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PARTIAL DIFFERENTIAL CHOW FORMS AND A TYPE OF PARTIAL DIFFERENTIAL CHOW VARIETIES

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Abstract. We first present an intersection theory of partial differential varieties with quasi-generic differential hypersurfaces. Then, based on the generic differential intersection theory, we define the partial differential Chow form for an irreducible partial differential variety V of Kolchin polynomial $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. And we establish for the partial differential Chow form most of the basic properties of the ordinary differential Chow form. Furthermore, we prove the existence of a type of partial differential Chow varieties.

1. Introduction

In their paper on Chow forms [3], Chow and van der Waerden described the motivation in these words:

> It is principally important to represent geometric objects by coordinates. Once this has been done for a specific kind of objects G, then it makes sense to speak of an algebraic manifold or an algebraic system of objects G, and to apply the whole theory of algebraic manifolds. It is desirable to provide the set of objects G with the structure of an algebraic variety (eventually, after a certain compactification), thus to characterise G by algebraic equations in the coordinates.

Through developing the theory of Chow forms, they managed to represent projective algebraic varieties or algebraic cycles by Chow coordinates, thus generalised Plücker coordinates and Grassmann coordinates; and they also provided the set of algebraic cycles of fixed dimension and degree with the structure of Chow variety.

To be more specific, given an algebraic cycle V of dimension d in a projective space, its Chow form is the unique homogenous polynomial F, which states the condition when V and $d+1$ hyperplanes have a point in common. The coefficients of the Chow form are defined to be the Chow coordinates of V. Chow proved that the set of all algebraic cycles of fixed dimension and degree in the coordinate space is a projective variety, called the Chow variety. So Chow varieties are simply parameter spaces of algebraic cycles of fixed dimension and degree. As basic concepts of algebraic geometry, Chow forms, as well as Chow varieties, play an important role in both theoretic and algorithmic aspects of algebraic geometry and have fruitful

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applications in many fields, such as intersection theory, transcendental number theory and algebraic computational complexity theory [1, 4, 5, 10, 22, 23, 28].

Differential algebra, founded by Ritt and Kolchin, is a branch of mathematics aiming to study algebraic ordinary or partial differential equations in a similar way in which polynomial equations are studied in algebraic geometry [14, 26]. The basic geometric objects of differential algebra are differential varieties. It is natural to ask how to represent differential varieties by coordinates and further provide specific sets of differential varieties with the structure of differential varieties. Also, in view of the importance of Chow forms and Chow varieties in algebraic geometry, it is desirable to develop the theory of differential Chow forms and differential Chow varieties in differential algebra and hope they play similar roles as their algebraic counterparts.

The work on differential Chow forms [9, 20] could be regarded as the beginning of such a systematic development, where the theory of differential Chow forms is established for ordinary differential varieties in both affine and projective cases and the existence of differential Chow varieties is proved in very special cases. Then the existence of ordinary differential Chow varieties in general cases is finally proved with a model-theoretical proof[6]. However, the theory of partial differential Chow forms is not yet developed for partial differential varieties.

But unlike the ordinary differential case, we encounter an insuperable obstacle in the course of defining partial differential Chow forms: due to the more complicated structure of partial differential characteristic sets, it is impossible to define differential Chow forms for most of the irreducible partial differential varieties (see Example 4.2). Then comes a natural question, that is, to explore in which conditions on partial differential varieties that we can define partial differential Chow forms and provide a specific kind of partial differential varieties (after taking Kolchin closure) with a structure of partial differential varieties. This is what we will deal with in this paper. More specifically, we will give a sufficient condition for the existence of partial differential Chow forms, and for those partial differential varieties, we will define partial differential Chow forms and prove the basic properties of partial differential Chow forms similar to those of their ordinary differential counterparts. And finally, we will show a type of partial differential Chow varieties exist.

To give the definition of partial differential Chow form, we need the generic intersection theory in the partial differential case which is also interesting in itself. Intersection theory is a fundamental issue in both algebraic geometry and differential algebra. The intersection theorem is a basic result in algebraic geometry, which claims that every component of the intersection of two irreducible varieties of dimension r and s in the n-dimensional affine space has dimension greater than or equal to $r + s - n$. However, as pointed out by Ritt, the intersection theorem fails for differential algebraic varieties [26]. Recently, we proved a generic intersection theorem for ordinary differential varieties and generic ordinary differential hypersurfaces [9]. Freitag generalised our result to the partial differential case using more geometric and model theoretical languages [7]. In this paper, we prove the intersection theorem of differential algebraic varieties with quasi-generic differential hypersurfaces (to be defined in Definition 3.1) using pure differential algebraic arguments. In particular, when the quasi-generic partial differential hypersurface is a generic one, the proof gives more elementary and simplified proofs for generic intersection theorems either in the ordinary differential case [9, Theorem 3.6] or in the partial differential case [7, Theorem 3.7].

The rest of the paper is organised as follows. In section 2, the basic notions and preliminary results that will be used in this paper are presented. Then an intersection theory for quasi-generic partial differential polynomials will be given in section 3. In section 4, the definition of the partial differential Chow form and a sufficient condition for its existence are introduced. Basic properties of partial differential Chow form will be explored in section 5. In section 6, we show that a special type of partial differential Chow varieties exist.

2. Preliminaries

In this section, some basic notation and preliminary results in differential algebra will be given. For more details about differential algebra, please refer to [14].

Let $\mathcal{F}$ be a differential field of characteristic 0 endowed with a finite set of derivation operators $\Delta = \{\delta_1, \ldots, \delta_m\}$, and let $\mathcal{E}$ be a fixed universal differential extension field of $\mathcal{F}$. If $m = 1$, $\mathcal{F}, \mathcal{E}$ are called *ordinary differential* fields; and if $m > 1$, they are called *partial differential* fields. Throughout the paper, unless otherwise indicated, all the differential fields (rings) we consider are partial differential fields (rings), and for simplicity, we shall use the prefix "Δ-" as a synonym of "partial differential" or "partial differentially" when the derivation operators in problem are exactly $\{\delta_1, \ldots, \delta_m\}$.

Let Θ be the free commutative semigroup (written multiplicatively) generated by $\delta_1, \ldots, \delta_m$. Every element $\theta \in \Theta$ is called a *derivative operator* and can be expressed uniquely in the form of a product $\prod_{i=1}^{m} \delta_i^{a_i}$ with $a_i \in \mathbb{N}$. The *order* of θ is defined to be $\mathrm{ord}(\theta) = \sum_{i=1}^{m} a_i$. The identity operator is of order 0. For ease of notation, we use Θ_s to denote the set of all derivative operators of order equal to s and $\Theta_{\leq s}$ denotes the set of all derivative operators of order not greater than s. For an element $u \in \mathcal{U}$, denote $u^{[s]} = \{\theta(u) : \theta \in \Theta_{\leq s}\}$.

A subset Σ of a Δ-extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be Δ-*dependent* over $\mathcal{F}$ if the set $(\theta\alpha)_{\theta\in\Theta,\alpha\in\Sigma}$ is algebraically dependent over $\mathcal{F}$, and is said to be Δ-*independent* over $\mathcal{F}$, or a family of Δ-$\mathcal{F}$-*indeterminates* in the contrary case. In the case Σ consists of one element α, we say that α is Δ-algebraic or Δ-transcendental over $\mathcal{F}$ respectively. The Δ-*transcendence degree* of $\mathcal{G}$ over $\mathcal{F}$, denoted by $\Delta\text{-tr.deg}\,\mathcal{G}/\mathcal{F}$, is the cardinality of any maximal subset of $\mathcal{G}$ which are Δ-independent over $\mathcal{F}$. And the transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ is denoted by $\text{tr.deg}\,\mathcal{G}/\mathcal{F}$.

Let $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[\Theta(\mathbb{Y})]$ be the Δ-polynomial ring with Δ-indeterminates $\mathbb{Y} = \{y_1, \ldots, y_n\}$ and coefficients in $\mathcal{F}$. A Δ-monomial in $\mathbb{Y}$ is just a monomial in $\Theta(\mathbb{Y})$. A Δ-ideal in $\mathcal{F}\{\mathbb{Y}\}$ is an ideal which is closed under the derivation operators. A prime (resp. radical) Δ-ideal is a Δ-ideal which is prime (resp. radical) as an ordinary algebraic ideal. Given $S \subset \mathcal{F}\{\mathbb{Y}\}$, we use $(S)_{\mathcal{F}\{\mathbb{Y}\}}$ and $[S]_{\mathcal{F}\{\mathbb{Y}\}}$ to denote the algebraic ideal and the Δ-ideal in $\mathcal{F}\{\mathbb{Y}\}$ generated by S respectively.

In this paper, by a Δ-affine space $\mathbb{A}^n$, we mean the set $\mathcal{E}^n$. A Δ-variety over $\mathcal{F}$ is $\mathbb{V}(\Sigma) = \{\eta \in \mathcal{E}^n : f(\eta) = 0, \forall f \in \Sigma\}$ for some set $\Sigma \subset \mathcal{F}\{\mathbb{Y}\}$. The Δ-varieties in $\mathbb{A}^n$ defined over $\mathcal{F}$ are the closed sets in a topology called the *Kolchin topology.* Given a Δ-variety V defined over $\mathcal{F}$, we denote $\mathbb{I}(V)$ to be the set of all Δ-polynomials in $\mathcal{F}\{\mathbb{Y}\}$ that vanish at every point of V. And we have a one-to-one correspondence between Δ-varieties (resp. irreducible Δ-varieties) and radical Δ-ideals (resp. prime Δ-ideal), that is, for any Δ-variety V over $\mathcal{F}$, $\mathbb{V}(\mathbb{I}(V)) = V$ and for any radical Δ-ideal $\mathcal{P}$ in $\mathcal{F}\{\mathbb{Y}\}$, $\mathbb{I}(\mathbb{V}(\mathcal{P}))=\mathcal{P}$. To distinguish from the notations in the differential case, for an algebraic ideal $\mathcal{P}\subset\mathcal{F}[\mathbb{Y}]$, we use $\mathcal{V}(\mathcal{P})$ to denote the algebraic variety in $\mathbb{A}^n$ defined by $\mathcal{P}$; and for an algebraic variety $V\subset\mathbb{A}^n$, we use $\mathcal{I}(V)$ to denote the radical ideal in $\mathcal{F}[\mathbb{Y}]$ corresponding to V. For a prime Δ-ideal $\mathcal{P}$, a point $\eta\in\mathbb{V}(\mathcal{P})$ is called a *generic point* of $\mathcal{P}$ (or $\mathbb{V}(\mathcal{P})$) if for any $f\in\mathcal{F}\{\mathbb{Y}\}$, $f(\eta)=0$ implies $f\in\mathcal{P}$. A Δ-ideal has a generic point if and only if it is prime.

A homomorphism φ from a differential ring $(\mathcal{R},\Delta)$ to a differential ring $(\mathcal{S},\Delta')$ with $\Delta'=\{\delta_1',\ldots,\delta_m'\}$ is a *differential homomorphism* if $\varphi\circ\delta_i=\delta_i'\circ\varphi\,(\forall i)$. Suppose $\Delta'=\Delta$ and $\mathcal{R}_0$ is a common Δ-subring of $\mathcal{R}$ and $\mathcal{S}$, φ is said to be a Δ-$\mathcal{R}_0$-homomorphism if φ leaves every element of $\mathcal{R}_0$ invariant. If, in addition $\mathcal{R}$ is a domain and $\mathcal{S}$ is a Δ-field, φ is called a Δ-*specialization* of $\mathcal{R}$ into $\mathcal{S}$. For Δ-specializations, we have the following lemma which generalizes the similar results both in the ordinary differential case ([9, Theorem 2.16]) and in the algebraic case ([12, p.168-169] and [9, Lemma 2.13]).

Lemma 2.1. *Let $P_i\in\mathcal{F}\{\mathbb{U},\mathbb{Y}\}\,(i=1,\ldots,m)$ be Δ-polynomials in the independent Δ-indeterminates $\mathbb{U}=(u_1,\ldots,u_r)$ and $\mathbb{Y}$. Let η be an n-tuple taken from some extension field of $\mathcal{F}$ free from $\mathcal{F}\langle\mathbb{U}\rangle$[1]. If $P_i(\mathbb{U},\eta)\,(i=1,\ldots,m)$ are Δ-dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any Δ-specialization $\mathbb{U}$ to $\overline{\mathbb{U}}\in\mathcal{F}^r$, $P_i(\overline{\mathbb{U}},\eta)\,(i=1,\ldots,m)$ are Δ-dependent over $\mathcal{F}$.*

Proof: Assume $k=\max_i\mathrm{ord}(P_i)$. Since $P_i(\mathbb{U},\eta)\,(i=1,\ldots,m)$ are Δ-dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, there exists $s\in\mathbb{N}$ such that the $\big(P_i(\mathbb{U},\eta)^{[s]}$ are algebraically dependent over $\mathcal{F}(\mathbb{U}^{[s+k]})$. When $\mathbb{U}$ Δ-specializes to $\overline{\mathbb{U}}\in\mathcal{F}^r$, $\mathbb{U}^{[s+k]}$ algebraically specializes to $\overline{\mathbb{U}}^{[s+k]}$. By [9, Lemma 2.13], $\big(P_i(\overline{\mathbb{U}},\eta)\big)^{[s]}$ are algebraically dependent over $\mathcal{F}$. Thus, $P_i(\overline{\mathbb{U}},\eta)\,(i=1,\ldots,m)$ are Δ-dependent over $\mathcal{F}$. $\square$

2.1. **Differential characteristic sets.** A *ranking* on $\mathcal{F}\{\mathbb{Y}\}$ is a total order on $\Theta(\mathbb{Y})=\{\theta y_j : j=1,\ldots,n;\theta\in\Theta\}$ which is compatible with the derivation operators: 1) for any $\theta y_j\in\Theta(\mathbb{Y})$ and δ_i, $\delta_i\theta y_j>\theta y_j$ and 2) $\theta_1 y_i>\theta_2 y_j\Longrightarrow\delta_k\theta_1 y_i>\delta_k\theta_2 y_j$ for $\theta_1 y_i,\theta_2 y_j\in\Theta(\mathbb{Y})$. By convention, $1<\theta y_j$ for all $\theta y_j\in\Theta(\mathbb{Y})$. Two important kinds of rankings are often used:

1) *Elimination ranking*: $y_i>y_j\Longrightarrow\theta_1 y_i>\theta_2 y_j$ for any $\theta_1,\theta_2\in\Theta$.

2) *Orderly ranking*: $k>l\Longrightarrow$ for any $\theta_1\in\Theta_k$, $\theta_2\in\Theta_l$ and $i,j\in\{1,\ldots,n\}$, we have $\theta_1 y_i>\theta_2 y_j$.

Let f be a Δ-polynomial in $\mathcal{F}\{\mathbb{Y}\}$ and $\mathscr{R}$ a ranking endowed on it. The greatest derivative θy_j w.r.t. $\mathscr{R}$ which appears effectively in f is called the *leader* of f, denoted by $\mathrm{ld}(f)$. Let d be the degree of f in $\mathrm{ld}(f)$. The *rank* of f is $\mathrm{ld}(f)^d$, denoted by $\mathrm{rk}(f)$. The coefficient of $\mathrm{rk}(f)$ in f is called the *initial* of f and denoted by I_f. The partial derivative of f w.r.t. $\mathrm{ld}(f)$ is called the *separant* of f, denoted by S_f. For any two Δ-polynomials f, g in $\mathcal{F}\{\mathbb{Y}\}\backslash\mathcal{F}$, f is said to be of *lower rank* than g if either $\mathrm{ld}(f)<\mathrm{ld}(g)$ or $\mathrm{ld}(f)=\mathrm{ld}(g)$ and $\deg(f,\mathrm{ld}(f))<\deg(g,\mathrm{ld}(f))$. By convention, any element of $\mathcal{F}$ is of lower rank than elements of $\mathcal{F}\{\mathbb{Y}\}\backslash\mathcal{F}$. We denote $f\preceq g$ if and only if either f is of lower rank than g or they have the same rank. Clearly, $\preceq$ is a totally ordering of $\mathcal{F}\{\mathbb{Y}\}$.

Let f and g be two Δ-polynomials and $\mathrm{rk}(f)=\theta(y_j)^d$. g is said to be *reduced* w.r.t. f if no proper derivatives of $\theta(y_j)$ appear in g and $\deg(g,\theta(y_j))<d$. Let $\mathcal{A}$

[1]By saying η free from $\mathcal{F}\langle\mathbb{U}\rangle$ (resp. Δ-free from $\mathcal{F}\langle\mathbb{U}\rangle$), we mean that $\mathbb{U}$ is a set of Δ-$\mathcal{F}\langle\eta\rangle$-indeterminates.

be a set of Δ-polynomials. $\mathcal{A}$ is said to be an *autoreduced set* if each Δ-polynomial of $\mathcal{A}$ is reduced w.r.t. any other element of $\mathcal{A}$. Every autoreduced set is finite.

Let $\mathcal{A}$ be an autoreduced set. We denote $\mathrm{H}_{\mathcal{A}}$ to be the set of all the initials and separants of $\mathcal{A}$ and $\mathrm{H}_{\mathcal{A}}^{\infty}$ to be the minimal multiplicative set containing $\mathrm{H}_{\mathcal{A}}$. The Δ-*saturation ideal* of $\mathcal{A}$ is defined to be

$$\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = \{p \in \mathcal{F}\{\mathbb{Y}\} \big| \exists h \in H_{\mathcal{A}}^{\infty}, \text{ s.t.}\, hp \in [A]\}.$$

The algebraic saturation ideal of $\mathcal{A}$ is denoted by $\mathrm{asat}(\mathcal{A}) = (\mathcal{A}) : \mathrm{H}_{\mathcal{A}}^{\infty}$.

Let $\mathcal{A} =< A_1, A_2, \ldots, A_s >$ and $\mathcal{B} =< B_1, B_2, \ldots, B_l >$ be two autoreduced sets with the A_i, B_j arranged in nondecreasing ordering. $\mathcal{A}$ is said to be of *lower rank* than $\mathcal{B}$, if either 1) there is some k ($\leq \min\{s, l\}$) such that for each $i < k$, A_i has the same rank as B_i, and $A_k \prec B_k$ or 2) $s > l$ and for each $i \in \{1, 2, \ldots, l\}$, A_i has the same rank as B_i. It is easy to see that the above definition introduces really a partial ordering among all autoreduced sets. Any sequence of autoreduced sets steadily decreasing in ordering $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \cdots \mathcal{A}_k \succ \cdots$ is necessarily finite.

Let $\mathcal{A} =< A_1, A_2, \ldots, A_t >$ be an autoreduced set with S_i and I_i as the separant and initial of A_i, and F any Δ-polynomial. Then there exists an algorithm, called Ritt's algorithm of reduction, which reduces F w.r.t. $\mathcal{A}$ to a Δ-polynomial R that is reduced w.r.t. $\mathcal{A}$, satisfying the relation

$$\prod_{i=1}^{t} \mathrm{S}_i^{d_i} \mathrm{I}_i^{e_i} \cdot F \equiv R, \mathrm{mod}\, [\mathcal{A}],$$

for $d_i, e_i \in \mathbb{N}\, (i = 1, 2, \ldots, t)$. We call R the *remainder* of P w.r.t. $\mathcal{A}$. We will need the following result in Section 3.

Proposition 2.2. [14, p.80, Proposition 2] Let $\mathcal{A}$ be an autoreduced set of $\mathcal{F}\{\mathbb{Y}\}$. If $F_1, \ldots, F_l \in \mathcal{F}\{\mathbb{Y}\}$, then there exist Δ-polynomials $E_1, \ldots, E_l \in \mathcal{F}\{\mathbb{Y}\}$, reduced with respect to $\mathcal{A}$ and of rank no higher than the highest of the ranks of $F_1, \ldots, F_l$, and there exist natural numbers j_A, t_A ($A \in \mathcal{A}$), such that

$$\prod_{A \in \mathcal{A}} \mathrm{S}_A^{j_A} \mathrm{I}_A^{t_A} \cdot F_j \equiv E_j, \mathrm{mod}\, [\mathcal{A}] \;\; (1 \leq j \leq l).$$

Let $\mathcal{J}$ be a Δ-ideal in $\mathcal{F}\{\mathbb{Y}\}$. An autoreduced set $\mathcal{C} \subset \mathcal{J}$ is said to be a *characteristic set* of $\mathcal{J}$, if $\mathcal{J}$ does not contain any nonzero element reduced w.r.t. $\mathcal{C}$. All the characteristic sets of $\mathcal{J}$ have the same and minimal rank among all autoreduced sets contained in $\mathcal{J}$. If $\mathcal{J}$ is prime, $\mathcal{C}$ reduces to zero only the elements of $\mathcal{J}$ and we have $\mathcal{J} = \mathrm{sat}(\mathcal{C})$. An autoreduced set $\mathcal{C}$ is called *coherent* if whenever $A, A' \in \mathcal{C}$ with $\mathrm{ld}(A) = \theta_1(y_j)$ and $\mathrm{ld}(A') = \theta_2(y_j)$, the remainder of $\mathrm{S}_{A'} \frac{\theta}{\theta_1}(A) - \mathrm{S}_A \frac{\theta}{\theta_2}(A')$ w.r.t. $\mathcal{C}$ is zero, where $\theta = \mathrm{lcm}(\theta_1, \theta_2)$. (Here, if $\theta_j = \prod_{i=1}^m \delta_i^{a_{ji}}$ ($j = 1, 2$) and $\max(a_{1i}, a_{2i}) = c_i$, then $\theta = \prod_{i=1}^m \delta_i^{c_i}$ and $\frac{\theta}{\theta_j} = \prod_{i=1}^m \delta_i^{c_i - a_{ji}}$.) The following result gives a criterion for an autoreduced set to be a characteristic set of a prime Δ-ideal.

Proposition 2.3. [14, p.167, Lemma 2] If $\mathcal{A}$ is a characteristic set of a prime Δ-ideal $\mathcal{P} \subset \mathcal{F}\{\mathbb{Y}\}$, then $\mathcal{P} = \mathrm{sat}(\mathcal{A})$, $\mathcal{A}$ is coherent, and $\mathrm{asat}(\mathcal{A})$ is a prime ideal not containing a nonzero element reduced w.r.t. $\mathcal{A}$. Conversely, if $\mathcal{A}$ is a coherent autoreduced set of $\mathcal{F}\{\mathbb{Y}\}$ such that $\mathrm{asat}(\mathcal{A})$ is a prime ideal not containing a nonzero element reduced w.r.t. $\mathcal{A}$, then $\mathcal{A}$ is a characteristic set of a prime Δ-ideal in $\mathcal{F}\{\mathbb{Y}\}$.

2.2. **Kolchin polynomials of prime differential ideals.** Let $\mathcal{P}$ be a prime Δ-ideal in $\mathcal{F}\{\mathbb{Y}\}$ with a generic point $\eta \in \mathbb{A}^n$. The *Δ-dimension of $\mathcal{P}$*, denoted by $\Delta\text{-}\dim(\mathcal{P})$, is defined as the Δ-transcendence degree of $\mathcal{F}\langle\eta\rangle$ over $\mathcal{F}$. Let $\mathcal{A}$ be a characteristic set of $\mathcal{P}$ w.r.t. some ranking. We use $\mathrm{ld}(\mathcal{A})$ to denote the set $\{\mathrm{ld}(F) : F \in \mathcal{A}\}$. Call y_j a *leading variable* of $\mathcal{A}$ if there exists some $\theta \in \Theta$ such that $\theta(y_j) \in \mathrm{ld}(\mathcal{A})$; otherwise, y_j is called a *parametric variable* of $\mathcal{A}$. The Δ-dimension of $\mathcal{P}$ is equal to the cardinality of the set of parametric variables of $\mathcal{A}$.

For a prime Δ-ideal, its Kolchin polynomial contains more quantitative information than the Δ-dimension. To recall the concept of Kolchin polynomial, we need an important numerical polynomial associated to a subset $E \subseteq \mathbb{N}^m$.

Lemma 2.4. [15, 16] *For every set $E = \{(e_{i1}, \ldots, e_{in}) : i = 1, \ldots, l\} \subseteq \mathbb{N}^m$ $(m \geq 1)$, let $V_E(t)$ denote the set of all elements $v \in \mathbb{N}^m$ such that v is not greater or equal to any element in E relative the the product order on $\mathbb{N}^m$. Then there exists a univariate numerical polynomial $\omega_E(t)$ such that $\omega_E(t) = \mathrm{card}(V_E(t))$ for all sufficiently large t. Moreover, $\omega_E(t)$ satisfies the following statements:*

1) $\deg(\omega_E) \leq m$, *and* $\deg(\omega_E) = m$ *if and only if* $E = \emptyset$. *And if* $E = \emptyset$, $\omega_E(t) = \binom{t+m}{m}$;
2) $\omega_E(t) \equiv 0$ *if and only if* $(0, \ldots, 0) \in E$;
3) *If* $\min_{i=1}^{l} e_{ik} = 0$ *for each k, then* $\deg(\omega_E(t)) < m - 1$.

Theorem 2.5. [15, Theorem 2] *Let $\mathcal{P}$ be a prime Δ-ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$. There exists a numerical polynomial $\omega_{\mathcal{P}}(t)$ with the following properties:*

1) *For sufficiently large $t \in \mathbb{N}$, $\omega_{\mathcal{P}}(t)$ equals the dimension of $\mathcal{P} \cap \mathcal{F}[\mathbb{Y}^{[t]}]$.*
2) $\deg(\omega_{\mathcal{P}}) \leq m = \mathrm{card}(\Delta)$.
3) *If we write $\omega_{\mathcal{P}}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$ where $a_i \in \mathbb{Z}$, then a_m equals the Δ-dimension of $\mathcal{P}$.*
4) *If $\mathcal{A}$ is a differential characteristic set of $\mathcal{P}$ with respect to an orderly ranking on $\mathcal{F}\{y_1, \ldots, y_n\}$ and if E_j denotes for any y_j the set of points $(l_1, \ldots, l_m) \in \mathbb{N}^m$ such that $\delta_1^{l_1} \cdots \delta_m^{l_m} y_j$ is the leader of an element of $\mathcal{A}$, then $\omega_{\mathcal{P}}(t) = \sum_{j=1}^{n} \omega_{E_j}(t)$.*

The numerical polynomial $\omega_{\mathcal{P}}(t)$ is defined to be the *Kolchin polynomial* of $\mathcal{P}$. Prime Δ-ideals whose characteristic sets consist of a single polynomial are of particular interest to us.

Lemma 2.6. *Let $\mathcal{P}$ be a prime Δ-ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$. Suppose $A \in \mathcal{F}\{y_1, \ldots, y_n\}$ constitutes a characteristic set of $\mathcal{P}$ under some orderly ranking $\mathscr{R}$. Then $\{A\}$ is also a characteristic set of $\mathcal{P}$ under an arbitrary ranking. In this case, we call $\mathcal{P}$ the* general component *of A.*

Proof: Suppose S_A is the separant of A under $\mathscr{R}$. Then $\mathcal{P} = [A] : \mathrm{S}_A^{\infty}$. Let $\mathscr{R}'$ be an arbitrary ranking and $\theta(y_k)$ be the leader of A under $\mathscr{R}'$. It suffices to show that there is no nonzero Δ-polynomial in $\mathcal{P}$ which is reduced with respect to A under the ranking $\mathscr{R}'$. Suppose the contrary and let $f \in \mathcal{P}\backslash\{0\}$ be a Δ-polynomial reduced with respect to A under $\mathscr{R}'$. Then f is free from the proper derivatives of $\theta(y_k)$. Since $f \in \mathcal{P}$, there exist $l \in \mathbb{N}$ and finitely many nonzero polynomials T_τ for $\tau \in \Theta$ such that $\mathrm{S}_A^l f = \sum_\tau T_\tau \tau(A)$. For each $\tau \neq 1$, $\tau(A) = S'_A \cdot \tau\theta(y_k) + L_\tau$ where S'_A is the separant of A under $\mathscr{R}'$. Substitute $\tau\theta(y_k) = -L_\tau/S'_A$ for $\tau > 1$ into both sides of the above identity and remove the denominators, then we get

$\mathrm{S}_A^l \mathrm{S}_{A'}^{l'} f = T_1 A$. Thus, A divides f which implies $f = 0$. The contradiction shows that A is a also a characteristic set of $\mathcal{P}$ under any ranking. □

Kolchin gave a criterion for a prime Δ-ideal to be the general component of some Δ-polynomial.

Lemma 2.7. [14, p. 160, Proposition 4] *Let $\mathcal{P}$ be a prime Δ-ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$. Then a necessary and sufficient condition that $\mathcal{P}$ is the general component of some polynomial A of order s is that the Kolchin polynomial of $\mathcal{P}$ is of the form*

$$\omega_{\mathcal{P}}(t) = n\binom{t+m}{m} - \binom{t+m-s}{m}.$$

The following result on algebraic ideals will be used later.

Lemma 2.8. *Let $\mathcal{P}$ be a prime ideal in the polynomial ring $\mathcal{F}[x_1, \ldots, x_n]$ of dimension $d > 0$. Assume $\mathcal{P} \cap \mathcal{F}[x_1] = \{0\}$. Then $\mathcal{J} = (\mathcal{P})_{\mathcal{F}(x_1)[x_2,\ldots,x_n]}$ is a prime ideal of dimension $d-1$.*

Proof: Since $\mathcal{P} \cap \mathcal{F}[x_1] = \{0\}$, $\mathcal{J} \neq \mathcal{F}(x_1)[x_2, \ldots, x_n]$. If $f_1, f_2 \in \mathcal{F}(x_1)[x_2, \ldots, x_n]$ and $f_1 f_2 \in \mathcal{J}$, then there exist $M_1, M_2 \in \mathcal{F}[x_1]$ such that $M_i f_i \in \mathcal{F}[x_1, \ldots, x_n]$ and $M_1 f_1 M_2 f_2 \in \mathcal{P}$. So either $M_1 f_1 \in \mathcal{P}$ or $M_2 f_2 \in \mathcal{P}$, which implies that either $f_1 \in \mathcal{J}$ or $f_2 \in \mathcal{J}$. Thus, $\mathcal{J}$ is a prime ideal.

Since $\dim(\mathcal{P}) = d$ and $\mathcal{P} \cap \mathcal{F}[x_1] = \{0\}$, without loss of generality, we suppose $\{x_1, x_2, \ldots, x_d\}$ is a parametric set of $\mathcal{P}$. We claim that $\{x_2, \ldots, x_d\}$ is a parametric set of $\mathcal{J}$, so $\dim(\mathcal{J}) = d-1$ follows. First, note that $\mathcal{J} \cap \mathcal{F}(x_1)[x_2, \ldots, x_d] = \{0\}$. For any other variable $x_k \in \{x_{d+1}, \ldots, x_n\}$, $\mathcal{P} \cap \mathcal{F}[x_1, x_2, \ldots, x_d, x_k] \neq \{0\}$, so $\mathcal{J} \cap \mathcal{F}(x_1)[x_2, \ldots, x_d, x_k] \neq \{0\}$. Thus, $\{x_2, \ldots, x_d\}$ is a parametric set of $\mathcal{J}$. □

3. Quasi-generic intersection theory in partial differential algebra

In this section, we will prove the quasi-generic intersection theorem with an elementary proof using pure differential algebraic languages, which generalises generic intersection theorems in both ordinary and partial differential cases [9, 7]. We should remark that the proof in the ordinary differential case could not be adapted here because of the complicated structure of differential characteristic sets in the partial differential case. However, the proof here we give could definitely simplify that of its ordinary differential analog.

Definition 3.1. A *generic Δ-polynomial of order s and degree g* is a Δ-polynomial which involves all Δ-monomials of order s and degree g with coefficients being Δ-$\mathcal{F}$-indeterminates. To be more precise, a generic Δ-polynomial $\mathbb{L}$ of order s and degree g is of the following form

$$\mathbb{L} = \sum_{M \in \mathscr{M}_{s,g}} u_M M,$$

where $\mathscr{M}_{s,g}$ is the set of all Δ-monomials of order bounded by s and degree bounded by g and all the coefficients u_M are Δ-$\mathcal{F}$-indeterminates. The Δ-zero set of a generic Δ-polynomial is called a *generic Δ-hypersurface*. And a *generic Δ-hyperplane* is defined to be the Δ-zero set of a generic Δ-polynomial of the form $u_0 + \sum_{j=1}^n u_j y_j$.

A *quasi-generic Δ-polynomial of order s* is a Δ-polynomial $\mathbb{L}$ of the form

$$\mathbb{L} = \sum_{M \in \mathscr{M}_{\mathbb{L}}} u_M M,$$

where the coefficients u_M are Δ-$\mathcal{F}$-indeterminates and its support $\mathscr{M}_{\mathbb{L}}$ of Δ-monomials appearing in $\mathbb{L}$ satisfies the following conditions:

- $1 \in \mathscr{M}_{\mathbb{L}}$;
- for each $j = 1, \ldots, n$, there exists some Δ-monomial $M_j(y_j) \in \mathscr{M}_{\mathbb{L}} \cap \mathcal{F}\{y_j\}$ with $\mathrm{ord}(M_j(y_j)) = s$.

Now, we give the main quasi-generic intersection theorem in partial differential algebra, which generalises the generic intersection theorem in the ordinary case [9].

Theorem 3.2. *Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety over $\mathcal{F}$. Let $\mathbb{L}$ be a quasi-generic Δ-polynomial of order s with the set of its coefficients $\mathbf{u}$. Then*

1) *over $\mathcal{F}\langle\mathbf{u}\rangle$, $V \cap \mathbb{V}(\mathbb{L}) \neq \emptyset$ if and only if Δ-$\mathrm{dim}(V) > 0$.*
2) *if Δ-$\mathrm{dim}(V) > 0$, then the intersection of V and $\mathbb{V}(\mathbb{L})$ is an irreducible Δ-variety over $\mathcal{F}\langle\mathbf{u}\rangle$ and its Kolchin dimension polynomial is*

$$\omega_{V\cap\mathbb{V}(\mathbb{L})}(t) = \omega_V(t) - \binom{t+m-s}{m}.$$

In particular, the Δ-dimension of $V \cap \mathbb{V}(\mathbb{L})$ is equal to $\triangle$-$\mathrm{dim}(V) - 1$.

Proof: Let $\mathcal{P} = \mathbb{I}(V) \subset \mathcal{F}\{\mathbb{Y}\}$ be the prime Δ-ideal corresponding to V and $\eta = (\eta_1, \ldots, \eta_n)$ be a generic point of $\mathcal{P}$ which is free from $\mathbf{u}$ (i.e., the $\mathbf{u}$ are Δ-$\mathcal{F}\langle\eta\rangle$-indeterminates). Let

$$\mathbb{L} = u_0 + \sum_{j=1}^{n} u_j M_j + \sum_{M_\alpha \in \mathscr{M}_{\mathbb{L}} \setminus \{1, M_1, \ldots, M_n\}} u_\alpha M_\alpha$$

where each M_j is a Δ-monomial in y_j of order s, whose existence is guaranteed by the definition of quasi-generic Δ-polynomials. Let $\mathbb{T} = \mathbb{L} - u_0$ and set $\zeta_0 = -\mathbb{T}|_{\mathbb{Y}=\eta}$.

1) Let $\mathcal{J}_0 = [\mathcal{P}, \mathbb{L}]_{\mathcal{F}_1\{\mathbb{Y}, u_0\}}$, where $\mathcal{F}_1 = \mathcal{F}\langle\mathbf{u}\backslash\{u_0\}\rangle$. Then it is easy to show that (η, ζ_0) is a generic point of $\mathcal{J}_0$, so $\mathcal{J}_0$ is a prime Δ-ideal. Let $\mathcal{J} = [\mathcal{P}, \mathbb{L}]_{\mathcal{F}\langle\mathbf{u}\rangle\{\mathbb{Y}\}}$. Clearly, $\mathcal{J} = [\mathcal{J}_0]_{\mathcal{F}\langle\mathbf{u}\rangle\{\mathbb{Y}\}}$ and $\mathcal{J} \cap \mathcal{F}_1\{\mathbb{Y}, u_0\} = \mathcal{J}_0$, so $\mathcal{J} \neq [1]$ if and only if $\mathcal{J}_0 \cap \mathcal{F}_1\{u_0\} = \{0\}$, or equivalently, ζ_0 is Δ-transcendental over $\mathcal{F}_1$. We show that $\mathcal{J} = [1]$ (i.e., $V \cap \mathbb{V}(\mathbb{L}) = \emptyset$) if and only if Δ-$\mathrm{dim}(V) = 0$.

If Δ-$\mathrm{dim}(V) = 0$, then for each $j = 1, \ldots, n$, η_j is Δ-algebraic over $\mathcal{F}$. So $\mathcal{F}_1\langle\eta\rangle$ is Δ-algebraic over $\mathcal{F}_1$. Since $\zeta_0 \in \mathcal{F}_1\langle\eta\rangle$, ζ_0 is Δ-algebraic over $\mathcal{F}_1$ and $\mathcal{J}_0 \cap \mathcal{F}_1\{u_0\} \neq [0]$, which implies $\mathcal{J} = [1]$. For the other direction, suppose $\mathcal{J} = [1]$, i.e., ζ_0 is Δ-algebraic over $\mathcal{F}_1$. For each j, by differentially specializing u_j to 1 and all the other elements in $\mathbf{u}\backslash\{u_0\, u_j\}$ to 0, by Lemma 2.1, $M_j(\eta_j)$, as well as η_j, is Δ-algebraic over $\mathcal{F}$. So Δ-$\mathrm{dim}(V) = 0$. Thus, $\mathcal{J} \neq [1]$ if and only if Δ-$\mathrm{dim}(V) > 0$.

2) Assume Δ-$\mathrm{dim}(V) > 0$. We will show that $\omega_{\mathcal{J}}(t) = \omega_V(t) - \binom{t+m-s}{m}$. For sufficiently large t, let $\mathcal{I}_t = \big(\mathcal{P} \cap \mathcal{F}[\mathbb{Y}^{[t]}], \mathbb{L}^{[t-s]}\big)_{\mathcal{F}_1[\mathbb{Y}^{[t]}, u_0^{[t-s]}]}$. We claim that

i) $\mathcal{I}_t \cap \mathcal{F}_1[u_0^{[t-s]}] = \{0\}$;

ii) $\mathcal{J} \cap \mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}] = (\mathcal{I}_t)_{\mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]}$.

If i) and ii) are valid, then by Lemma 2.8, we have

$$\omega_{V\cap\mathbb{V}(\mathbb{L})}(t) = \dim(\mathcal{J} \cap \mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]) = \dim\big((\mathcal{I}_t)_{\mathcal{F}_1(u_0^{[t-s]})[\mathbb{Y}^{[t]}]}\big) = \omega_{\mathcal{P}}(t) - \binom{t+m-s}{m}.$$

So it remains to show the validity of claims i) and ii).

First note that $(\eta^{[t]}, \zeta_0^{[t-s]})$ is a generic point of $\mathcal{I}_t$. Claim i) is equivalent to say that the $\zeta_0^{[t-s]}$ are algebraically independent over $\mathcal{F}_1$. This is indeed valid, for ζ_0 is Δ-transcendental over $\mathcal{F}_1$ by 1).

For claim ii), it suffices to show that for each $f \in \mathcal{J} \cap \mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]$, f can be written as a linear combination of polynomials in $\mathcal{P} \cap \mathcal{F}[\mathbb{Y}^{[t]}]$ and $\mathbb{L}^{[t-s]}$ with coefficients in $\mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]$. Let $f \in \mathcal{J} \cap \mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]$. Multiplying f by some nonzero polynomial in $\mathcal{F}_1\{u_0\}$ when necessary, we can assume $f \in \mathcal{F}_1[\mathbb{Y}^{[t]}, u_0^{[t-s+k]}]$ for some $k \in \mathbb{N}$. So, $f \in \mathcal{J}_0$ and $f(\eta^{[t]}, \zeta_0^{[t-s+k]}) = 0$ follows. Let $Z = \cup_{i=1}^k \Theta_{t-s+i}$. Rewrite f as a polynomial in $\big(\theta(u_0)\big)_{\theta\in Z}$ with coefficients in $\mathcal{F}_1[\mathbb{Y}^{[t]}, u_0^{[t-s]}]$, and suppose

$$f = \sum_\alpha g_\alpha M_\alpha$$

where $g_\alpha \in \mathcal{F}_1[\mathbb{Y}^{[t]}, u_0^{[t-s]}]$ and the M_α are finitely many distinct monomials in the variables $\big(\theta(u_0)\big)_{\theta\in Z}$. So $f(\eta^{[t]}, \zeta_0^{[t-s+k]}) = 0$ implies that

$$\sum_\alpha g_\alpha(\eta^{[t]}, \zeta_0^{[t-s]}) M_\alpha\big((\theta(\zeta_0))_{\theta\in Z}\big) = 0.$$

If we can show that

$$\big(\theta(\zeta_0)\big)_{\theta\in Z}$$

are algebraically independent over $\mathcal{F}_1(\eta^{[t]})$, then obviously, $g_\alpha(\eta^{[t]}, \zeta_0^{[t-s]}) = 0$ for each α and $g_\alpha \in \mathcal{I}_t$ which implies that $f \in (\mathcal{I}_t)_{\mathcal{F}\langle\mathbf{u}\rangle[\mathbb{Y}^{[t]}]}$.

So it suffices to show that

$$\big(\theta(\zeta_0)\big)_{\theta\in Z}$$

are algebraically independent over $\mathcal{F}_1(\eta^{[t]})$. Suppose the contrary, then $\big(\theta(\zeta_0)\big)_{\theta\in Z}$ are algebraically dependent over $\mathcal{F}_1(\eta^{[t]})$. Let $\mathcal{A}$ be a Δ-characteristic set of $\mathcal{P}$ with respect to some orderly ranking. Since $\Delta\text{-}\dim(V) > 0$, there exists at least one j_0 such that y_{j_0} is a parametric variable of $\mathcal{A}$. By algebraically specializing u_{j_0} to 1 and all the other derivatives in $\Theta_{\le t-s+k}(\mathbf{u}\backslash\{u_0\})$ to 0, and by the algebraic version of Lemma 2.1, $\big(\theta(M_{j_0}(\eta_{j_0}))\big)_{\theta\in Z}$ are algebraically dependent over $\mathcal{F}(\eta^{[t]})$. By multiplying some $D(\eta^{[t]}) \in \mathcal{F}[\eta^{[t]}]$ when necessary, we get a nonzero polynomial $G(\mathbb{Y}) = \sum_l g_l(\mathbb{Y}^{[t]}) T_l(M_{j_0}(y_{j_0}))$ vanishing at η, where the $T_l(M_{j_0}(y_{j_0}))$ are distinct monomials in $\big(\theta(M_{j_0}(y_{j_0}))\big)_{\theta\in Z}$ and for each l, $g_l(\eta^{[t]}) \ne 0$. By Proposition 2.2, there exist $h_l \in \mathcal{F}[\mathbb{Y}^{[t]}]$, reduced with respect to $\mathcal{A}$, and natural numbers j_A, k_A ($A \in \mathcal{A}$) such that

$$\prod_{A\in\mathcal{A}} \mathrm{I}_A^{j_A} \mathrm{S}_A^{k_A} \cdot g_l \equiv h_l \bmod [\mathcal{A}], \text{ for all } l\text{'s}.$$

Thus, $H(\mathbb{Y}) = \sum_l T_l(M_{j_0}(y_{j_0})) h_l(\mathbb{Y}^{[t]})$ is a nonzero polynomial which is reduced with respect to $\mathcal{A}$ and satisfies $H(\eta) = 0$, a contradiction. Thus, $\big(\theta(\zeta_0)\big)_{\theta\in Z}$ are algebraically independent over $\mathcal{F}_1(\eta^{[t]})$ and claim 2) is valid. Consequently, we have proved that $\omega_{V\cap\mathbb{L}}(t) = \omega_V(t) - \binom{t+m-s}{m}$. $\square$

Remark 3.3. By the proof of Theorem 3.2, once we know a variable y_{i_0} which is a parametric variable of a characteristic set of $\mathbb{I}(V)$ under some orderly ranking, for those $\mathbb{L}$ whose support contains 1 and a Δ-monomial in y_{i_0} of order s with coefficients Δ-$\mathcal{F}$-indeterminates, we could still get $\omega_{V\cap\mathbb{L}}(t) = \omega_V(t) - \binom{t+m-s}{m}$.

When $\mathbb{L}$ is a generic Δ-polynomial, as a corollary, we get the partial differential analog of [9, Theorem 1.1], which was proven by Freitag [7] with a model-theoretical proof.

Corollary 3.4. Let V be an irreducible Δ-variety over $\mathcal{F}$ with $\omega_V(t) > \binom{t+m}{m}$. Let $\mathbb{L}$ be a generic Δ-polynomial of order s and degree g with coefficient set $\mathbf{u}$. Then the intersection of V and $\mathbb{L} = 0$ is a nonempty irreducible Δ-variety over $\mathcal{F}\langle\mathbf{u}\rangle$ and its Kolchin polynomial is

$$\omega_{V\cap\mathbb{L}}(t) = \omega_V(t) - \binom{t+m-s}{m}.$$

The following result gives the information of the intersection of several quasi-generic Δ-polynomials.

Corollary 3.5. Let $\mathbb{L}_i\,(i = 1,\ldots,r; r \leq n)$ be independent quasi-generic Δ-polynomials of order s_i respectively. Suppose $\mathbf{u}_i$ is the set of coefficients of $\mathbb{L}_i$. Then $[\mathbb{L}_1,\ldots,\mathbb{L}_n]_{\mathcal{F}\langle\mathbf{u}_1,\ldots,\mathbf{u}_r\rangle}\{\mathbb{Y}\}$ is a prime Δ-ideal with its Kolchin polynomial equal to

$$\omega(t) = \sum_{i=1}^{n}\Big[\binom{t+m}{m} - \binom{t+m-s_i}{m}\Big].$$

In particular, its Δ-dimension is 0, the differential type is $m-1$ and the typical Δ-dimension is $\sum_{i=1}^{n} s_i$.

4. Partial Differential Chow forms

In this section, we will introduce the definition of partial Δ-Chow forms and show for a specific kind of Δ-varieties, their Δ-Chow forms exist.

Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety over $\mathcal{F}$ with Δ-dimension d. Let

$$\mathbb{L}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n\,(i = 0,1,\ldots,d)$$

be $d+1$ independent generic Δ-hyperplanes with coefficient vector $\mathbf{u}_i = (u_{i0}, u_{i1},\ldots,u_{in})$. Let $\mathcal{J} = [\mathbb{I}(V),\mathbb{L}_0,\ldots,\mathbb{L}_d]_{\mathcal{F}\{\mathbb{Y},\mathbf{u}_0,\ldots,\mathbf{u}_d\}}$.

Lemma 4.1. *$\mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is a prime Δ-ideal of codimension 1.*

Proof: Let $\eta = (\eta_1,\ldots,\eta_n)$ be a generic point of V free from each $\mathbf{u}_i$ and let $\zeta_i = -\sum_{k=1}^{n} u_{ik}\eta_k\,(i = 0,\ldots,d)$. Denote $\zeta = (\zeta_0, u_{01},\ldots,u_{0n},\ldots,\zeta_d, u_{d1},\ldots,u_{dn})$ and $\mathbf{u} = \cup_{i=0}^{d}\mathbf{u}_i\backslash\{u_{i0}\}$. It is easy to show that (η,ζ) is a generic point of $\mathcal{J}$, so $\mathcal{J}$ is a prime Δ-ideal. Thus, $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is a prime Δ-ideal with a generic point ζ. Since the Δ-dimension of $\mathcal{P}$ is d, by Lemma 2.1, any d of the ζ_i are Δ-independent over $\mathcal{F}\langle\mathbf{u}\rangle$. Note that $\mathcal{F}\langle\zeta\rangle \subset \mathcal{F}\langle\mathbf{u},\eta\rangle$. So Δ-tr.deg$\mathcal{F}\langle\zeta\rangle/\mathcal{F} = (d+1)n + d$, i.e., the codimension of $\mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is 1. □

In the ordinary differential case, there always exists a unique irreducible δ-polynomial such that $\mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}$ is the general component of this polynomial. This unique polynomial is defined to be the δ-Chow form of V. However, unlike the ordinary differential case, for a prime Δ-ideal of codimension 1, it may not be the general component of any single Δ-polynomial, as Example 4.2 shows.

Example 4.2. Let $m = 2$ and $V = \mathbb{V}(\delta_1(y),\delta_2(y)) \subset \mathbb{A}^1$. Let $\mathbb{L}_0 = u_{00} + u_{01}y$ and $\mathcal{J} = [\mathbb{I}(V),\mathbb{L}_0] \subset \mathcal{F}\{y,u_{00},u_{01}\}$. Then

$$\mathcal{J} \cap \mathcal{F}\{u_{00},u_{01}\} = \mathrm{sat}\big(u_{01}\delta_1(u_{00}) - u_{00}\delta_1(u_{01}),\, u_{01}\delta_2(u_{00}) - u_{00}\delta_2(u_{01})\big),$$

which is of codimension 1 but not the general component of a single Δ-polynomial.

The above fact makes it impossible to define Δ-Chow forms for all the irreducible Δ-varieties. Below, we define Δ-Chow forms for irreducible Δ-varieties satisfying certain properties.

Definition 4.3. If $\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\dots,\mathbf{u}_d\}$ is the general component of some irreducible Δ-polynomial, that is, there exists an irreducible Δ-polynomial $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ such that

$$\mathcal{J}\cap\mathcal{F}\{\mathbf{u}_0,\dots,\mathbf{u}_d\}=\mathrm{sat}(F),$$

then we say the Δ-Chow form of V exists and we call F the Δ-*Chow form* of V or its corresponding prime Δ-ideal $\mathbb{I}(V)$.

Following this definition, a natural question is to explore in which conditions on Δ-varieties such that their Δ-Chow forms exist. Now, we proceed to give a sufficient condition for the existence of Δ-Chow forms.

Lemma 4.4. *Let $\mathcal{P}$ be a prime Δ-ideal in $\mathcal{F}\{y_1,\dots,y_n\}$ and $\mathcal{A}$ a characteristic set of $\mathcal{P}$ with respect to an orderly ranking $\mathscr{R}$. Suppose the Kolchin polynomial of $\mathcal{P}$ is $\omega_{\mathcal{P}}(t)=(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$ for some $d,s\in\mathbb{N}$. Then there exist $n-d$ distinct variables $y_{i_1},\dots,y_{i_{n-d}}$ such that $\mathrm{ld}(\mathcal{A})=\{y_{i_1},\dots,y_{i_{n-d-1}},\theta(y_{i_{n-d}})\}$ for some $\theta\in\Theta_s$.*

Proof: For each $j=1,\dots,n$, let E_j denote the matrix whose row vectors are $(a_1,\dots,a_m)\in\mathbb{N}^m$ such that $\delta_1^{a_1}\cdots\delta_m^{a_m}y_j$ is the leader of an element of $\mathcal{A}$. Here, if y_j is not a leading variable, then set $E_j=\emptyset$. Suppose the leading variables of $\mathcal{A}$ are $y_{i_1},\dots,y_{i_l}$. By Theorem 2.5, $\omega_{\mathcal{P}}(t)=\sum_{j=1}^n\omega_{E_j}(t)=(n-l)\binom{t+m}{m}+\sum_{j=1}^l\omega_{E_{i_j}}=(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$. Since $E_{i_j}\neq\emptyset$, the degree of $\omega_{E_{i_j}}$ is less than m. Comparing the coefficient of t^m of the both sides of the above equality, we get $l=n-d$.

For $j=1,\dots,n-d$, let $e_{i_j}=(e_{i_j1},\dots,e_{i_jn})\in\mathbb{N}^m$ be a vector constructed from E_{i_j} with each e_{i_jk} the minimal element of the k-th column of E_{i_j}, and let H_{i_j} be the matrix whose row vectors are the corresponding row vectors of E_{i_j} minus e_{i_j} respectively. Denote $s_{i_j}=\sum_{k=1}^n e_{i_jk}$. Then clearly, $\omega_{E_{i_j}(t)}=\omega_{e_{i_j}}(t)+\omega_{H_{i_j}}(t-s_{i_j})$. By item 3) of Lemma 2.4, the degree of $\omega_{H_{i_j}}(t-s_{i_j})$ is strictly less than $m-1$. Thus, $\omega_{\mathcal{P}}(t)=(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}=d\binom{t+m}{m}+\sum_{j=1}^{n-d}\omega_{e_{i_j}}(t)+\sum_{j=1}^{n-d}\omega_{H_{i_j}}(t-s_{i_j})$. So $\binom{t+m}{m}-\binom{t+m-s}{m}=\sum_{j=1}^{n-d}[\binom{t+m}{m}-\binom{t+m-s_{i_j}}{m}]+\sum_{j=1}^{n-d}\omega_{H_{i_j}}(t-s_{i_j})$. Comparing the coefficients of t^{m-1} and t^{m-2} on the both sides and use the fact $\binom{t+m}{m}-\binom{t+m-s}{m}=\frac{s}{(m-1)!}t^{m-1}+\frac{s(m+1)-s^2}{2\cdot(m-2)!}t^{m-2}+o(t^3)$, we get

$$\begin{cases} s=\sum_{j=1}^{n-d}s_{i_j},\\ -s^2/2=-\sum_{j=1}^{n-d}s_{i_j}^2/2+(m-2)!\cdot\sum_{j=1}^{n-d}\mathrm{coeff}\big(\omega_{H_{i_j}},t^{m-2}\big).\end{cases}$$

If two of the s_{i_j} are nonzero, then obviously $-s^2/2<-\sum_{j=1}^{n-d}s_{i_j}^2/2$, which implies that the above system of equations is not valid. Thus, there exists only one i_j such that $s_{i_j}=s$ and all the other $n-d-1$ of the s_{ij} is equal to zero. Without loss of generality, suppose $s_{i_{n-d}}=s$. So $\binom{t+m}{m}-\binom{t+m-s}{m}=\binom{t+m}{m}-\binom{t+m-s}{m}+\sum_{j=1}^{n-d}\omega_{H_{i_j}}(t-s_{i_j})$. As a consequence, $H_{i_j}=\{(0,\dots,0)\}$. Thus, each E_{i_j} has only one row vector, and $\mathrm{ld}(\mathcal{A})=\{y_{i_1},\dots,y_{i_{n-d-1}},\theta(y_{i_{n-d}})\}$ for some $\theta\in\Theta_s$. □

The following result gives a sufficient condition for the existence of Δ-Chow forms.

Theorem 4.5. *Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety over $\mathcal{F}$ with Kolchin polynomial*

$$\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$$

for some $s \in \mathbb{N}$. Then the Δ-Chow form of V exists. And the order of the Δ-Chow form of V is s.

Proof: Let $\mathcal{P} = \mathbb{I}(V) \subset \mathcal{F}\{\mathbb{Y}\}$. Let $\mathcal{P}^\star = [\mathcal{P}, \mathbb{L}_1, \ldots, \mathbb{L}_d] \subset \mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d\rangle\{\mathbb{Y}\}$. Then by Corollary 3.4, $\mathcal{P}^\star$ is a prime Δ-ideal of Δ-dimension 0 and $\omega_{\mathcal{P}^\star} = \omega_V(t) - d\binom{t+m}{m} = \binom{t+m}{m} - \binom{t+m-s}{m}$. Let $\mathcal{J}_0 = [\mathcal{P}^\star, \mathbb{L}_0]_{\mathcal{F}\langle \mathbf{u}_1,\ldots,\mathbf{u}_d\rangle\{\mathbb{Y},\mathbf{u}_0\}} \cap \mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d\rangle\{\mathbf{u}_0\}$. Recall $\mathcal{J} = [\mathcal{P}, \mathbb{L}_0, \ldots, \mathbb{L}_d]_{\mathcal{F}\{\mathbb{Y},\mathbf{u}_0,\ldots,\mathbf{u}_d\}} \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$. Then $\mathcal{J}$ and $\mathcal{J}_0$ have such relations: $\mathcal{J}_0 = [\mathcal{J}]_{\mathcal{F}\langle \mathbf{u}_1,\ldots,\mathbf{u}_d\rangle\{\mathbf{u}_0\}}$ and $\mathcal{J} = \mathcal{J}_0 \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$. So $\mathcal{J} = \mathrm{sat}(F)_{\mathcal{F}\{\mathbf{u}_0,\ldots,\mathbf{u}_d\}}$ for some Δ-polynomial F if and only if $\mathcal{J}_0 = \mathrm{sat}(F)_{\mathcal{F}\langle \mathbf{u}_1,\ldots,\mathbf{u}_d\rangle\{\mathbf{u}_0\}}$. Thus, it suffices to consider for the case $\dim(V) = 0$, that is, to show the Δ-Chow form of V exists if $\omega_V(t) = \binom{t+m}{m} - \binom{t+m-s}{m}$ for some $s \in \mathbb{N}$.

Now suppose $\dim(V) = 0$ and let $\eta = (\eta_1, \ldots, \eta_n)$ be a generic point of V free from $\mathbf{u}_0$. Let $\zeta_0 = -\sum_{j=1}^n u_{0j}\eta_j$. Then $(\zeta_0, u_{01}, \ldots, u_{0n})$ is a generic point of $\mathcal{J} = [\mathbb{I}(V), \mathbb{L}_0] \cap \mathcal{F}\{\mathbf{u}_0\}$.

On the one hand, since $\zeta_0^{[t]} \subset \mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]}, \eta^{[t]})$, we have

$$\omega_{(\zeta_0,u_{01},\ldots,u_{0n})}(t) \leq \omega_{(u_{01},\ldots,u_{0n},\eta)}(t) = (n+1)\binom{t+m}{m} - \binom{t+m-s}{m}.$$

On the other hand, by Lemma 4.4, $\omega_V(t) = \binom{t+m}{m} - \binom{t+m-s}{m}$ implies that the leading variables of a characteristic set of $\mathcal{A}$ with respect to an orderly ranking is $\{y_{i_1}, \ldots, y_{i_{n-1}}, \theta(y_{i_n})\}$ with $\theta \in \Theta_s$. So $\{\tau(\eta_{i_n}) : \tau \in \Theta_{\leq t}, \theta \nmid \tau\}$ is algebraically independent over $\mathcal{F}$. By the contrapositive of the algebraic version of Lemma 2.1, $S := \{\tau(\zeta_0) : \tau \in \Theta_{\leq t}, \theta \nmid \tau\}$ is algebraically independent over $\mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]})$. Note that $\mathrm{card}(S) = \binom{t+m}{m} - \binom{t+m-s}{m}$. Thus, we have

$$\begin{aligned} & \omega_{(\zeta_0,u_{01},\ldots,u_{0n})}(t) = \mathrm{tr.deg}\,\mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]}, \zeta_0^{[t]})/\mathcal{F} \\ = \quad & \mathrm{tr.deg}\,\mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]})/\mathcal{F} + \mathrm{tr.deg}\,\mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]})(\zeta_0^{[t]})/\mathcal{F}(u_{01}^{[t]}, \ldots, u_{0n}^{[t]}) \\ \geq \quad & n\binom{t+m}{m} + \binom{t+m}{m} - \binom{t+m-s}{m}. \end{aligned}$$

Thus, $\omega_{(\zeta_0,u_{01},\ldots,u_{0n})}(t) = (n+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. By Lemma 2.7, there exists an irreducible Δ-polynomial F of order s such that $\mathcal{J} = \mathrm{sat}(F)$, so the Δ-Chow form of V exists. $\square$

We conjecture that for the existence of Δ-Chow form of V, $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ is also a necessary condition:

Conjecture 4.6. Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety over $\mathcal{F}$ of differential dimension d. Then a necessary and sufficient condition such that the Δ-Chow form of V exists is that the Kolchin polynomial of V is

$$\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m} \quad \text{for some } s \in \mathbb{N}.$$

In the remaining sections of the paper, we focus on irreducible Δ-varieties of Kolchin polynomial $\omega(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ whose Δ-Chow forms exist guaranteed by Theorem 4.5. The following result is an easy fact, which could be used to compute Δ-Chow forms by pure algebraic computations.

Lemma 4.7. *Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety of Kolchin dimension polynomial $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. Let $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ be the Δ-Chow form of V. Then*

$$\Big(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}], \mathbb{L}_0^{[s]}, \ldots, \mathbb{L}_d^{[s]}\Big) \cap \mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}] = \big(F(\mathbf{u}_0, \ldots, \mathbf{u}_d)\big). \tag{4.1}$$

Proof: Let $\mathcal{Q} = \big(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}], \mathbb{L}_0^{[s]}, \ldots, \mathbb{L}_d^{[s]}\big) \cap \mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}]$. By Definition 4.3, $[\mathbb{I}(V), \mathbb{L}_0, \ldots, \mathbb{L}_d] \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\} = \mathrm{sat}(F)$ and $\mathrm{ord}(F) = s$. So $(F) = \mathrm{sat}(F) \cap \mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}] = [\mathbb{I}(V), \mathbb{L}_0, \ldots, \mathbb{L}_d] \cap \mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}]$. It remains to show $F \in \mathcal{Q}$. Regard $\mathcal{A} =< \mathbb{L}_0^{[s]}, \ldots, \mathbb{L}_d^{[s]} >$ as an algebraic autoreduced set with $\mathrm{ld}(\theta(\mathbb{L}_i)) = \theta(u_{i0})$, and let F_1 be the algebraic remainder of F with respect to $\mathcal{A}$, then $F_1 \in [\mathbb{I}(V), \mathbb{L}_0, \ldots, \mathbb{L}_d] \cap \mathcal{F}[\mathbb{Y}^{[s]}, \mathbf{u}^{[s]}] = \big(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]\big)$, where $\mathbf{u} = \cup_i \mathbf{u}_i \backslash \{u_{i0}\}$. So $F \in \mathcal{Q}$ and (4.1) follows. $\square$

Below is an example of Δ-Chow forms.

Example 4.8. Let $\mathcal{P} = [\delta_1(y_1), y_2 - y_1^2] \subset \mathcal{F}\{y_1, y_2\}$. Clearly, $\omega_{\mathcal{P}}(t) = \binom{t+2}{2} - \binom{t+1}{2} = t+1$. The Δ-Chow form of $\mathcal{P}$ is

$$F(\mathbf{u}_0) = \delta_1(u_{00})^2 u_{02}^2 - 2\delta_1(u_{00}) u_{00} \delta_1(u_{02}) u_{02} - \delta_1(u_{00}) \delta_1(u_{01}) u_{01} u_{02} + u_{00}^2 (\delta_1(u_{02}))^2 +$$

$$\delta_1(u_{00}) u_{01}^2 \delta_1(u_{02}) + u_{00} (\delta_1(u_{01}))^2 u_{02} - u_{00} \delta_1(u_{01}) u_{01} \delta_1(u_{02}) + \delta_1(u_{00}) \delta_1(u_{01}) u_{01} u_{02}.$$

5. Properties of the partial differential Chow form

In this section, we will prove basic properties of Δ-Chow forms. In particular, we will show the Δ-Chow forms are Δ-homogenous and prove the Δ-Chow form has a Poisson-type product formula similar to its ordinary differential counterpart.

5.1. Partial differential Chow forms are differentially homogenous.

In this section, we will show that the Δ-Chow form is also Δ-homogenous. Recall that $\mathcal{F}$ is a Δ-field with the set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and set of derivative operators Θ. Given two derivatives $\theta_1 = \prod_{i=1}^m \delta_i^{a_i}$ and $\theta_2 = \prod_{i=1}^m \delta_i^{b_i} \in \Theta$, if $a_i \leq b_i$ for each i, then we denote $\theta_1 | \theta_2$. In case $\theta_1 | \theta_2$, we denote $\frac{\theta_2}{\theta_1} = \prod_{i=1}^m \delta_i^{b_i - a_i}$, and denote the product of binomial coefficients $\prod_{i=1}^m \binom{b_i}{a_i}$ by $\binom{\theta_2}{\theta_1}$. It is easy to verify that $\theta(fg) = \sum_{\tau | \theta} \binom{\theta}{\tau} \cdot \frac{\theta}{\tau}(f) \cdot \tau(g)$ for all $f, g \in \mathcal{F}$.

Definition 5.1. A Δ-polynomial $f \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is said to be of Δ-*homogenous* of degree r if $f(\lambda y_0, \lambda y_1, \ldots, \lambda y_n) = \lambda^r f(y_0, y_1, \ldots, y_n)$ holds for a Δ-indeterminate λ over $\mathcal{F}\{y_0, y_1, \ldots, y_n\}$.

The following lemma is a partial differential analog of the Euler's criterion on homogenous polynomials, which was listed as an exercise in [14, p.71].

Lemma 5.2. *A necessary and sufficient condition that $f \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ be Δ-homogenous of degree r is that f satisfies the following system of equations:*

$$\sum_{\tau \in \Theta} \sum_{j=0}^{n} \binom{\tau\theta}{\theta} \tau(y_j) \cdot \frac{\partial f}{\partial \tau\theta(y_j)} = \begin{cases} rf, & \theta = 1 \\ 0, & \theta \in \Theta,\, \theta \neq 1. \end{cases} \tag{5.1}$$

Proof: Denote $\mathbb{Y} = (y_0, \ldots, y_n)$ temporarily for convenience. Let λ be a Δ-indeterminate over $\mathcal{F}\{\mathbb{Y}\}$.

First we show the necessity. Since f is Δ-homogenous of degree r, then $f(\lambda\mathbb{Y}) = \lambda^r f(\mathbb{Y})$. Differentiating both sides of this equality w.r.t. $\theta(\lambda)$, we get

$$\begin{aligned}
&\sum_{j=0}^{n}\sum_{\tau\in\Theta}\frac{\partial\tau\theta(\lambda y_j)}{\theta(\lambda)}\frac{\partial f}{\partial\tau\theta(y_j)}(\lambda\mathbb{Y}) = \sum_{\tau\in\Theta}\sum_{j=0}^{n}\binom{\tau\theta}{\theta}\tau(y_j)\cdot\frac{\partial f}{\partial\tau\theta(y_j)}(\lambda\mathbb{Y})\\
=\ &\frac{\partial f(\lambda\mathbb{Y})}{\partial\theta(\lambda)} = \begin{cases} rf(\mathbb{Y})\lambda^{r-1}, & \theta = 1\\ 0, & \theta\in\Theta, \theta\neq 1.\end{cases}
\end{aligned}$$

Setting $\lambda = 1$, we get (5.1).

For the sufficiency, we first choose an orderly ranking $\mathscr{R}$. Obviously, $\frac{\partial}{\partial\tau(\lambda)}f(\lambda\mathbb{Y}) = 0$ for all $\tau\in\Theta$ and $\mathrm{ord}(\tau) > \mathrm{ord}(f)$. Suppose $\theta\in\Theta$ satisfies that for all $\tau\in\Theta$, if $\theta|\tau$ and $\tau\neq\theta$, then $\frac{\partial}{\partial\tau(\lambda)}f(\lambda\mathbb{Y}) = 0$. Then

$$\begin{aligned}
\lambda\cdot\frac{\partial}{\partial\theta(\lambda)}f(\lambda\mathbb{Y}) &= \sum_{\tau\in\Theta}\binom{\tau\theta}{\theta}\tau(\lambda)\frac{\partial}{\partial\tau\theta(\lambda)}f(\lambda\mathbb{Y})\\
&= \sum_{\tau\in\Theta}\binom{\tau\theta}{\theta}\tau(\lambda)\sum_{j=0}^{n}\sum_{\xi\in\Theta}\binom{\xi\tau\theta}{\tau\theta}\xi(y_j)\frac{\partial}{\partial\xi\tau\theta(y_j)}f(\lambda\mathbb{Y})\\
&= \sum_{j=0}^{n}\sum_{\tau\in\Theta}\sum_{\xi\in\Theta,\xi|\tau}\binom{\frac{\tau}{\xi}\theta}{\theta}\frac{\tau}{\xi}(\lambda)\binom{\tau\theta}{\frac{\tau}{\xi}\theta}\xi(y_j)\frac{\partial}{\partial\tau\theta(y_j)}f(\lambda\mathbb{Y})\\
&= \sum_{j=0}^{n}\sum_{\tau\in\Theta}\binom{\tau\theta}{\theta}\Big(\sum_{\xi\in\Theta,\xi|\tau}\binom{\tau}{\xi}\frac{\tau}{\xi}(\lambda)\xi(y_j)\Big)\frac{\partial}{\partial\tau\theta(y_j)}f(\lambda\mathbb{Y})\\
&= \sum_{j=0}^{n}\sum_{\tau\in\Theta}\binom{\tau\theta}{\theta}\tau(\lambda y_j)\frac{\partial}{\partial\tau\theta(y_j)}f(\lambda\mathbb{Y})\\
&= \Big(\sum_{\tau\in\Theta}\sum_{j=0}^{n}\binom{\tau\theta}{\theta}\tau(y_j)\cdot\frac{\partial f}{\partial\tau\theta(y_j)}\Big)_{\mathbb{Y}=\lambda\mathbb{Y}}
\end{aligned}$$

By (5.1), if $\theta\neq 1$, then $\frac{\partial}{\partial\theta(\lambda)}f(\lambda\mathbb{Y}) = 0$, so $f(\lambda\mathbb{Y})$ is free from $\theta(\lambda)$ for all $1\neq\theta\in\Theta$; and if $\theta = 1$, we get $\lambda\cdot\frac{\partial}{\partial\lambda}f(\lambda\mathbb{Y}) = rf(\lambda\mathbb{Y})$, so $\frac{\partial\lambda^{-r}f(\lambda\mathbb{Y})}{\partial\lambda} = -r\lambda^{-r-1}f(\lambda\mathbb{Y}) + \lambda^{-r}\frac{\partial f(\lambda\mathbb{Y})}{\partial\lambda} = 0$ and $f(\lambda\mathbb{Y}) = \lambda^r f(\mathbb{Y})$ follows. Thus, $f(\mathbb{Y})$ is Δ-homogenous of degree r. □

Now, we show the Δ-homogeneity of Δ-Chow forms.

Theorem 5.3. *Let $V\subset\mathbb{A}^n$ be an irreducible Δ-variety of Kolchin polynomial $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. Let $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$ be the Δ-Chow form of V. Then $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$ is Δ-homogenous of the same degree r in each $\mathbf{u}_i$.*

Proof: By the definition of Δ-Chow form, $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$ has the symmetric property in the sense that interchanging $\mathbf{u}_i$ and $\mathbf{u}_j$ in F, the resulting polynomial and F differ at most by a sign. In particular, F is of the same degree in each $\mathbf{u}_i$. So it suffices to show the Δ-homogeneity of F for $\mathbf{u}_0$.

Let $\eta = (\eta_1,\ldots,\eta_n)$ be a generic point of V and $\zeta_i = -\sum_{j=1}^{n}u_{ij}\eta_j$. From the definition of the Δ-Chow form, $F(\zeta_0, u_{01},\ldots,u_{0n};\ldots;\zeta_d, u_{d1},\ldots,u_{dn}) = 0$. For each $j=1,\ldots,n$ and $\theta\in\Theta$ with $\mathrm{ord}(\theta)\le s=\mathrm{ord}(F)$, take the partial derivatives of both sides of this equality with respect to $\theta(u_{0j})$, then we get

$$\overline{\frac{\partial F}{\partial\theta(u_{0j})}}-\sum_{\tau\in\Theta}\overline{\frac{\partial F}{\partial\tau\theta(u_{00})}}\binom{\tau\theta}{\theta}\tau(\eta_j)=0,\ \ (j=1,\ldots,n) \tag{5.2}$$

where $\overline{\frac{\partial F}{\partial\theta(u_{0j})}}$ is obtained by substituting ζ_i for u_{i0} $(i=0,\ldots,d)$ in $\frac{\partial F}{\partial\theta(u_{0j})}$.

Fix a $\theta_1\in\Theta$. For each $\theta\in\Theta$ with $\theta_1|\theta$, multiply (5.2) by $\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})$ on both sides, and add these equalities together for all these θ and $j=1,\ldots,n$, then we get

$$\sum_{j=1}^{n}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta}}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\overline{\frac{\partial F}{\partial\theta(u_{0j})}}-\sum_{j=1}^{n}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta}}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\sum_{\tau\in\Theta}\overline{\frac{\partial F}{\partial\tau\theta(u_{00})}}\binom{\tau\theta}{\theta}\tau(\eta_j)=0. \tag{5.3}$$

Note that

$$\begin{aligned}
&-\sum_{j=1}^{n}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta}}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\sum_{\tau\in\Theta}\overline{\frac{\partial F}{\partial\tau\theta(u_{00})}}\binom{\tau\theta}{\theta}\tau(\eta_j)\\
\overset{\tau\to\frac{\tau}{\theta}}{=}\ &-\sum_{j=1}^{n}\sum_{\substack{\tau\in\Theta\\ \theta_1|\tau}}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta,\theta|\tau}}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\overline{\frac{\partial F}{\partial\tau(u_{00})}}\binom{\tau}{\theta}\frac{\tau}{\theta}(\eta_j)\\
=\ &-\sum_{j=1}^{n}\sum_{\substack{\tau\in\Theta\\ \theta_1|\tau}}\binom{\tau}{\theta_1}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta,\theta|\tau}}\binom{\tau/\theta_1}{\theta/\theta_1}\frac{\theta}{\theta_1}(u_{0j})\frac{\tau}{\theta}(\eta_j)\overline{\frac{\partial F}{\partial\tau(u_{00})}}\\
=\ &\sum_{\substack{\tau\in\Theta\\ \theta_1|\tau}}\binom{\tau}{\theta_1}\Big(-\sum_{j=1}^{n}\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta,\theta|\tau}}\binom{\tau/\theta_1}{\theta/\theta_1}\frac{\theta}{\theta_1}(u_{0j})\frac{\tau}{\theta}(\eta_j)\Big)\overline{\frac{\partial F}{\partial\tau(u_{00})}}\\
=\ &\sum_{\substack{\theta\in\Theta\\ \theta_1|\theta}}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(\zeta_0)\overline{\frac{\partial F}{\partial\theta(u_{00})}}.
\end{aligned}$$

So (5.3) is reduced to $\sum_{j=0}^{n}\sum_{\theta_1|\theta}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\overline{\frac{\partial F}{\partial\theta(u_{0j})}}=0$. Thus, the polynomial $G_{\theta_1}=\sum_{j=0}^{n}\sum_{\theta_1|\theta}\binom{\theta}{\theta_1}\frac{\theta}{\theta_1}(u_{0j})\frac{\partial F}{\partial\theta(u_{0j})}$ vanishes at $(\zeta_0,u_{01},\ldots,u_{0n};\ldots;\zeta_d,u_{d1},\ldots,u_{dn})$, which implies that $G_{\theta_1}\in\mathrm{sat}(F)$. Since $\mathrm{ord}(G_{\theta_1})\le\mathrm{ord}(F)$ and $\deg(G_{\theta_1})=\deg(F)$, $G_{\theta_1}=r\cdot F$ for some $r\in\mathcal{F}$. For a fixed orderly ranking $\mathscr{R}$ on $\mathbf{u}_0$, we consider the lex monomial ordering induced by $\mathscr{R}$. When $\theta_1\ne1$, note that the leading monomial of F will definitely not appear in G_{θ_1}, so G_{θ_1} must be a zero polynomial. And when $\theta_1=1$, G_1 and F can only differ by a nonnegative integer, so $r\in\mathbb{N}$. Thus, by Lemma 5.2, F is differentially homogenous in $\mathbf{u}_0$ of degree r. □

Definition 5.4. The number r in Theorem 5.3 is defined to be the Δ-*degree* of the Δ-variety V or its corresponding prime Δ-ideal.

5.2. **Factorization of partial differential Chow forms.** In this section, we fix an orderly ranking $\mathscr{R}$ on $\mathbf{u}_0,\ldots,\mathbf{u}_n$ with u_{00} greater than any other u_{ij}. Suppose $\mathrm{ld}(F)=\theta(u_{00})$ and θ is reserved for this derivative temporarily in this section. Let

$$\mathcal{F}_{\mathbf{u}}=\mathcal{F}\langle\mathbf{u}_1,\ldots,\mathbf{u}_d,u_{01},\ldots,u_{0n}\rangle\ \text{ and }\ \mathcal{F}_0=\mathcal{F}_{\mathbf{u}}\big(\tau(u_{00}):\tau\in\Theta,\theta\nmid\tau\big).$$

Regard F as a univariate polynomial $f\big(\theta(u_{00})\big)$ in $\theta(u_{00})$ with coefficients in $\mathcal{F}_0$ and suppose $g = \deg(F, \theta(u_{00}))$. Then $f\big(\theta(u_{00})\big)$ is irreducible over $\mathcal{F}_0$ and in a suitable algebraic extension field of $\mathcal{F}_0$, $f(\theta(u_{00})) = 0$ has g roots $\gamma_1, \ldots, \gamma_g$. Thus

$$f(\theta(u_{00})) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{l=1}^{g} \big(\theta(u_{00}) - \gamma_l\big) \tag{5.4}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ is free from $\theta(u_{00})$.

For each $l = 1, \ldots, g$, let

$$\mathcal{F}_l = \mathcal{F}_0\big(\gamma_l\big) \tag{5.5}$$

be an algebraic extension of $\mathcal{F}_0$ defined by $f\big(\theta(u_{00})\big) = 0$. We will define derivations $\delta_{l,1}, \ldots, \delta_{l,m}$ on $\mathcal{F}_l$ so that $\big(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\}\big)$ becomes a partial differential field. This can be done step by step in a very natural way. For the ease of notation, for each $\tau = \prod_{k=1}^m \delta_k^{d_k}$ with $(d_1, \ldots, d_m) \in \mathbb{N}^m$, we denote $\tau_l = \prod_{k=1}^m \delta_{l,k}^{d_k}$. In step 1, for each $a \in \mathcal{F}_{\mathbf{u}}$, define $\tau_l(a) = \tau(a)$, in particular, $\delta_{l,k}(a) = \delta_k(a)$ for each $k = 1, \ldots, m$. In step 2, we need to define the derivatives of u_{00}. For all $\tau \in \Theta$ with $\theta \nmid \tau$ or $\tau = \theta$, define $\tau_l(u_{00})$ as follows:

$$\tau_l(u_{00}) = \begin{cases} \tau(u_{00}) \in \mathcal{F}_l, & \theta \nmid \tau \\ \gamma_l \in \mathcal{F}_l, & \tau = \theta. \end{cases}$$

And for all $\tau \in \Theta$ with $\theta | \tau$ and $\tau \neq \theta$, we define $\tau_l(u_{00})$ inductively on the ordering of $\Theta(u_{00})$ induced by $\mathscr{R}$. Since F, regarded as a univariate polynomial f in $\theta(u_{00})$, is a minimal polynomial of γ_l, $\mathrm{S}_f = \frac{\partial f}{\partial \theta(u_{00})}$ does not vanish at $\theta(u_{00}) = \gamma_l$. First, for the minimal $\tau = \delta_k \theta$ for some $k \in \{1, \ldots, m\}$, define

$$\tau_l(u_{00}) = \delta_{l,k}(\gamma_l) = -\, T/\mathrm{S}_f\big|_{\theta(u_{00})=\gamma_l},$$

where $\delta_k(f) = S_f \cdot \delta_k \theta(u_{00}) + T$. This is reasonable, since all the derivatives of u_{00} involved in S_f and T have been defined in the former steps and we should have $\delta_{l,k}\big(f(\gamma_l)\big) = \mathrm{S}_f\big|_{\theta(u_{00})=\gamma_l} \delta_{l,k}(\gamma_l) + T\big|_{\theta(u_{00})=\gamma_l} = 0$. Suppose all the derivatives of u_{00} less than $\tau(u_{00}) = \prod_{k=1}^m \delta_k^{d_k} \theta(u_{00})$ have been defined, we can proceed in the similar way to define $\tau_l(u_{00}) = \prod_{k=1}^m \delta_{l,k}^{d_k}(\gamma_l)$. Namely, use the differential polynomial $\tau(f) = S_f \cdot \tau(u_{00}) + T_\tau$ and define $\tau_l(u_{00}) = -T_\tau/\mathrm{S}_f\big|_{\pi\theta(u_{00})=\pi(\gamma_l),\, \pi\theta<\tau}$. In this way, $(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\})$ is a partial differential field which can be considered as a finitely differential extension field of $(\mathcal{F}_{\mathbf{u}}, \Delta)$.

Since $\mathcal{F}_{\mathbf{u}}$ is a finitely generated Δ-extension field of $\mathcal{F}$ contained in $\mathcal{E}$. By the definition of universal differential extension fields, there exists a Δ-extension field $\mathcal{F}^* \subset \mathcal{E}$ of $\mathcal{F}_{\mathbf{u}}$ and a differential $\mathcal{F}_{\mathbf{u}}$-isomorphism φ_l from $(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\})$ to $(\mathcal{F}^*, \Delta)$. For a polynomial $G \in \mathcal{F}\{\mathbb{Y}\}$ and a point $\eta \in \mathcal{F}_l^n$, $G(\eta) = 0$ implies $G(\varphi_l(\eta)) = 0$. For convenience, by saying η is in a Δ-variety V over $\mathcal{F}$, we mean $\varphi_l(\eta) \in V$. Summing up the above results, we have

Lemma 5.5. $(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\})$ *is a finitely differential extension field of* $(\mathcal{F}_{\mathbf{u}}, \Delta)$, *which is differentially* $\mathcal{F}_{\mathbf{u}}$*-isomorphic to a differential subfield of* $\mathcal{E}$.

Note that the above defining steps give a differential homomorphism ϕ_l from $(\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}, \Delta)$ to the differential field $(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\})$ for each l by mapping $\tau(u_{ij})$ to $\tau_l(u_{ij})$. That is, for a Δ-polynomial $p \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$, $\phi_l(p)$ is obtained from p by substituting $\tau\theta(u_{00}) = \tau_l(\gamma_l)$. Then we have the following result.

Lemma 5.6. *Let $P \in \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$. Then $P \in \mathrm{sat}(F)$ if and only if $\phi_l(P) = 0$.*

Proof: If $P \in \mathrm{sat}(F)$, then there exists $m \in \mathbb{N}$ such that $S_F^m P \in [F]$. Since ϕ_l is a differential homomorphism and $\phi_l(F) = 0$, $\phi_l(S_F^m P) = 0$. Note from the above that $\phi_l(S_F) \neq 0$, so $\phi_l(P) = 0$ follows. For the other side, suppose $\phi_l(P) = 0$. Let R be the differential remainder of P w.r.t. F under the ranking $\mathscr{R}$. Since $\phi_l(P) = 0$, $\phi_l(R) = 0$. Note that R is free from all the proper derivatives of $\theta(u_{00})$ and $\deg(R, \theta(u_{00})) < g$. So $R|_{\theta(u_{00})=\gamma_l} = 0$, which implies from the irreducibility of F that R is divisible by F. Thus, $R = 0$ and $P \in \mathrm{sat}(F)$. □

Remark 5.7. Similar to the ordinary differential case, in order to make $\mathcal{F}_l$ a partial differential field, we need to introduce differential operator $\delta_{l,1}, \ldots, \delta_{l,m}$ related to γ_l and there does not exist a unique set of differential operators to make all $\mathcal{F}_l (l = 1, \ldots, g)$ differential fields.

Below, we now give the following Poisson-type product formula.

Theorem 5.8. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ be the Δ-Chow form of an irreducible Δ-variety over $\mathcal{F}$ of Kolchin polynomial $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. Fix an orderly ranking with $u_{00} > u_{ij}$ and suppose $\mathrm{ld}(F) = \theta(u_{00})$ and $g = \deg\big(F, \theta(u_{00})\big)$. Then, there exist $\xi_{l1}, \ldots, \xi_{ln}$ in a differential extension field $(\mathcal{F}_l, \{\delta_{l,1}, \ldots, \delta_{l,m}\})$ of $(\mathcal{F}_{\mathbf{u}}, \Delta)$ such that*

$$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{l=1}^{g} \theta\big(u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{l\rho}\big) \tag{5.6}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is in $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$. Note that equation (5.6) is formal and should be understood in the following precise meaning: $\theta(u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{l\rho}) \stackrel{\triangle}{=} \theta(u_{00}) + \theta_l(\sum_{\rho=1}^{n} u_{0\rho}\xi_{l\rho})$.

Proof: We will follow the notations above. By Lemma 5.6, $\phi_l(S_F) \neq 0$. Let $\xi_{lj} = \phi_l(\frac{\partial F}{\partial \theta(u_{0j})}) / \phi_l(S_F)$ for $j = 1, \ldots, n$ and $\xi_l = (\xi_{l1}, \ldots, \xi_{ln}) \in \mathcal{F}_l^n$. We will prove

$$\gamma_l = -\theta_l\Big(\sum_{j=1}^{n} u_{0j}\xi_{lj}\Big).$$

Differentiating $F(\zeta_0, u_{01}, \ldots, u_{0n}; \ldots; \zeta_d, u_{d1}, \ldots, u_{dn}) = 0$ w.r.t. $\theta(u_{0j})$ on both sides, we have

$$\overline{\frac{\partial F}{\partial \theta(u_{0j})}} + \overline{\frac{\partial F}{\partial \theta(u_{00})}} \cdot (-\xi_j) = 0, \tag{5.7}$$

where the $\overline{\frac{\partial F}{\partial \theta(u_{0j})}}$ are obtained by substituting ζ_i to $u_{i0}\,(i = 0, \ldots, d)$ in $\frac{\partial F}{\partial \theta(u_{0j})}$. Multiplying u_{0j} to the above equation and for j from 1 to n, adding them together, we have

$$\sum_{j=1}^{n} u_{0j} \overline{\frac{\partial F}{\partial \theta(u_{0j})}} + \overline{\frac{\partial F}{\partial \theta(u_{00})}} \cdot (-\sum_{j=1}^{n} u_{0j}\xi_j) = \sum_{j=1}^{n} u_{0j} \overline{\frac{\partial F}{\partial \theta(u_{0j})}} + \overline{\frac{\partial F}{\partial \theta(u_{00})}} \cdot \zeta_0 = 0.$$

Thus, $\sum_{j=0}^{n} u_{0j} \frac{\partial F}{\partial \theta(u_{0j})} \in \mathrm{sat}(F)$. By Lemma 5.6,

$$\sum_{j=1}^{n} u_{0j}\phi_l\big(\frac{\partial F}{\partial \theta(u_{0j})}\big) + \phi_l(u_{00})\phi_l\big(\frac{\partial F}{\partial \theta(u_{00})}\big) = 0,$$

so $\phi_l(u_{00}) = -\sum_{j=1}^n u_{0j}\xi_{lj}$. Thus, $\theta_l(\phi_l(u_{00})) = \phi_l(\theta(u_{00})) = \gamma_l = -\theta_l(\sum_{j=1}^n u_{0j}\xi_{lj})$. Substituting them into equation (5.4), (5.6) is proved. $\square$

Theorem 5.9. *The points $(\xi_{l1}, \ldots, \xi_{ln})\,(l = 1, \ldots, g)$ in (5.6) are generic points of the Δ-variety V over $\mathcal{F}$. If $d > 0$, they also satisfy the equations*

$$u_{\sigma 0} + \sum_{\rho=1}^{n} u_{\sigma\rho} y_\rho = 0\,(\sigma = 1, \ldots, d).$$

Proof: Suppose $P(y_1, \ldots, y_n) \in \mathcal{F}\{\mathbb{Y}\}$ is any Δ-polynomial vanishing on V. Then $P(\xi_1, \ldots, \xi_n) = 0$. From (5.7), $\xi_\rho = \overline{\frac{\partial f}{\partial\theta(u_{0\rho})}} \Big/ \overline{\frac{\partial f}{\partial\theta(u_{00})}}$, so we have

$$P\Big(\overline{\frac{\partial F}{\partial\theta(u_{01})}} \Big/ \overline{\frac{\partial F}{\partial\theta(u_{00})}}, \ldots, \overline{\frac{\partial F}{\partial\theta(u_{0n})}} \Big/ \overline{\frac{\partial F}{\partial\theta(u_{00})}}\Big) = 0,$$

where $\overline{\frac{\partial F}{\partial\theta(u_{0\rho})}}$ are obtained by substituting ζ_i to $u_{i0}\,(i = 0, 1, \ldots, d)$ in $\frac{\partial f}{\partial\theta(u_{0\rho})}$. Thus, there exists an $m \in \mathbb{N}$, such that

$$(\frac{\partial F}{\partial\theta(u_{00})})^m \cdot P\Big(\frac{\partial F}{\partial\theta(u_{01})} \Big/ \frac{\partial F}{\partial\theta(u_{00})}, \ldots, \frac{\partial F}{\partial\theta(u_{0n})} \Big/ \frac{\partial F}{\partial\theta(u_{00})}\Big) \in \mathrm{sat}(F).$$

By Lemma 5.6, we have $P(\xi_{l1}, \ldots, \xi_{ln}) = 0$, which means that $(\xi_{l1}, \ldots, \xi_{ln}) \in V$.

Conversely, for any $Q \in \mathcal{F}\{\mathbb{Y}\}$ such that $Q(\xi_{l1}, \ldots, \xi_{ln}) = 0$, by Lemma 5.6, there exists an $l \in \mathbb{N}$ such that $\widetilde{Q} = (\frac{\partial F}{\partial\theta(u_{00})})^l Q(\frac{\partial F}{\partial\theta(u_{01})} \Big/ \frac{\partial F}{\partial\theta(u_{00})}, \ldots, \frac{\partial F}{\partial\theta(u_{0n})} \Big/ \frac{\partial f}{\partial\theta(u_{00})}) \in \mathrm{sat}(F)$. So $Q(\xi_1, \ldots, \xi_n) = 0$. Thus, $(\xi_{l1}, \ldots, \xi_{ln})$ is a generic point of V.

By equation (5.7), $\overline{\frac{\partial F}{\partial\theta(u_{0j})}} + \overline{\frac{\partial F}{\partial\theta(u_{00})}} \cdot (-\xi_j) = 0$, so we have $\sum_{j=1}^n u_{\sigma j} \overline{\frac{\partial F}{\partial\theta(u_{0j})}} + \zeta_\sigma \overline{\frac{\partial F}{\partial\theta(u_{00})}} = 0$. Thus, $\sum_{j=0}^n u_{\sigma j} \frac{\partial F}{\partial\theta(u_{0j})} \in \mathrm{sat}(F)$. If $\sigma \neq 0$, then $\sum_{j=0}^n u_{\sigma j} \phi_l(\frac{\partial F}{\partial\theta(u_{0j})}) = 0$. Consequently, $u_{\sigma 0} + \sum_{j=1}^n u_{\sigma j}\xi_{\tau j} = 0\,(\sigma = 1, \ldots, d)$. $\square$

Remark 5.10. The leading differential degree could not be defined in the partial differential case, for the number g in Theorem 5.8 depends on the ranking we choose to get the Poisson-type product formula. Also, it may happen that under any orderly ranking, the leaders of the Δ-Chow forms of two irreducible Δ-varieties with the same Kolchin polynomial are alway different, so it is difficult to define partial differential cycles as we did in the ordinary differential case.

We conclude this section by showing that the vanishing of the Δ-Chow form gives a necessary and "sufficient" condition (in the sense of Kolchin closure) such that V and $d + 1$ number of Δ-hyperplanes have a nonempty intersection.

Theorem 5.11. *Let V be an irreducible Δ-variety of Kolchin polynomial $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ and $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ the Δ-Chow form of V. The following assertions hold.*

1) *Let $\mathscr{R}$ be some elimination ranking satisfying $u_{ij} < u_{00} < y_1 \cdots < y_n$. Let $\mathrm{ld}(F) = \theta(u_{00})$ and S_F the separant of F. Then*

$$\{F, \mathrm{S}_F y_1 - \frac{\partial F}{\partial\theta(u_{01})}, \ldots, \mathrm{S}_F y_n - \frac{\partial F}{\partial\theta(u_{0n})}\}$$

is a characteristic set of $[\mathbb{I}(V), \mathbb{L}_0, \ldots, \mathbb{L}_d]_{\mathcal{F}\{\mathbb{Y}, \mathbf{u}_0, \ldots, \mathbf{u}_d\}}$ w.r.t. $\mathscr{R}$.

2) *For any given* $(\mathbf{v}_0,\dots,\mathbf{v}_d)\in(\mathbb{P}^n)^{d+1}$*, if* $V\bigcap\cap_i\mathbb{V}(v_{i0}+v_{i1}y_1+\cdots+v_{in}y_n)\neq\emptyset$*, then* $F(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$*. And if* $F(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$ *and* $\mathrm{S}_F(\mathbf{v}_0,\dots,\mathbf{v}_d)\neq 0$*, then* V *and* $v_{i0}+v_{i1}y_1+\cdots+v_{in}y_n=0\,(i=0,\dots,d)$ *have at least one point in common.*

Proof: The proof of item 1) is similar to the ordinary differential case. And item 2) is a direct consequence of item 1). □

6. The existence of a type of partial differential Chow varieties

As mentioned in the introduction, to study a specific kind of geometric objects, it is important and useful to represent them by coordinates and further show that the set of objects is actually an algebraic system. For us, this specific kind of objects are irreducible Δ-varieties with Kolchin dimension polynomial $(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$. As in the ordinary differential case, we could give these Δ-varieties coordinate representations via their Δ-Chow forms.

Definition 6.1. Let V be an irreducible Δ-variety over $\mathcal{F}$ of Kolchin polynomial $\omega_V(t)=(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$ and of Δ-degree r. Let $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ be the Δ-Chow form of V. The coefficient vector of F, regarded as a point in a projective space determined by (n,d,s,r), is defined to be the Δ-*Chow coordinate* of V.

Definition 6.2. Fix an index (n,d,s,r). Let $G_{(n,d,s,r)}$ be a functor from the category of Δ-fields to the category of sets which associates each Δ-field $\mathcal{K}$ with the set $G_{(n,d,s,r)}(\mathcal{K})$, consisting of all irreducible Δ-varieties $V\subset\mathbb{A}^n$ over $\mathcal{K}$ with $\omega_V(t)=(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$ and Δ-degree r. If this functor is represented by some Δ-constructible set, meaning that there is a Δ-constructible set and a natural isomorphism between the functor $G_{(n,d,s,r)}$ and the functor given by this Δ-constructible set (regarded also as a functor from the category of Δ-fields to the category of sets), then we call this Δ-constructible set the Δ-*Chow variety* of index (n,d,s,r) of $\mathbb{A}^n$, and also say the Δ-Chow variety of index (n,d,s,r) exists.

In this section, we will show that Δ-Chow varieties of index (n,d,s,r) exist for all chosen n,d,s,r. Similar to the ordinary differential case, the main idea is to first definably embed $G_{(n,d,s,r)}$ into a finite disjoint union $\mathcal{C}$ of the chosen algebraic Chow varieties and then show the image of $G_{(n,d,s,r)}$ is a definable subset of $\mathcal{C}$. So, the language from model theory of partial differentially closed fields (see [21, 24, 27]) will be used and we assume $\mathcal{E}$ is a Δ-closed field of characteristic 0 (i.e., $\mathcal{E}\models\mathrm{DCF}_{0,m}$) throughout this section.

6.1. **Definable properties and Prolongation admissible varieties.**

Here are some basic notions and results from model theory that we will be used in the proof of the main theorem. For more details and explantations, see [6].

We say that a family of sets $\{X_a\}_{a\in B}$ is a *definable family* if there are formulae $\psi(x;y)$ and $\phi(y)$ so that B is the set of realizations of ϕ (i.e., $B=\{\bar{e}\in\mathcal{E}^n:\mathcal{E}\models\phi(\bar{e})\}$) and for each $a\in B$, X_a is the set of realizations of $\psi(x;a)$.

Given a property $\mathcal{P}$ of definable sets, we say that $\mathcal{P}$ is definable in families if for any family of definable sets $\{X_a\}_{a\in B}$ given by the formulae $\psi(x;y)$ and $\theta(y)$, there is a formula $\phi(y)$ so that the set $\{a\in B\,:\,X_a$ has property $\mathcal{P}\}$ is defined by ϕ.

Given an operation $\mathcal{F}$ which takes a set and returns another set, we say that $\mathcal{F}$ is definable in families if for any family of definable sets $\{X_a\}_{a\in B}$ given by the formulae $\psi(x;y)$ and $\theta(y)$, there is formula $\phi(z;y)$ so that for each $a \in B$, the set $\mathcal{F}(X_a)$ is defined by $\phi(z;a)$.

We will require the following facts about definability in algebraically closed fields.

Fact 6.3. [6] Relative to the theory of algebraically closed fields (ACF), we have the the following statements.

(1) The Zariski closure is definable in families.
(2) The dimension and degree of the Zariski closure of a set are definable in families.
(3) Irreducibility of the Zariski closure is a definable property.
(4) If the Zariski closure is an irreducible hypersurface given by the vanishing of some nonzero polynomial, then the degree of that polynomial in any particular variable is definable in families.
(5) The set of irreducible varieties in $\mathbb{A}^n$ of dimension d and degree g is a definable family.

We also need to generalise results on prolongation admissible varieties [6] to the partial differential case. Notations τ_l, ∇_l, B_l should be specified beforehand. For an algebraic variety $X = \mathcal{V}(f_1,\dots,f_o) \subset \mathbb{A}^n$ defined by polynomials $f_i \in \mathcal{F}[y_1,\dots,y_n]$, $\tau_l(X) \subseteq \mathbb{A}^{n\binom{l+m}{m}}$ denotes the algebraic variety defined by $(\theta(f_i))_{\theta\in\Theta_{\leq l}}$ considered as algebraic polynomials in $\mathcal{F}[\Theta_{\leq l}(\mathbb{Y})]$ with $\mathbb{Y} = (y_1,\dots,y_n)$. Thus, $\tau_l\mathbb{A}^n = \mathbb{A}^{n\binom{l+m}{m}}$ with coordinates corresponding to variables $(\mathbb{Y}, \Theta_1(\mathbb{Y}),\dots,\Theta_l(\mathbb{Y}))$. Given a point $\bar{a} \in \mathbb{A}^n$, $\nabla_l(\bar{a})$ denotes the point $\big(\bar{a}, \Theta_1(\bar{a}),\dots,\Theta_l(\bar{a})\big) \in \tau_l\mathbb{A}^n$, and for a Δ-variety $W \subset \mathbb{A}^n$, $B_l(W)$ is the Zariski closure of the set $\{\nabla_l(\bar{a}) : \bar{a} \in \mathbb{W}\}$. In other words, $B_l(W) = \mathcal{V}\big(\mathbb{I}(W) \cap \mathcal{F}[\Theta_{\leq l}(\mathbb{Y})]\big) \subseteq \tau_l\mathbb{A}^n$.

Definition 6.4. Let $V \subset \tau_s\mathbb{A}^n$ be an algebraic variety. We say V is *prolongation admissible* if $B_s\big(\mathbb{V}(\mathcal{I}(V))\big) = V$.

Lemma 6.5. *Let $V \subset \tau_s\mathbb{A}^n$ be an irreducible prolongation admissible variety and $\mathcal{A}$ a characteristic set of V w.r.t. an ordering induced by some orderly ranking $\mathscr{R}$ on $\Theta(\mathbb{Y})$. For each $k = 1,\dots,n$, let*

$$E_k = \{\theta y_k \in \mathrm{ld}(\mathcal{A}) :\ \forall \tau y_k \in \mathrm{ld}(\mathcal{A}), \tau|\theta \Rightarrow \tau(y_k) = \theta(y_k)\}.$$

If $E_k \neq \emptyset$, then for each $\tau y_k \in \Theta_{\leq s}(y_k)$ which is a proper derivative of some element of E_k, there exists $A_{\tau,k} \in \mathcal{A}$ such that $\mathrm{ld}(A_{\tau,k}) = \tau y_k$ and $A_{\tau,k}$ is linear in τy_k.

Proof: Let $W = \mathbb{V}\big(\mathcal{I}(V)\big) \subset \mathbb{A}^n$ and $W = \cup_{i=1}^{l} W_i$ be the irreducible decomposition of W. Since V is prolongation admissible, $B_s(W) = V$. So there exists some i_0 such that $B_s(W_{i_0}) = V$. Suppose $\mathcal{B}$ is a Δ-characteristic set of W_{i_0} w.r.t. $\mathscr{R}$. Let $\mathcal{C} = \Theta(\mathcal{B}) \cap \mathcal{F}[\Theta_{\leq s}(\mathbb{Y})]$, $\mathcal{C}$ is a characteristic set of $B_s(W_{i_0}) = V$. Since $\mathcal{C}$ and $\mathcal{A}$ have the same rank, $\mathcal{A}$ should satisfy the desired property. $\square$

We now show that prolongation admissibility is a definable property.

Lemma 6.6. *Prolongation admissibility is definable in families.*

Proof: Let $(V_b)_{b\in B}$ be a definable family of algebraic varieties in $\tau_s\mathbb{A}^n$ with V_b defined by $f_i\big(b, (\theta(y_j)_{\theta\in\Theta_{\leq s}, 1\leq j\leq n})\big) = 0$, $i = 1,\dots,\ell$. By abuse of notation, let $B_s(V_b)$ be the Zariski closure of $\{\nabla_s(\bar{a}) : \nabla_s(\bar{a}) \in V_b\}$ in $\tau_s\mathbb{A}^n$. Then $\deg(B_s(V_b))$ has a uniform bound T in terms of the degree bound D of the f_i, m, n, ℓ and s. Indeed, let $z_{j,\theta}\,(j = 1,\dots,n; \theta \in \Theta_{\leq s})$ be new Δ-variables and replace $\theta(x_j)$ by

$z_{j,\theta}$ in each f_i to get a new differential polynomial g_i. Consider the new differential system $S := \{g_1, \ldots, g_\ell, \delta_k(z_{j,\theta}) - z_{j,\delta_k\theta} :\ k = 1, \ldots, m; \theta \in \Theta_{\leq s-1}\}$. Regard S as a pure algebraic polynomial system in $z_{j\theta}$ and $\delta_k(z_{j\theta})$ temporarily, and let U be the Zariski closed set defined by S in $\tau(\tau_s\mathbb{A}^n)$. Let $Z = \{\bar{c} \in \tau_s\mathbb{A}^n : \nabla(\bar{c}) \in U\}$. Clearly, $Z = \{\nabla_s(\bar{a}) : \nabla_s(\bar{a}) \in V_b\}$. By [8, Corollary 4.5], the degree of the Zariski closure of Z, namely $B_s(V_b)$, is bounded by some number D_1 which depend on D, m, n, ℓ and s.

By [11, Proposition 3], an irreducible algebraic variety $V \in \tau_s\mathbb{A}^n$ can be defined by $n\big(\binom{s+m}{m}+1\big)+1$ polynomials of degree bounded by the degree of V. So $B_s(V_b)$ can be defined by at most $n\big(\binom{s+m}{m}+1\big)+1$ polynomials of degree bounded by D_1^2. Hence, $(B_s(V_b))_{b\in B}$ is a definable family. Since V_b is prolongation admissible if and only if $V_b = B_s(V_b)$, which implies that $\{b : V_b \text{ is prolongation admissible}\}$ is a definable set. Thus, prolongation admissibility is definable in families. $\square$

Definition 6.7. Let $V \subset \tau_l\mathbb{A}^n$ be an irreducible prolongation admissible variety and $W = \mathbb{V}(I(V))$ be the Δ-variety defined by defining equations of V. A component W_1 of W is called a *dominant component* if $B_s(W_1) = V$.

The following result shows how to get the desired unique irreducible Δ-varieties from irreducible prolongation admissible varieties.

Lemma 6.8. *Let $V \subseteq \tau_s(\mathbb{A}^n)$ be an irreducible prolongation admissible variety of dimension $(d+1)\binom{s+m}{m}-1$ and in the case $s>0$, suppose $\pi_{s,0}(V)$ is of dimension $d+1$. Then $W = \mathbb{V}(I(V))$ has a unique dominant component W_1 and $\omega_{W_1}(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$.*

Proof: Two cases should be considered according to whether $s = 0$ or not.

Case 1) $s = 0$. In this case, $\mathcal{P} = I(V)$ is a prime ideal of $\mathcal{F}[y_1, \ldots, y_n]$ of dimension d. By [14, p.200, Proposition 10], $\{\mathcal{P}\}$ is a prime Δ-ideal of $\mathcal{F}\{y_1, \ldots, y_n\}$ with Kolchin Δ-polynomial $\omega_{\{\mathcal{P}\}}(t) = d\binom{t+m}{m}$. Thus, $W = \mathbb{V}(\mathcal{P})$ itself is its dominant component and satisfies the desired property.

Case 2) $s > 0$. Fix an orderly ranking $\mathscr{R}$ on $\mathcal{F}\{\mathbb{Y}\}$ and denote $\mathscr{R}_l$ to be the ordering on $\Theta_{\leq l}(\mathbb{Y})$ induced by $\mathscr{R}$. Since $\pi_{s,0}(V)$ is of dimension $d+1$, a characteristic set of $\overline{\pi_{s,0}(V)}$ w.r.t. $\mathscr{R}_0$ is of the form $B_1, \ldots, B_{n-d-1}$ where $\mathrm{ld}(B_i) = y_{\sigma_i}$ for each i. Since V is irreducible and prolongation admissible, by Lemma 6.5, $S = \{\theta(y_{\sigma_i}) :\ \mathrm{ord}(\theta) \leq s, i = 1, \ldots, n-d-1\}$ is a subset of the leaders of a characteristic set $\mathcal{A}$ of V w.r.t. $\mathscr{R}_s$. Since the dimension of V is $(d+1)\binom{s+m}{m}-1$, $\mathrm{ld}(\mathcal{A}) = S \cup \{\tau(y_{\sigma_{n-d}})\}$ for some $\tau \in \Theta_s$ and $\sigma_{n-d} \in \{1, \ldots, n\}\backslash\{\sigma_1, \ldots, \sigma_{n-d-1}\}$. So there exists $B_{n-d} \in \mathcal{A}$ s.t. $\mathrm{ld}(B_{n-d}) = \tau(y_{\sigma_{n-d}})$.

Let $\mathcal{B} =< B_1, \ldots, B_{n-d} >$. Clearly, $\mathcal{B}$ is an irreducible coherent autoreduced set of $\mathcal{F}\{y_1, \ldots, y_n\}$, by [14, Lemma 2, p.167], $\mathcal{B}$ is a Δ-characteristic set of a prime Δ-ideal $\mathcal{P} \subset \mathcal{F}\{y_1, \ldots, y_n\}$ w.r.t. $\mathscr{R}$. Clearly, $\mathcal{P} = \mathrm{sat}(\mathcal{B})$ has Kolchin polynomial $\omega_{\mathcal{P}}(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. We now show that $\mathbb{V}(\mathcal{P}) \subseteq W$ and $B_s(\mathbb{V}(\mathcal{P})) = V$. Since V is an irreducible prolongation admissible variety, there exists a point $\bar{a} \in \mathbb{A}^n$ such that $\nabla_s(\bar{a})$ is a generic point of V. So as Δ-polynomials, B_i vanishes at $\bar{a}$ while $H_{\mathcal{B}}$ does not. Thus, $\mathcal{P}$ vanishes at $\bar{a}$, and consequently, $\nabla_s(\bar{a}) \in B_s(\mathbb{V}(\mathcal{P}))$. So $V \subset B_s(\mathbb{V}(\mathcal{P}))$. Since both V and $B_s(\mathbb{V}(\mathcal{P}))$ are of the same dimension, $B_s(\mathbb{V}(\mathcal{P})) = V$. So, $\mathcal{I}(V) = \mathcal{P} \cap \mathcal{F}[\Theta_{\leq s}(\mathbb{Y})] \subset \mathcal{P}$, as a consequence, $\mathbb{V}(\mathcal{P}) \subset \mathbb{V}(I(V)) = W$.

Suppose W_0 is a dominant component of W. Given a generic point $\xi \in W_0$, $\nabla_s(\xi)$ is a generic point of V. So, $\mathcal{B}$ vanishes at ξ and $H_{\mathcal{B}}$ does not vanish at ξ.

Thus, $\mathbb{V}(\mathcal{P})$ vanishes at ξ, i.e., $W_0 \subseteq \mathbb{V}(\mathcal{P})$. So $W_0 = \mathbb{V}(\mathcal{P})$. Thus, $\mathbb{V}(\mathcal{P})$ is the unique dominant component W and $\omega_{W_1}(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$. □

6.2. **Proof of the main theorem.** Before proving the main theorem, we need to bound the degree of $B_s(V)$ to get the candidates of the algebraic Chow varieties which can be used to paramertrize Δ-varieties in $G_{(n,d,s,r)}$.

Lemma 6.9. *Let $V \subset \mathbb{A}^n$ be an irreducible Δ-variety in $G_{(n,d,s,r)}$. Then $B_s(V)$ is an irreducible variety in $\tau_s(\mathbb{A}^n)$ of dimension $(d+1)\binom{s+m}{m} - 1$ and the degree of $B_s(V)$ satisfies*

$$r\Big/\binom{s+m}{m} \leq \deg(B_s(V)) \leq \big[(s+1)(d+1)r\big]^{n(s+1)\binom{s+m}{m}+1}.$$

Proof: Clearly, $B_s(V)$ is an irreducible variety in $\tau_s(\mathbb{A}^n)$ of dimension $(d+1)\binom{s+m}{m} - 1$. For the degree bound, we first show $\deg(B_s(V)) \leq \big[(s+1)(d+1)r\big]^{n(s+1)\binom{s+m}{m}+1}$. Since $V \in \mathbb{V}_{(n,d,s,r)}$, the Δ-Chow form $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ of V exists, and we have $\mathrm{ord}(F) = s$ and $\deg(F, \mathbf{u}_0^{[s]}) = r$. Let $\mathcal{J} = [\mathbb{I}(V), \mathbb{L}_0, \ldots, \mathbb{L}_d]_{\mathcal{F}\{\mathbb{Y}, \mathbf{u}_0, \ldots, \mathbf{u}_d\}}$. Let $\mathscr{R}$ be a ranking on $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d, \mathbb{Y}\}$ satisfying 1) $\theta(u_{ij}) < \tau(y_k)$ for any θ and τ, and 2) $\mathscr{R}$ restricted to $\mathbf{u}_0, \ldots, \mathbf{u}_d$ is an orderly ranking. Let $\mathscr{R}_s$ be the ordering on $\mathbf{u}_0^{[2s]}, \ldots, \mathbf{u}_d^{[2s]}$ and $\mathbb{Y}^{[s]}$ induced by $\mathscr{R}$. Suppose $\mathrm{ld}(F) = \theta(u_{00})$ for some $\theta \in \Theta_s$ and $\mathrm{S}_F = \frac{\partial F}{\partial \theta(u_{00})}$.

By Theorem 5.11, the polynomial $G_j = \mathrm{S}_F y_j - \frac{\partial F}{\partial \theta(u_{0j})} \in \mathcal{J}$ and note that $\deg(G_j) = (d+1)r$. We construct polynomials $G_{j,\theta} \in \mathcal{J}$ for $\theta \in \Theta_{\leq s}$ with $\mathrm{rk}(G_{j,\theta}) = \theta(y_j)$ and $\deg(G_{j,\theta}) \leq (\mathrm{ord}(\theta)+1)(d+1)r$ inductively on the order of θ. Set $G_{j,1} = G_j$. Let $G_{j,\delta_i} = \mathrm{rem}(\delta_i(G_{j,1}), G_{j,1})$ be the algebraic remainder of $\delta_i(G_{j,1})$ with respect to $G_{j,1}$. Clearly, $G_{j,\delta_i} \in \mathcal{J}$ and is of the form $G_{j,\delta_i} = \mathrm{S}_F^2 \delta_i(y_j) + T_{j,\delta_i}$ for some $T_{j,\delta_i} \in \mathcal{F}[\mathbf{u}^{[s+1]}]$. An easy calculation shows that $\deg(G_{j,\delta_i}) \leq 2(d+1)r$. Suppose the desired $G_{j,\tau} = \mathrm{S}_F^{\mathrm{ord}(\tau)+1}\tau(y_j) + T_{j,\tau}$ ($\tau \in \Theta_{\leq k}$) have been constructed, we now define $G_{j,\tau}$ ($\tau \in \Theta_{k+1}$). For $\tau \in \Theta_{k+1}$, let $G_{j,\tau}$ be the algebraic remainder of $\tau(G_j)$ with respect to $< G_{j,\tau} : \tau \in \Theta_{\leq k}; k \leq s >$. Then $G_{j,\tau} \in \mathcal{J}$ and $G_{j,\tau} = \mathrm{S}_F^{k+2}\tau(y_j) + T_{j,\tau}$,where $T_{j,\tau} \in \mathcal{F}[\mathbf{u}^{[k+1+s]}]$ satisfies $\deg(T_{j,\tau}) \leq (k+2)(d+1)r$. In this way, polynomials $G_{j,\tau} \in \mathcal{J}$ ($\tau \in \Theta_{\leq s}$) are constructed.

Clearly, $< F^{[s]}, G_{j,\tau} : \tau \in \Theta_{\leq s} >$ is an irreducible ascending chain under $\mathscr{R}_s$, so $\mathcal{J}_s = \big(F^{[s]}, (G_{j,\tau})_{\tau \in \Theta_{\leq s}}\big) : \mathrm{S}_F^{\infty}$ is a prime ideal in $\mathcal{F}[\mathbb{Y}^{[s]}, \mathbf{u}_0^{[2s]}, \mathbf{u}_1^{[2s]}, \ldots, \mathbf{u}_d^{[2s]}]$, which is a component of $\mathcal{V}(F, (G_{j,\tau})_{\tau \in \Theta_{\leq s}})$. By Beźout Theorem [11, Theorem 1],

$$\begin{aligned}
\deg(\mathcal{J}_s) &\leq [(d+1)r]^{\binom{s+m}{m}} \cdot \prod_{j=1}^{n} \prod_{\theta \in \Theta_{\leq s}} \deg(G_{j\theta}) \\
&\leq [(d+1)r]^{\binom{s+m}{m}} \cdot \prod_{j=1}^{n} \prod_{l=0}^{s} [(l+1)(d+1)r]^{\binom{s+m}{m}} \\
&\leq [(s+1)(d+1)r]^{n(s+1)\binom{l+m}{m}+1}.
\end{aligned}$$

Let $\mathcal{J}_s' = \mathcal{J}_s \cap \mathcal{F}[\mathbb{Y}^{[s]}]$. We claim that $\mathcal{J}_s' = \mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]$. Indeed, on the one hand, $\mathcal{J}_s' \subset \mathcal{J} \cap \mathcal{F}[\mathbb{Y}^{[s]}] = \mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]$; on the other hand, for any polynomial $H \in \mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]$, the algebraic remainder of H with respect to $< G_{j,\tau} : \tau \in \Theta_{\leq s} >$ is a polynomial $H_1 \in \mathcal{J} \cap \mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\} = \mathrm{sat}(F)$ with $\mathrm{ord}(H_1) \leq 2s$.

Thus, $H_1 \in \text{asat}(F^{[s]})$ and $H \in \mathcal{J}_s$. So by [11, Lemma 2] or [19, Theorem 2.1], $\deg(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]) = \deg(\mathcal{J}_s') \leq \deg(\mathcal{J}_s)$.

Now, we show $\deg(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]) \geq r/\binom{s+m}{m}$. By Lemma 4.7, $\big[\mathcal{I} \cap \mathcal{F}[\mathbb{Y}^{[s]}], \mathbb{L}_0^{[s]}, \ldots, \mathbb{L}_d^{[s]}\big] \cap \mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}] = (F)$. Similar to the procedures in [18, Theorem 6.25], the Δ-Chow form of $\mathbb{I}(V)$ could be obtained from the algebraic Chow form of $\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]$ by algebraic specializations. So $(d+1)r \leq (d+1)\binom{s+m}{m}\deg(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}])$ and $\deg(B_s(V)) = \deg(\mathbb{I}(V) \cap \mathcal{F}[\mathbb{Y}^{[s]}]) \geq r/\binom{s+m}{m}$. $\square$

Remark 6.10. In the ordinary differential case, the construction of $G_{j,k}$ is much easier and each $G_{j,k}$ $(k \leq s)$ could be chosen from $\mathcal{F}[\mathbf{u}_0^{[s]}, \ldots, \mathbf{u}_d^{[s]}, \mathbb{Y}^{[s]}]$. However, we could not construct $G_{j,\theta}$ in that way for there may exist $\tau \in \Theta_{\leq s}$ such that any derivative of $\tau(u_{00})$ does not appear in F. Also, here $G_{j,\theta} \in \mathcal{F}[\mathbf{u}_0^{[2s]}, \ldots, \mathbf{u}_d^{[2s]}, \mathbb{Y}^{[s]}]$ for $\theta \in \Theta_{\leq s}$.

Now, we are ready to prove that Δ-Chow varieties of index (n,d,s,r) exist for all n,d,s,r. As mentioned in the beginning of this section, we will use certain algebraic Chow varieties to parametrize Δ-varieties in $G_{(n,d,s,r)}$. For the sake of later use, we briefly recall the concept of algebraic Chow varieties here. For an irreducible variety $V \subseteq \mathbb{P}^n$ of dimension d, the algebraic Chow form of V is the polynomial $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ whose vanishing gives a necessary and sufficient condition for V and $d+1$ hyperplanes having a nonempty intersection in $\mathbb{P}^n$. The Chow form of a d-cycle W in $\mathbb{P}^n$, $W = \sum_{i=1}^{l} t_i W_i$ with $t_i \in \mathbb{N}$ and $\dim(W_i) = d$, is the product of Chow forms of W_i with multiplicity t_i. Its degree in each $\mathbf{u}_i$ is called the degree of W and its coefficient vector is defined to be the Chow coordinate of W. The set of Chow coordinates of all d-cycles in $\mathbb{P}^n$ of degree e is a projective variety in the Chow coordinate space [3], called the Chow variety, and denoted by $\text{Chow}_n(d,e)$. However, the affine Chow variety of all d-cycles in $\mathbb{A}^n$ of degree e is not closed in the Chow coordinate space, but it is always a constructible set [6, Proposition 3.4], also denoted by $\text{Chow}_n(d,e)$. All the Chow varieties we use here are affine ones.

Let $\text{Chow}_{n\binom{s+m}{m}}\big((d+1)\binom{s+m}{m}-1, e\big)$ be the algebraic Chow variety in $\tau_s(\mathbb{A}^n)$ $(s > 0)$ which is of dimension $(d+1)\binom{s+m}{m}-1$ and degree e. Consider the disjoint union of algebraic constructible sets

$$\mathcal{C} = \bigcup_{D_1 \leq e \leq D_2} \text{Chow}_{n\binom{s+m}{m}}\Big((d+1)\binom{s+m}{m} - 1, e\Big)$$

where D_1, D_2 are the lower and upper bounds given in Lemma 6.9. So each point $a \in \mathcal{C}$ represents a $[(d+1)\binom{s+m}{m} - 1]$-cycle in $\tau_s\mathbb{A}^n$. To represent an irreducible Δ-variety V of the desired Kolchin polynomial and Δ-degree by a point in $\mathcal{C}$, we only need to consider irreducible varieties with Chow coordinates in $\mathcal{C}$.

Let $\mathcal{C}_1$ be the subset consisting of all points $a \in \mathcal{C}$ such that a is the Chow coordinate of an irreducible variety W which is prolongation admissible and additionally satisfies the following conditions:

(1) $\pi_{s,0}(W)$ is of dimension $d+1$;
(2) the unique dominant component of the Δ-variety defined by equations of W is of Δ-degree g.

Theorem 6.11. *The set $\mathcal{C}_1$ is a Δ-constructible set and the map which associates an irreducible Δ-variety $V \subseteq \mathbb{A}^n$ in $G_{(n,d,s,r)}$ with the Chow coordinate of the irreducible variety $B_s(V) \subseteq \tau_l(\mathbb{A}^n)$ identifies $G_{(n,d,s,r)}$ with $\mathcal{C}_1$. In particular, the*

Δ-Chow variety of all irreducible Δ-varieties of Kolchin polynomial $(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$ *and Δ-degree r exists.*

Proof: In the case $s=0$, the Δ-Chow form of each $V\in G_{(n,d,0,r)}$ is equal to the Chow form of $B_0(V)\subseteq\mathbb{A}^n$, so the set of Δ-Chow coordinates of Δ-varieties in $G_{(n,d,s,r)}$ is just the same as the set of Chow coordinates of all irreducible varieties in $\mathbb{A}^n$ of dimension d and degree r. By item 5) of Fact 6.3, the latter set is a definable subset of $\mathrm{Chow}_n(d,r)$, so $G_{(n,d,0,r)}$ is definable. Below, we suppose $s>0$.

First, we show $\mathcal{C}_1$ is a Δ-constructible set. From the definition of Chow coordinates, we know each $\mathrm{Chow}_{n\binom{s+m}{m}}\big((d+1)\binom{s+m}{m}-1,e\big)$ actually represents a definable family $S_e:=(F_c)_{c\in\mathrm{Chow}_{n\binom{s+m}{m}}\left((d+1)\binom{s+m}{m}-1,e\right)}$ of homogenous polynomials which are Chow forms of algebraic cycles in $\tau_s\mathbb{A}^n$ of dimension $(d+1)\binom{s+m}{m}-1$ and degree e. The algebraic cycle whose Chow coordinate is c is irreducible if and only if its Chow form F_c is irreducible. Since irreducibility is a definable property, the set $\mathcal{C}_0=\big\{c\in\mathrm{Chow}_{n\binom{s+m}{m}}((d+1)\binom{s+m}{m}-1,e): F_c \text{ is irreducible}\big\}$ is a definable set. Take an arbitrary $c\in\mathcal{C}_0$ and the corresponding polynomial $F_c\in S_e$ for an example. Let V_c be the corresponding irreducible variety with Chow coordinate c. By item 5) of Fact 6.3, $(V_c)_{c\in\mathcal{C}_0}$ is a definable family. And by Lemma 6.6 and Fact 6.3, $\mathcal{C}_2=\big\{c\in\mathcal{C}_0: V_c \text{ is prolongation admissible and} \dim(\pi_{s,0}(V_c))=d+1\big\}$ is a definable set. Then by Lemma 6.8, for each $c\in\mathcal{C}_2$, the Δ-variety corresponding to V_c has a unique dominant component W_c and the Kolchin polynomial of W_c is $(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$.

Since the Kolchin polynomial of W_c is $(d+1)\binom{t+m}{m}-\binom{t+m-s}{m}$, the Δ-Chow form of W_c exists. Let U be the algebraic variety in $\tau_s\mathbb{A}^n\times\big(\mathbb{P}^{(n+1)\binom{s+m}{m}}\big)^{d+1}$ defined by the defining formulae of V_c and $\theta(L_i)=0$ for $\theta\in\Theta_{\leq s}$ and $i=0,\ldots,d$ with each $\theta(L_i)=\theta(u_{i0})+\sum_{k=1}^n\sum_{\tau|\theta}\binom{\theta}{\tau}\frac{\theta}{\tau}(u_{ik})\tau(y_k)$ regarded as a polynomial in variables $\Theta_{\leq s}(y_k)$ and $\Theta_{\leq s}(u_{ik})$. Since $B_s(W_c)=V_c$, by Lemma 4.7, the Zariski closure of the image of U under the following projection map

$$\pi:\tau_s\mathbb{A}^n\times\Big(\mathbb{P}^{(n+1)\binom{s+m}{m}}\Big)^{d+1}\longrightarrow\Big(\mathbb{P}^{(n+1)\binom{s+m}{m}}\Big)^{d+1}$$

is an irreducible variety of codimension 1, and the defining polynomial F of $\overline{\pi(U)}$ is the Δ-Chow form of W_c. By item 4) of Fact 6.3, the total degree of F is definable in families; this quantity is just the Δ-degree of W_c. So the Δ-degree of W_c is definable in families. Hence, $\mathcal{C}_1$ is a definable set, and also a Δ-constructible set due to the fact that the theory $\mathrm{DCF}_{0,m}$ eliminates quantifiers [21, 24, 27].

By Lemma 6.8 and its proof, each irreducible variety V corresponding to a point of $\mathcal{C}_1$ determines an irreducible Δ-variety $W\in G_{(n,d,s,r)}$, where W is the unique dominant component of the Δ-variety corresponding to the prolongation admissible variety V. And on the other hand, each $W\in G_{(n,d,s,r)}$ determines the corresponding algebraic irreducible variety $B_s(W)$, whose Chow coordinate is a point of $\mathcal{C}_1$ guaranteed by Lemma 6.9. So we have established a natural one-to-one correspondence between $G_{(n,d,s,r)}$ and $\mathcal{C}_1$. Thus, $G_{(n,d,s,r)}$ is represented by the Δ-constructible set $\mathcal{C}_1$. □

7. Conclusion

In this paper, a quasi-generic partial differential intersection theorem is first given. Namely, the intersection of an irreducible partial differential variety V with a quasi-generic differential hypersurface of order s is shown to be an irreducible differential variety with Kolchin polynomial $\omega_V(t) - \binom{t+s+m}{m}$. Then partial differential Chow forms are defined for irreducible partial differential varieties of Kolchin polynomial $(d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ and basic properties similar to their algebraic and ordinary differential counterparts are presented. Finally, differential Chow coordinate representations are defined for such partial differential varieties, and the set of all irreducible partial differential varieties of fixed Kolchin polynomial and differential degree is shown to have a structure of differentially constructible set.

The above results have generalized the generic differential intersection theory and the theory of differential Chow forms and differential Chow varieties obtained for the ordinary differential case [9, 6] to their partial differential analogs. However, the theory of partial differential Chow forms and partial differential Chow varieties far more complete and there are several problems left open for further research. As stated in Conjecture 4.6, we conjecture that Kolchin polynomial of the form $(d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ for some $d, s \in \mathbb{N}$ gives not only a sufficient condition, but also a necessary condition for the existence of partial differential Chow forms. Another problem left is how to represent general irreducible partial differential varieties by coordinates and further how to provide a set of partial differential varieties of fixed characteristics with a structure of differential constructible set.

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