A Reduction Approach to Creative Telescoping

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http://www.mmrc.iss.ac.cn/~schen/Talks/TutorialISSAC2019.pdf

Outline

- Introduction to Creative Telescoping
- Creative Telescoping via Reductions
    - Rational case
    - Hyperexponential case
    - Hypergeometric case

Part 1. Introduction to Creative Telescoping

- What is creative telescoping
- Existence problems in creative telescoping
- Algorithms for creative telescoping

Wilf–Zeilberger theory

In the early 1990s, Wilf and Zeilberger developed an algorithmic theory for proving identities in combinatorics and special functions.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

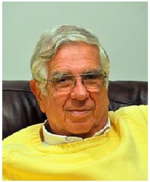
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2$$

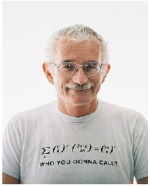
$$\int_0^\infty x^{\alpha-1} Li_n(-xy)dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_{-1}^{+1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}} dx = (-1)^n \pi I_n(p)$$

$$\ldots$$



Herbert Wilf



Doron Zeilberger

Telescoping

Problem. For a sequence $f(k)$ in some class $\mathfrak{S}(k)$, decide whether there exists $g(k) \in \mathfrak{S}(k)$ s.t.

$$f(k) = g(k+1) - g(k) = \Delta_k(g)$$

$$\Downarrow$$

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(b) + g(b) - g(b-1) + \cdots + g(a+1) - g(a)$$
$$= g(b+1) - g(a)$$

Example.

Telescoping

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Example.
Rational sums

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Delta_k \left(-\frac{1}{k} \right) = 1 - \frac{1}{n+1}$$

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$$= g(b+1) - g(a)$$

Example.
Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k \left(\frac{4k\binom{2k}{k}^2}{4^{2k}} \right) = \frac{(n+1)\binom{2n+2}{n+1}^2}{4^{2n+1}}$$

Telescoping: Gosper's algorithm

Definition. $H(n) : \mathbb{N} \to \mathbb{F}$ is hypergeometric over $\mathbb{F}(n)$ if

$$\frac{H(n+1)}{H(n)} \in \mathbb{F}(n).$$

Examples. $n^2, \quad \frac{1}{n^2+2n+1}, \quad 2^n, \quad n!, \ldots$

Telescoping Problem. Given hypergeometric $H(n)$, decide whether $\exists$ hypergeometric $T(n)$ s.t.

$$H(n) = T(n+1) - T(n).$$

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Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
Mathematics

Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

R. WILLIAM GOSPER, JR.

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$$\sum_{n=1}^{m} \frac{\prod_{j=1}^{n-1} aj^3 + bj^2 + cj + d}{\prod_{j=1}^{n} aj^3 + bj^2 + cj + e} = \frac{1 - \prod_{j=1}^{m} \frac{aj^3 + bj^2 + cj + d}{aj^3 + bj^2 + cj + e}}{e - d}$$

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Thursday 18th July 2019

Time	Conference Room 2
09:00 – 10:00	**William Chen** - The Art of Telescoping **Chair:** Dongming Wang
10:00 – 10:30	**Coffee Break**

The name “creative telescoping”

The phrase “creative telescoping ” was first mentioned in an expositional paper by van der Poorten on Apéry proof of the irrationality of $\zeta(3)$ in 1979.

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(1916-1994)

Alfred van der Poorten
(1942-2010)

A Proof that Euler Missed ...

Apéry's Proof of the Irrationality of $\zeta(3)$

An Informal Report

Alfred van der Poorten

$$\zeta(3) =: \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k},$$

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

Then $a_0 = 0, a_1 = 6; b_0 = 1, b_1 = 5$ and each sequence $\{a_n\}$ and $\{b_n\}$ satisfies the recurrence (2).

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geqslant 2. \qquad (2)$$

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$$B_{n,k} = 4(2n+1)\,(k(2k+1)-(2n+1)^2)\binom{n}{k}^2\binom{n+k}{k}^2,$$

with the motive that

$$B_{n,k}-B_{n,k-1} = (n+1)^3\binom{n+1}{k}^2\binom{n+1+k}{k}^2 - (34n^3+51n^2+27n+5)\binom{n}{k}^2\binom{n+k}{k}^2 + n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2,$$

and, *0 mirabile dictu,* the sequence $\{b_n\}$ does indeed satisfy the recurrence (2) by virtue of the method of *creative telescoping* (by the usual conventions: $B_{nk} = 0$ for $k < 0$ or $k > n$; note also that $P(n) = 34n^3 + 51n^2 + 27n + 5$ implies $P(n-1) = -P(-n)$.)

Algebraic setting

Notation.

- $\mathbb{F}$: a field of characteristic zero;
- $\mathbb{F}(\mathbf{v})$: the field of rational functions in $\mathbf{v} = v_1, \ldots, v_n$ over $\mathbb{F}$;
- D_{v_i}: the partial derivation defined by

$$D_{v_i}(f(\mathbf{v})) = \frac{\partial f(\mathbf{v})}{\partial v_i}.$$

- S_{v_i}: the partial shift operator defined by

$$S_{v_i}(f(\mathbf{v})) = f(v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_n);$$

Algebraic setting

$\mathbb{F}(\mathbf{v})\langle D_{v_1}, \ldots, D_{v_n}\rangle$: the ring of linear differential operators over $\mathbb{F}(\mathbf{v})$

$$L := \sum_{0 \le i_1, \ldots, i_d \le N} f_{i_1,\ldots,i_n} D_{v_1}^{i_1} \cdots D_{v_n}^{i_n} \quad \text{with } f_{i_1,\ldots,i_n} \in \mathbb{F}(\mathbf{v}),$$

in which $D_{v_i} \cdot D_{v_j} = D_{v_j} \cdot D_{v_i}$ for $i,j \in \{1, \ldots, n\}$ and

$$D_{v_i} \cdot f = f \cdot D_{v_i} + \frac{\partial f}{\partial v_i} \quad \text{for any } f \in \mathbb{F}(\mathbf{v}).$$

$\mathbb{F}(\mathbf{v})\langle S_{v_1}, \ldots, S_{v_n}\rangle$: the ring of linear recurrence operators over $\mathbb{F}(\mathbf{v})$

$$L := \sum_{0 \le i_1, \ldots, i_d \le N} f_{i_1,\ldots,i_n} S_{v_1}^{i_1} \cdots S_{v_n}^{i_n} \quad \text{with } f_{i_1,\ldots,i_n} \in \mathbb{F}(\mathbf{v}),$$

in which $S_{v_i} \cdot S_{v_j} = S_{v_j} \cdot S_{v_i}$ for $i,j \in \{1, \ldots, n\}$ and

$$S_{v_i} \cdot f(\mathbf{v}) = f(v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_n) \cdot S_{v_i} \quad \text{for any } f \in \mathbb{F}(\mathbf{v}).$$

Creative telescoping: the discrete case

Problem. For a sequence $f(n,k)$ in some class $\mathfrak{S}(n,k)$, find a linear recurrence operator $L \in \mathbb{F}(n)\langle S_n \rangle$ and $g \in \mathfrak{S}(n,k)$ s.t.

$$\underbrace{L(n,S_n)}_{\text{Telescoper}}(f) = S_k(g) - g \triangleq \Delta_k(g)$$

Call g the certificate for L.

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Example. Let $f(n,k) = \binom{n}{k}^2$. Then a telescoper for f and its certificate g are respectively

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2}{(k-n-1)^2} \cdot f$$

Proving combinatorial identities

$$F(n) := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

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Creative telescoping for $f = \binom{n}{k}^2$: $L(f) = \Delta_k(g)$, where

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Since $f(n,k) = 0$ when $k < 0$ or $k > n$, we have

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2$$

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Taking sums on both sides of $L(f) = \Delta_k(g)$:

$$\sum_{k=-\infty}^{+\infty} L(f) = L\left(\sum_{k=-\infty}^{+\infty} f\right) = g(n,+\infty) - g(n,-\infty) = 0$$

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The sequence $F(n)$ satisfies

$$(n+1)F(n+1) - (4n+2)F(n) = 0$$

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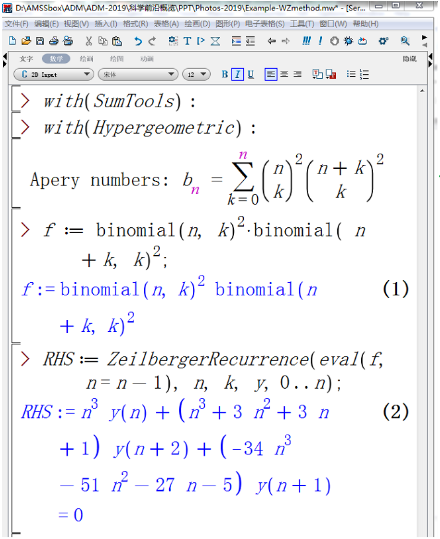
Verify the initial condition:

$$F(1) = 2 = \binom{2}{1}$$

Then the identity is proved!

Example: Recurrence for Apéry numbers



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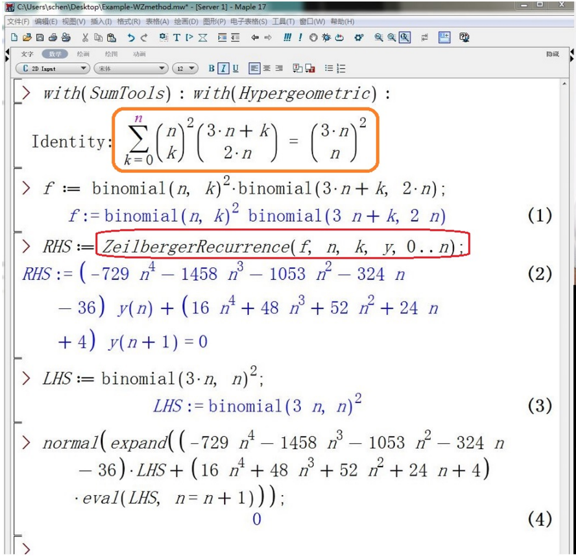
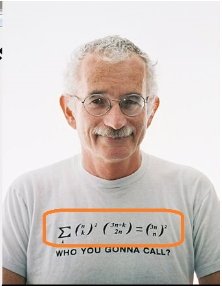
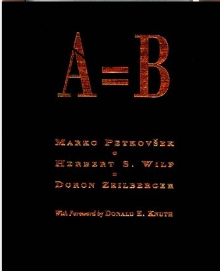
Example: Identity on T-shirt







Creative telescoping: the continuous case

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Call g the certificate for L.

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Example. Let $f(x,y) = \exp(-(x/y)^2 - y^2)$. Then a telescoper for f and its certificate are

$$L = D_x^2 - 4 \quad \text{and} \quad g = \frac{2}{y} \cdot f$$

Proving integral identities

$$F(x) := \int_{-\infty}^{+\infty} \exp(-(x/y)^2 - y^2)\, dy = \sqrt{\pi} \exp(-2x).$$

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Creative telescoping for $f = \exp(-(x/y)^2 - y^2)$: $\quad L(f) = D_y(g)$, where

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Taking integrals on both sides of $L(f) = D_y(g)$:

$$\int_{-\infty}^{+\infty} L(f)\, dy = L\left(\int_{-\infty}^{+\infty} f\, dy\right) = g(x, +\infty) - g(x, -\infty) = 0$$

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The function $F(x)$ satisfies

$$y''(x) - 4y(x) = 0$$

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$$L = D_x^2 - 4 \quad \text{and} \quad g = \frac{2}{y} \cdot f$$

Verify the initial conditions:

$$F(0) = \sqrt{\pi} \quad \text{and} \quad F'(0) = -2\sqrt{\pi}$$

Then the identity is proved!

Example: differential equations for integrals

In 1958, Manin gave a method for showing that

$$F(x) = \oint_{\Gamma} f(x,y)dy, \quad \text{where} \quad f = \frac{1}{\sqrt{y(y-1)(y-x)}}$$

satisfies the Picard-Fuchs differential equation

$$u''(x) + \frac{2x-1}{x(x-1)}u'(x) + \frac{1}{4x(x-1)}u(x) = 0.$$

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ALGEBRAIC CURVES OVER FIELDS WITH DIFFERENTIATION

Ju. I. MANIN

A differential-algebraic homomorphism is constructed from the group of divisor classes of degree zero on a curve defined over a constant field with differentiation into the additive group of a finite-dimensional vector space over the constant field. A partial study of the kernel of this homomorphism is made.

. . . .

12. Elliptic curves. General case. In general, when there are no special restrictions on a, b the formulas obtained are much more complicated.

We seek a linear relation

$$D^2\bar{\omega} + p^1 D\bar{\omega} + p^0\bar{\omega} = 0$$

with undetermined coefficients p^1, p^0. Separating total differentials gives

$$D^2\omega + p^1 D\omega + p^0\omega = d\Big[-\frac{1}{2}\sum_{\alpha=1}^{3}\frac{(De_\alpha)^2}{(3e_\alpha^2+a)^2}\frac{Y^3}{(X-e_\alpha)^3}$$

$$+\sum_{\alpha=1}^{3}\frac{-\frac{3}{2}(f_2e_\alpha^2+f_1e_\alpha+f_0)+p^1(e_\alpha Da+Db)+e_\alpha D^2a+D^2b}{(3e_\alpha^2+a)^2}\cdot\frac{Y}{X-e_\alpha}+\frac{1}{2}g_2Y\Big]$$

$$+\Big[p^0+\frac{3}{4}\Big(g_0+h_0-\frac{ag_2}{3}\Big)-\frac{1}{2}p^1k_0-\frac{1}{2}l_0\Big]\frac{dX}{Y}$$

$$+\Big[\frac{3}{4}(g_1+h_1)-\frac{1}{2}p^1k_1-\frac{1}{2}l_1\Big]\frac{XdX}{Y}.$$

. . . .

Abbreviating, we write

$$g_2 = \sum_{\alpha=1}^{3}\frac{(De_\alpha)^2}{(3e_\alpha^2+a)^2}, \quad f_2 = -3\sum_{\alpha=1}^{3}\frac{e_\alpha(De_\alpha)^2}{3e_\alpha^2+a}, \quad h_1 = \frac{2a^2f_2+9bf_1-6af_0}{\delta},$$

$$g_1 = \sum_{\alpha=1}^{3}\frac{e_\alpha(De_\alpha)^2}{(3e_\alpha^2+a)^2}, \quad f_1 = -\sum_{\alpha=1}^{3}\frac{(6e_\alpha^2+a)(De_\alpha)^2}{3e_\alpha^2+a}, \quad h_0 = \frac{3abf_2-2a^2f_1-9bf_0}{\delta},$$

$$g_0 = -2\sum_{\alpha=1}^{3}\frac{e_\alpha^2(De_\alpha)^2}{(3e_\alpha^2+a)^2}, \quad f_0 = -\frac{a}{b}\sum_{\alpha=1}^{3}\frac{(De_\alpha)^2e_{\alpha+1}^2e_{\alpha+2}^2}{3e_\alpha^2+a}+\frac{(Db)^2}{b},$$

$$k_1 = \frac{9bDa-6aDb}{\delta}, \quad l_1 = \frac{9bD^2a-6aD^2b}{\delta},$$

$$k_0 = -\frac{2a^2Da+9bDb}{\delta}, \quad l_0 = -\frac{2a^2D^2a+9bD^2b}{\delta}.$$

The coefficients p^1 and p^0 can be found from the relations

$$p^0+\frac{3}{4}\Big(g_0+h_0-\frac{ag_2}{3}\Big)-\frac{1}{2}p^1k_0-\frac{1}{2}l_0 = 0,$$

$$\frac{3}{4}(g_1+h_1)-\frac{1}{2}p^1k_1-\frac{1}{2}l_1 = 0.$$

For a curve C_u of the special form $Y^2 = X(X-1)(X-u)$ (see [1]) the relation on the differentials is quite simple:

$$D^2\bar{\omega} + \frac{2u-1}{u(u-1)}D\bar{\omega} + \frac{1}{4u(u-1)}\bar{\omega} = 0,$$

although even here $Z(\mathfrak{p})$ appears to be fairly complicated.

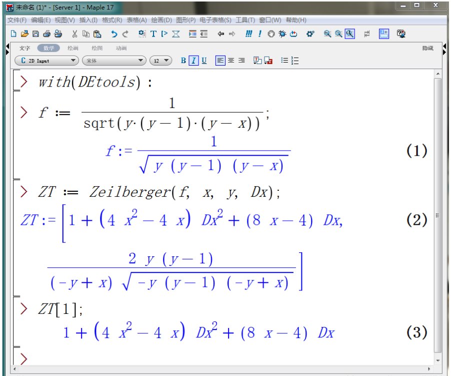
Example: differential equations for integrals

In 1958, Manin gave a method for showing that

$$F(x) = \oint_{\Gamma} f(x,y)dy, \quad \text{where} \quad f = \frac{1}{\sqrt{y(y-1)(y-x)}}$$

satisfies the Picard-Fuchs differential equation

$$u''(x) + \frac{2x-1}{x(x-1)}u'(x) + \frac{1}{4x(x-1)}u(x) = 0.$$



Creative telescoping: the mixed case

Problem. For a term $f(x,k)$ in some class $\mathfrak{S}(x,k)$, find a linear differential operator $L \in \mathbb{F}(x)\langle D_x \rangle$ and $g \in \mathfrak{S}(x,k)$ s.t.

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f) = \Delta_k(g)$$

Call g the certificate for L.

Example. Let $f(x,k) = \binom{2k}{k} \cdot x^k$. Then a telescoper for f and its certificate are

$$L = (1-4x)D_x - 2 \quad \text{and} \quad g = -\frac{k}{x} \cdot f.$$

$$\Downarrow$$

$$\sum_{k=0}^{+\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}.$$

Creative telescoping: the mixed case

Problem. For a term $f(n,y)$ in some class $\mathfrak{S}(n,y)$, find a linear recurrence operator $L \in \mathbb{F}(n)\langle S_n \rangle$ and $g \in \mathfrak{S}(n,y)$ s.t.

$$\underbrace{L(n,S_n)}_{\text{Telescoper}}(f) = D_y(g)$$

Call g the certificate for L.

Example. Let $f(n,y) = y^{n-1}\exp(-y)$. Then a telescoper for f and its certificate are

$$L = S_n - n \quad \text{and} \quad g = -y \cdot f.$$

$$\Downarrow$$

$$\Gamma(n) = \int_0^{+\infty} f(n,y)\,dy \quad \text{satisfies} \quad \Gamma(n+1) = n\Gamma(n).$$

Handbooks of identities

Dixon's identity

$$\sum_{k=-a}^{a} (-1)^k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$$

Handbooks of identities

Dixon's identity

$$\sum_{k=-a}^{a}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}=\frac{(a+b+c)!}{a!b!c!}$$

Hille-Hardy's identity

$$\sum_{n=0}^{\infty}\sum_{k_1}\sum_{k_2}\frac{u^n n!}{(a+1)_n}\binom{n+a}{n-k_1}\frac{(-x)^{k_1}}{k_1!}\binom{n+a}{n-k_2}\frac{(-y)^{k_2}}{k_2!}$$
$$=(1-u)^{-a-1}\exp\left\{-\frac{(x+y)^u}{1-u}\right\}\sum_n\frac{1}{n!(a+1)_n}\left(\frac{xyu}{(1-u)^2}\right)^n$$

Handbooks of identities

Dixon's identity

$$\sum_{k=-a}^{a}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}=\frac{(a+b+c)!}{a!b!c!}$$

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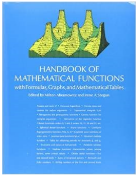
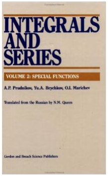
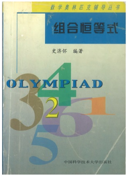
$$=(1-u)^{-a-1}\exp\left\{-\frac{(x+y)^u}{1-u}\right\}\sum_n\frac{1}{n!(a+1)_n}\left(\frac{xyu}{(1-u)^2}\right)^n$$







...

Solving conjectures in combinatorics

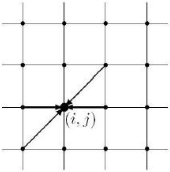
PNAS 2009

Proof of Ira Gessel's lattice path conjecture

Manuel Kauers[a], Christoph Koutschan[a], and Doron Zeilberger[b,1]

Theorem. *Let $f(n;i,j)$ denote the number of Gessel walks going in n steps from $(0,0)$ to (i,j). Then $f(n;0,0)=0$ if n is odd and*

$$f(2n;0,0) = 16^n \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} \quad (n \geq 0),$$



PNAS 2011

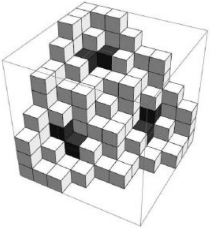
Proof of George Andrews's and David Robbins's *q*-TSPP conjecture

Christoph Koutschan[a,1], Manuel Kauers[b,2], and Doron Zeilberger[c]

Theorem 1. *Let π/S_3 denote the set of orbits of a totally symmetric plane partition π under the action of the symmetric group S_3. Then the orbit-counting generating function (ref. 3, p. 200, and ref. 2, p. 106) is given by*

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n.



7

Fundamental problems

Creative telescoping

Fundamental problems

Creative telescoping

$$\underbrace{L(n, S_n)}_{\text{Telescoper}}(f(n,k)) = \Delta_k(g(n,k))$$

Fundamental problems

Creative telescoping

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,y)) = D_y(g(x,y))$$

Fundamental problems

Creative telescoping

$$\int_{-\infty}^{+\infty} \exp(-x^2/y^2 - y^2)\, dy = \sqrt{\pi}\exp(-2x)$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,k)) = \Delta_k(g(x,k))$$

Fundamental problems

Creative telescoping

$$\sum_{k=0}^{+\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x,\partial_x)}_{\text{Telescoper}}(f(x,y_1,\ldots,y_m)) = \sum_{i=1}^{m} \partial_{y_i}(g_i(x,y_1,\ldots,y_m))$$

Fundamental problems

Creative telescoping

$$\underbrace{L(x, \partial_x)}_{\text{Telescoper}} (f(x, y_1, \ldots, y_m)) = \sum_{i=1}^{m} \partial_{y_i} (g_i(x, y_1, \ldots, y_m))$$

Existence problem.

For a class of functions, decide whether telescopers exist?

Fundamental problems

Creative telescoping

$$\underbrace{L(x, \partial_x)}_{\text{Telescoper}} (f(x, y_1, \ldots, y_m)) = \sum_{i=1}^{m} \partial_{y_i}(g_i(x, y_1, \ldots, y_m))$$

Existence problem.
For a class of functions, decide whether telescopers exist?

Construction problem.
For a class of functions, how to computer telescopers if exist?

Fundamental problems

Creative telescoping

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Existence problem.
For a class of functions, decide whether telescopers exist?

Construction problem.
For a class of functions, how to computer telescopers if exist?

Tools: Combining Computer Algebra with

- Holonomic D-modules
- Differential and difference algebra
- Non-commutative polynomials
- ...

D-finite functions

Definition. A function $f(x_1,\ldots,x_d)$ is D-finite over $\mathbb{F}(x_1,\ldots,x_d)$ if for each $i\in\{1,\ldots,d\}$, f satisfies a LPDE:

$$p_{i,r_i}\frac{\partial^{r_i}f}{\partial x_i^{r_i}}+p_{i,r_i-1}\frac{\partial^{r_i-1}f}{\partial x_i^{r_i-1}}+\cdots+p_{i,0}f=0,$$

where $p_{i,j}\in\mathbb{F}[x_1,\ldots,x_d]$.

D-finite functions

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where $p_{i,j} \in \mathbb{F}[x_1,\ldots,x_d]$.

R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1: 175–188, 1980.

L. Lipshitz. D-Finite Power Series. *Journal of Algebra*, 122: 353–373, 1989.

D-finite functions

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$$p_{i,r_i}\frac{\partial^{r_i} f}{\partial x_i^{r_i}} + p_{i,r_i-1}\frac{\partial^{r_i-1} f}{\partial x_i^{r_i-1}} + \cdots + p_{i,0} f = 0,$$

where $p_{i,j} \in \mathbb{F}[x_1,\ldots,x_d]$.

Elimination Lemma. (Lipshitz1988)

Let $f(x,y)$ be D-finite over $\mathbb{F}(x,y)$. Then

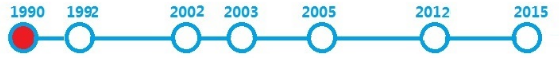
$$\begin{cases} P(x,y,D_x)(f) = 0 \\ Q(x,y,D_y)(f) = 0 \end{cases} \quad \rightsquigarrow \quad A(x,D_x,D_y)(f) = 0 \quad \text{with } \deg_{D_y}(A) \text{ minimal}$$

$$\Downarrow$$

$A(x,D_x,0)$ is a telescoper for f

Existence problem: the bivariate case

Existence problem: the bivariate case



1990: Zeilberger proved that telescopers always exist for holonomic functions:

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence problem: the bivariate case

1992: Wilf and Zeilberger proved that telescopers always exist for proper hypergeometric terms:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "*q*") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence problem: the bivariate case

2002: Abramov and Le solved the existence problem for rational functions in two discrete variables:

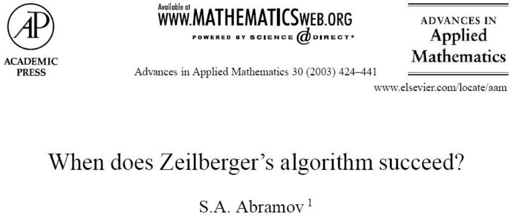
N·H
ELSEVIER
Discrete Mathematics 259 (2002) 1–17
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A criterion for the applicability of Zeilberger's algorithm to rational functions ☆

S.A. Abramov[a], H.Q. Le[b,*]

Existence problem: the bivariate case

2003: Abramov solved the existence problem for bivariate hypergeometric terms: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Advances in Applied Mathematics 30 (2003) 424–441

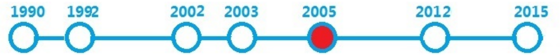
ADVANCES IN Applied Mathematics

www.elsevier.com/locate/aam

When does Zeilberger's algorithm succeed?

S.A. Abramov [1]

Existence problem: the bivariate case



2005: W.Y.C. Chen, Hou and Mu solved the existence problem for bivariate q-hypergeometric terms:

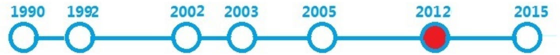
ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Symbolic Computation 39 (2005) 155–170
Journal of Symbolic Computation
www.elsevier.com/locate/js

Applicability of the q-analogue of Zeilberger's algorithm

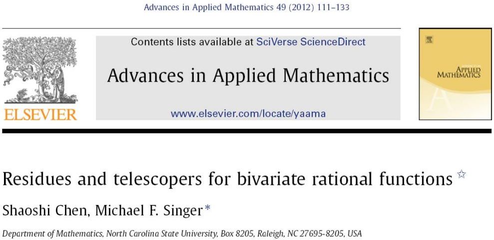
William Y.C. Chen*, Qing-Hu Hou, Yan-Ping Mu

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, PR China

Existence problem: the bivariate case



2012: C. and Singer solved the existence problem for bivariate rational functions in the mixed cases:



Advances in Applied Mathematics 49 (2012) 111–133

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

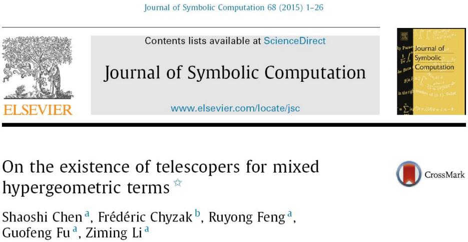
Residues and telescopers for bivariate rational functions ☆

Shaoshi Chen, Michael F. Singer *

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205, USA

Existence problem: the bivariate case

2015: C., Chyzak, Feng, Fu and Li solved the existence problem for bivariate mixed hypergeometric terms:


Journal of Symbolic Computation 68 (2015) 1–26

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

On the existence of telescopers for mixed hypergeometric terms ☆

Shaoshi Chen [a], Frédéric Chyzak [b], Ruyong Feng [a], Guofeng Fu [a], Ziming Li [a]

Existence problem: the trivariate rational case

$$L(x, \partial_x)(f) = \partial_y(g) + \partial_z(h)$$

Existence problem: the trivariate rational case

$$L(x, \partial_x)(f) = \partial_y(g) + \partial_z(h)$$

L \ (∂_y, ∂_z)	(D_y, D_z)	(Δ_y, D_z)	(Δ_y, Δ_z)
D_x	● always exist		
S_x			

Remark. In the pure continuous case, Zeilberger in 1990 showed that telescopers always exist for rational functions.

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

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D_x	○ always exist		
S_x			● solved

Remark. In the pure discrete case, existence problem of telescopers has been solved by Chen et al in 2016.

Existence Problem of Telescopers: Beyond the Bivariate Case *

Shaoshi Chen[1,2], Qing-Hu Hou[3], George Labahn[2], Rong-Hua Wang[4]

Existence problem: the trivariate rational case

$$L(x,\partial_x)(f) = \partial_y(g) + \partial_z(h)$$

L \ (∂_y,∂_z)	(D_y,D_z)	(Δ_y,D_z)	(Δ_y,Δ_z)
D_x	○ always exist	problem 3	problem 4
S_x	problem 1	problem 2	○ solved

Remark. In the four mixed discrete case, the existence problem is solved by Chen, Du and Zhu.

Thursday 18th July 2019

	Conference Room 2	Conference Room 8
	Special Functions	Symbolic-Numeric I
	Chair: Frederic Chyzak	Chair: Anna Bigatti
10:30 – 10:55	**Shaoshi Chen, Lixin Du and Chaochao Zhu** - Existence Problem of Telescopers for Rational Functions in Three Variables: the Mixed Cases	**Kisun Lee, Anton Leykin and Michael Burr** - Effective certification of approximate solutions to systems of equations involving analytic functions

Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)

2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)

3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)

4. Reduction-based methods ($\approx$ 2010)

Zeilberger's algorithm

Input: A proper hypergeometric term $H(n,k)$
Output: A telescoper $L \in \mathbb{F}[n]\langle S_n \rangle$ s.t.

$$L(n,S_n)(H) = \Delta_k(G)$$

Zeilberger's algorithm

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$$L(n,S_n)(H) = \Delta_k(G)$$

- Pick some $r \in \mathbb{N}$ and set $L_r = \sum_{i=0}^{r} c_i S_n^i$

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$$L(n,S_n)(H) = \Delta_k(G)$$

- Pick some $r \in \mathbb{N}$ and set $L_r = \sum_{i=0}^{r} c_i S_n^i$
- Consider the hypergeometric term

$$L_r(H) := \sum_{i=0}^{r} c_r H(n+i,k)$$

Zeilberger's algorithm

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Output: A telescoper $L \in \mathbb{F}[n]\langle S_n \rangle$ s.t.

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- Pick some $r \in \mathbb{N}$ and set $L_r = \sum_{i=0}^{r} c_i S_n^i$
- Consider the hypergeometric term

$$L_r(H) := \sum_{i=0}^{r} c_r H(n+i,k)$$

- Call Gosper's algorithm on $L_r(H)$ to check whether $\exists\ c_0, \ldots, c_r \in \mathbb{F}[n]$ s.t.

$$L_r(H) = \Delta_k(G_r)$$

Zeilberger's algorithm

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- If all c_i's are zero, increase r and try again

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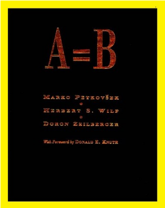

Petkovsek, Wilf & Zeilberger

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Guess the certificate of L?

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2 - 84n^3 - 68460k^6n^4 - 840n^4 - 3720n^5 + 140700k^4n^6 - 9480n^6 - 15024n^7 - 10500k^2n^8 - 14808n^8 - 8400n^9 - 79590n^2k^7 + 284235n^4k^5 - 143640n^6k^3 + 210nk^8 - 26250n^3k^6 + 133035n^5k^4 - 35700n^7k^2 + 252k^{11} + 18900k^9n - 213780k^7n^3 + 368340k^5n^5 - 110460k^3n^7 - 2100n^{10} + 1890k^9 - 1764k^7 + 1260k^5 - 378k^3 - 1260k^{10} - 294nk^2 + 700nk^4 - 588nk^6 + 63504k^{11}n^5 + 52920k^{11}n^4 + 30240k^{11}n^3 + 11340k^{11}n^2 - 2940n^2k^2 - 13080n^3k^2 - 33780n^4k^2 - 55116n^5k^2 - 57348n^6k^2 - 17360k^3n^2 - 48860k^3n^3 - 94920k^3n^4 - 135156k^3n^5 - 55440k^3n^8 - 13860k^3n^9 - 3780k^3n + 7000n^2k^4 + 31185n^3k^4 + 80850n^4k^4 + 90090n^7k^4 + 27720n^8k^4 + 57141k^5n^2 + 155610k^5n^3 + 347886k^5n^6 + 238392k^5n^7 + 110880k^5n^8 + 27720k^5n^9 + 12600k^5n - 5880n^2k^6 - 114114n^5k^6 - 123816n^6k^6 - 83160n^7k^6 - 27720n^8k^6 - 379830k^7n^4 - 469128k^7n^5 - 411840k^7n^6 - 257400k^7n^7 - 110880k^7n^8 - 27720k^7n^9 - 17640k^7n + 9405n^3k^8 + 24750n^4k^8 + 42075n^5k^8 + 47520n^6k^8 + 34650n^7k^8 + 13860n^8k^8 + 85085k^9n^2 + 398475k^9n^4 + 23100k^9n^9 + 480480k^9n^5 + 92400k^9n^8 + 235620k^9n^7 + 227150k^9n^3 + 404250k^9n^6 - 12628k^{10}n - 13860k^{10}n^9 - 152460k^{10}n^3 - 60060k^{10}n^8 - 267960k^{10}n^4 - 157080k^{10}n^7 - 271656k^{10}n^6 - 56980k^{10}n^2 - 323400k^{10}n^5 + 2520k^{11}n + 2520k^{11}n^9 + 11340k^{11}n^8 + 30240k^{11}n^7 + 52920k^{11}n^6)$$

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Very often, certificates are not needed!

4th generation: the reduction approach

Goal. Separating the computations of telescopers and certificates

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- Differential case:
  - Bostan, C., Chyzak, Li (2010): bivariate rational functions
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  - Bostan, Lairez, Salvy (2013): multivariate rational functions
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Outline

- Introduction to Creative Telescoping
  - What is creative telescoping
  - Fundamental problems
  - Algorithms and applications
- Creative Telescoping via Reductions
  - Rational case
  - Hyperexponential case
  - Hypergeometric case

Part 2. Rational Telescoping via Reductions

- Rational Telescoping: the continuous case
  - Hermite–Ostrogradsky reduction
  - Telescoping via Hermite–Ostrogradsky reduction
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  - Abramov's reduction
  - Existence criterion for telescopers
  - Telescoping via Abramov's reduction

Rational Telescoping: the continuous case

Telescoping Problem. For $f \in \mathbb{F}(x,y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) \quad \text{for some } g \in \mathbb{F}(x,y).$$

Existence Theorem. Telescopers always exist for rational functions in $\mathbb{F}(x,y)$.

Integrability Problem. For $f \in \mathbb{F}(x,y)$, decide whether

$$f = D_y(g) \quad \text{for some } g \in \mathbb{F}(x,y).$$

If such a g exists, f is said to be D_y-integrable in $\mathbb{F}(x,y)$.

Hermite–Ostrogradsky Reduction

Hermite
(1822-1901)

Ostrogradsky
(1801-1862)

Additive Decomposition. Let $f \in \mathbb{E}(y)$ with $\mathbb{E} = \mathbb{F}(x)$. Then

$$f = D_y(g) + \frac{a}{b},$$

where $g \in \mathbb{E}(y)$ and $a, b \in \mathbb{E}[y]$ with $\deg_y(a) < \deg_y(b)$ and b being squarefree. Moreover

$$f \text{ is } D_y\text{-integrable in } \mathbb{E}(y) \quad \Leftrightarrow \quad a = 0$$

If $\mathbb{E} = \mathbb{C}$, then

$$\int f\,dy = \underbrace{g}_{\textit{Rational}} + \underbrace{\sum_{b(\beta_i)=0} c_i \log(y - \beta_i)}_{\textit{Transcendental}}$$

Hermite–Ostrogradsky Reduction

Hermite
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Ostrogradsky
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Step 1. Squarefree partial fraction decomposition:

$$f = p_0 + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{p_{i,j}}{q_i^j},$$

where $\deg_y(p_{i,j}) < \deg_y(q_i)$ and q_i squarefree.

Step 2. Reducing the multiplicity:

$$\begin{aligned}
\frac{p}{q^m} &= \frac{sq + tD_y(q)}{q^m} = \frac{s}{q^{m-1}} + \frac{tD_y(q)}{q^m} \\
&= \frac{s}{q^{m-1}} + D_y\left(\frac{t(1-m)^{-1}}{q^{m-1}}\right) - \frac{(1-m)^{-1}D_y(t)}{q^{m-1}} \\
&= \frac{s-(1-m)^{-1}D_y(t)}{q^{m-1}} + D_y\left(\frac{t(1-m)^{-1}}{q^{m-1}}\right) \\
&= \cdots = D_y\left(\frac{p_1}{q^{m-1}}\right) + \frac{p_2}{q}.
\end{aligned}$$

Telescoping via Reductions

Reduction w.r.t. y: Let $f = P/Q \in \mathbb{F}(x,y)$. Let

$$Q^* = \text{the sqfr. part of } Q \quad \text{and} \quad d_y^* = \deg_y(Q^*).$$

By Hermite-Ostrogradsky reduction

$$f = D_y(g) + \frac{a}{Q^*}, \quad \deg_y a < d_y^*,$$

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Idea: For $i = 0, 1, 2, \ldots$, compute

- $D_x^i(f) = D_y(g_i) + a_i/Q^*, \quad \deg_y(a_i) < d_y^*,$
- until $\exists\ \eta_0, \ldots, \eta_i \in \mathbb{F}(x)$ with $\eta_i \neq 0$ s.t.

$$\sum_{j=0}^{i} \eta_j a_j = 0 \quad \Longleftrightarrow \quad \underbrace{\sum_{j=0}^{i} \eta_j D_x^j(f)}_{\text{telescoper}} = D_y \underbrace{\left(\sum_{j=0}^{i} \eta_j g_j \right)}_{\text{certificate}}.$$

Features of the Reduction Approach

- Order bound: Given $f = P/Q \in \mathbb{F}(x,y)$,

  its minimal telescoper has order at most d_y^* $(\le \deg_y Q)$.

- Separating the computations of telescopers and certificates:

$$a_j \in \mathbb{F}(x)[y], \text{ with } \deg_y(a_j) < d_y^*$$

$$\sum_{j=0}^{i} \eta_j(x) \cdot a_j = 0 \quad \Longrightarrow \quad \left\{ \text{Telescoper:} \quad \sum_{j=0}^{i} \eta_j D_x^j; \right.$$

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$$\sum_{j=0}^{i} \eta_j(x) \cdot a_j = 0 \quad \Longrightarrow \quad \begin{cases} \text{Telescoper:} & \sum_{j=0}^{i} \eta_j D_x^j; \\ \text{Certificate:} & \sum_{j=0}^{i} \eta_j g_j. \end{cases}$$

Enumerating 2D Rook walks

The Rook moves in a straight line in first quadrant of a plane, and it will not revisit the place it walked.

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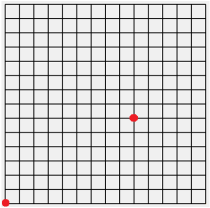

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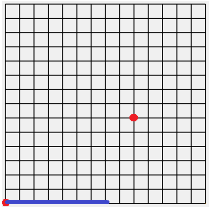

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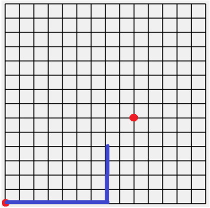

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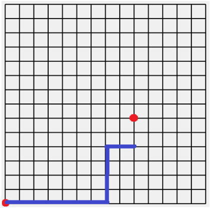

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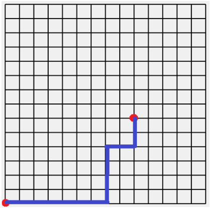

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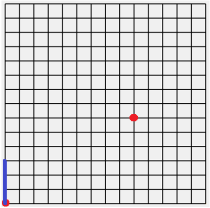

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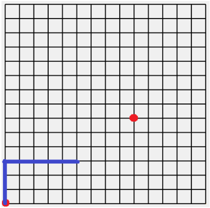

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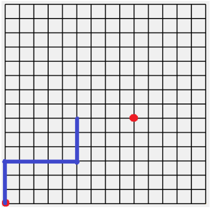

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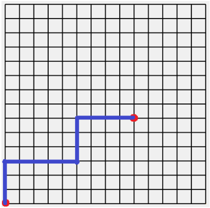

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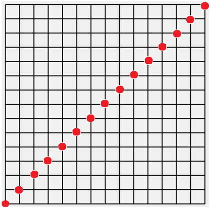

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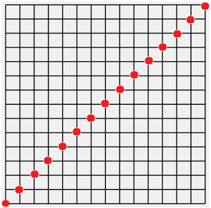

Problem. How to count the number R_n of different Rook walks from $(0,0)$ to (n,n)?

Diagonals

$r(m,n)$: the number of different Rook walks from $(0,0)$ to (m,n).

$$F(x,y) = \sum_{m,n\geq 0} r(m,n)\, x^m y^n = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

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The diagonal of $F(x,y)$ is

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Lemma: Let $f := y^{-1} \cdot F(y, x/y)$ and $L(x, D_x)$ be a linear differential operator with coefficients in $\mathbb{F}(x)$. Then

$$L(x,D_x)(f) = D_y(g) \quad \text{with } g \in \mathbb{F}(x,y) \quad \Rightarrow \quad L(\text{diag}(F)) = 0$$

Telescopers for 2D Rook walks

$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1}F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

Telescopers for 2D Rook walks

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$$D_x^2(f) = D_y(g_2) + \frac{4(9x-7)y}{(9x-1)(9x^2 - 10x + 1)(y(3xy - 2y^2 - 2x + y))}$$

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Telescoping via reductions:

$$c_0(x) \cdot \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

$$+ c_1(x) \cdot \frac{-2y}{(9x-1)(y(3xy - 2y^2 - 2x + y))}$$

$$+ c_2(x) \cdot \frac{4(9x-7)y}{(9x-1)(9x^2 - 10x + 1)(y(3xy - 2y^2 - 2x + y))}$$

$$= 0$$

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Telescoping via reductions:

$$(0) \cdot \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

$$+ (18x - 14) \cdot \frac{-2y}{(9x-1)(y(3xy - 2y^2 - 2x + y))}$$

$$+ (9x^2 - 10x + 1) \cdot \frac{4(9x-7)y}{(9x-1)(9x^2 - 10x + 1)(y(3xy - 2y^2 - 2x + y))}$$

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Telescoping via reductions:

Therefore,

- the minimal telescoper for f is

$$L = (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x$$

- the corresponding certificate is

$$G = \frac{(-36x+28)y^3 + (27x^2 + 42x - 45)y^2 + (-36x^2 - 12x + 24)y + 12x^2 - 4}{2(3xy - 2y^2 - 2x + y)^2}$$

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$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1}F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

Therefore,

- the minimal telescoper for f is

$$L = (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x$$

- Then the generating function of the sequence R_n satisfies

$$L(x, D_x)\left(\sum_{n\geq 0} R(n)x^n\right) = 0.$$

Telescopers for 2D Rook walks

$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1} F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

Therefore,

- the minimal telescoper for f is

$$L = (9x^2 - 10x + 1) D_x^2 + (18x - 14) D_x$$

- From the differential equation, we get the explicit form:

$$\sum_{n \geq 0} R(n) x^n = \frac{1}{2} + \frac{1-x}{2\sqrt{1 - 10x + 9x^2}}$$

Telescopers for 2D Rook walks

$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1}F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

Therefore,

- the minimal telescoper for f is

$$L = (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x$$

- From the differential equation, we get the linear recurrence:

$$R_{n+2} = \frac{(10n+4)R_{n+1} - (9n-9)R_n}{n+1} \quad (R_0 = 1, R_1 = 2).$$

Telescopers for 2D Rook walks

$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1}F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

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$$L = (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x$$

- Running the recurrence, $R(n)$ is as follows.

$1, 2, 14, 106, 838, 6802, 56190, 470010, 3968310, \ldots$ OEIS:A051708

Telescopers for 2D Rook walks

$$F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \quad \Rightarrow \quad f = y^{-1}F(y, x/y) = \frac{xy - y^2 - x + y}{y(3xy - 2y^2 - 2x + y)}$$

Telescoping via reductions:

Therefore,

- the minimal telescoper for f is

$$L = (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x$$

- From the differential equation, we get the asymptotic estimate:

$$R_n \sim \sqrt{\frac{2}{\pi n}} \cdot 3^{2n-1} \quad (n \to +\infty)$$

Rational Telescoping: the discrete case

Telescoping Problem. For $f \in \mathbb{F}(x,y)$, find $L \in \mathbb{F}(x)\langle S_x \rangle$ such that

$$L(x, S_x)(f) = \Delta_y(g) \quad \text{for some } g \in \mathbb{F}(x,y).$$

Remark. Telescopers may not exist in this case.

$$f = 1/(x^2 + y^2) \text{ has no telescoper.}$$

Summability Problem. For $f \in \mathbb{F}(x,y)$, decide whether

$$f = \Delta_y(g) \quad \text{for some } g \in \mathbb{F}(x,y).$$

If such a g exists, f is said to S_y-summable in $\mathbb{F}(x,y)$.

Dispersion and shift-free polynomials

Definition. For $p \in \mathbb{E}[y]$, the dispersion of p in y is

$$\text{disp}_y(p) = \max\{i \in \mathbb{Z} \mid \gcd(p(y), p(y+i)) \neq 1\}$$
$$= \max\{i \in \mathbb{Z} \mid \exists \alpha \in \overline{\mathbb{E}} \text{ s.t. } p(\alpha) = p(\alpha+i) = 0\}$$

Example. Let $p = y(y-x)(y-x+3)(y+x)$. Then $\text{disp}_y(p) = 3$.

Definition. $p \in \mathbb{E}[y]$ is shift-free in y if $\text{disp}_y(p) = 0$.

Prop. Let $f = p/q \in \mathbb{E}(y)$ with $\gcd(p,q) = 1$ and $\deg_y(p) < \deg_y(q)$.

- If $f = \Delta_y(g)$ for $g = a/b \in \mathbb{E}(y)$, then $\text{disp}_y(q) = \text{disp}_y(b) + 1$;
- If $\text{disp}_y(q) = 0$, then f is not S_y-summable in $\mathbb{E}(y)$.

Abramov's Reduction

Additive Decomposition. Let $f \in \mathbb{E}(y)$ with $\mathbb{E} = \mathbb{F}(x)$. Then

$$f = \Delta_y(g) + \frac{a}{b},$$

where $g \in \mathbb{E}(y)$ and $a, b \in \mathbb{E}[y]$ with $\deg_y(a) < \deg_y(b)$ and b being shift-free in y. Moreover

$$f \text{ is } S_y\text{-summable in } \mathbb{E}(y) \quad \Leftrightarrow \quad a = 0$$

Step 1. Irreducible partial fraction decomposition:

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{\ell=0}^{\lambda_{i,j}} \frac{a_{i,j,\ell}}{S_y^{\ell}(d_i)^j},$$

where $\deg_y(a_{i,j,\ell}) < \deg_y(d_i)$ and $\text{disp}_y(d_1 \cdots d_n) = 0$.

Step 2. Reducing the dispersion:

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Additive Decomposition. Let $f \in \mathbb{E}(y)$ with $\mathbb{E} = \mathbb{F}(x)$. Then

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$$\frac{a}{S_y^{\ell}(d^m)} = \frac{a}{S_y^{\ell}(d^m)} - \frac{S_y^{-1}(a)}{S_y^{\ell-1}(d^m)} + \frac{S_y^{-1}(a)}{S_y^{\ell-1}(d^m)}$$

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Step 2. Reducing the dispersion:

$$\frac{a}{S_y^{\ell}(d^m)} = \quad \cdots \quad = \Delta_y(g_{\ell,m}) + \frac{S_y^{-\ell}(a)}{d^m}$$

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$$f = \Delta_y(g) + \frac{a}{d_1^{m_1} \cdots d_n^{m_n}}.$$

Existence criterion

Definition. $p \in \mathbb{F}[x,y]$ is integer-linear over $\mathbb{F}$ if $\exists q \in \mathbb{F}[z]$ and $m,n \in \mathbb{Z}$ s.t. $p = q(mx+ny)$.

Theorem. (AbramovLe2002) Let $f \in \mathbb{F}(x,y)$. Then f has a telescoper in $\mathbb{F}(x)\langle S_x \rangle$ if and only if

$$f = \Delta_y(g) + \frac{a}{b_1 \cdots b_n},$$

where $g \in \mathbb{F}(x,y), a, b_i \in \mathbb{F}[x,y]$ and the b_i's are integer-linear.

Examples. $f_1 = 1/(x^2+y^2)$ has no telescoper since x^2+y^2 is not integer-linear and $f_2 = 1/((x+y)(3x+2y))$ has a telescoper.

Telescoping via reduction

$$f = \frac{1}{(y+x)(2y+3x)} = \Delta_y(\cdots) + \underbrace{\frac{1}{(y+x)(2y+3x)}}_{r_0}$$

$$S_x(f) = \frac{1}{(y+x+1)(2y+3x+3)} = \Delta_y(\cdots) + \underbrace{\frac{x+3}{(x+1)(y+x)(2y+3x+3)}}_{r_1}$$

$$S_x^2(f) = \frac{1}{(y+x+2)(2y+3x+6)} = \Delta_y(\cdots) + \underbrace{\frac{x}{(x+2)(y+x)(2y+3x)}}_{r_2}$$

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$$S_x(f) = \frac{1}{(y+x+1)(2y+3x+3)} = \Delta_y(\cdots) + \underbrace{\frac{x+3}{(x+1)(y+x)(2y+3x+3)}}_{r_1}$$

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Finding linear dependency:

$$x \cdot r_0 + 0 \cdot r_1 - (x+2) \cdot r_2 = 0$$

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Finding linear dependency:

$$x \cdot r_0 + 0 \cdot r_1 - (x+2) \cdot r_2 = 0$$

$$\Downarrow$$

$L = -(x+2)S_x^2 + x$ is a telescoper for f.

How the reduction approach work?

Reduction map. Let $\mathscr{M}$ be a $\mathbb{F}(x,y)\langle \partial_x, \partial_y \rangle$-module.

$$\begin{array}{rccc} [\cdot]: & \mathscr{M} & \to & \mathscr{M} \\ & f & \mapsto & [f] \end{array}$$

satisfies the properties

1. Normality:

$$f = \partial_y(g) \quad \Leftrightarrow \quad [f] = 0$$

2. Finite-dimensionality:

$$\dim_{\mathbb{F}(x)} \mathsf{span}_{\mathbb{F}(x)}\{[\partial_x^i(f)] \mid i \in \mathbb{N}\} < +\infty$$

3. Linearity:

$$[f_1 + f_2] = [f_1] + [f_2]$$

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$$[f_1 + f_2] = [f_1] + [f_2]$$

Reduction-based creative telescoping.

$$c_0[f] + \cdot + c_r[\partial_x^i(f)] = 0 \qquad \Leftrightarrow \qquad \underbrace{(c_0 + \cdots + c_r\partial_x^r)}_{\text{Telescoper}}(f) = \partial_y(g)$$

Outline

- Introduction to Creative Telescoping
- Creative Telescoping via Reductions
  - Rational case
  - Hyperexponential case
  - Hypergeometric case

Part 3. Hyperexponential Telescoping via Reductions

- Hyperexponential Integrability
- Hermite Reduction
- Telescoping via Hermite Reduction
- Example: Counting 3D Rook Walks

Univariate hyperexponential functions

Definition. $H(y)$ is hyperexponential over $\mathbb{E}(y)$ if

$$f := \frac{D_y(H)}{H} \in \mathbb{E}(y).$$

Write informally

$$H = \exp\left(\int f(y)\,dy\right).$$

Examples. $1/(1+y)$, $\exp(2+y^2)$, $(1+y^2)^c$, $\frac{1}{\sqrt{1+y^2}}, \ldots$

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Structural form.

$$\left(\int f(y)\,dy\right) = u_0 + \sum_{i=1}^{m} c_i \log(u_i)$$

$$\Downarrow$$

$$H = \exp\left(\int f(y)\,dy\right) = \exp(u_0) \cdot \prod_{i=1}^{m} u_i^{c_i}$$

Hyperexponential integrability

Integrability Problem. For a hyperexp. $H(y)$, decide whether

$$H = D_y(G) \quad \text{for hyperexp. } G \text{ over } \mathbb{E}(y).$$

If such a G exists, H is said to be hyperexp. D_y-integrable. Note that $G = u \cdot H$ for some $u \in \mathbb{E}(y)$.

Example. $\exp(y)$ is hyperexp. D_y-integrable but not $\exp(y^2)$.

Fact. Let $f = D_y(H)/H \in \mathbb{E}(y)$. Then

$$H \text{ is hyperexp. } D_y\text{-integrable}$$

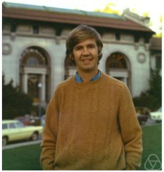
$$\Updownarrow$$

$$D_y(u(y)) + f \cdot u(y) = 1 \text{ has a solution in } \mathbb{E}(y).$$

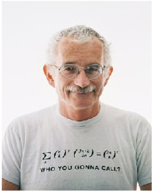
Almkvist–Zeilberger's algorithm

Let $f = D_y(H)/H$. Find a rational solution of

$$D_y(u(y)) + f \cdot u(y) = 1.$$



Gert Almkvist



Doron Zeilberger

1 Decompose

$$f = \frac{D_y(p)}{p} + \frac{q}{r},$$

where $p, q, r \in \mathbb{E}[y]$ and q, r satisfies

$$\gcd(r, q - jD_y(r)) = 1 \quad \text{for all } j \in \mathbb{N}.$$

2 Find a polynomial solution of

$$p = (q + D_y(r))v(y) + rD_y(v(y))$$

3 If $v \in \mathbb{E}[y]$ exists, return $u := (rv/p)$.

Multiplicative factorization

Definition. A pair $(S,K) \in \mathbb{E}(y)^2$ is called the canonical form of $f \in \mathbb{E}(y)$ if

$$f = \frac{D_y(S)}{S} + K,$$

where $S = u/v$ and $K = p/q$ s.t. $\gcd(q,v) = 1$ and

$$\gcd(q, p - i \cdot D_y(q)) = 1 \quad \text{for all } i \in \mathbb{Z}.$$

S is called the shell and K the kernel of f.

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S is called the shell and K the kernel of f.

Multiplicative form. Let (S,K) be the canonical form of $f := D_y(H)/H$. Then

$$H = \exp\left(\int f\,dy\right) = \underbrace{S}_{\text{rational part}} \cdot \exp\left(\int K\,dy\right)$$

Remark. $K = 0 \quad \Rightarrow \quad H \in \mathbb{E}(y)$.

Geddes-Le-Li's reduction

Additive Decomposition. Let $H = S \cdot T$ with $T = \exp\left(\int K\,dy\right)$ and

$$S = \frac{p}{q} \quad \text{and} \quad K = \frac{a}{b}.$$

Then $\exists g \in \mathbb{E}(y), u, v \in \mathbb{E}[y]$ s.t.

$$H = D_y(g \cdot T) + \left(\frac{u}{q^*} + \frac{v}{b}\right) \cdot T,$$

where q^* is the squarefree part of q and $\deg_y(u) < \deg_y(q^*)$.

Remark. If $H \in \mathbb{E}(y)$, then G-L-L = H-O.

Proposition. H is hyperexp. D_y-integrable if and only if

- $u = 0$;
- $bD_y(w) + aw = v$ has a polynomial solution in $\mathbb{E}[y]$.

Polynomial reduction

Given $K = a/b \in \mathbb{E}(y)$ with $K \neq 0$, define

$$\begin{array}{rccc} \phi_K: & \mathbb{E}[y] & \longrightarrow & \mathbb{E}[y] \\ & p & \longmapsto & b \cdot D_y(p) + ap. \end{array}$$

Call ϕ_K the polynomial reduction map w.r.t. K.

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Write

$$\mathbb{E}[y] = \mathrm{im}(\phi_K) \oplus \mathcal{N}_K,$$

where

$$\mathcal{N}_K = \mathrm{span}_{\mathbb{E}}\{y^i \mid \forall q \in \mathrm{im}(\phi_K), \deg_y(q) \neq i\}.$$

Call $\mathcal{N}_K$ the standard complement of $\mathrm{im}(\phi_K)$.

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Call $\mathcal{N}_K$ the standard complement of $\mathrm{im}(\phi_K)$.

Proposition.

- $\dim_{\mathbb{E}}(\mathcal{N}_K) \leq \max\{\deg_y(a), \deg_y(b) - 1\}$;
- $\mathrm{im}(\phi_K)$ has an $\mathbb{E}$-basis $\{\phi_K(y^i) \mid i \in \mathbb{N}\}$.

Hermite reduction

For a hyperexp. function $H = S \cdot T$, where

$$S = \frac{p}{q} \quad \text{and} \quad T = \exp\left(\int K\,dy\right) \quad \text{with} \quad K = \frac{a}{b},$$

Step 1. Geddes-Le-Li's reduction:

$$H = D_y(g \cdot T) + \left(\frac{u}{q^*} + \frac{v}{b}\right) \cdot T$$

Step 2. Polynomial reduction: Let $v = v_1 + v_2$ with

$$v_1 = \phi_K(w) \text{ and } v_2 \in \mathscr{N}_K.$$

Then

$$\frac{v}{b} \cdot T = D_y\left(w \cdot T\right) + \frac{v_2}{b} \cdot T.$$

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Then

$$\frac{v}{b} \cdot T = D_y\left(w \cdot T\right) + \frac{v_2}{b} \cdot T.$$

$$\Downarrow$$

$$H = D_y((g + w) \cdot T) + \left(\frac{u}{q^*} + \frac{v_2}{b}\right) \cdot T$$

Properties of Hermite reduction

For a hyperexp. function $H = S \cdot T$, where

$$S = \frac{p}{q} \quad \text{and} \quad T = \exp\left(\int K\, dy\right) \quad \text{with} \quad K = \frac{a}{b},$$

Hermite reduction decomposes H as

$$H = S \cdot T = \underbrace{D_y\,(g \cdot T)}_{\text{integrable}} + \underbrace{\frac{r}{b \cdot q^*} \cdot T}_{\text{non-integrable}}$$

where

- $\deg_y(r) < \deg_y(q^*) + \max\{\deg_y(a), \deg_y(b) - 1\}$ with q^* being squarefree;
- $\frac{r}{b \cdot q^*}$ is unique;
- h is hyperexp. D_y- integrable $\quad \Leftrightarrow \quad r = 0.$

We call $r/(b \cdot q^*) \cdot T$ the residual form of H.

Bivariate hyperexponential functions

Definition. $H(x,y)$ is hyperexponential over $\mathbb{F}(x,y)$ if

$$g := \frac{D_x(H)}{H}, \quad f := \frac{D_y(H)}{H} \in \mathbb{F}(x,y).$$

Write informally

$$H = \exp\left(\int g\,dx + f\,dy\right)$$

Examples.

$$\frac{1}{x+y}, \quad \exp(x^2+y^2), \quad (x^2+y^2)^{\pi}, \quad \frac{1}{\sqrt{x+y}}, \quad \dots$$

Fact. If H_1, H_2 are hyperexp., so is $H_1 \cdot H_2$.

$D_x D_y(H) = D_y D_x(H)$

Bivariate hyperexponential functions

Definition. $H(x,y)$ is hyperexponential over $\mathbb{F}(x,y)$ if

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$$\frac{1}{x+y}, \quad \exp(x^2+y^2), \quad (x^2+y^2)^{\pi}, \quad \frac{1}{\sqrt{x+y}}, \quad \ldots$$

Fact. If H_1, H_2 are hyperexp., so is $H_1 \cdot H_2$.

$$D_x D_y(H) = D_y D_x(H) \quad \Rightarrow \quad D_x(f) = D_y(g)$$

Bivariate hyperexponential functions

Definition. $H(x,y)$ is hyperexponential over $\mathbb{F}(x,y)$ if

$$g := \frac{D_x(H)}{H}, \quad f := \frac{D_y(H)}{H} \in \mathbb{F}(x,y).$$

Write informally

$$H = \exp\left(\int g\,dx + f\,dy\right)$$

Examples.

$$\frac{1}{x+y}, \quad \exp(x^2+y^2), \quad (x^2+y^2)^{\pi}, \quad \frac{1}{\sqrt{x+y}}, \quad \ldots$$

Fact. If H_1, H_2 are hyperexp., so is $H_1 \cdot H_2$.

$D_x D_y(H) = D_y D_x(H) \quad \Rightarrow \quad D_x(f) = D_y(g) \quad \Rightarrow \quad \text{denom}(g) \mid \text{denom}(f)$

Hermite reduction for $D_x^i(H)$

Hermite reduction in y decomposes H as

$$H = D_y\left(g_0 \cdot T\right) + \frac{r_0}{b \cdot q^*} \cdot T \quad \text{with } T = \exp\left(J\,dx + K\,dy\right) ,$$

where $b = \text{denom}(K)$ and $\text{denom}(J) \mid b$. Then

$$D_x(H) = D_y(D_x(g_0 \cdot T)) + \left(D_x\left(\frac{r_0}{b \cdot q^*}\right) + \frac{r_0}{b \cdot q^*} \cdot J\right) \cdot T$$

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$$\vdots$$

$$D_x^i(H) = D_y\left(g_i \cdot T\right) + \frac{r_i}{b \cdot q^*} \cdot T$$

Telescoping via reductions

Given a hyperexp. function $H = S \cdot T$, where

$$S = \frac{p}{q} \quad \text{and} \quad T = \exp\left(\int J\,dx + K\,dy\right) \quad \text{with} \quad K = \frac{a}{b},$$

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$$D_x^i(H) = D_y\left(g_i T\right) + \frac{r_i}{b \cdot q^*} T$$

with $\deg_y(r_i) < \underbrace{\deg_y(q^*) + \max\{\deg_y(a), \deg_y(b) - 1\}}_{\text{a fixed number depending on } H}$.

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2 Find linear dependency: until $\exists\ c_0, \ldots, c_i \in \mathbb{F}(x)$ with $c_i \neq 0$ s.t.

$$\sum_{j=0}^{i} c_j r_j = 0 \quad \Longleftrightarrow \quad \underbrace{\left(\sum_{j=0}^{i} c_j D_x^j\right)}_{\text{telescoper } L}(H) = D_y \underbrace{\left(\sum_{j=0}^{i} c_j g_j T\right)}_{\text{certificate } g}.$$

Enumerating 3D Rook walks

The Rook moves in a straight line in first quadrant of 3D lattice

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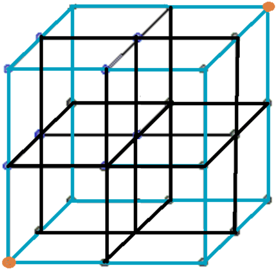

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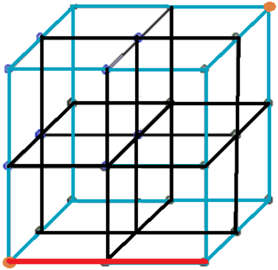

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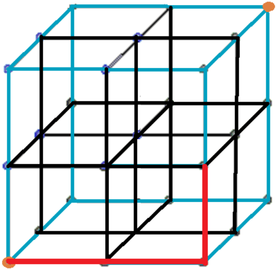

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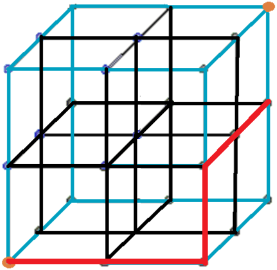

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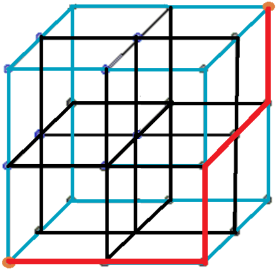

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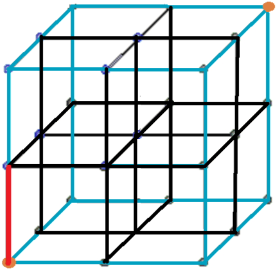

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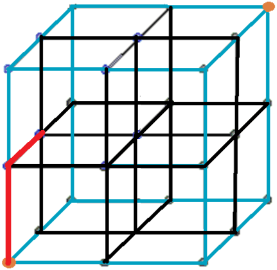

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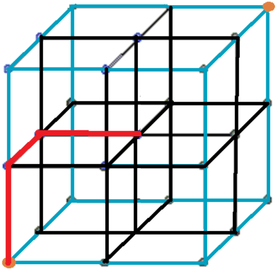

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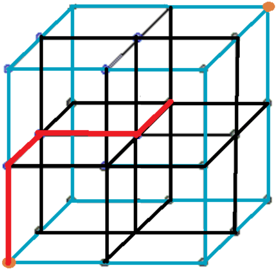

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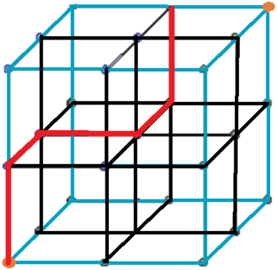

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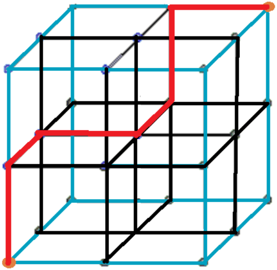

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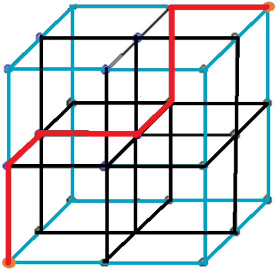

Problem. How to count the number R_n of different Rook walks from $(0,0,0)$ to (n,n,n)?

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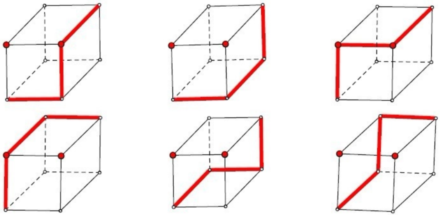

$R_1 = 6$

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The Rook moves in a straight line in first quadrant of 3D lattice

$$R_8 = ?$$

Problem. How to count the number R_n of different Rook walks from $(0,0,0)$ to (n,n,n)?

Diagonals

$r(m,n,k)$: the number of different Rook walks from $(0,0,0)$ to (m,n,k).

$$F(x,y,z) = \sum_{m,n \geq 0} r(m,n,k)\, x^m y^n z^k = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y} - \frac{z}{1-z}}.$$

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Lemma: Let $f := (yz)^{-1}F(y,z/y,x/z)$ and $L \in \mathbb{F}(x)\langle D_x\rangle$. Then

$L(x,D_x)(f) = D_y(g) + D_z(h)$ with $g,h \in \mathbb{F}(x,y,z)$ $\Rightarrow$ $L(\mathrm{diag}(F)) = 0$.

Residues

Definition. Let $f \in \mathbb{F}(x,y)(z)$. The residue of f at β_i w.r.t. z, denoted by $\text{res}_z(f, \beta_i)$, is the coefficient $\alpha_{i,1}$ in

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(z-\beta_i)^j}, \quad \text{where } \alpha_{i,j}, \beta_i \in \overline{\mathbb{F}(x,y)}.$$

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Lemma. Let $f \in \mathbb{F}(x,y,z)$ and $\beta \in \overline{\mathbb{F}(x,y)}$. Then

- $D_v(\text{res}_z(f,\beta)) = \text{res}_z(D_v(f),\beta)$ for $D_v \in \{D_x, D_y\}$.
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Remark. The second assertion is not true for algebraic functions!

Equivalence between two telescoping problems

Theorem (Picard1912). Let $f = p/q \in \mathbb{F}(x,y,z)$ and $L \in \mathbb{F}(x)\langle D_x \rangle$. Then

$$L(x, D_x)(f) = D_y(g) + D_z(h) \text{ for } g, h \in \mathbb{F}(x,y,z)$$

$$\Updownarrow$$

$$L(x, D_x)(\alpha_i) = D_y(\gamma_i) \text{ for all residues } \alpha_i = \text{res}_z(f, \beta_i),$$

$$\text{where } \beta_i, \gamma_i \in \overline{\mathbb{F}(x,y)} \text{ with } q(x,y,\beta_i) = 0.$$

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Remark.

$$L_i(x, D_x)(\alpha_i) = D_y(\beta_i),\ 1 \leq i \leq n$$

$$\Downarrow$$

$$L = \text{LCLM}(L_1, L_2, \ldots, L_n) \text{ is a telescoper for all } \alpha_i.$$

Telescopers for 3D Rook walks

For 3D Rook walks, the rational function is

$$f := \frac{1}{yz}F(y, z/y, x/z) = \frac{(-1+y)(y-z)(-z+x)}{zy((3y-2)z^2+(y+3x-2y^2-4xy)z+3xy^2-2xy)}$$

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$$r_1 = \frac{y-1}{y(3y-2)},\ r_2 = -r_3 = \underbrace{\frac{(y-1)^2}{y(3y-2)\sqrt{-4y^3+16xy^2+4y^2-y-24xy+9x}}}_{\text{hyperexponential over } \mathbb{F}(x,y)}.$$

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Telescopers. $L_1 = D_x$ and $L_2 = L_3$ with

$$\begin{aligned} L_2 =& {D_x}^3 + \frac{\left(4608\,x^4-6372\,x^3+813\,x^2+514\,x-4\right){D_x}^2}{x\left(-2+121\,x+475\,x^2-1746\,x^3+1152\,x^4\right)} \\ &+ \frac{4\left(576\,x^3-801\,x^2-108\,x+74\right)D_x}{x\left(-2+121\,x+475\,x^2-1746\,x^3+1152\,x^4\right)} \end{aligned}$$

Recurrences for diagonal 3D Rook walks

$L = \mathsf{LCLM}(L_1, L_2, L_3)$ is a telescoper for $f(x,y,z)$.

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Recurrence. From the differential equation, we get

$$(1152n^2 + 1152n^3)R_n + (-7830n - 3204 - 6372n^2 - 1746n^3)R_{n+1} + (2957n + 762 + 2238n^2 + 475n^3)R_{n+2} + (4197n + 4698 + 1240n^2 + 121n^3)R_{n+3} + (-22n^2 - 80n - 96 - 2n^3)R_{n+4} = 0.$$

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Using the initial values $R_0 = 1, R_1 = 6, R_2 = 222, R_3 = 9918$, we get

$$R_8 = 4223303759148.$$

Outline

- Introduction to Creative Telescoping
  - What is creative telescoping
  - Fundamental problems
  - Algorithms and applications
- Creative Telescoping via Reductions
  - Rational case
  - Hyperexponential case
  - Hypergeometric case

Part 4. Hypergeometric Telescoping via Reductions

- Abramov–Petkovšek Reduction
- Existence of Telescopers
- Construction of Telescopers

Univariate hypergeometric terms

Definition. $H(k)$ is hypergeometric over $\mathbb{E}(k)$ if

$$\frac{H(k+1)}{H(k)} \triangleq \frac{S_k(H)}{H} \in \mathbb{E}(k).$$

Examples.

$$1/(1+k), \quad 2^k, \quad k!, \quad \Gamma(2k+1), \ldots$$

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Fact. If H_1, H_2 are hypergeometric, so is $H_1 \cdot H_2$.

$$f = \frac{S_k(r)}{r} \cdot \mu \cdot \frac{(k-\alpha_1)\cdots(k-\alpha_n)}{(k-\beta_1)\cdots(k-\beta_m)} \quad \text{with } \alpha_i - \beta_j \notin \mathbb{Z}$$

$$\Downarrow$$

$$H = r \cdot \mu^k \cdot \frac{\Gamma(k-\alpha_1+1)\cdots\Gamma(k-\alpha_n+1)}{\Gamma(k-\beta_1+1)\cdots\Gamma(k-\beta_m+1)}$$

Hypergeometric summability

Summability Problem. For a hypergeom. $H(k)$, decide whether

$$H = \Delta_k(G) \quad \text{for hypergeom. } G \text{ over } \mathbb{E}(k).$$

If such a G exists, H is said to be hypergeom. S_k-summable. Note that $G = u \cdot H$ for some $u \in \mathbb{E}(k)$.

Example. $k \cdot k! = \Delta_k(k!)$ is hypergeom. S_k-summable but not $k!$.

Fact. Let $f = S_k(H)/H \in \mathbb{E}(k)$. Then

$$H \text{ is hypergeom. } S_k\text{-summable}$$

$$\Updownarrow$$

$$f \cdot S_k(u(k)) - u(k) = 1 \text{ has a solution in } \mathbb{E}(k).$$

Gosper's algorithm

Let $f = S_k(H)/H \in \mathbb{E}(k)$. Find a rational solution of

$$f \cdot S_k(u(k)) - u(k) = 1.$$

1 Compute Gosper's form

$$f = \frac{S_k(p)}{p} \cdot \frac{q}{r},$$

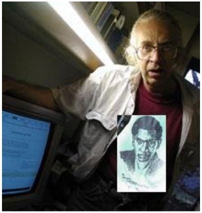
where $p, q, r \in \mathbb{E}[k]$ and q, r satisfies

$$\gcd(q(k), r(k+j)) = 1 \quad \text{for all } j \in \mathbb{N}.$$

2 Find a polynomial solution of

$$p = q \cdot S_k(v(k)) - S_k^{-1}(r) \cdot v(y)$$

3 If $v \in \mathbb{E}[k]$ exists, return $u := S_k^{-1}(r)v/p$.



Bill Gosper

Multiplicative factorization

Defn. A pair $(S,K) \in \mathbb{E}(k)^2$ is called the canonical form of $f \in \mathbb{E}(k)$ if

$$f = \frac{S(k+1)}{S(k)} \cdot K, \quad \text{where } S = \frac{u}{v} \text{ and } K = \frac{p}{q}$$

satisfying $\gcd(q,v) = 1$ and K is shift-reduced, i.e.,

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S is called a shell and K a kernel of f.

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Multiplicative form. Let (S,K) be the canonical form of $f := S_k(H)/H$. Then

$$H = \underbrace{S}_{\text{rational part}} \cdot T \quad \text{with} \quad \frac{S_k(T)}{T} = K.$$

Remark. $K = 1 \quad \Rightarrow \quad H \in \mathbb{E}(k)$.

Abramov–Petkovšek reduction

Additive Decomposition. Let $H = S \cdot T$ with

$$S = \frac{p}{q} \quad \text{and} \quad \frac{S_k(T)}{T} = K = \frac{a}{b}.$$

Then $\exists g \in \mathbb{E}(k)$ and $u, v \in \mathbb{E}[k]$ s.t.

$$H = \Delta_k(g \cdot T) + \left(\frac{u}{\tilde{q}} + \frac{v}{b}\right) \cdot T,$$

where $\tilde{q}$ is shift-free, $\deg_k(u) < \deg_k(\tilde{q})$, and

$$\gcd\left(\tilde{q}, S_k^{-\ell}(a)\right) = \gcd\left(\tilde{q}, S_k^{\ell}(b)\right) = 1 \quad \text{for all } \ell \geq 0.$$

Remark. If $H \in \mathbb{E}(k)$, then Abramov–Petkovšek = Abramov.

Proposition. H is hypergeom. S_y-summable if and only if

- $u = 0$;
- $aS_k(w) - bw = v$ has a polynomial solution in $\mathbb{E}[k]$.

Polynomial reduction

Given $K = a/b \in \mathbb{E}(k)$ with $K \neq 0$, define

$$\begin{array}{rccc} \phi_K : & \mathbb{E}[k] & \longrightarrow & \mathbb{E}[k] \\ & p & \mapsto & a \cdot S_k(p) - b \cdot p. \end{array}$$

Call ϕ_K the polynomial reduction map w.r.t. K.

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Write

$$\mathbb{E}[k] = \mathrm{im}(\phi_K) \oplus \mathcal{N}_K,$$

where

$$\mathcal{N}_K = \mathrm{span}_{\mathbb{E}}\{k^i \mid \forall q \in \mathrm{im}(\phi_K), \deg_k(q) \neq i\}.$$

Call $\mathcal{N}_K$ the standard complement of $\mathrm{im}(\phi_K)$.

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Proposition.

- $\dim_{\mathbb{E}}(\mathcal{N}_K) \leq \max\{\deg_k(a), \deg_k(b)\}$;
- $\mathrm{im}(\phi_K)$ has an $\mathbb{E}$-basis $\{\phi_K(k^i) \mid i \in \mathbb{N}\}$.

Modified Abramov–Petkovšek (MAP) reduction

For a hypergeom. term $H = S \cdot T$, where

$$S = \frac{p}{q} \quad \text{and} \quad \frac{S_k(T)}{T} = K = \frac{a}{b},$$

Step 1. Abramov–Petkovšek's reduction:

$$H = \Delta_k(g \cdot T) + \left(\frac{u}{\tilde{q}} + \frac{v}{b}\right) \cdot T,$$

Step 2. Polynomial reduction: Let $v = v_1 + v_2$ with

$$v_1 = \phi_K(w) \text{ and } v_2 \in \mathcal{N}_K.$$

Then

$$\frac{v}{b} \cdot T = \Delta_k\left(w \cdot T\right) + \frac{v_2}{b} \cdot T.$$

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Then

$$\frac{v}{b} \cdot T = \Delta_k(w \cdot T) + \frac{v_2}{b} \cdot T.$$

$$\Downarrow$$

$$H = \Delta_k((g+w) \cdot T) + \left(\frac{u}{\tilde{q}} + \frac{v_2}{b}\right) \cdot T$$

Properties of MAP reduction

For a hypergeom. term $H = S \cdot T$, where

$$S = \frac{p}{q} \quad \text{and} \quad \frac{S_k(T)}{T} = K = \frac{a}{b},$$

MAP reduction decomposes H as

$$H = S \cdot T = \underbrace{\Delta_k(g \cdot T)}_{\text{summable}} + \underbrace{\frac{r}{b \cdot \tilde{q}} \cdot T}_{\text{non-summable}}$$

where

- $\deg_k(r) < \deg_k(\tilde{q}) + \max\{\deg_y(a), \deg_y(b)\}$ with $\tilde{q}$ being shift-free;
- h is hypergeom. S_y-summable $\quad \Leftrightarrow \quad r = 0$.

We call $r/(b \cdot \tilde{q}) \cdot T$ the residual form of H, which is not unique.

Linearity adjustment of residual forms

residual form + residual form $\neq$ residual form.

Example. Let the kernel $K = 1$.

$$\frac{1}{y+1} + \frac{1}{y} = \frac{2y+1}{y(y+1)}$$

Linearity adjustment of residual forms

$$\text{residual form} + \text{residual form} \neq \text{residual form}.$$

Example. Let the kernel $K = 1$.

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Proposition. Let $K \in \mathbb{F}(y)$ be shift-reduced and $\sigma_y(T)/T = K$. Let $r_1 \cdot T, r_2 \cdot T$ be two residual forms w.r.t. K. Then $\exists$ a residual form r_1' w.r.t. K s.t.

$$r_1 T = \Delta_y(g \cdot T) + r_1' T \quad \text{and} \quad (r_1' + r_2) \cdot T \text{ is a residual form.}$$

Bivariate hypergeometric terms

Definition. $H(n,k)$ is hypergeometric over $\mathbb{F}(n,k)$ if

$$g := \frac{S_n(H)}{H}, \quad f := \frac{S_k(H)}{H} \in \mathbb{F}(n,k).$$

Examples.

$$\frac{1}{n+k}, \quad 2^n 3^k, \quad \binom{n}{k}, \quad (n+k)!, \quad \Gamma(2n+3k), \ldots$$

Ore–Sato Theorem.

$$H = f(n,k)\lambda^n \mu^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + b_i k + c_i)}{\Gamma(u_i n + v_i k + w_i)},$$

where $f \in \mathbb{F}(n,k), \lambda, \mu, c_i, w_i \in \mathbb{F}$ and $a_i, b_i, u_i, v_i \in \mathbb{Z}$.

Existence criterion for telescopers

Hypergeometric Telescoping. Given hypergeom. H over $\mathbb{F}(n,k)$, find $L \in \mathbb{F}(n)\langle S_n \rangle$ s.t.

$$L(n,S_n)(H) = \Delta_k(G) \quad \text{for hypergeom. } G \text{ over } \mathbb{F}(n,k).$$

Remark. Telescopers may not exist for hypergeometric terms, e.g., $H = \binom{n}{k}/(n^2+k^2)$.

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Definition. H is proper if it is of the form

$$H = p(n,k)\lambda^n \mu^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + b_i + c_i)}{\Gamma(u_i n + v_i k + w_i)},$$

where p is polynomial in $\mathbb{F}[n,k]$.

Abramov's criterion.

$$H \text{ has a telescoper} \quad \Leftrightarrow \quad H = \Delta_k(H_1) + H_2, \text{ where } H_2 \text{ is proper.}$$

Telescoping via reductions

Consider

$$H = \frac{1}{n+k} \cdot k!$$

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- $T = H/S = k!$

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$$= \Delta_k(g_1 \cdot T) + \left(-\frac{1/n}{n+k} + \frac{1}{n} \right) T$$

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Telescoping via reductions

Consider

$$H = \frac{1}{n+k} \cdot k!$$

$$c_0(n) \cdot \frac{1}{n+k}$$

$$+ c_1(n) \cdot \left(-\frac{1/n}{n+k} + \frac{1}{n} \right)$$

$$+ c_2(n) \cdot \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)} \right)$$

$$= 0$$

Telescoping via reductions

Consider

$$H = \frac{1}{n+k} \cdot k!$$

$$-1 \cdot \frac{1}{n+k}$$

$$+(1-n) \cdot \left(-\frac{1/n}{n+k} + \frac{1}{n}\right)$$

$$+(n+1) \cdot \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)}\right)$$

$$= 0$$

Telescoping via reductions

Consider

$$H = \frac{1}{n+k} \cdot k!$$

Therefore,

- the minimal telescoper for T w.r.t. k is

$$L = (n+1) \cdot S_n^2 - (n-1) \cdot S_n - 1$$

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- the corresponding certificate is

$$G = ((n+1) \cdot g_2 - (n-1) \cdot g_1 - 1 \cdot g_0) \cdot T$$

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$$\begin{aligned} G &= ((n+1) \cdot g_2 - (n-1) \cdot g_1 - 1 \cdot g_0) \cdot T \\ &= \frac{k!}{(n+k)(n+k+1)} \end{aligned}$$

Softwares

- MAPLE:
  1. EKHAD by Zeilberger
  2. DEtools:-Zeilberger by Le
  3. SumTools[Hypergeometric]:-Zeilberger by Le
  4. Mgfun:-creative_telescoping by Chyzak
  5. ReductionCT by C., Huang, Kauers, and Li
  6. ...
- MATHEMATICA:
  1. fastZeil: Zb by Paule and Schorn
  2. HolonomicFunctions: CreativeTelescoping by Koutschan
  3. ...
- Maxima: Zeilberger by Fabrizio Caruso
- Reduce: zeilberg by Wolfram Koepf
- ...

4th generation: the reduction approach

Goal. Separating the computations of telescopers and certificates

- Differential case:
  - Bostan, C., Chyzak, Li (2010): bivariate rational functions
  - Bostan, C, Chyzak, Li, Xin (2013): bivariate hyperexp. funs
  - Bostan, Lairez, Salvy (2013): multivariate rational functions
  - C., Kauers, Koutschan (2016): bivariate algebraic functions
  - C., van Hoeij, Kauers, Koutschan (2018): fuchsian D-finite
  - van der Hoeven (2017, 2018), Bostan, Chyzak, Lairez, Salvy (2018): D-finite functions
- Shift case:
  - C., Huang, Kauers, Li (2015): bivariate hypergeom. terms
  - Huang (2016): new bounds for hypergeom. creative telescoping
  - Giesbrecht, Huang, Labahn, Zima (2019): faster algorithm
  - C., Hou, Huang, Labahn, Wang (2019): trivariate rational

Open problems

J Syst Sci Complex (2017) 30: 154–172

Some Open Problems Related to Creative Telescoping*

CHEN Shaoshi · KAUERS Manuel

Open problems

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Open Problem 1. Multivariate extension of Gosper's algorithm

Given a multivariate hypergeometric term $H(k_1,\ldots,k_d)$ over $\mathbb{F}(k_1,\ldots,k_d)$, decide whether there exist hypergeometric terms $G_1,\ldots,G_d$ such that

$$H = \Delta_{k_1}(G_1) + \cdots + \Delta_{k_d}(G_d).$$

Remark. Bivariate rational case: ChenSinger (2014), HouWang (2015). Multivariate rational case: ChenDu (2019).

Open problems

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Open Problem 2. Picard's problem (1889)

Given a rational function $f \in \mathbb{C}(x,y,z)$, decide whether there exist $u,v,w \in \mathbb{C}(x,y,z)$ such that

$$f = D_x(u) + D_y(v) + D_z(w).$$

Remark. The bivariate case was solved by Picard in 1889.

Open problems

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Open Problem 3. Inverse creative telescoping problem

1. Given an $L \in \mathbb{F}(n)\langle S_n \rangle$, decide whether there exists a hypergeometric term $H(n,k)$ s.t. L is a telescoper for H.

2. Given an $L \in \mathbb{F}(x)\langle D_x \rangle$, decide whether there exists a hyperexponential function $H(x,y)$ s.t. L is a telescoper for H.

Remark. Petkovsek recently made some progress on this problem.

Open problems

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Open Problem 4. Computational challenges

For $d = 4, 5, \ldots, 12$, prove recurrence equations for the diagonal of the rational series $1/(1 - \sum_{i=1}^{d} \frac{x_i}{1-x_i})$ conjectured in the paper

Advances in Applied Mathematics 47 (2011) 813–819

Contents lists available at ScienceDirect

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ELSEVIER

The computational challenge of enumerating high-dimensional rook walks

Manuel Kauers[a,*,1], Doron Zeilberger[b,2]

Summary

Reduction algorithms solve simultaneously

- Existence problem of telescopers
- Construction problem of telescopers

photo from internet

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Thank you!