mathcal{I} = \mathrm{sat}(\mathcal{A})$ be a prime differential ideal of differential dimension d with $\mathcal{A} = A_1, \ldots, A_{n-d}$ a characteristic set under an arbitrary ranking. Set $m_i = \deg(A_i)$, $m = \max_i\{m_i\}$, $e_i = \mathrm{ord}(A_i)$, and $e = \max_i\{e_i\}$. Algorithm 3 computes the differential Chow form of $\mathcal{I}$ with*

$$(\mathrm{Jac}(\mathcal{A}) + n + 3)\big[(n-d)(m+1)^{(2n-d+1)(\mathrm{Jac}(\mathcal{A})+1)+3}\big]^{2.376 n_1 (ne+n+e+2)}$$

$\mathcal{F}$-arithmetic operations in the worst cases, where $n_1 = (e + \mathrm{Jac}(\mathcal{A}) + 1)[(d+1)(n+1)+n]$.

Algorithm 3 DChowform($\mathcal{A}$).

Input: A characteristic set $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ of a nonzero prime differential ideal $\mathcal{I}$ under an arbitrary ranking $\mathscr{R}$.

Output: The differential Chow form $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ of $\mathcal{I}$.

1. For $i = 0, \ldots, d$, let $\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n$ and $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$.
2. Set $\widehat{h} = \mathrm{Jac}(\mathcal{A})$.
3. Set $F = 0$ and $t = 1$.
4. While $t \le \prod_{i=1}^{n-d} \deg(A_i)^{\widehat{h}+1} 2^{(\widehat{h}+1)(d+1)} \big(2\sum_{i=1}^{n-d}(\deg(A_i) - 1) + 1\big)$ do
    4.1. Set $h = \mathrm{ord}(\mathcal{A})$.
    4.2. While $h \le \widehat{h}$ do
        4.2.1. Set F_0 to be a homogeneous GPol of degree t in $\mathbf{v} = \cup_{i=0}^{d} \mathbf{u}_i^{[h]}$.
        4.2.2. Set $\mathbf{c} = \mathrm{coeff}(F_0, \mathbf{v})$.
        4.2.3. Substitute $u_{i0}^{(k)} = -(u_{i1}y_1 + \cdots + u_{in}y_n)^{(k)}$ $(i = 0, \ldots, d; 0 \le k \le h)$ into F_0 to get F_1.
        4.2.4. Compute $F_2 = \delta\text{-rem}(F_1, \mathcal{A})$ under ranking $\mathscr{R}$.
        4.2.5. Set $\mathcal{P} = \mathrm{coeff}(F_2, \Theta(\mathbb{Y}) \cup \mathbf{v})$. Note $\mathcal{P}$ is a set of linear polynomials in $\mathbf{c}$.
        4.2.6. Solve the linear homogeneous equation system $\mathcal{P} = 0$.
        4.2.7. If $\mathcal{P} = 0$ has non-zero solutions, then select one and substitute it into F_0 to get F;
        4.2.8. If $F \neq 0$, then
            4.2.8.1. If F is not differentially homogeneous, then $F = 0$, $\widehat{h} = h - 1$, goto step 4.3.
            4.2.8.2. For $1 \le i \le n-d$, compute $\alpha_i = \delta\text{-rem}(A_i, \mathcal{C}_F)$, if $\alpha_i \neq 0$ then $F = 0$, $\widehat{h} = h - 1$, goto step 4.3; else $i = i + 1$.
            4.2.8.3. For $0 \le i \le d$, compute $\beta_i = \delta\text{-rem}(\mathbb{P}_i, \mathcal{C}_F)$, if $\beta_i \neq 0$ then $F = 0$, $\widehat{h} = h - 1$, goto step 4.3; else $i = i + 1$.
            4.2.8.4. Compute $\gamma = \delta\text{-rem}(\mathrm{I}_{\mathcal{A}}\mathrm{S}_{\mathcal{A}}, \mathcal{C}_F)$, if i$\gamma = 0$, then $F = 0$, $\widehat{h} = h - 1$, goto step 4.3.
            4.2.8.5. Return F.
        4.2.9. $h := h + 1$.
    4.3. $t := t + 1$.

/*/ $\mathcal{C}_F = \{F, \frac{\partial F}{\partial u_{00}^{(h)}} y_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \ldots, \frac{\partial F}{\partial u_{00}^{(h)}} y_n - \frac{\partial F}{\partial u_{0n}^{(h)}}\}$.
/*/ Pol and GPol stand for algebraic polynomial and generic algebraic polynomial.
/*/ coeff(F, V) returns the set of coefficients of F as an algebraic polynomial in V.

Proof. First, we claim that $(*)$ for each fixed degree t, if there exists $h \le \mathrm{Jac}(\mathcal{A})$ such that 1) $\mathcal{P}_{t,h}$ has at least one nonzero solution and 2) $\mathcal{P}_{<t,\le h}$ and $\mathcal{P}_{t,<h}$ has only the trivial solution $\mathbf{0}$, then either the obtained nonzero F is the differential Chow form, or $\mathrm{ord}(\mathcal{I}) \le h-1$. Indeed, if the obtained F does not satisfy conditions in steps 4.2.8.1 to 4.2.8.4, then

F is the differential Chow form by Theorem 33. Otherwise, F is a nonzero differential polynomials contained in $\text{sat}(\text{Chow}(\mathcal{I})) = [\text{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ which is not the differential Chow form. Here, $\text{Chow}(\mathcal{I})$ is the differential Chow form of $\mathcal{I}$. So $\text{ord}(\text{Chow}(\mathcal{I})) \le h$. But if $\text{ord}(\text{Chow}(\mathcal{I})) = h$, $\text{Chow}(\mathcal{I})$ divides F, a contradiction to the hypothesis that each $\mathcal{P}_{<t,\le h}$ does not have nonzero solutions. Thus, $\text{ord}(\text{Chow}(\mathcal{I})) < h$, and the claim $(*)$ is proved.

Second, we claim that it is enough to pick any one of the nonzero solutions in step 4.2.7. Suppose there are two distinct solutions $\mathbf{c}_1$ and $\mathbf{c}_2$ of $\mathcal{P} = 0$ obtained in step 4.2.6. Let F_1 and F_2 be the polynomials obtained by substituting $\mathbf{c}_1$ and $\mathbf{c}_2$ into F_0 respectively. Equivalently, we need to show that F_1 does not satisfy steps 4.2.8.1 to 4.2.8.4 if and only if F_2 does not. Suppose F_1 does not satisfy steps 4.2.8.1 to 4.2.8.4, then by Theorem 33, F_1 is the differential Chow form of $\text{sat}(\mathcal{A})$. Since F_2 has the same degree as F_1 and the same order guaranteed by claim $(*)$, $F_2 = a \cdot F_1$ $(a \in \mathcal{F})$ must be the differential Chow form, which proves the claim.

The above two claims, as well as the order and degree bounds given in Theorems 18 and 30, guarantee that the algorithm finds a nonzero polynomial $F \in [\text{sat}(\mathcal{A}), \mathbb{P}_0, \ldots, \mathbb{P}_d] \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ satisfying the conditions in Theorem 33 with minimal degree. This F must be irreducible, for $\mathcal{P}_{i,j} = 0$ only possess zero solutions for $i < t$ and $j \le h$. By Theorem 33, the output F must be the differential Chow form of $\mathcal{I}$.

We estimate the complexity of the algorithm below. It suffices to estimate the complexity step 4.2.4, step 4.2.6, and step 4.2.8 in the algorithm. Similarly as in the proof of Theorem 28, for fixed t and h, step 4.2.4 and step 4.2.6 can be done with at most

$$T_1^{(h,t)} = 2^{2.376}\big[(m+1)^{(n-d)(h+1)+1}(2t+1)\big]^{2.376(e+h+1)[n+(d+1)(n+1)]}$$

and

$$T_2^{(h,t)} = [(m+1)^{(n-d)(h+1)}(2t+1)]^{2.376(e+h+1)[n+(d+1)(n+1)]}$$

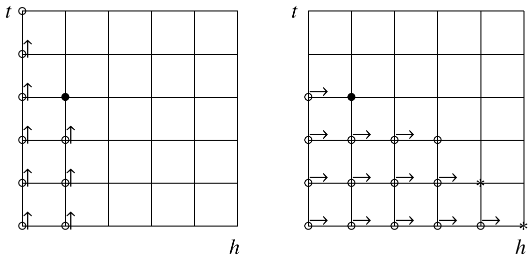
arithmetic operations respectively. For each fixed t, step 4.2.8 will be executed at most once. In step 4.2.8.2, we need to compute the differential remainder of A_i with respect to $\mathcal{C}_F$. By Lemma 27, this step can be done with at most

$$T_3^{(t,h)} = \sum_{i=1}^{n-d} 2^{2.376}[(m_i+1)(t+1)^{(n+1)(e_i+1)+1}]^{2.376[n+(d+1)(n+1)](h+e_i+1))}$$

arithmetic operations. Similarly, we get step 4.2.8.3 and step 4.2.8.4 can be done with at most

$$T_4^{(t,h)} = \sum_{i=0}^{d} 2^{2.376}[(2+1)(t+1)^{n+2}]^{2.376[n+(d+1)(n+1)](h+1)}$$

and



(a) Algorithm 2 (b) Algorithm 3

Fig. 1. "∘" means the algorithm is executed for the corresponding (t, h) but $\mathcal{P}_{t,h}$ has only a zero solution, and "$*$" means $\mathcal{P}_{t,h}$ has nonzero solutions but the corresponding nonzero F is not the differential Chow form, while "•" means the corresponding F is the output.

$$T_5^{(t,h)} = 2^{2.376}\Big[\big(2\sum_{i=1}^{n-d}(m_i-1)+1\big)(t+1)^{(n+1)(e+1)+1}\Big]^{2.376[n+(d+1)(n+1)](h+e+1)}$$

arithmetic operations respectively. From Theorem 30, we know that step 4 may loop from 1 to $D = 2^{(\mathrm{Jac}(\mathcal{A})+1)(d+1)}\big(2\sum_{i=1}^{n-d}(m_i-1)+1\big)\prod_{i=1}^{n-d} m_i^{\mathrm{Jac}(\mathcal{A})+1}$, and for each fixed t, from Theorem 18, step 4.2 may loop from $\mathrm{ord}(\mathcal{A})$ to $\mathrm{Jac}(\mathcal{A})$. Thus, the differential Chow form can be computed with less than

$$\begin{aligned}
&\sum_{t=1}^{D}\sum_{h=\mathrm{ord}(\mathcal{A})}^{\mathrm{Jac}(\mathcal{A})}\big[T_1^{(t,h)}+T_2^{(t,h)}+\delta_{h,h_0(t)}(T_3^{(t,h_0(t))}+T_4^{(t,h_0(t))}+T_5^{(t,h_0(t))})\big]\\
&< (\mathrm{Jac}(\mathcal{A})+1)\big[(n-d)(m+1)^{(2n-d+1)(\mathrm{Jac}(\mathcal{A})+1)+3}\big]^{2.376n_1+1}\\
&\quad + (n+2)\big[2(n-d)(m+1)^{(\mathrm{Jac}(\mathcal{A})+1)(n+1)+3}\big]^{2.376n_1(ne+n+e+2)}\\
&< (\mathrm{Jac}(\mathcal{A})+n+3)\big[(n-d)(m+1)^{(2n-d+1)(\mathrm{Jac}(\mathcal{A})+1)+3}\big]^{2.376n_1(ne+n+e+2)}
\end{aligned}$$

$\mathcal{F}$-arithmetic operations, where we set $n_1 = (n-d)(e+\mathrm{Jac}(\mathcal{A})+1)[(d+1)(n+1)+n]$. Here, $h_0(t)$ is the smallest h such that $\mathcal{P}_{t,h}=0$ has a nonzero solution, and $\delta_{h,h_0(t)}$ is the Kronecker delta. □

Remark 2. We use Fig. 1 to illustrate the searching strategies of Algorithm 2 and Algorithm 3. Both algorithms have their own advantages and defects in different situations. Fig. 1 shows Algorithm 2 has higher efficiency than Algorithm 3 in some cases. And it may happen that Algorithm 3 has higher efficiency than Algorithm 2 in certain cases. For example, let $n=2$ and $\mathcal{A}=\{(y_1')^2y_2''-y_1\}$ with $\mathscr{R}$ the elimination ranking $y_2<y_1$. Here, the differential Chow form of $\mathrm{sat}(\mathcal{A})$ is of order 2 and total degree 14. We use Fig. 2 to show the steps which are needed to execute in Algorithm 2 and Algorithm 3 respectively for this example. It is clear that Algorithm 3 is of higher efficiency than Algorithm 2 in this particular example.

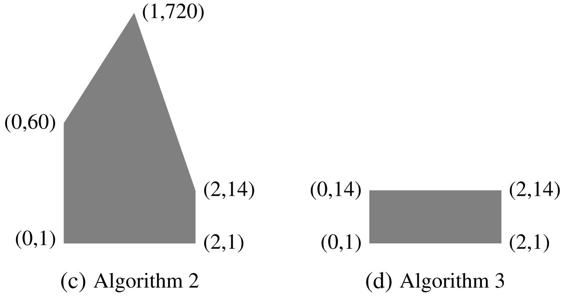


Fig. 2. Both Algorithm 2 and Algorithm 3 return the same differential polynomial F with $h = 2$ and $t = 14$. Algorithm 2 is executed at all the integer lattice points (h, t) which lie in the gray convex polygon as shown in the figure (c), while Algorithm 3 is executed at all the integer lattice points (h, t) of the gray convex polygon in the figure (d).

Remark 3. When using Algorithm 3 to compute the differential Chow form, in step 4.2.8.1, we can examine whether the current nonzero differential polynomial F satisfies the symmetric properties described in Theorem 10. If it is not symmetric, goto step 4.3. This will improve the efficiency of the algorithm.

A few words should be said about the algorithms in this paper. All the algorithms are theoretical ones and we use them to estimate the computational complexity of differential Chow forms in the worst case. They are not implemented in computer algebra systems.

Remark 4. For a fixed prime differential ideal $\mathcal{I}$, transforming characteristic sets from one ranking to another will not affect the results of the algorithms. That is because the differential Chow form is uniquely determined by the differential ideal $\mathcal{I}$ itself and it does not depend on the choice of ranking or the choice of characteristic set under a certain ranking. But the computational complexity in practice will definitely be affected. In general, if giving a characteristic set of $\mathcal{I}$ under an orderly ranking, the differential Chow form is more easily be computed using Algorithm 1.

We conclude this section by giving an application of the algorithms in this paper. Given a characteristic set $\mathcal{A}$ of a prime differential ideal $\mathcal{I}$ under an arbitrary ranking, Theorem 18 shows that $\mathrm{ord}(\mathcal{I}) \leq \mathrm{Jac}(\mathcal{A})$. But what is the precise order of $\mathcal{I}$? And how to compute it?

Since $\mathrm{ord}(\mathcal{I}) = \mathrm{ord}(\mathrm{Chow}(\mathcal{I}))$, if the differential Chow form of $\mathcal{I}$ has been computed, then clearly we can read off the order of $\mathcal{I}$. Thus, the above problem can be solved by computing the differential Chow form of $\mathcal{I}$.

Acknowledgments

We are highly grateful to Professor Xiao-Shan Gao for supporting us to work on this problem, also for helpful discussions and comments on this paper. We appreciate the helpful suggestions of the referees very much.

The first author would like to thank the University of California, Berkeley to provide such a good research environment during her appointment as a Visiting Scholar. The first author also would like to thank Atoshi Chowdhury for providing me with helpful suggestions in rewriting the introduction and revising the paper.

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