D-finite Functions

Algorithmic, Analytic and Arithmetic Aspects

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KLMM, AMSS
Chinese Academy of Sciences

Computer Mathematics 2017
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based on joint works with Jason P. Bell, Manuel Kauers
Ziming Li, Michael F. Singer and Yi Zhang

Outline

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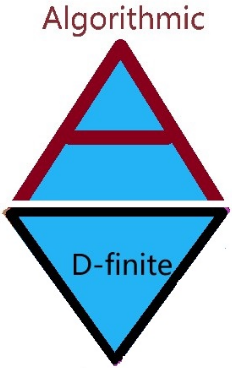

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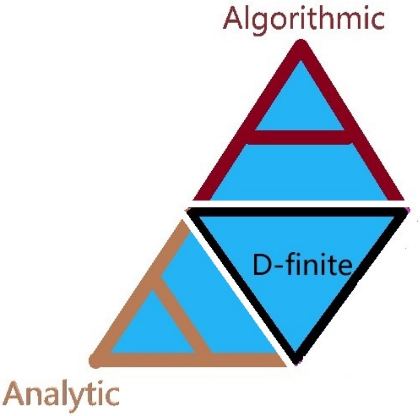

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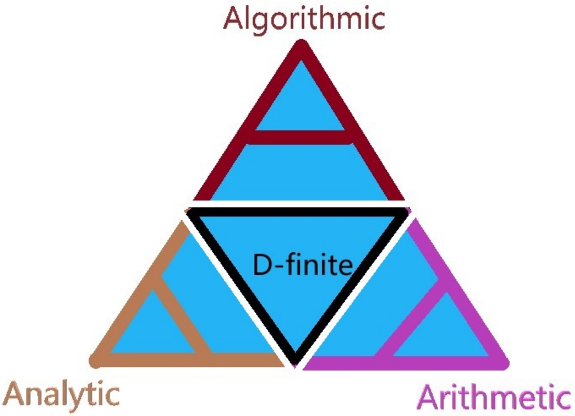

D-finite functions

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Definition. A function $f(x_1,\ldots,x_d)$ is D-finite over $\mathbb{K}(x_1,\ldots,x_d)$ if for each $i \in \{1,\ldots,d\}$, f satisfies a LPDE:

$$p_{i,r_i}\frac{\partial^{r_i} f}{\partial x_i^{r_i}} + p_{i,r_i-1}\frac{\partial^{r_i-1} f}{\partial x_i^{r_i-1}} + \cdots + p_{i,0} f = 0,$$

where $p_{i,j} \in \mathbb{K}[x_1,\ldots,x_d]$.

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R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1: 175–188, 1980.

L. Lipshitz. D-Finite Power Series. *Journal of Algebra*, 122: 353–373, 1989.

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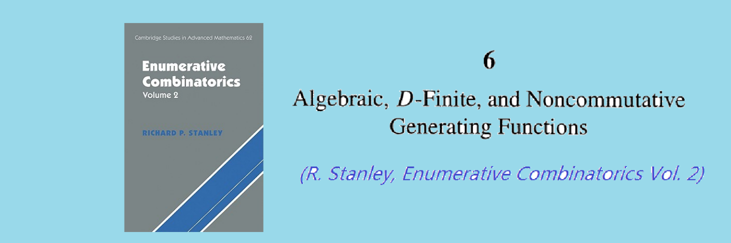

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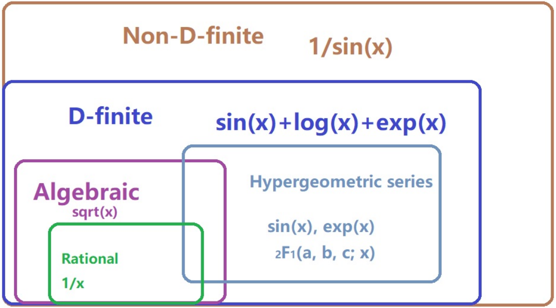
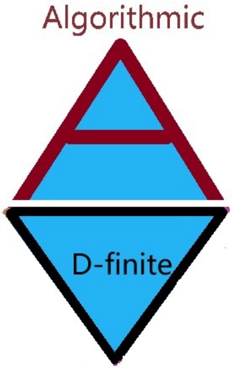


Algorithmic aspect of D-finite functions



Ring of linear differential operators

Definition. Let $\mathbb{F} := \mathbb{K}(x_1, \ldots, x_d)$. The ring $\mathfrak{D} := \mathbb{F}\langle D_{x_1}, \ldots, D_{x_d}\rangle$ of linear differential operators over $\mathbb{F}$ consists of all polynomials

$$L := \sum_{0 \le i_1, \ldots, i_d \le N} f_{i_1,\ldots,i_d} D_{x_1}^{i_1} \cdots D_{x_d}^{i_d} \quad \text{with } f_{i_1,\ldots,i_d} \in \mathbb{F},$$

in which $D_{x_i} \cdot D_{x_j} = D_{x_j} \cdot D_{x_i}$ for $i, j \in \{1, \ldots, d\}$ and

$$D_{x_i} \cdot f = f \cdot D_{x_i} + \frac{\partial f}{\partial x_i} \quad \text{for any } f \in \mathbb{F}.$$

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Definition. Let h be a infinitely differentiable function and $L \in \mathfrak{D}$. We define the action of L on h by

$$L(h) = \sum_{0 \le i_1,\ldots,i_d \le N} f_{i_1,\ldots,i_d} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_d}}{\partial x_1^{i_d}} (h).$$

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Definition. Let $\mathfrak{M}$ be a left $\mathfrak{D}$-module (e.g. $\mathbb{K}[[x_1, \ldots, x_d]]$) and $h \in \mathfrak{M}$. The annihilating ideal of h is the set

$$\mathrm{Ann}_{\mathfrak{D}}(h) := \{L \in \mathfrak{D} \mid L(h) = 0\}.$$

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Remark. $\mathfrak{D}$ is a non-commutative ring.

Proposition.

$$h \text{ is } D\text{-finite over } \mathbb{F}$$

$$\Updownarrow$$

$$\dim_{\mathbb{F}}(\mathfrak{D}/\mathrm{Ann}_{\mathfrak{D}}(h)) < +\infty$$

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$$\mathrm{Ann}_{\mathfrak{D}}(h) \text{ is of dimension zero.}$$

Closure properties

Let $\mathbf{n} = n_1, \ldots, n_d$, $\mathbf{x} = x_1, \ldots, x_d$, and $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$.

Definition. Let $f = \sum a(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ and $g = \sum b(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ be in $\mathbb{K}[[\mathbf{x}]]$. The Hadamard product of f and g is

$$f \odot g = \sum a(\mathbf{n})b(\mathbf{n})\mathbf{x}^{\mathbf{n}}.$$

The diagonal of f is defined as $\mathrm{diag}(f) = \sum a(n, \ldots, n)x^n \in \mathbb{K}[[x]]$.

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Theorem (Lipshitz1989). Let $\mathscr{D} := \{f \in \mathbb{K}[[\mathbf{x}]] \mid f \text{ is D-finite}\}$. Then

(i) if $f, g \in \mathscr{D}$, then $f + g$, $f \cdot g$, and $f \odot g$ are in $\mathscr{D}$;

(ii) if $f \in \mathscr{D}$, $\mathrm{diag}(f)$ is D-finite in $\mathbb{K}[[x]]$;

(iii) if $f \in \mathscr{D}$, and $\alpha_1, \ldots, \alpha_d \in K[[\mathbf{y}]]$ are algebraic over $K(\mathbf{y})$ and the substitution makes sense, then $f(\alpha_1, \ldots, \alpha_d)$ is D-finite.

Algorithms for D-finite Functions

Let $\mathfrak{D} := \mathbb{K}(x)\langle D_x \rangle$ and $\mathfrak{M}$ be a left $\mathfrak{D}$-module.

Univariate case. Let $f_1, f_2 \in \mathfrak{M}$ be D-finite with annihilating operators $L_1, L_2 \in \mathfrak{D}$ resp. An annihilating operator for $f_1 + f_2$ is

$$L := \mathsf{LCLM}(L_1, L_2) = R_1 L_1 = R_2 L_2, \quad R_1, R_2 \in \mathfrak{D}.$$

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Example. Let $f_1 = \exp(x)$ and $f_2 = \sqrt{x}$. Then their annihilating operators are $L_1 := D_x - 1$ and $L_2 := 2xD_x - 1$, resp. We get an annihilating operator for $f_1 + f_2$ by

$$L := \mathrm{LCLM}(L_1, L_2) = 2x(2x-1)D_x^2 - (4x^2+1)D_x + (2x+1).$$

Remark. Abramov, Le and Li developed the Maple package **OreTools** for manipulating univariate D-finite functions.

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S. Abramov, H. Le and Z. Li. Univariate Ore polynomial rings in computer algebra. *Journal of Mathematical Sciences*, 131(5), 5885-5903, 2005.

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A. Bostan, F. Chyzak, Z. Li and B. Salvy. Fast computation of common left multiples of linear ordinary differential operators, *Proceedings of ISSAC2012*, pp. 99–106, ACM Press, 2012.

Algorithms for D-finite Systems

Let $\mathfrak{D} := \mathbb{K}(x_1, \ldots, x_d)\langle D_{x_1}, \ldots, D_{x_d}\rangle$ and $\mathfrak{M}$ be a left $\mathfrak{D}$-module.

Multivariate case. Let $f_1, f_2 \in \mathfrak{M}$ be D-finite with annihilating ideals $I_1, I_2 \subseteq \mathfrak{D}$ resp. An annihilating ideal for $f_1 + f_2$ is

$$I := I_1 \cap I_2.$$

Remark. F. Chyzak's Maple library **Mgfun** and C. Koutschan's Mathematica package **HolonomicFunctions** for manipulating multivariate D-finite functions.

F. Chyzak and B. Salvy. Non-commutative Elimination in Ore Algebras Proves Multivariate Identities. *Journal of Symbolic Computation*, 26 (2) :187-227, 1998.

Creative Telescoping

Theorem. Let $f(x,y)$ be D-finite over $\mathbb{K}(x,y)$. Then there exists a nonzero $L \in \mathbb{K}(x)\langle D_x \rangle$ such that

$$\underbrace{L(x,D_x)}_{\text{Telescoper}}(f) = D_y(Q(f)) \quad \text{for some } Q \in \mathbb{K}(x,y)\langle D_x, D_y \rangle.$$

Call Q the certificate for L.

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Application. The core step in the Wilf-Zeilberger theory of algorithmic proving of combinatorial identities.

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Example. Let $f(x,y) = 1/\sqrt{y(y-1)(y-x)}$. Then

$$L = 4(x-1)xD_x^2 + 4(2x-1)D_x + 1, \quad Q = \frac{2y(y-1)}{x-y}.$$

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Problem. How to develop efficient algorithms for computing telescopers for D-finite functions?

Reduction-based Algorithms: Rational case

A. Bostan, S. Chen, F. Chyzak, and Z. Li. Complexity of Creative Telescoping for Bivariate Rational Functions. *Proceeding of ISSAC2010*, pp. 203–210, ACM Press, 2010.

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Theorem. Let $f(x,y)$ be D-finite over $\mathbb{K}(x,y)$. Then there exists a nonzero $L \in \mathbb{K}(x)\langle D_x \rangle$ such that

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Problem. How to develop efficient algorithms for computing telescopers for D-finite functions?

Reduction-based Algorithms: Hyperexponential case

A. Bostan, S. Chen, F. Chyzak, Z. Li, and G. Xin. Hermite Reduction and Creative Telescoping for Hyperexponential Functions. *Proceedings of ISSAC2013*, pp. 77–84, ACM Press, 2013.

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Reduction-based Algorithms: Hypergeometric case

S. Chen, H. Huang, M. Kauers, and Z. Li. A Modified Abramov-Petkovsek Reduction and Creative Telescoping for Hypergeometric Terms. *Proceedings of ISSAC2015* , pp. 117–124, ACM Press, 2015.

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Reduction-based Algorithms: Algebraic case

S. Chen, M. Kauers, and C. Koutschan. Reduction-Based Creative Telescoping for Algebraic Functions. *Proceedings of ISSAC2016*, pp. 175–182, ACM Press, 2016.

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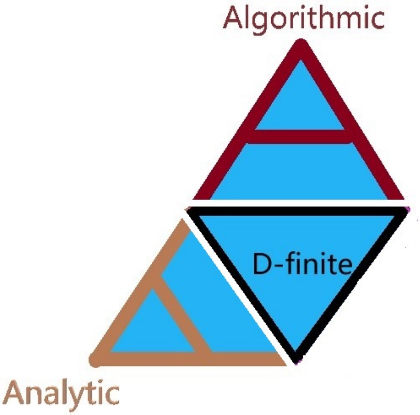
Reduction-based Algorithms: Fuchsian D-finite case

S. Chen, M. van Hoeij, M. Kauers, and C. Koutschan. Reduction-Based Creative Telescoping for Fuchsian D-finite Functions. *Journal of Symbolic Computation*, 85:108–127, 2018.

Analytic aspect of D-finite functions



Apparent singularities

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- The converse is not true: the equation may have singularities where all solutions are regular.

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Solutions in this case:

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How to distinguish apparent and non-apparent singularities when we don't have closed form solutions?

Applications. Asymptotic estimates of the coefficient growth of D-finite power series.

Removable singularities

Definition. Let $L := \sum_{i=0}^{r} \ell_i D^i \in \mathbb{C}[x]\langle D\rangle$. A factor $q \in \mathbb{C}[x]$ of $\operatorname{lc}(L)$ is removable if there exists $Q \in \mathbb{C}(x)\langle D\rangle$ such that

$$QL \in C[x]\langle D\rangle \quad \text{and} \quad \operatorname{lc}(QL) = \frac{1}{q}\operatorname{lc}(L).$$

Removable singularities

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non-removable singularity

removable singularity

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Example. $L = (x-1)xD - (x-2)$, $q = x$, $Q = \frac{1}{x}D^2$.

Problem. How to decide removability and how to remove a removable singularity?

Theorem.

$$x - \zeta \text{ is removable} \quad \Leftrightarrow \quad x = \zeta \text{ is an apparent singularity}$$

Removability criterion

$$L = p_r(x)D^r + p_{r-1}(x)D^{r-1} + \cdots + p_1(x)D + p_0(x)$$

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$$\begin{array}{ll} & \text{The factor } x \text{ is removable} \\ \iff & \text{The singularity } x = 0 \text{ is apparent} \end{array}$$

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Thus, to check removability, just compute the first few terms of the power series solutions of L, and see how many there are:

$$\bigcirc + \bigcirc x + \bigcirc x^2 + \bigcirc x^3 + \cdots$$
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Algorithm for removing x from L (if possible):

- If L has less than r power series solutions, return NO_WAY
- Determine the gaps $e_1, \ldots, e_m$ in the exponent pattern for L

$$
\begin{array}{l}
1 + \bigcirc x + \bigcirc x^2 + \bigcirc x^3 + \bigcirc x^4 + \cdots \\
0 + 0\,x + 1\,x^2 + \bigcirc x^3 + \bigcirc x^4 + \cdots \\
0 + 0\,x + 0\,x^2 + 0\,x^3 + 1\,x^4 + \cdots
\end{array}
$$

$$
\begin{array}{ccc}
\Downarrow & \qquad & \Downarrow \\
e_1 = 1 & & e_2 = 3
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Compute $\operatorname{lclm}(L, M)$.

Example $L = (x-1)xD^2 - (x^2+2x-2)D - (2x^2-8x+4)$.

$$\begin{aligned}
\operatorname{lclm}(L,M) = &(x-1)(2x^4-10x^3+11x^2-8x+2)\,D^4\\
&-(2x^5-2x^4-27x^3+44x^2-28x+8)\,D^3\\
&-4x^2(x^3-10x^2+24x-12)\,D^2\\
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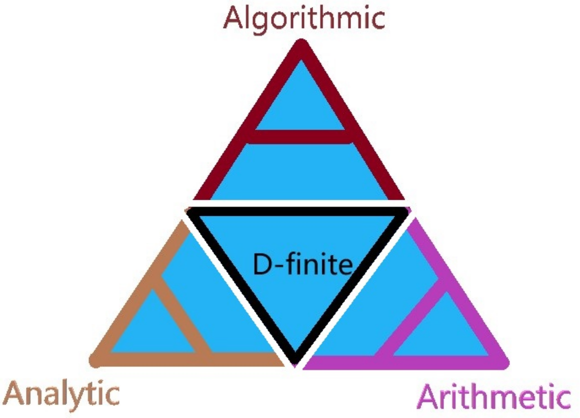
S. Chen, M. Kauers, and M. Singer. Desingularization of Ore Operators. *Journal of Symbolic Computation*, 74(C): 617-626, 2016.

S. Chen, M. Kauers, Z. Li and Y. Zhang. Apparent Singularities of D-finite Systems. Sumitted to *Journal of Symbolic Computation*, 2017.

Arithmetic aspect of D-finite functions



Hadamard’s problem on power series

In 1892, Hadamard in his thesis said that

> *“Indeed, the Taylor expansion does not reveal the properties of the function represented, and even seems to mask them completely. ”*

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Hadamard then considered the following problem:

> *What relationships are there between the coefficients of a power series and the singularities of the function it represents?*

Hadamard's problem on power series

In 1892, Hadamard in his thesis said that

"Indeed, the Taylor expansion does not reveal the properties of the function represented, and even seems to mask them completely. "

Hadamard then considered the following problem:

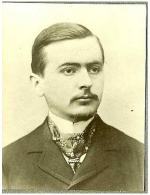
What relationships are there between the coefficients of a power series and the singularities of the function it represents?

Two special cases of the problem have been studied:

- Power series with integral coefficients;
- Power series with finitely distinct coefficients.

Power series with integral coefficients

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \mathbb{Z}.$$



Pierre Fatou (1878-1929)

Pierre Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

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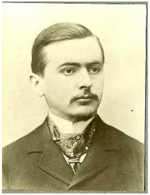

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Fatou's Theorem. If $f(x)$ converges inside the unit disk, then it is either rational or transcendental over $\mathbb{Q}(x)$.

Power series with integral coefficients

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George Pólya (1887-1985)

George Pólya, *Uber Potenzreihen mit ganzzahligen Koeffizienten*, Math. Ann. **77** (1916), no. 4, 497–513.

Fritz Carlson, *Über Potenzreihen mit ganzzahligen Koeffizienten*, Math. Z. **9** (1921), no. 1-2, 1–13.

Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

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Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

Corollary. If $f(x)$ is algebraic, then it is rational.

Power series with finitely distinct coefficients

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \Delta \text{ with } |\Delta| < +\infty.$$

Gábor Szegő (1895-1985)

From 1917 to 1922, there are four papers with the same title:

Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten.

Power Series with Finitely Distinct Coefficients

1. G. Polya in 1917, Math. Ann.
2. R. Jentzsch in 1918, Math. Ann.
3. F. Carlson in 1919, Math. Ann.
4. G. Szego in 1922, Math Ann.

Szegö's Theorem (1922)
A power series with finitely distinct coefficients in $\mathbb{C}$ is either rational or has the unit circle as its natural boundary.

Arithmetical aspects of power series

Problem. Decide whether a given power series is rational, algebraic, transcendental, or hyper-transcendental?

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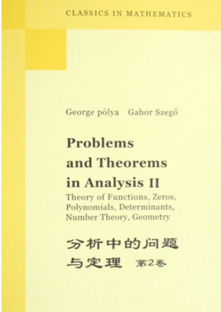
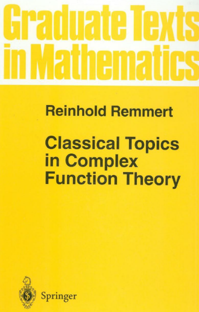


Chapter 3. Arithmetical Aspects of Power Series

Arithmetical aspects of power series

Problem. Decide whether a given power series is rational, algebraic, transcendental, or hyper-transcendental?



Power series with integral coefficients
(the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

Power series with integral coefficients (the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

- André Martineau, *Extension en n-variables d'un théorème de Pólya-Carlson concernant les séries de puissances à coefficients entiers*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A1127–A1129. MR 0291495

- V. P. Šeĭnov, *Transfinite diameter and certain theorems of Pólya in the case of several complex variables*, Sibirsk. Mat. Ž. **12** (1971), 1382–1389.

- Emil J. Straube, *Power series with integer coefficients in several variables*, Comment. Math. Helv. **62** (1987), no. 4, 602–615. MR 920060

Power series with integral coefficients (the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

- André Martineau, *Extension en n-variables d'un théorème de Pólya-Carlson concernant les séries de puissances à coefficients entiers*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A1127–A1129. MR 0291495
- V. P. Šeĭnov, *Transfinite diameter and certain theorems of Pólya in the case of several complex variables*, Sibirsk. Mat. Ž. **12** (1971), 1382–1389.
- Emil J. Straube, *Power series with integer coefficients in several variables*, Comment. Math. Helv. **62** (1987), no. 4, 602–615. MR 920060

Rationality Theorem (BellChen, 2016) If the power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Power series with finitely distinct coefficients (the multivariate case)

Theorem (van der Poorten & Shparlinsky, 1994).
Let $a_n : \mathbb{N} \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function $f(x) = \sum_n a_n x^n$ is D-finite, then it is rational.

Remark. This follows from Szegö's theorem by the fact that a D-finite power series can only have finitely many singularities.

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Rationality Theorem (BellChen, 2016). Let $a_{n_1,\ldots,n_d} : \mathbb{N}^d \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function

$$f(x_1,\ldots,x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is D-finite, then it is rational.

J. P. Bell, S. Chen. Power Series with Coefficients from a Finite Set. Journal of Combinatorial Theory, Series A, 151, pp. 241–253, 2017.

Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1,\ldots,x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

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We may ask the following questions:

When F_V is zero?

Remark. This is Hilbert Tenth Problem when K is $\mathbb{Q}$. In 1970, Matiyasevich proved that this problem is undecidable.

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We may ask the following questions:

When F_V is a polynomial?

Remark. In 1929, Siegel proved that a smooth algebraic curve C of genus $g \geq 1$ has only finitely many integer points over a number field K.

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We may ask the following questions:

When F_V is a rational function?

Theorem. If F_V is rational, then $V\cap\mathbb{N}^d$ is semilinear, i.e., $\exists n\in\mathbb{N}$ and finite subsets $V_0,\ldots V_n$ of $\mathbb{N}^d$, and $b_1,\ldots,b_n\in\mathbb{N}^d$ such that

$$E = V_0 \bigcup \left\{ \bigcup_{i=1}^{n} \left(b_i + \sum_{v\in V_i} v\cdot\mathbb{N} \right) \right\}.$$

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When F_V is a D-finite function?

Corollary.

$$F_V \text{ is } D\text{-finite} \quad \Leftrightarrow \quad F_V \text{ is rational}.$$

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Theorem.

The problem of testing whether F_V is rational is undecidable!

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We may ask the following questions:

When F_V is a differentially algebraic function?

Definition. $F \in K[[x_1,\ldots,x_d]]$ is differentially algebraic if the transcendence degree of the filed generated by the derivatives $D_{x_1}^{i_1}\cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1,\ldots,x_d)$ is finite.

Nonnegative integer points on algebraic curves

Theorem. If V is defined by linear polynomials over $\mathbb{Q}$, then F_V is rational.

Theorem. Let $p(x,y) \in \mathbb{C}[x,y]$. If the generating function

$$F_p(x,y) := \sum_{(n,m) \in V(p) \cap \mathbb{N}^2} x^n y^m$$

is rational. Then $p = f \cdot g$, where $f, g \in \mathbb{C}[x,y]$ s.t.

$$f = \prod_i (s_i \cdot x + t_i \cdot y + c_i) \quad \text{with } s_i, t_i \in \mathbb{Z} \text{ and } c_i \in \mathbb{C}$$

and g has only finite zeros in $\mathbb{N}^2$.

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and g has only finite zeros in $\mathbb{N}^2$.

Example. Let $p = x^2 - y$. Since p is not a product of integer-linear polynomials, the power series $F_p(x,y)$ is not D-finite.

Open problems

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

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Example. Let $p = x^2 - y$. Then the power series

$$F_p(x,y) := \sum_{m\geq 0} x^m y^{m^2}$$

is not differentially algebraic, otherwise, $F_p(x,2) = \sum 2^{m^2} x^m$ is differentially algebraic. By Mahler's lemma, we get a contradiction

$$2^{m^2} \ll (m!)^c \quad \text{for any positive constant } c.$$

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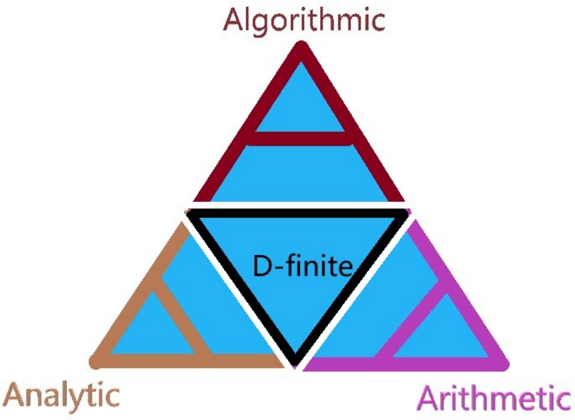
Conjecture (Chowla-Chowla-Lipshitz-Rubel). The power series

$$f := \sum_{n\in\mathbb{N}} x^{n^3} \in \mathbb{C}[[x]]$$

is not differentially algebraical, i.e., satisfies no ADE.

Remark. The power series $\sum x^{n^2}$ is differentially algebraic.

Summary



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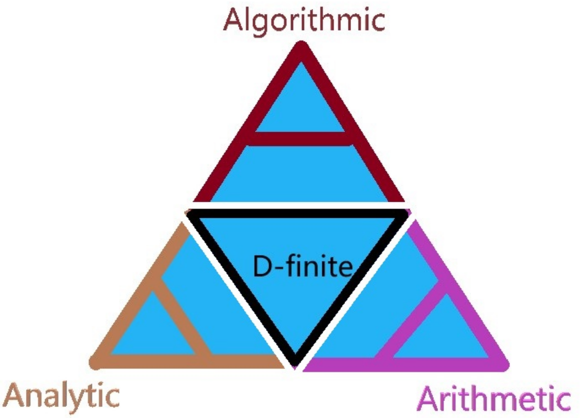

Thank you!