Recall Let (K,δ) be a differential field of char 0 and $K\{Y\}=K\{y_1,\cdots,y_n\}$ be the diff poly ring over K in the diff variables $y_1,\cdots,y_n$. Let (E,δ) be a fixed diff closed field extention of (K,δ). The affine space $\mathbb{A}^n=E^n$.

Ritt-Raudenbush Basis Theorem

Every radical diff ideal in $K\{Y\}$ is finitely generated as a radical diff ideal.

Differential Nullstellensatz

$$\mathbb{I}(\mathbb{V}(S)) = \{S\} = \sqrt{[S]} \quad \text{for } \forall S \subseteq K\{Y\}.$$

In particular, $\mathbb{V}(S)=\phi \Leftrightarrow 1\in\{S\}$.

Irreducible decomposition of diff varieties

Every diff variety $V\subseteq E^n$ is a finite intersection of irreducible diff varieties. If $V=\bigcup_{i=1}^{l} V_i$ is such an irredundant/minimal decomposition, each V_i is called an irreducible component of V.

Let $A\in K\{y_1,\cdots,y_n\}\setminus K$ be an irreducible diff polynomial (irreducible as a poly in $K[y_i^{(j)}: j\in\mathbb{N}, i=1,\cdots,n]$). Select an arbitrary ranking $\mathcal{R}$ on $\Theta(Y)$. Let S_A be the separant of A under $\mathcal{R}$. Then

Lemma 3.3.3 $P_1=\{A\}:S_A=\{f\in K\{Y\} \mid S_A f\in\{A\}\}$ is a prime diff ideal and A is a diff characteristic set of P_1 under $\mathcal{R}$.

Prop 3.3.4 $\{A\}=P_1\cap\{A,S_A\}$.

Let $\{A,S_A\}=Q_1\cap\cdots\cap Q_t$ be the minimal prime decomposition of $\{A,S_A\}$. Then $\{A\}=P_1\cap Q_1\cap\cdots\cap Q_t$. Suppressing those Q_i with $P_1\subseteq Q_i$ and denote the left Q_i's by $P_2,\cdots,P_r$. Then

$\{A\} = P_1 \cap \cdots \cap P_r$ is the minimal prime decomposition of $\{A\}$.

Claim For each separant S of A under any arbitrary ranking,

$$S \notin P_1 = \{A\}:S_A \text{ and } S \in P_2, \cdots, P_r.$$

Proof. $S \notin P_1$ follows from Lemma 3.3.3 and the fact $A \nmid S$.

Since $\{A, S_A\} \subseteq P_2, \cdots, P_r$, $S_A \in P_2, \cdots, P_r$.

$S \in P_2, \cdots, P_r$ follows from the fact that $\{P_1, \cdots, P_r\}$ are the unique irreducible components of $\{A\}$. 証.

Remark: A is the δ-characteristic set of $P_1 = \{A\}:S_A = \{A\}:S$ (S is the separant of A under any other ranking) $\{A\}:S_A = \mathrm{sat}(A)$.

P_1 or $\mathbb{V}(P_1)$ is called the general component of $A=0$.

$P_2, \cdots, P_r$ are called singular components of $A=0$.

Example $n=1$, $A = (y')^2 - 4y$, $S_A = 2y'$

$\{A, S_A\} = \{(y')^2 - 4y, 2y'\} = [y] \leftarrow$ prime diff ideal

$A' = 2y'(y''-2)$ $\therefore$ $y''-2 \in \{A\}:S_A$, $y''-2 \notin [y]$.

$\{A\}:S \supseteq [(y')^2-4y, y''-2] = ((y')^2-4y, y''-2, y''', \cdots) \underset{\text{prime}}{=} I$ (for $K\{y\}/I \cong K[y,y']/(A)$)

$\Rightarrow$ $\{A\}:S = [(y')^2-4y, y''-2]$ is the general component of A and $[y]$ is the singular component of A.

To solve $(y')^2 - 4y$ over $K = (\mathbb{R}(x), \frac{d}{dx})$:

$$\frac{dy}{dx} = \pm 2\sqrt{y} \Rightarrow \frac{dy}{2\sqrt{y}} = \pm dx \Rightarrow \sqrt{y} = \pm x + c$$

So $y = (x+c)^2$ or $y = 0$. (c an arbitrary constant).

Def. A δ-zero $\eta \in E^n$ of A is called a nonsingular zero if $\exists$ a separant S of A s.t. $S(\eta) \neq 0$. And if $S(\eta)=0$ for all separants of A, η is called a singular solution/zero of $A=0$

Nonsingular zeros belong to general component of A, but general component of A may contain singular solutions of A.

Example: $A=(y')^2-y^3 \in K\{y\}$. $S_A = 2y'$

Since $\mathbb{V}(A, S_A) = \{0\}$, $\eta=0$ is the only singular solution of $A=0$.

$A' = 2y'y'' - 3y^2y' = 2y'(y'' - \frac{3}{2}y^2)$

$\Rightarrow \{A\} = \{A, y''-\frac{3}{2}y^2\} \cap [y] = \{A, y''-\frac{3}{2}y^2\} = \mathrm{sat}(A)$

Thus, $\eta=0$ is embedded in the general component of $A(=0)$.

(Geometrically, $K=(\mathbb{C}(t), \frac{d}{dt})$. $\eta_c = \frac{1}{4(t+c)^2}$ is a one-parameter family of nonsingular solutions (c arbitrary constant). $\lim_{c\to\infty} \eta_c = 0$.)

Ritt's Problem (still open!) Given $A \in K\{y_1, \cdots, y_n\}$ irreducible with $A(0, \cdots, 0) = 0$, decide whether $(0, \cdots, 0) \in \mathbb{V}(\mathrm{Sat}(A))$?

With deeper results (low power theorem) not covered in our course, we have the following Ritt's component theorem.

Theorem 3.3.5 Let $A \in K\{y_1, \cdots, y_n\}$ be a δ-poly not in K.

Let $\{A\} = p_1 \cap \cdots \cap p_r$ be the minimal prime decomposition of $\{A\}$, then $\exists\, B_i \in K\{y_1, \cdots, y_n\}$ irreducible s.t. $p_i = \mathrm{Sat}(B_i)$, $i=1, \cdots, r$.

In particular, if A is irreducible, then $\exists\, i_0$ s.t. $B_{i_0} = aA$ ($a \in K^*$) and for $i \neq i_0$, A involves a proper derivative of the leader of each B_i w.r.t. any ranking and $\mathrm{ord}(B_i) < \mathrm{ord}(A)$.

Chapter 4 Extensions of differential fields

Let (K,δ) be a differential field of char 0. Let X be an indeterminate over K. Then δ can be extended to a derivation δ_0 on $K[X]$ s.t. $\delta_0(X)=0$ given by $\delta_0(\sum_{i=0}^{l} r_i X^i) = \sum_{i=0}^{l} \delta(r_i) X^i$. There is also a derivation on $K[X]$ s.t. $\frac{d}{dX}(K)=0$ and $\frac{d}{dX}(X)=1$ given by $\frac{d}{dX}(\sum_{i=0}^{l} r_i X^i) = \sum_{i=1}^{l} i r_i X^{i-1}$.

Any derivation δ_1 on $K[X]$ which extends δ is given by

$\delta_1 = \delta_0 + \delta_1(X)\frac{d}{dX}$; Conversely, by defining $\delta_1(X) = p(X) \in K[X]$,

$\delta_1 = \delta_0 + p(X)\frac{d}{dX}$ is a derivation on $K[X]$ extending δ.

(Proof. First suppose δ_1 is a derivation on $K[X]$ extending δ. Then $\forall\ f = \sum_i r_i X^i \in K[X]$, $\delta_1(f) = \sum_i \delta_1(r_i)X^i + \sum_{i=1}^{r} i r_i X^{i-1}\delta(X) = \delta_0(f) + \delta(X)\frac{d}{dX}(f)$

So $\delta_1 = \delta_0 + \delta_1(X)\frac{d}{dX}$. Now let $\delta_1 : K[X] \to K[X]$ be defined by $\delta_1(f) = \delta_0(f) + \delta_1(X)\frac{d}{dX}(f)$. Then $\forall\ a \in K$, $\delta_1(a) = \delta_0(a) + \delta_1(X)\frac{d}{dX}(a) = \delta(a)$;

$\forall\ f,g \in K[X]$, $\delta_1(f+g) = \delta_0(f+g) + \delta_1(X)\frac{d}{dX}(f+g) = \delta_1(f) + \delta_1(g)$ $\Rightarrow$ δ_1 is a derivation which extends δ.)

$\delta_1(fg) = \delta_0(fg) + \delta_1(X)\frac{d}{dX}(fg) = \delta_1(f)g + f\delta_1(g)$

<u>Theorem 4.1</u> Let $K \subseteq L$ be fields of char 0. Then any derivation on K could be extended to a derivation on L. This extension is unique iff L is algebraic over K.

Proof. Let δ be a derivation on K. First suppose $L = K(\gamma)$.

If γ is transcendental over K, then there exists a derivation δ_0 on $K[\gamma]$ s.t. $\delta_0|_K = \delta_K$ and $\delta_0(\gamma)=0$. δ_0 Now extends to a derivation on $L = K(\gamma)$.

If γ is algebraic over K, let $F(X)$ be the minimal polynomial of γ over K. Let $g(X) \in K[X]$ be a polynomial to be determined. δ extends to a derivation δ_0 on $K[X]$ by setting $\delta_0(X)=0$. So

$\delta_1 = \delta_0 + g(X)\frac{d}{dX}$ is a derivation on $K[X]$.

We want to choose $g(x)$ s.t. δ_1 maps the ideal $F\cdot K[x]$ to itself.
The condition for this is that $\delta_1(F)(\gamma)=0$, or

$$\delta_0(F)(\gamma)+g(\gamma)\frac{dF}{dx}(\gamma)=0.$$

$$\begin{tikzcd} K[x] \arrow[r, "\delta"] \arrow[d] & K[x] \text{ w/ } \delta(F\cdot K[x]) \subseteq F\cdot K[x] \\ L=K[\gamma]\cong K[x]/F\cdot K[x] \arrow[r, "\bar{\delta}"] & K[x]/F\cdot K[x] \end{tikzcd}$$

Since $\frac{dF}{dx}(\gamma)\neq 0$, $g(\gamma)=-\dfrac{\delta_0(F)(\gamma)}{\frac{dF}{dx}(\gamma)}$. $K(\gamma)=K[\gamma]$ implies that we can find $g(x)\in K[x]$ with desired property. Choose $g(x)\in K[x]$ s.t. δ_1 maps $F\cdot K[x]$ to itself. Now δ_1 induces a map $\bar{\delta}_1$ on $K[x]/F\cdot K[x]$ by $\bar{\delta}_1(A(x)+F\cdot K[x])=\delta_1(A(x))+F\cdot K[x]$ and this $\bar{\delta}_1$ is the desired derivation on $K(\gamma)=K[\gamma]$. ($\bar{\delta}_1(\gamma)=g(\gamma)=-\delta_0(F)(\gamma)/F'(\gamma)$.)

For the general case, let $\mathcal{E}=\{(K_1,\delta_1)\mid K\subseteq K_1\subseteq L \text{ and } \delta_1|_K=\delta_K\}$. Then $\mathcal{E}\neq\phi$. Let $(K_1,\delta_1)\subseteq(K_2,\delta_2)\subseteq\cdots\subseteq(K_n,\delta_n)\subseteq\cdots$ be an ascending chain in $\mathcal{E}$. Then $(\bigcup_i K_i,\bar{\delta})$ w/ $\forall a\in K_i$, $\bar{\delta}(a)=\delta_i(a)$ is in $\mathcal{E}$.
By Zorn's lemma, $\exists$ a maximal elt (M,δ_M) in $\mathcal{E}$. Clearly, $M=L$.

<u>Uniqueness</u> If L is not algebraic over K, then $\exists\,\gamma\in L$ trans. over K. There will be more than one derivation on $K[\gamma]$ which extends δ on K. If L is algebraic over K, for each $\gamma\in L$, let $F(x)=\sum_{i=0}^{d}\gamma_i x^i\in K[x]$ be the minimal polynomial of γ over K. Let D be the derivation on L which extends δ on K. $F(\gamma)=0\Rightarrow 0=D(F(\gamma))=D(\sum_{i=0}^{d}\gamma_i x^i)=\sum_{i=0}^{d}\delta(\gamma_i)\gamma^i+(\sum_{i=1}^{d}i\gamma_i\gamma^{i-1})D(\gamma)\Rightarrow D(\gamma)=-(\sum_{i=0}^{d}\delta(\gamma_i)\gamma^i)/(\sum_{i=1}^{d}i\gamma_i\gamma^{i-1})$ which is unique. ☒

<u>Corollary 4.2</u> If $K\subseteq L$ are fields of char 0 and δ be a derivation on L s.t. $\delta(K)\subseteq K$. If $\gamma\in L$ is alg over K, then $\delta(\gamma)\in K(\gamma)$. In particular, if $\gamma\in L$ is alg over a constant subfield of L, then γ is a constant ($\gamma'=0$).

With the language of diff polys, Definition 2.1 can be restated as:

Def 4.3. Let $K \subseteq L$ be differential field extensions and $\gamma \in L$. If $\exists\, 0 \neq p(y) \in K\{y\}$ s.t. $p(\gamma)=0$, then γ is said to be <u>differential algebraic</u> over K. Otherwise, γ is called <u>differentially transcendental</u> over K.
Let $\gamma_1, \dots, \gamma_n \in K$. we call $\gamma_1, \dots, \gamma_n$ <u>differentially algebraically dependent</u> over K if $\exists\, F(y_1, \dots, y_n) \in K\{y_1, \dots, y_n\}^*$ s.t. $F(\gamma_1, \dots, \gamma_n)=0$. Otherwise, they are said to be differentially transcendental over K.

<u>Lemma 4.4</u> Let $K \subseteq L$ be differential fields of char 0 and $\gamma \in L$. Then γ is diff algebraic over K
$\iff$ tr.deg $K\langle\gamma\rangle/K < \infty$.

Proof. "$\Rightarrow$" Sps γ is diff algebraic over K. Let $A(y) \in K\{y\}$ be a <u>characteristic</u> set of $\mathbb{I}(\gamma) \subseteq K\{y\}$. Assume $\text{ord}(A) = n$.

$A(y)$ is of minimal order and minimal degree under the desired order

<u>Claim: tr.deg $K\langle\gamma\rangle/K = n$.</u>

Clearly, $\gamma, \gamma', \dots, \gamma^{(n-1)}$ are alg. indep. over K and $\gamma^{(n)}$ is algebraic over $K(\gamma, \gamma', \dots, \gamma^{(n-1)})$. And $A(\gamma)=0 \Rightarrow S_A(\gamma)\cdot \gamma^{(n+1)} + T_A(\gamma) = 0$, where $T_A(\gamma) \in K(\gamma, \dots, \gamma^{(n)})$. $\Rightarrow \gamma^{(n+1)} = -T_A(\gamma)/S_A(\gamma) \in K(\gamma, \gamma', \dots, \gamma^{(n)})$.
$\Rightarrow \forall\, k \in \mathbb{N}$, $\gamma^{(n+k)} \in K(\gamma, \dots, \gamma^{(n-1)}, \gamma^{(n)})$.
So $K\langle\gamma\rangle = K(\gamma, \dots, \gamma^{(n-1)}, \gamma^{(n)})$ and tr.deg $K\langle\gamma\rangle/K = n$.
"$\Leftarrow$" $n=$ tr.deg $K\langle\gamma\rangle/K < \infty$ implies that $\gamma, \gamma', \gamma'', \dots, \gamma^{(n)}$ are alg. dep. over K.
So γ is diff algebraic over K. ∎.

Remark: 1) If γ is diff alg over K and $f(y) \neq 0$ is a diff poly of minimal order which vanishes at γ, then tr.deg $K\langle\gamma\rangle/K = \text{ord}(f)$.
2) The result "$\Rightarrow$" is false in the partial differential case $(K, \{\delta_1, \dots, \delta_m\})$, where tr.deg $K\langle\gamma\rangle/K$ might be infinity but the differential type of $K\langle\gamma\rangle$ is $\leq m-1$.