Symbolic Summation of Multivariate Rational Functions via Shift Equivalence Testing *

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Abstract

The Shift Equivalence Testing (SET) of polynomials is deciding whether two polynomials $p(x_1, \ldots, x_m)$ and $q(x_1, \ldots, x_m)$ satisfy the relation $p(x_1 + a_1, \ldots, x_m + a_m) = q(x_1, \ldots, x_m)$ for some $a_1, \ldots, a_m$ in the coefficient field. The SET problem is one of the basic computational problems in computer algebra and algebraic complexity theory, which was reduced by Dvir, Oliveira and Shpilka in 2014 to the Polynomial Identity Testing (PIT) problem. This paper presents a general scheme for designing algorithms to solve the SET problem which includes Dvir-Oliveira-Shpilka's algorithm as a special case. With the algorithms for the SET problem over integers, we give complete solutions to two challenging problems in symbolic summation of multivariate rational functions, namely the rational summability problem and the existence problem of telescopers for multivariate rational functions. Our approach is based on the structure of isotropy groups of polynomials introduced by Sato in 1960s. Our results can be used to detect the applicability of the Wilf-Zeilberger method to multivariate rational functions.

Keywords: Dispersion set; isotropy group; shift equivalence; symbolic summation; telescoper

MSC classes: 68W30, 12H05, 12H10

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1 Introduction

Symbolic summation is a classical and active research topic in symbolic computation, whose central problem is evaluating and simplifying different types of sums arising from combinatorics and theoretical physics [13,70] and other areas. For a given sequence in a certain specific class, the indefinite summation problem (in the univariate case) is to determine whether the given sequence is the difference of another sequence in the same class, which is a discrete analogue of indefinite integration problem. For instance, $-1/(n^2+n)$ is the difference of $1/n$, but $1/n$ is not the difference of any rational sequence. The definite summation problem is to find a closed form for the sum $\sum_{i=a}^{b-1} f(i)$ assuming that the function $f(x)$ is well-defined in the interval $[a, b]$. The two summation problems are connected by the discrete Leibniz–Newton formula. Since the early 1970s, efficient algorithms have been developed for symbolic summation [78, Chapter 23]. Abramov's algorithm [1–3] solves the indefinite summation problem for univariate rational functions. A Hermite-like reduction algorithm for rational summation was developed by Paule via greatest factorial factorizations in [10,56,60,65]. The indefinite summation problem for hypergeometric terms is handled by Gosper's algorithm [36]. For sequences in a general difference field, the corresponding problem is studied by Karr in [46,47] with significant improvements by Schneider [69] and recent fruitful applications in quantum field theory [13,72]. Most of existing complete algorithms are mainly applicable to the summation problem with univariate inputs. A long-term project in symbolic computation is to develop theories, algorithms and software for symbolic summation of multivariate functions. Along this way, some algorithms were developed to deal with doubles sums [30] and binomial multiple sums [15]. In this paper, we will present a first algorithm for symbolic summation of multivariate rational functions.

In the multivariate case, the stimulating problem was first raised by Andrews and Paule in [8]: "is it possible to provide any algorithmic device for reducing multiple sums to single sums?"In order to address the problem of Andrews and Paule, a first try is to solve the summability problem. For a multiple sum of the form

$$F = \sum_{x_1=a_1}^{b_1} \cdots \sum_{x_m=a_m}^{b_m} f(x_1, \ldots, x_m),$$

one would try to detect whether f is summable, i.e., whether f can be written as

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} g(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_m) - g(x_1, \ldots, x_m)$$

for some functions g_i's. If so, the m multiple sum can be reduced to several $m-1$ multiple sums:

$$\begin{aligned} F = &\sum_{x_2=a_2}^{b_2} \cdots \sum_{x_m=a_m}^{b_m} g(b_1+1, x_2, \ldots, x_m) - g(a_1, x_2, \ldots, x_m) + \cdots \\ &+ \sum_{x_1=a_1}^{b_1} \cdots \sum_{x_{m-1}=a_{m-1}}^{b_{m-1}} g(x_1, \ldots, x_{n-1}, b_m+1) - g(x_1, \ldots, x_{m-1}, a_m). \end{aligned}$$

If the above summands are summable again (with respect to $m-1$ variables), one can similarly reduce the $m-1$ multiple sums to several simpler sums. Finally, one may succeed in finding a closed form of the multiple sum.

From now on, we will focus on the symbolic-summation problems of multivariate rational functions. To this end, we assume that $\mathbb{F}$ is a computational field of characteristic zero and let $\mathbb{F}(\mathbf{x})$

denote the field of rational functions in variables $\mathbf{x} = \{x_1, \ldots, x_m\}$ over $\mathbb{F}$. We define the shift operator σ_{x_i} with respect to x_i by

$$\sigma_{x_i}(f(x_1, \ldots, x_m)) = f(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_m)$$

for all $f \in \mathbb{F}(\mathbf{x})$. The summability problem for multivariate rational functions is as follows.

(1) **Rational Summability Problem**: Given a rational function $f(\mathbf{x}) \in \mathbb{F}(\mathbf{x})$, decide whether there exist rational functions $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}) \in \mathbb{F}(\mathbf{x})$ such that

$$f = \sigma_{x_1}(g_1) - g_1 + \cdots + \sigma_{x_m}(g_m) - g_m.$$

If such g_i's exist, we say that f is $(\sigma_{x_1}, \ldots, \sigma_{x_m})$-*summable* in $\mathbb{F}(\mathbf{x})$ and the g_i's are the *certificates* of f.

When $m = 1$, Abramov [2] introduced the notion of dispersion in rational summation. The Abramov's dispersion of a univariate polynomial is defined as the maximal integer distance among all its roots. There are several summation algorithms [2, 4, 10, 56, 60, 65]. The common idea is to decompose $f(x_1) = \sigma_{x_1}(g(x_1)) - g(x_1) + r(x_1)$ such that the dispersion of the denominator of r is zero. Then f is σ_{x_1}-summable in $\mathbb{F}(x_1)$ if and only if $r = 0$. When $m = 2$, by exploring the summability of algebraic functions in $\overline{\mathbb{F}(x_2)}(x_1)$, a summability criterion for rational functions in $\mathbb{F}(x_1, x_2)$ was given by Chen and Singer [28]. Later it was adapted by Hou and Wang [44] into a practical bivariate rational summation algorithm without algebraic extensions. In more than two variables, there is no complete algorithm for deciding the summability of rational functions.

Let $\mathbb{K}$ be a subfield of $\mathbb{F}$. If $\mathbb{F} = \mathbb{K}(t)$ for some transcendental $t \in \mathbb{F}$ over $\mathbb{K}$, let σ_t be the shift operator with respect to t. For a multiple sum of the form

$$F(t) = \sum_{x_1=a_1}^{b_1} \cdots \sum_{x_m=a_m}^{b_m} f(t, x_1, \ldots, x_m),$$

if f is not summable, one may try the method of creative telescoping [83] to find a recurrence relation of $F(t)$. To do this, we shall ask the following existence problem of telescopers for multivariate rational functions.

(2) **Existence Problem of Telescopers**: Given a rational function $f(t, \mathbf{x}) \in \mathbb{F}(\mathbf{x})$ with $\mathbb{F} = \mathbb{K}(t)$, decide whether there exists a nonzero linear recurrence operator $L = \sum_{i=0}^{r} \ell_i \sigma_t^i$ with $\ell_i \in \mathbb{F}$ such that

$$L(f) = \sigma_{x_1}(g_1) - g_1 + \cdots + \sigma_{x_m}(g_m) - g_m \quad \text{for some } g_1, \ldots, g_m \in \mathbb{F}(\mathbf{x}).$$

If such an operator L exists, then L is called a *telescoper* for f of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_m})$ and the g_i's are called the *certificates* of L.

We say that Problems (1) and (2) can be decided *constructively* if the g_i's in (1), or L and the g_i's in (2), can be computed explicitly. For example, using our Algorithm 6.13, one can find that the following rational function

$$f(t, x, y, z) = \frac{(2y - t)(2x - t)(2z - t)}{(y + t + 1)(-2t + y - 1)(x + t + 1)(-2t + x - 1)(z + t + 1)(-2t + z - 1)}.$$

has a telescoper $L = \sigma_t - 1$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ with the certificates $(u, v, w) \in \mathbb{K}(t, x, y, z)^3$ in explicit expressions. This can be used to show

$$F(t) = \sum_{x=0}^{t} \sum_{y=0}^{t} \sum_{z=0}^{t} f(t, x, y, z) = 0,$$

because $F(t)$ satisfies the recurrence relation $F(t+1) - F(t) = 0$ with the initial value $F(0) = 0$. See more details in Example 6.16.

Creative telescoping is the core of the Wilf–Zeilberger theory of computer-generated proofs of combinatorial identities [61, 80, 81]. The existence problem of telescopers is equivalent to the termination of Zeilberger's algorithm [82, 83] and can be used to detect the hypertranscendence and algebraic dependency of functions defined by indefinite sums or integrals [42, 71]. A sufficient condition, namely holonomicity, on the existence of telescopers was first given by Zeilberger in 1990 using Bernstein's theory of holonomic D-modules [11]. Wilf and Zeilberger in [81] proved that telescopers exist for proper hypergeometric terms. However, holonomicity and properness are only sufficient conditions. Abramov and Le [6] solved the existence problem of telescopers for rational functions in two discrete variables. This work was soon extended to the hypergeometric case by Abramov [5], the q-hypergeometric case in [29], and the mixed rational and hypergeometric case in [20, 27]. The criteria on the existence of telescopers for rational functions in three variables were given in [21, 22, 24]. In arbitrary number of variables, there is no available algorithm for deciding the existence of telescopers for rational functions.

In this paper we will solve algorithmically the summability problem and the existence problem of telescopers for general rational functions in several discrete variables. From the univariate case to the multivariate case, especially when $m > 2$, it becomes difficult to find the structure of all solutions $(g_1, \ldots, g_m) \in \mathbb{F}(\mathbf{x})^m$ of the following difference equation:

$$\sigma_{x_1}(g_1) - g_1 + \cdots + \sigma_{x_m}(g_m) - g_m = 0. \tag{1.1}$$

When $m = 2$, the structure theorem was discovered in [19] using the summability criterion for univariate rational functions. The dimension of its solution space over $\mathbb{F}$ becomes infinite if $m > 1$. For example, if $m = 3$, then for every $f \in \mathbb{F}(x_1, x_2, x_3)$, $(\Delta_{x_2}\Delta_{x_3}(f), -2\Delta_{x_1}\Delta_{x_3}(f), \Delta_{x_1}\Delta_{x_2}(f))$ is a solution of (1.1), where $\Delta_{x_i} = \sigma_{x_i} - 1$. In the process of deciding the summability of a rational function f, we avoid finding all possible tuples of the certificates, but focus on finding one tuple $(g_1, \ldots, g_m)$ of the certificates if f is summable. In this direction, we will explore some shift-invariant properties and shift-invariant subspaces of $\mathbb{F}(\mathbf{x})$.

Similar to the cases in smaller variables [24, 28, 44], we need a generalized version of a partial fraction decomposition (taking into account the shift operators σ_{x_i}). This requires to compute the dispersion set of multivariate polynomials. The latter is equivalent to solving the shift equivalence testing problem (SET) of polynomials, see its definition in the next section.

Different from the cases in smaller variables, we use the notion of isotropy group under shift operations to formulate our criteria for summability and existence of telescopers. The isotropy group of a non-constant polynomial in m variables forms a free abelian group whose rank is less than m. This property helps us to reduce the summability problem in m variables to that in fewer variables. An algorithm for solving the SET problem can be used to compute this rank.

1.1 The SET problem on multivariate polynomials

Polynomials are basic arithmetic structures in mathematics and computer sciences. Efficient algorithms have been developed for manipulating polynomials in computer algebra [35, 51, 78, 85] with extensive complexity studies in [18, 75, 77]. Let $\mathbb{F}[\mathbf{x}]$ be the ring of polynomials in m variables $\mathbf{x} = x_1, \ldots, x_m$ over $\mathbb{F}$. One can ask several basic computational questions on polynomials: Given $p, q \in \mathbb{F}[\mathbf{x}]$ and $\mathbf{P}, \mathbf{Q} \in \mathbb{F}[\mathbf{x}]^n$,

(1) **Polynomial Identity Testing (PIT)**: Is $p(\mathbf{x})$ identically zero?

(2) **Fast Evaluation and Interpolation (FEI)**: How fast can we evaluate $p(\mathbf{x})$ at many points and interpolate it from values at many points?

(3) **Fast Multiplication and Factorization (FMF)**: How fast can we multiply $p(\mathbf{x})$ by $q(\mathbf{x})$ and factor $p(\mathbf{x})$ into a product of irreducible polynomials over $\mathbb{F}$?

(4) **Polynomial Equivalence Testing (PET)**: Decide whether there exists some invertible matrix $A \in GL_m(\mathbb{F})$ such that $p(\mathbf{x}) = q(\mathbf{x} \cdot A)$.

(5) **Shift Equivalence Testing (SET)**: Decide whether there exists some vector $\mathbf{b} \in \mathbb{F}^m$ such that $q(\mathbf{x}) = p(\mathbf{x} + \mathbf{b})$.

(6) **Isomorphism of Polynomials (IP)**: Decide whether there exists a pair $(A, B) \in GL_m(\mathbb{F}) \times GL_n(\mathbb{F})$ such that $\mathbf{Q} = \mathbf{P}(\mathbf{x} \cdot A) \cdot B$.

(7) **Affine Projection of Polynomials (APP)**: Decide whether there exists a polynomial r in $n < m$ variables such that $p(\mathbf{x}) = r(\mathbf{x} \cdot A + \mathbf{b})$ for some $n \times m$ matrix A over $\mathbb{F}$ and some vector $\mathbf{b} \in \mathbb{F}^n$.

The answers to these questions may depend on the way in which how we model polynomials. A randomized polynomial-time algorithm for PIT was given independently by Schwartz [73] and Zippel [84], whose derandomization is still a long-standing open problem in algebraic complexity theory with impressive progress in the last three decades (see surveys [67,68,74]). When polynomials are modelled as arithmetic circuits, partial derivatives of polynomials are used extensively and essentially in most of the above questions (see the comprehensive survey [31]). Kayal presented a deterministic algorithm for the first question in the case where the input circuit is a sum of powers of sums of univariate polynomials and a randomized polynomial-time algorithm for some special cases of the fourth question in [49]. Fast algorithms for the second and third questions are fundamental for solving many computational problems in computer algebra [78, 85]. The fifth question was originally motivated by sparse interpolation of polynomials [37,38,53,54] and answered in several works [32,33,39,40,48] with different methods. The sixth question was first introduced by Patarin [59] and has rich applications in multivariate cryptography [12, 16, 34, 41]. In 2012, Kayal proved that the seventh question is NP-hard in general but admits randomized polynomial-time algorithms for special classes of polynomials including permanent and determinant polynomials [31, 50]. Beside the above-mentioned results, research and extensive work on these questions have been done by combing tools from symbolic computation and algebraic complexity theory. The above seven dwarfs build an exchanging bridge between mathematics and computer science.

In this paper, we will show that the SET problem plays a crucial role in symbolic summation of multivariate rational functions and present a general scheme for designing algorithms to solve the SET problem which includes Dvir-Oliveira-Shpilka's algorithm in [32, 33] as a special case.

1.2 The main results

We now present our main results on the SET problem, rational summability problem, and existence problem of telescopers for rational functions in several variables.

1.2.1 Algorithms for the SET problem

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{N}^+$ be the set of positive integers. Given two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$, we say that p is *shift equivalent* to q over $\mathbb{F}$ if there exist $s_1, \dots, s_m \in \mathbb{F}$ such that

$$p(x_1 + s_1, \dots, x_m + s_m) = q(x_1, \dots, x_m).$$

We call the set $\{\mathbf{s} \in \mathbb{F}^m \mid p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x})\}$ the *dispersion set* of p and q over $\mathbb{F}$, denoted by $F_{p,q}$. The Shift Equivalence Testing (SET) problem is to decide whether the dispersion set $F_{p,q}$ is empty or not. Introducing m new variables $\mathbf{a} = a_1, \dots, a_m$ in this section, we consider the polynomial $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})$ in $\mathbb{F}[\mathbf{a}, \mathbf{x}]$ and write it as $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\Lambda}} c_{\boldsymbol{\alpha}}(\mathbf{a})\mathbf{x}^{\boldsymbol{\alpha}}$ with $\boldsymbol{\Lambda}$ being a finite subset of $\mathbb{N}^m$. In general, the coefficients $c_{\boldsymbol{\alpha}}(\mathbf{a})$ are polynomials in $\mathbb{F}[\mathbf{a}]$ that may not be linear. So it seems that we need to solve a polynomial system in order to determine the set $F_{p,q}$. However, Grigoriev (G) in [39, 40] proved that $F_{p,q}$ is actually a linear variety and he also gave a recursive algorithm for determining this variety using the following relation

$$F_{p,q} = \left(\bigcap_{i=1}^{m} F_{\partial_{x_i}(p), \partial_{x_i}(q)} \right) \cap \{\mathbf{s} \in \mathbb{F}^m \mid p(\mathbf{s}) = q(\mathbf{0})\},$$

where ∂_{x_i} denotes the partial derivative with respect to x_i. Since partial derivations decrease the degree of polynomials, the SET problem boils down to solving a linear system. Another way to derive the linear system that defines $F_{p,q}$ was given by Kauers and Schneider (KS) in [48] with applications in solving linear partial difference equations. The idea is to compute the radical of the ideal I generated by the set $\{c_{\boldsymbol{\alpha}}(\mathbf{a})\}_{\boldsymbol{\alpha} \in \boldsymbol{\Lambda}}$ in $\mathbb{F}[\mathbf{a}]$ via Gröbner basis method. A more efficient algorithm was given by Dvir, Oliveira and Shpilka (DOS) in [32,33]. They reduced the SET problem to the PIT problem, then solved the latter one by randomized algorithms. Inspired by the DOS algorithm, we now present a general scheme for designing algorithms to solve the SET problem.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$ be two vectors in $\mathbb{N}^m$. We say $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ if $\alpha_i \geq \beta_i$ for all $1 \leq i \leq m$ and we denote the sum $\sum_{i=1}^m \alpha_i$ by $|\boldsymbol{\alpha}|$. Let $\mathrm{Supp}_{\mathbf{x}}(p)$ denote the support of p consisting of all monomials $\mathbf{x}^{\boldsymbol{\alpha}}$ whose corresponding coefficients in p are nonzero.

Definition 1.1 (Admissible cover)**.** *Let $S_{p,q} = \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \mid \mathbf{x}^{\boldsymbol{\alpha}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))\} \subseteq \mathbb{F}[\mathbf{a}]$. A collection $\{S_0, S_1, \dots, S_k\}$ of subsets is called a* cover *of $S_{p,q}$ if $S_{p,q}$ is the union of $S_0, S_1, \dots, S_k$. Such a cover $\{S_0, S_1, \dots, S_k\}$ is called an* admissible cover *of $S_{p,q}$ if it satisfies the following two conditions:*

(1) All polynomials in S_0 are of degree in $\mathbf{a}$ at most one.

(2) For all $\ell = 1, 2, \dots, k$, if $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_\ell$, then $c_{\boldsymbol{\beta}}(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$ for all $\boldsymbol{\beta} \in \mathbb{N}^m$ with $\boldsymbol{\beta} > \boldsymbol{\alpha}$ and $\mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))$.

Without loss of generality, we may assume that the two given polynomials p and q in the SET problem are of the same degree d in $\mathbf{x}$.

Definition 1.2 (Linearization)**.** *Let $p = p_0 + p_1 + \cdots + p_d$ be the homogeneous decomposition of $p \in \mathbb{F}[\mathbf{x}]$ in $\mathbf{x}$. For a vector $\mathbf{s} \in \mathbb{F}^m$, we call the linear polynomial $p_0(\mathbf{x}) + p_1(\mathbf{x}) + \sum_{i=2}^d p_i(\mathbf{s})$ the* linearization *of p at $\mathbf{s}$, denoted by $L_{\mathbf{x}=\mathbf{s}}(p)$. Note that $L_{\mathbf{x}=\mathbf{s}}(p) = p$ if $d \leq 1$.*

For a polynomial set $P \subseteq \mathbb{F}[\mathbf{x}]$, we let $L_{\mathbf{x}=\mathbf{s}}(P) = \{L_{\mathbf{x}=\mathbf{s}}(p) \mid p \in P\}$ and $\mathbb{V}_{\mathbb{F}}(P) = \{\mathbf{s} \in \mathbb{F}^m \mid p(\mathbf{s}) = 0 \text{ for all } p \in P\}$. Our first main result says that any admissible cover of $S_{p,q}$ leads to an algorithm for solving the polynomial system $S_{p,q}$ which only requires solving several linear systems.

Theorem 1.3. *Let $S_{p,q} = \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \mid \mathbf{x}^{\boldsymbol{\alpha}} \in \operatorname{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))\}$. If $\{S_0, S_1, \ldots, S_k\}$ is an admissible cover of $S_{p,q}$, then for all $\ell = 1, \ldots, k$, we have either $\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right) = \varnothing$ or*

$$\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} S_i\right) = \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}}(S_i)\right) \quad \text{for any } \mathbf{s} \in \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right).$$

In particular, the covers $\{S_0^D, S_1^D, \ldots, S_d^D\}$ and $\{S_0^H, S_1^H, \ldots, S_d^H\}$ of $S_{p,q}$ are admissible, where

$$S_i^D := \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_{p,q} \mid \deg_{\mathbf{a}}(c_{\boldsymbol{\alpha}}(\mathbf{a})) = i\} \quad \text{and} \quad S_i^H := \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_{p,q} \mid |\boldsymbol{\alpha}| = d - i\}.$$

We call the above two typical admissible covers $\mathbf{a}$-*degree cover* and $\mathbf{x}$-*homogeneous cover* of $S_{p,q}$ respectively. The former one will lead to a new algorithm called ADC in Section 8 and the latter one corresponds to the DOS algorithm. We illustrate these two admissible covers via a concrete example.

Example 1.4. *Let $p = x^4 + x^3y + xy^2 + z^2$ and $q = p(x, y+1, z+2) + xy$. By collecting the coefficients of $p(x+a, y+b, z+c) - q(x, y, z)$ with respect to the variables x, y, z, we get the set $S_{p,q}$. Then the $\mathbf{a}$-degree cover and $\mathbf{x}$-homogeneous cover of $S_{p,q}$ are*

$S_{p,q}$	S_1^H	S_2^H	S_3^H	S_4^H
S_4^D				$a^4 + a^3b + ab^2 + c^2 - 4$
S_3^D			$4a^3 + 3a^2b + b^2 - 1$, $a^3 + 2ab$	
S_2^D		$6a^2 + 3ab$, $3a^2 + 2b - 3$		
S_1^D	$4a + b - 1$, $3a$	a	$2c - 4$	
S_0^D				

1.2.2 Reduction for rational summability

The rational summability problem has been solved in the univariate and bivariate cases [1,2,28,44]. In order to address the problem in the general multivariate case, it suffices to provide a method that reduces the problem in m variables to that in fewer variables. The reduction method relies on the theory of isotropy groups of polynomials introduced by Sato in 1960s [66]. The computation of isotropy groups needs solving the SET problem over integers, for which we can use polynomial-time algorithms for computing the Hermite normal forms of an integer matrix [45].

Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_m} \rangle$ be the free abelian multiplicative group generated by the shift operators $\sigma_{x_1}, \ldots, \sigma_{x_m}$ that acts on $\mathbb{F}(\mathbf{x})$. For any $\tau \in G$, define the difference operator $\Delta_\tau(g) = \tau(g) - g$ for any $g \in \mathbb{F}(\mathbf{x})$. Let $f \in \mathbb{F}[\mathbf{x}]$ and H be a subgroup of G. The set

$$[f]_H := \{\sigma(f) \mid \sigma \in H\}$$

is called the H-*orbit* at f. The *isotropy* group H_f of f in H is defined as

$$H_f := \{\sigma \in H \mid \sigma(f) = f\}.$$

Note that H_f is a free abelian group and the quotient group H/H_f is also free by [66, Lemma A-3]. The isotropy groups of polynomials will play an important role in the reduction for rational summability. We will show in Section 4.1 that a basis of the isotropy group of a polynomial can be computed.

Similar to the bivariate case, we also use Abramov's reduction [2, 3] repeatedly to decompose $f \in \mathbb{F}(\mathbf{x})$ into the form

$$f = \Delta_{\sigma_{x_1}}(u_1) + \cdots + \Delta_{\sigma_{x_m}}(u_m) + r \text{ with } r = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{1.2}$$

where $u_1, \ldots, u_m \in \mathbb{F}(\mathbf{x})$, $a_{i,j} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\hat{\mathbf{x}}_1 = \{x_2, \ldots, x_m\}$, $d_i \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$ and the d_i's are monic irreducible polynomials in distinct $\langle \sigma_{x_1}, \ldots, \sigma_{x_m}\rangle$-orbits. We will explain in Section 5.1 how to obtain the decomposition (1.2) in details. The following lemma reduces the rational summability problem from general rational functions to simple fractions.

Lemma 1.5. *Let f be as in (1.2). Then f is summable in $\mathbb{F}(\mathbf{x})$ if and only if each $a_{i,j}/d_i^j$ is summable in $\mathbb{F}(\mathbf{x})$.*

We now only need to study the rational summability problem for rational functions of the form

$$f = \frac{a}{d^j}, \tag{1.3}$$

where $j \in \mathbb{N}^+$, $a \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $d \in \mathbb{F}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(a) < \deg_{x_1}(d)$. The following theorem further reduces the problem in m variables to another similar problem in r variables, where r is the rank of the isotropy group that is strictly less than m.

Theorem 1.6 (Summability criterion)**.** *Let $f = a/d^j \in \mathbb{F}(\mathbf{x})$ be of the form (1.3). Let $\{\tau_i\}_{i=1}^r (1 \leq r < m)$ be a basis of the free group G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Then f is summable in $\mathbb{F}(\mathbf{x})$ if and only if*

$$a = \Delta_{\tau_1}(b_1) + \cdots + \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for all $1 \leq i \leq r$.

Note that the above reduced problem is related to the operators $\tau_1, \ldots, \tau_r$ in the isotropy group G_d. In order to turn back to the usual shifts, using Proposition 5.12, we can construct an $\mathbb{F}$-automorphism ϕ of $\mathbb{F}(\mathbf{x})$ such that a is $(\tau_1, \ldots, \tau_r)$-summable in $\mathbb{F}(\mathbf{x})$ if and only if each $\phi(a)$ is $(\sigma_{x_1}, \ldots, \sigma_{x_r})$-summable in $\mathbb{F}(\mathbf{x})$. So the rational summability problem in m variables can be completely reduced to the same problem in fewer variables. Combining the existing methods in the univariate case, we now obtain a complete solution to the rational summability problem for multivariate rational functions.

1.2.3 Reduction for the existence of telescopers

The existence problem of telescoper can be viewed as a parameterization of the rational summability problem. The latter problem is equivalent to testing whether the identity operator is a telescoper or not. Similar to the strategy used in the rational summability problem, we shall provide a method for reducing the existence problem of telescoper in $m+1$ variables to that in fewer variables.

For a rational function $f(t, \mathbf{x}) \in \mathbb{F}(\mathbf{x})$ with $\mathbb{F} = \mathbb{K}(t)$, the existence problem of telescopers for f can also be reduced to simple fractions of the form a/d^j as in (1.3). The second reduction of the number of variables also relies on the structure of isotropy groups. Let $G_t = \langle \sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_m} \rangle$ be the group generated by $\sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_m}$ and $G_{t,d}$ be the isotropy group of d in G_t. Then the quotient group $G_{t,d}/G_d$ is still a free abelian group with $\operatorname{rank}(G_{t,d}/G_d) \le 1$. If $\operatorname{rank}(G_{t,d}/G_d) = 0$, then we show that f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_m})$ if and only if f is $(\sigma_{x_1}, \ldots, \sigma_{x_m})$-summable in $\mathbb{F}(\mathbf{x})$. So in this case, the existence problem of telescopers for f is equivalent to the rational summability problem. If $\operatorname{rank}(G_{t,d}/G_d) = 1$, we have the following existence criterion.

Theorem 1.7 (Existence criterion)**.** *Let $f = a/d^j \in \mathbb{K}(t, \mathbf{x})$ as above with $\operatorname{rank}(G_{t,d}/G_d) = 1$. Let $\{\tau_0, \tau_1, \ldots, \tau_r\}(1 \le r < m)$ be a basis of $G_{t,d}$ such that $G_{t,d}/G_d = \langle \bar{\tau}_0 \rangle$ and let $\{\tau_1, \ldots, \tau_r\}$ be a basis of G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_m})$ if and only if there exists a nonzero operator $L = \sum_{i=0}^{\rho} \ell_i \tau_0^i$ with $\ell_i \in \mathbb{K}(t)$ such that*

$$L(a) = \Delta_{\tau_1}(b_1) + \cdots + \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ and $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for $1 \le i \le r$.

Similar to the summability problem, after a suitable transformation of rational functions, the existence problem of telescopers in $m + 1$ variables can also be reduced to that in fewer variables. Since the bivariate case has been solved in [6], we now have a complete solution to the existence problem of telescopers for rational functions in several discrete variables.

1.2.4 Complexity results

We also provide the detailed complexity analysis of the algorithms for the SET problem and the summability problem and creative telescoping for multivariate rational functions. We show that the complexity of all of these algorithms is polynoamial in the output size (for the details, see Theorems 3.8, 5.14 and 6.14).

1.3 An example

We now show an example to illustrate the main steps of deciding the rational summability problem with the help of algorithms for the SET problem over integers.

Let f be a rational function in $\mathbb{Q}(x, y, z)$ of the form

$$f = \frac{-z^2 + x}{x^2 + 2xy + z^2} + \frac{x - y - 2z}{x^2 + 2xy + z^2 + 2x} + \frac{z^2 + y}{x^2 + 2xy + z^2 + 8x + 2y - 2z + 8} + \frac{x + z}{(x - 3y)^2(y + z) + 1} + \left(y + \frac{z}{y^2 + z - 1} - \frac{1}{y^2 + z} \right) \frac{1}{(x + 2y + z)^2}.$$

Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ be the free abelian group generated by the shift operators $\sigma_x, \sigma_y, \sigma_z$. In order to decide whether f is $(\sigma_x, \sigma_y, \sigma_z)$-summable in $\mathbb{Q}(x, y, z)$, the first step is the so-called *orbital decomposition*, where we first compute the irreducible partial fraction decomposition of f with respect to x and then classify all irreducible factors of the denominator of f according to the shift equivalence relation. Applying algorithms for the partial fraction decomposition and the SET problem over integers, we obtain the orbital decomposition $f = f_1 + f_2 + f_3$, where

$$f_1 = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} + \frac{y + z^2}{\sigma_x \sigma_y^3 \sigma_z^{-1}(d_1)}, \quad f_2 = \frac{x + z}{d_2} \quad \text{and} \quad f_3 = \left(y + \frac{z}{y^2 + z - 1} - \frac{1}{y^2 + z} \right) \frac{1}{d_3^2}$$

with $d_1 = x^2 + 2xy + z^2$, $d_2 = (x-3y)^2(y+z)+1$ and $d_3 = x + 2y + z$. Here f_1, f_2, f_3 are three orbital components of f, since any two elements of d_1, d_2, d_3 are not shift equivalent. By Lemma 1.5, we have that f is $(\sigma_x, \sigma_y, \sigma_z)$-summable in $\mathbb{Q}(x, y, z)$ if and only if each f_i is summable.

The second step is using Abramov's reduction to reduce the summability problem from a general rational function to simple fractions. Since f_2, f_3 are already simple fractions, we only need to reduce f_1. For any $a, d \in \mathbb{F}(x, y, z)$ and any integer $k \in \mathbb{Z}$, Abramov's reduction decomposes $a/\sigma^k(b)$ as

$$\frac{a}{\sigma^k(b)} = \sigma(h) - h + \frac{\sigma^{-k}(a)}{b},$$

where $h = 0$ if $k = 0$, $h = \sum_{i=0}^{k-1} \frac{\sigma^{i-k}(a)}{\sigma^i(b)}$ if $k > 0$ and $h = -\sum_{i=0}^{-k-1} \frac{\sigma^i(a)}{\sigma^{i+k}(b)}$ if $k < 0$. Applying the reduction formula to f_1 with $\sigma = \sigma_x, \sigma_y, \sigma_z$ successively yields

$$f_1 = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1) + r_1 \text{ with } r_1 = \frac{2x-1}{d_1},$$

where $u_1 = \frac{y+z^2}{\sigma_y^3\sigma_z^{-1}(d_1)}$, $v_1 = \frac{x-y+1-2z}{d_1} + \sum_{\ell=0}^{2} \frac{y+\ell-3+z^2}{\sigma_y^\ell\sigma_z^{-1}(d_1)}$ and $w_1 = -\frac{y-3+z^2}{\sigma_z^{-1}(d_1)}$. Then f_1 is summable if and only if r_1 is $(\sigma_x, \sigma_y, \sigma_z)$-summable.

The third step is using the summability criterion to reduce the summability problem into few variables. For r_1, the isotropy group of d_1 in G is $G_{d_1} = \{\mathbf{1}\}$. By Theorem 1.6, r_1 is summable if and only if its numerator is zero. Hence r_1 is not $(\sigma_x, \sigma_y, \sigma_z)$-summable and neither are f_1 and f. For f_2, the isotropy group of d_2 in G is $G_{d_2} = \{\tau\}$ with $\tau = \sigma_x^3\sigma_y\sigma_z^{-1}$. By Theorem 1.6, we see that f_2 is summable in $\mathbb{Q}(x, y, z)$ if and only if $a_2 = x + z$ is (τ)-summable in $\mathbb{Q}(x, y, z)$. Since $a_2 = x + z = \Delta_\tau(b)$ with $b = \frac{1}{9}(x-3)(2x+3z)$, so a_2 is (τ)-summable, which implies that f_2 is $(\sigma_x, \sigma_y, \sigma_z)$-summable. Since $f_2 = \Delta_\tau(\frac{b}{d_2})$, its certificates can be obtained by Abramov's reduction:

$$f_2 = \Delta_x(u_2) + \Delta_y(v_2) + \Delta_z(w_2),$$

where $u_2 = \sum_{\ell=0}^{2} \sigma_x^\ell\sigma_y\sigma_z^{-1}\left(\frac{b}{d_2}\right)$, $v_2 = \sigma_z^{-1}\left(\frac{b}{d_2}\right)$ and $w_2 = -\sigma_z^{-1}\left(\frac{b}{d_2}\right)$. For f_3, a basis of the isotropy group G_{d_3} is $\{\tau_1, \tau_2\}$, where $\tau_1 = \sigma_x^2\sigma_y^{-1}$ and $\tau_2 = \sigma_x\sigma_z^{-1}$. Finally, we construct a $\mathbb{Q}$-automorphism ϕ_3 of $\mathbb{Q}(x, y, z)$ as follows

$$\phi_3(h(x, y, z)) = h(2x + y, -x, -y + z),$$

for any $h \in \mathbb{Q}(x, y, z)$. It can be checked that $\phi_3 \circ \tau_1 = \sigma_x \circ \phi_3$ and $\phi_3 \circ \tau_2 = \sigma_y \circ \phi_3$. So $a_3 = f_3 d_3^2$ is (τ_1, τ_2)-summable in $\mathbb{Q}(x, y, z)$ if and only if $\phi_3(a_3)$ is (σ_x, σ_y)-summable in $\mathbb{Q}(x, y, z)$. This reduces the summability problem in three variables to the summability problem in two variables. By induction it follows that $\phi_3(a_3)$ is not (σ_x, σ_y)-summable. Therefore f_3 is not $(\sigma_x, \sigma_y, \sigma_z)$-summable. In this case, f_3 can be decomposed into the sum of a summable part and a non-summable one:

$$f_3 = \Delta_x(u_3) + \Delta_y(v_3) + \Delta_z(w_3) + \frac{z}{(y^2+z)d_3^2},$$

where $u_3 = \sum_{\ell=0}^{1} \sigma_x^\ell\sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right) + \sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$, $v_3 = -\sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right)$, $w_3 = -\sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$ with $b_1 = -\frac{1}{2}y(y+1)$ and $b_2 = \frac{z+1}{y^2+z}$.

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we define the existence problem of telescopers and the summability problem precisely and recall some basic complexity estimates for later use. We present a general scheme for designing algorithms to solve the shift equivalence testing problem in Section 3, and compare our new algorithms with the other known algorithms in Appendix. In Section 4, we first recall the notion of isotropy groups of polynomials and their basic properties, and then introduce orbital decompositions for rational functions. We apply orbital decompositions in Section 5 to reduce the rational summability problem for general rational functions to that for simple fractions. After this, we present a criterion on the summability of such simple fractions. We not only decide the summability of a rational function but can also derive it explicitly. In Section 6, we again use the structure of isotropy groups and orbital decompositions to derive a criterion for the existence of telescopers for rational functions in variables t and $\mathbf{x}$. Moreover, we present an algorithm for computing a telescoper if it exists.

2 Preliminaries

In this section, we will recall some basic terminologies of symbolic summation and creative telescoping and overview some complexity results for later use.

2.1 Telescopers and summability of rational functions

Through out the paper, let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}(t, \mathbf{x})$ be the field of rational functions in t and $\mathbf{x} = \{x_1, \ldots, x_m\}$ over $\mathbb{K}$. For each $v \in \mathbf{v} = \{t, x_1, \ldots, x_m\}$, the shift operator σ_v with respect to v is defined as the $\mathbb{K}$-automorphism of $\mathbb{K}(\mathbf{v})$ such that

$$\sigma_v(v) = v + 1 \text{ and } \sigma_v(w) = w \text{ for all } w \in \mathbf{v} \setminus \{v\}.$$

Let $\mathcal{R} := \mathbb{K}(\mathbf{v})\langle S_t, S_{x_1}, \ldots, S_{x_m}\rangle$ denote the ring of linear recurrence operators over $\mathbb{K}(\mathbf{v})$, in which $S_{v_i} S_{v_j} = S_{v_j} S_{v_i}$ for all $v_i, v_j \in \mathbf{v}$ and $S_v f = \sigma_v(f) S_v$ for any $f \in \mathbb{K}(\mathbf{v})$ and $v \in \mathbf{v}$. The action of an operator $L = \sum_{i_0, i_1, \ldots, i_m \geq 0} a_{i_0, i_1, \ldots, i_m} S_t^{i_0} S_{x_1}^{i_1} \cdots S_{x_m}^{i_m} \in \mathcal{R}$ on a rational function $f \in \mathbb{K}(\mathbf{v})$ is defined as

$$L(f) = \sum_{i_0, i_1, \ldots, i_m \geq 0} a_{i_0, i_1, \ldots, i_m} \sigma_t^{i_0} \sigma_{x_1}^{i_1} \cdots \sigma_{x_m}^{i_m}(f).$$

For each $v \in \mathbf{v}$, the difference operator Δ_v with respect to v is defined by $\Delta_v = S_v - \mathbf{1}$, where $\mathbf{1}$ stands for the identity map on $\mathbb{K}(\mathbf{v})$.

We now introduce the notion of telescopers for rational functions in $\mathbb{K}(t, \mathbf{x})$.

Definition 2.1 (Telescoper)**.** *Let n be a positive integer such that $1 \leq n \leq m$ and let $f \in \mathbb{K}(t, \mathbf{x})$ be a rational function. A nonzero linear recurrence operator $L \in \mathbb{K}(t)\langle S_t\rangle$ is called a* telescoper *of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for f if there exist $g_1, \ldots, g_n \in \mathbb{K}(t, \mathbf{x})$ such that*

$$L(f) = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n).$$

The rational functions $g_1, \ldots, g_n$ are called the certificates *of L.*

Problem 2.2 (Existence Problem of Telescopers)**.** *Given a rational function $f \in \mathbb{K}(t, \mathbf{x})$ and an integer n with $1 \leq n \leq m$, decide whether or not f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$. If so, find a telescoper L and its certificates $g_1, \ldots, g_n$.*

In order to decide the existence of telescopers for $f \in \mathbb{K}(t, \mathbf{x})$, one may first use the shortcut to decide whether $L = \mathbf{1}$ is a telescoper for f. This is equivalent to the following summability problem of f in $\mathbb{F}(\mathbf{x})$ with $\mathbb{F} = \mathbb{K}(t)$.

Definition 2.3 (Summability). *Let $\mathbb{F}$ be a field of characteristic zero and n be a positive integer such that $1 \leq n \leq m$. A rational function $f \in \mathbb{F}(\mathbf{x})$ is called $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-*summable* in $\mathbb{F}(\mathbf{x})$ if there exist $g_1, \ldots, g_n \in \mathbb{F}(\mathbf{x})$ such that*

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n).$$

The rational functions $g_1, \ldots, g_n$, if they exists, are called the certificates *of f.*

Problem 2.4 (Rational Summability Problem). *Given a rational function $f \in \mathbb{F}(\mathbf{x})$ and an integer n with $1 \leq n \leq m$, decide whether or not f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable in $\mathbb{F}(\mathbf{x})$. If so, find a tuple $(g_1, \ldots, g_n)$ such that the g_i's are the certificates of f.*

In practice, the certificate tends to be much larger than the telescoper and we might only focus on the evaluation of the certificate. So we output the certificate as a sum of the products of several rational functions applied by shift operations and difference isomorphisms (see definition below). Such a certificate is called an *unnormalised certificate*.

The main idea of solving the summability problem is using mathematical induction to reduce the number of difference operators in this problem. To say explicitly, we shall reduce the $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summability problem for $f \in \mathbb{F}(\mathbf{x})$ to the $(\sigma_{x_1}, \ldots, \sigma_{x_r})$-summability problem for another rational function $a \in \mathbb{F}(\mathbf{x})$, where r is smaller than n and the base field $\mathbb{F}(\mathbf{x})$ in the summability problem is unchanged. Similarly, we shall reduce the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for $f \in \mathbb{K}(t, \mathbf{x})$ to the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_r})$ for some rational function $a \in \mathbb{K}(t, \mathbf{x})$.

We introduce below a general definition of the summability problem and existence problem of telescopers, which plays a central role to set up the reduction process for solving Problems 2.4 and 2.2. Let $G_t = \langle \sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ be the group generated by the shift operators $\sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n}$ under the operation of composition of functions. Then G_t is a free abelian group. For any $\tau \in G_t$, the difference operator Δ_τ is defined by

$$\Delta_\tau = S_t^{i_0} S_{x_1}^{i_1} \cdots S_{x_n}^{i_n} - \mathbf{1} \quad \text{if } \tau = \sigma_t^{i_0} \sigma_{x_1}^{i_1} \cdots \sigma_{x_n}^{i_n}.$$

For short, we use Δ_v to denote Δ_{σ_v} for each $v \in \mathbf{v}$. A finite subset $\{\tau_1, \ldots, \tau_r\}$ of G_t is said to be $\mathbb{Z}$-*linearly independent* if for all $a_1, \ldots, a_r \in \mathbb{Z}$, we have

$$\tau_1^{a_1} \cdots \tau_r^{a_r} = \mathbf{1} \quad \Rightarrow \quad a_1 = a_2 = \cdots = a_r = 0.$$

Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ be the subgroup of G_t generated by shift operators $\sigma_{x_1}, \ldots, \sigma_{x_n}$. Let $\{\tau_1, \ldots, \tau_r\} (1 \leq r \leq n)$ be a family of $\mathbb{Z}$-linearly independent elements in G. In general, a rational function $f \in \mathbb{F}(\mathbf{x})$ is called $(\tau_1, \ldots, \tau_r)$-*summable* in $\mathbb{F}(\mathbf{x})$ if

$$f = \Delta_{\tau_1}(g_1) + \cdots + \Delta_{\tau_r}(g_r)$$

for some $g_1, \ldots, g_r \in \mathbb{F}(\mathbf{x})$. Choose an element $\tau_0 = \sigma_t^{k_0} \sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n} \in G_t$ such that k_0 is nonzero. Then $\tau_0, \tau_1, \ldots, \tau_r$ are $\mathbb{Z}$-linearly independent in G_t. Let $T_0 = S_t^{k_0} S_{x_1}^{k_1} \cdots S_{x_n}^{k_n} \in \mathfrak{R}$ be the operator corresponding to τ_0. We say that a nonzero operator $L \in \mathbb{K}(t)\langle T_0 \rangle$ is a *telescoper of type* $(\tau_0; \tau_1, \ldots, \tau_r)$ for $f \in \mathbb{K}(t, \mathbf{x})$ if $L(f)$ is $(\tau_1, \ldots, \tau_r)$-summable in $\mathbb{K}(t, \mathbf{x})$.

Let R be a ring and $\sigma \colon R \to R$ be a ring automorphism of R. The pair (R, σ) is called a *difference ring*. If R is a field, we call the pair (R, σ) a *difference field*. Let (R_1, σ_1) and (R_2, σ_2)

be two difference rings and $\phi\colon R_1 \to R_2$ be a ring homomorphism. If ϕ satisfies the property that $\phi \circ \sigma_1 = \sigma_2 \circ \phi$, that means the following diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\phi} & R_2 \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ R_1 & \xrightarrow{\phi} & R_2 \end{array}$$

commutes, then ϕ is called a *difference homomorphism*. If in addition ϕ is a bijection, then its inverse ϕ^{-1} is also a difference homomorphism. In this case, we call ϕ a *difference isomorphism*. The notion of difference isomorphisms will be used to state our summability criteria and the existence criteria of telescopers.

An operator $L \in \mathbb{K}(t)\langle S_t\rangle$ is called a *common left multiple* of operators $L_1, \ldots, L_r \in \mathbb{K}(t)\langle S_t\rangle$ if there exist $R_1, \ldots, R_r \in \mathbb{K}(t)\langle S_t\rangle$ such that

$$L = R_1 L_1 = \cdots = R_r L_r.$$

Since $\mathbb{K}(t)\langle S_t\rangle$ is a left Euclidean domain, such an operator L always exists. Among all of such multiples, the monic one of smallest degree in S_t is called the *least common left multiple* (LCLM). Efficient algorithms for computing LCLM have been developed in [7, 14, 17].

Remark 2.5. *Let $f = f_1 + \cdots + f_r$ with $f_i \in \mathbb{K}(t, \mathbf{x})$. If each f_i has a telescoper L_i of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for $i = 1, \ldots, r$, then the LCLM of L_i's is a telescoper of the same type for f. This fact follows from the commutativity between operators in $\mathbb{K}(t)\langle S_t\rangle$ and the difference operators Δ_{x_i}'s.*

2.2 Complexity estimates

All complexity estimates of the algorithms in this paper are in terms of arithmetical operations in $\mathbb{K}$, denoted by "ops". The notation $\tilde{O}$ indicates the complexity estimates with hidden polylogarithmic factors.

Let $\mathbf{y} = \{y_1, y_2, \ldots, y_r\}$ be a subset of $\mathbf{v} = \{t, x_1, \ldots, x_m\}$ and $\mathbf{d} = (d_1, d_2, \ldots, d_r)$ be a vector in $\mathbb{N}^r$. Let $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$ denote the set of polynomials in $\mathbb{K}[\mathbf{y}]$ whose degrees in y_i are no more than d_i for $i = 1, 2, \ldots, r$. Let $\mathbb{K}(\mathbf{y})_{\mathbf{d}}$ denote the set of rational functions in $\mathbb{K}(\mathbf{y})$ with numerators and denominators in $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$. In particular, we denote $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$ (resp. $\mathbb{K}(\mathbf{y})_{\mathbf{d}}$) by $\mathbb{K}[\mathbf{y}]_d$ (resp. $\mathbb{K}(\mathbf{y})_d$) for simplicity if $d_1 = d_2 = \cdots = d_r = d$. For a rational function $f(\mathbf{y}) = p(\mathbf{y})/q(\mathbf{y}) \in \mathbb{K}(\mathbf{y})$, where $p(\mathbf{y})$ and $q(\mathbf{y})$ are coprime polynomials, the degree of $f(\mathbf{y})$ in y_i is defined as $\max\{\deg_{y_i}(p(\mathbf{y})), \deg_{y_i}(q(\mathbf{y}))\}$.

We first recall some complexity estimates of the basic operations on univatiate polynomials and rational functions (see the books [18, 78] for their proofs).

Fact 2.6. *Let d be an integer in $\mathbb{N}$. The following operations can be performed in $\tilde{O}(d)$ ops in $\mathbb{K}$:*

(1) addition, multiplication and differentiation of elements in $\mathbb{K}[x]_d$ and $\mathbb{K}(x)_d$;

(2) computing the greatest common divisor of two elements in $\mathbb{K}[x]_d$;

(3) partial fraction decomposition of an element in $\mathbb{K}(x)_d$ with a given factorization of its denominator.

Efficient algorithms for basic operations on multivariate polynomials have been developed in [58, 76]. We summarize the needed results as follows.

Fact 2.7. *For a vector* $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$*, the following operations can be performed in* $\tilde{O}(md_1 \cdots d_m)$ *ops in* $\mathbb{K}$*:*

(1) multipoint evaluation and interpolation in $\mathbb{K}[\mathbf{x}]_\mathbf{d}$ *from the given values on* $O(d_1 \cdots d_m)$ *points which form an* m*-dimensional tensor product grid;*

(2) expansion of $f(\mathbf{x}+\mathbf{s})$ *into the form* $f(\mathbf{x}+\mathbf{s}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$ *for* $f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]_\mathbf{d}$ *and* $\mathbf{s} \in \mathbb{K}^m$*.*

The following result about factorization of multivariate polynomials is from [55, Theorem 3.26].

Fact 2.8. *For a vector* $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$*, a polynomial in* $\mathbb{Q}[\mathbf{x}]_\mathbf{d}$ *can be factored into the product of irreducible factors in* $\tilde{O}((\min(\mathbf{d}))^{m-1}(d_1 \cdots d_m)^6)$ *ops in* $\mathbb{Q}$*.*

Let $\omega \in (2, 3]$ be a feasible exponent of matrix multiplication in $\mathbb{K}$, i.e., two square matrices of order r can be multiplied using $O(r^\omega)$ ops. Solving a system of linear equations is almost as hard as multiplying two matrices, see [18] for more details.

Fact 2.9. *A* $\mathbb{K}$*-linear system of equations of size* r *can be solved in* $O(r^\omega)$ *ops in* $\mathbb{K}$*.*

The complexity estimates of computing LCLM can be found in [14].

Fact 2.10. *(See [14, Theorem 1]) Let* $L_1, \ldots, L_k$ *be operators in* $\mathbb{K}[t]\langle S_t \rangle$ *whose bidegrees in* (t, S_t) *are at most* (d, r)*. Then the LCLM of* $L_1, \ldots, L_k$ *has bidegree at most* $(dk(kr - r + 1), kr)$ *in* (t, S_t)*, and it can be computed in* $\tilde{O}(k^{2\omega} r^\omega d)$ *ops in* $\mathbb{K}$*.*

3 Shift equivalence testing of polynomials

In this section, we first state the problem of Shift Equivalence Testing (SET) and give an overview of our algorithm for solving SET problem in Section 3.1. The idea of our algorithm is inspired by the DOS algorithm [32, 33]. Then we develop a general scheme for designing algorithms to solve the SET problem, whose proof is given in Section 3.2. More precisely, we introduce admissible covers of the associated polynomial system with the SET problem and prove that every admissible cover corresponds to an algorithm for solving the SET problem. In Section 3.3, we give two special admissible covers in practice, one of which corresponds to the DOS algorithm.

3.1 Overview of the general algorithm

Let $\mathbb{F}$ be a field of characteristic zero and let $\mathbb{F}[\mathbf{x}]$ be the ring of polynomials in $\mathbf{x} = \{x_1, \ldots, x_n\}$ over $\mathbb{F}$. Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are said to be *shift equivalent* if there exist $s_1, \ldots, s_n \in \mathbb{F}$ such that

$$p(x_1 + s_1, \ldots, x_n + s_n) = q(x_1, \ldots, x_n).$$

The set $\{\mathbf{s} \in \mathbb{F}^n \mid p(\mathbf{x}+\mathbf{s}) = q(\mathbf{x})\}$ is called the *dispersion set* of p and q over $\mathbb{F}$, denoted by $F_{p,q}$. Recall basic properties of the dispersion set in [32].

Lemma 3.1. *(See [32, Observation 4.2 and Lemma 4.4]) Let* $p, q \in \mathbb{F}[\mathbf{x}]$*. Then*

(1) $F_{p,p}$ *is a linear subspace of* $\mathbb{F}^n$ *over* $\mathbb{F}$*.*

(2) $F_{p,q} = \mathbf{s} + F_{p,p}$ *for any* $\mathbf{s} \in F_{p,q}$ *if* $F_{p,q} \neq \varnothing$*.*

The problem of Shift Equivalence Testing can be stated as follows.

Problem 3.2 (Shift Equivalence Testing Problem)**.** *Given $p, q \in \mathbb{F}[x_1, \ldots, x_n]$, decide whether there exists $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{F}^n$ such that*

$$p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x}).$$

If such a vector $\mathbf{s}$ exists, compute the dispersion set $F_{p,q}$ of p and q over $\mathbb{F}$. In this case, by Lemma 3.1, it suffices to find a special solution $\mathbf{s}$ in $F_{p,q}$ and a basis of $F_{p,p}$ over $\mathbb{F}$.

A related problem is testing the shift equivalence over integers, i.e. deciding whether there exists a vector $\mathbf{s} \in \mathbb{Z}^n$ such that $p(\mathbf{x}+\mathbf{s}) = q(\mathbf{x})$. We denote the set $\{\mathbf{s} \in \mathbb{Z}^n \mid p(\mathbf{x}+\mathbf{s}) = q(\mathbf{x})\}$ by $Z_{p,q}$. The computation of $Z_{p,q}$ will play an important role in the next sections where we study the rational summability problem and the existence problem of telescopers. By Lemma 3.1, we know $F_{p,q}$ is a linear variety over $\mathbb{F}$. Once the computation of $F_{p,q}$ boils down to solving linear systems, we can also compute $Z_{p,q}$ by combining the same methods for the SET problem over $\mathbb{F}$ and any algorithm for computing integer solutions of linear systems.

In the univariate case, the SET problem was solved by computing the resultant of two polynomials [1]. In the multivariate case, there are three different methods for solving the SET problem in the literature. In 1996, Grigoriev first gave a recursive algorithm (G) for the SET problem in [39, 40]. In 2010, motivated by solving linear partial difference equations, another algorithm (KS) for computing $Z_{p,q}$ via the Gröbner basis method was given by Kauers and Schneider in [48]. In 2014, a new algorithm with better complexity was given by Dvir, Oliveira and Shpilka (DOS) in [32, 33]. We have implemented all of the three algorithms in Maple and the experimental comparison is tabulated in the appendix. The timings indicate that the DOS algorithm is the most efficient one among the three methods in practice.

In this section, we introduce n new variables $\mathbf{a} = \{a_1, \ldots, a_n\}$. The SET problem is equivalent to finding the zeros of the polynomial $p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}) \in \mathbb{F}[\mathbf{a}, \mathbf{x}]$ with respect to $\mathbf{a}$. Collecting its coefficients in $\mathbf{x}$, we obtain a polynomial system. A direct approach to the SET problem is solving this polynomial system. Without exploring the hidden structure of the polynomial system, this naive approach could be very in-efficient. The common idea of the above three methods is to find the defining linear system of $F_{p,q}$, which avoids solving the polynomial system directly. To do this, the DOS algorithm finds an appropriate finite cover of the polynomial system. Then it reduces the SET problem to solving several linear systems successively by evaluating the non-linear part of polynomials. This kind of evaluation is called the linearization of polynomials, whose definition will be strictly stated below.

We first introduce some notations for later use. For any two vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, we say $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ if and only if $\alpha_i \geq \beta_i$ for all $1 \leq i \leq n$. This defines a partial order on $\mathbb{N}^n$. For a subset $\mathbf{y} = \{y_1, y_2, \ldots, y_m\}$ of $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ with $\hat{\mathbf{y}} := \mathbf{x} \setminus \mathbf{y}$, let $f(\mathbf{x}) = \sum_\alpha c_{\boldsymbol{\alpha}}(\hat{\mathbf{y}})\mathbf{y}^{\boldsymbol{\alpha}} \in \mathbb{F}[\hat{\mathbf{y}}][\mathbf{y}]$. Let $H^d_{\mathbf{y}}(f(\mathbf{x}))$ denote the homogeneous component of $f(\mathbf{x})$ of degree d in $\mathbf{y}$ and let $\mathrm{Supp}_{\mathbf{y}}(f)$ denote the set $\{\mathbf{y}^{\boldsymbol{\alpha}} \mid c_{\boldsymbol{\alpha}}(\hat{\mathbf{y}}) \neq 0\}$, which consists of nonzero monomials of $f(\mathbf{x})$. For simplicity, when $\mathbf{y} = \mathbf{x}$, we write $H^\ell_{\mathbf{y}}(f(\mathbf{x}))$ as $H^\ell(f(\mathbf{x}))$ and $\mathrm{Supp}_{\mathbf{y}}(f)$ as $\mathrm{Supp}(f)$. For a subset $S \subseteq \mathbb{F}[\mathbf{x}]$, let $\mathbb{V}_{\mathbb{F}}(S)$ be the zero set $\{\mathbf{s} \in \mathbb{F}^n \mid f(\mathbf{s}) = 0, \forall f \in S\}$.

Definition 3.3 (Linearization, Definition 1.2, restated)**.** *Let $f(\mathbf{x}) = H^0_{\mathbf{y}}(f)(\mathbf{y}) + H^1_{\mathbf{y}}(f)(\mathbf{y}) + \cdots + H^d_{\mathbf{y}}(f)(\mathbf{y})$ be the homogeneous decomposition of $f \in \mathbb{F}[\mathbf{x}] = \mathbb{F}[\hat{\mathbf{y}}][\mathbf{y}]$. For a vector $\mathbf{s} \in \mathbb{F}^m$, we call the linear polynomial $H^0_{\mathbf{y}}(f)(\mathbf{y}) + H^1_{\mathbf{y}}(f)(\mathbf{y}) + \sum_{i=2}^d H^i_{\mathbf{y}}(f)(\mathbf{s})$ the* linearization *of f at $\mathbf{s}$ with respect to $\mathbf{y}$, denoted by $L_{\mathbf{y}=\mathbf{s}}(f)$. Note that $L_{\mathbf{y}=\mathbf{s}}(f) = f$ if $d \leq 1$. For a polynomial set $S \subseteq \mathbb{F}[\mathbf{x}]$, let $L_{\mathbf{y}=\mathbf{s}}(S) := \{L_{\mathbf{y}=\mathbf{s}}(f) \mid f \in S\}$ be the* linearization *of S.*

In the following we will present the main idea how our new algorithm works. In order to compute $F_{p,q}$, we first write

$$p(\mathbf{x}+\mathbf{a})-q(\mathbf{x})=\sum_{\boldsymbol{\alpha}\in\Lambda}c_{\boldsymbol{\alpha}}(\mathbf{a})\mathbf{x}^{\boldsymbol{\alpha}},$$

where $c_{\boldsymbol{\alpha}}(\mathbf{a})\in\mathbb{F}[\mathbf{a}]$ and Λ is a finite subset of $\mathbb{N}^n$. Let

$$S:=\{c_{\boldsymbol{\alpha}}(\mathbf{a})\in\mathbb{F}[\mathbf{a}]\mid c_{\boldsymbol{\alpha}}(\mathbf{a})\text{ is a nonzero coefficient of }\mathbf{x}^{\boldsymbol{\alpha}}\text{ in }p(\mathbf{x}+\mathbf{a})-q(\mathbf{x})\}\,. \tag{3.1}$$

Then $F_{p,q}=\mathbb{V}_{\mathbb{F}}(S)$ is the zero set of S in $\mathbb{F}^n$. First, we classify all polynomials in S according to their total degrees in $\mathbf{a}$ and write $S=S_0^D\cup\cdots\cup S_{d'}^D$, where $d'=\deg_{\mathbf{a}}(p(\mathbf{x}+\mathbf{a})-q(\mathbf{x}))$ and

$$S_i^D=\{c_\alpha(\mathbf{a})\in S\mid \deg_{\mathbf{a}}(c_\alpha(\mathbf{a}))=i\}$$

for $i=0,\ldots,d'$. Then $\mathbb{V}_{\mathbb{F}}(S)=\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{d'}S_i^D)$. We may assume that $S_0^D=\varnothing$, otherwise p,q are not shift equivalent and return $F_{p,q}=\varnothing$. If $S_0^D\cup S_1^D$ has no solution in $\mathbb{F}^n$, return $F_{p,q}=\varnothing$. Otherwise take an arbitrary solution $\mathbf{s}^{(0)}\in\mathbb{V}_{\mathbb{F}}(S_0^D\cup S_1^D)$. Note that all polynomials in $S_0^D\cup S_1^D$ are linear and thus such an element $\mathbf{s}^{(0)}$ can be computed straightforwardly. We shall prove that the nonlinear system $S_0^D\cup S_1^D\cup S_2^D$ has the same solutions as its linearization $S_0^D\cup S_1^D\cup L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_2^D)$ at the point $\mathbf{s}^{(0)}$. If the latter linear system has no solution, return $F_{p,q}=\varnothing$. Otherwise, take an arbitrary solution $\mathbf{s}^{(1)}\in\mathbb{V}_{\mathbb{F}}(S_0^D\cup S_1^D\cup L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_2^D))$ by solving the linear system. Then consider the linearization of $\cup_{i=0}^3S_i^D$ at $\mathbf{s}^{(1)}$ and we shall prove that $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^3 S_i^D)=\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^3 L_{\mathbf{a}=\mathbf{s}^{(1)}}(S_i^D))$. Continuing the above process, we will finally find an equivalent linear system of the polynomial system $S=\cup_{i=0}^{d'}S_i^D$ by linearization.

Example 3.4. *Let $p=x^2+2xy+y^2+2x+6y$ and $q=x^2+2xy+y^2+4x+8y+11$ be two polynomials in $\mathbb{Q}[x,y]$. Decide whether p,q are shift equivalent with respect to x,y. Since*

$$p(x+a,y+b)-q(x,y)=(2a+2b-2)\cdot x+(2a+2b-2)\cdot y+(a^2+2ab+b^2+2a+6b-11),$$

we have $S=S_1^D\cup S_2^D$, where $S_1^D=\{2a+2b-2\}$ and $S_2^D=\{a^2+2ab+b^2+2a+6b-11\}$. Take an arbitrary solution $(a,b)=(1,0)$ of S_1^D. The linearization of S_2^D at $(1,0)$ is

$$L_{(a,b)=(1,0)}(S_2^D)=\{1^2+2\cdot1\cdot0+0^2+2a+6b-11\}=\{2a+6b-10\}.$$

In this example, the linear system $S_1^D\cup L_{(a,b)=(1,0)}(S_2^D)$ is indeed equivalent to the polynomial system $S_1^D\cup S_2^D$. So $F_{p,q}=\mathbb{V}_{\mathbb{F}}(S_1^D\cup L_{(a,b)=(0,1)}(S_2^D))=\{(-1,2)\}$.

Since shift operations do not change the total degree in $\mathbf{x}$, the homogeneous components of both sides of $p(\mathbf{x}+\mathbf{a})=q(\mathbf{x})$ with respect to $\mathbf{x}$ must be equal. The homogeneous decomposition of $p(\mathbf{x}+\mathbf{a})-q(\mathbf{x})$ yields another cover $\{S_0^H,S_1^H,\ldots,S_d^H\}$ of S, where $d=\max\{\deg_{\mathbf{x}}(p(\mathbf{x})),\deg_{\mathbf{x}}(q(\mathbf{x}))\}$ and

$$S_i^H:=\{c_{\boldsymbol{\alpha}}(\mathbf{a})\in S\mid c_{\boldsymbol{\alpha}}(\mathbf{a})\text{ is the coefficient of }\mathbf{x}^{\boldsymbol{\alpha}}\text{ in }H_{\mathbf{x}}^{d-i}(p(\mathbf{x}+\mathbf{a}))-H_{\mathbf{x}}^{d-i}(q(\mathbf{x}))\}$$

for $i=0,1,\ldots,d$. In the DOS algorithm, they first introduced the above method of linearization to solve the polynomial system $S=S_0^H\cup S_1^H\cup\cdots\cup S_d^H$ and proved the correctness of their algorithm by using formal partial derivatives. In Example 3.4, $S=S_1^D\cup S_2^D=S_1^H\cup S_2^H$, where $S_i^H=S_i^D$ for $i=1,2$. In general, these two covers are different, see Example 1.4. A natural question is for which cover, we can use the method of linearization to compute the dispersion set. One answer is the admissible cover defined below. In fact, the above two covers $\{S_0^D,S_1^D,\ldots,S_{d'}^D\}$ and $\{S_0^H,S_1^H,\ldots,S_d^H\}$, called by $\mathbf{a}$-*degree cover* and $\mathbf{x}$-*homogeneous cover* respectively, are both admissible, which will be proved in Section 3.3.

Definition 3.5 (Admissible cover, Definition 1.1, restated)**.** *Let $S \subseteq \mathbb{F}[\mathbf{a}]$ be as in (3.1). A collection $\{S_0, S_1, \dots, S_m\}$ of subsets is called a* cover *of S if S is the union of $S_0, S_1, \dots, S_m$. Such a cover $\{S_0, S_1, \dots, S_m\}$ is called an* admissible cover *of S if it satisfies the following two conditions:*

(1) All polynomials in S_0 are of degree at most one in $\mathbf{a}$.

(2) For every $\ell = 1, 2, \dots, m$, if $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_\ell$, then $c_{\boldsymbol{\beta}}(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$ for all $\boldsymbol{\beta} \in \mathbb{N}^n$ with $\boldsymbol{\beta} > \boldsymbol{\alpha}$ and $\mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))$.

A general algorithm for solving the SET problem via the method of linearization is as follows. This algorithm inherits one feature of the DOS algorithm: it could be early terminated when p, q are not shift equivalent. If two nonzero polynomials $p(\mathbf{x})$ and $q(\mathbf{x})$ are shift equivalent, then they have the same degree d in x and $H^d(p(\mathbf{x})) = H^d(q(\mathbf{x}))$, which means $\deg(p(\mathbf{x}) - q(\mathbf{x})) < \deg(p(\mathbf{x}))$. Therefore, we can check the degree condition at the beginning of the algorithm for better efficiency.

Algorithm 3.6 (Shift Equivalence Testing)**. ShiftEquivalent(**p**,** q**,** $[x_1, \dots, x_n]$**).**
INPUT: two multivariate polynomials $p, q \in \mathbb{F}[\mathbf{x}]$;
OUTPUT: a special solution of $F_{p,q}$ and an $\mathbb{F}$-basis of $F_{p,p}$ if p and q are shift equivalent; $\{\}$ otherwise.

1 *if* $p(\mathbf{x}) = q(\mathbf{x}) = 0$, **return** $\mathbb{F}^n$.
2 *if* $\deg(p(\mathbf{x}) - q(\mathbf{x})) \geq \deg(p(\mathbf{x}))$, **return** $\{\}$.
3 *set* $S := \mathrm{Coefficients}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}), \mathbf{x}) \subseteq \mathbb{F}[\mathbf{a}]$.
4 *let* $\{S_0, S_1, \dots, S_m\}$ *be an admissible cover of* S.
5 *set* $\mathbf{s}^{(0)} := \mathbf{0}$.
6 **for** $\ell = 0, \dots, m$ **do**
7   *set* $L^{(\ell)} := \cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)$.
8   *solve the linear system in* $\mathbf{a}$ *defined by* $L^{(\ell)}$.
9   *if the linear system* $L^{(\ell)}$ *has no solution,* **return** $\{\}$.
10   *else there is a special solution* $\mathbf{s}' \in \mathbb{F}^n$ *by evaluating each free variable at* 0, *set* $\mathbf{s}^{(\ell+1)} := \mathbf{s}'$.
11 **return** $\mathbf{s}^{(m+1)}$ *and an* $\mathbb{F}$*-basis of the solution space of the homogeneous linear equations induced by* $L^{(m)}$.

The correctness of Algorithm 3.6 is guaranteed by the following theorem.

Theorem 3.7 (Theorem 1.3, restated)**.** *If the cover $\{S_0, S_1, \dots, S_m\}$ of S is admissible, then for all $\ell = 0, 1, \dots, m$, we have either $\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right) = \varnothing$ or*

$$\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} S_i\right) = \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) \quad \text{for any } \mathbf{s}^{(\ell)} \in \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right).$$

The proof of Theorem 3.7 will be given in the next subsection.

Theorem 3.8. *For a vector $\mathbf{d} \in \mathbb{N}^n$, let $p(\mathbf{x})$ and $q(\mathbf{x})$ be two multivariate polynomials in $\mathbb{F}[\mathbf{x}]_{\mathbf{d}}$. Then Algorithm 3.6 can test whether p and q are shift equivalent and output a special solution of $F_{p,q}$ with an $\mathbb{F}$-basis of $F_{p,p}$ if $F_{p,q} \neq \varnothing$ using $\tilde{O}(d_1^{\omega} \cdots d_n^{\omega})$ ops in $\mathbb{F}$.*

Proof. The first three steps take $\tilde{O}(2nd_1^2 \cdots d_n^2)$ ops by Fact 2.7. Since both $\mathbf{a}$-degree cover and $\mathbf{x}$-homogeneous cover can be obtained by traversing elements in S, Step 4 can be performed in

$\tilde{O}(d_1^2 \cdots d_n^2)$ ops. By the definitions of **a**-degree cover and **x**-homogeneous cover, we have that m is of the size $\tilde{O}(d_1 + \cdots + d_n)$. Note that by Lemma 3.16 below, in each iteration of the loop in Step 6, Step 7 can be replaced by setting $L^{(\ell)}$ to be the union of $L^{(\ell-1)}$ and $L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_\ell)$ if $\ell \geq 1$, and hence the size of the linear system in Step 8 is no more than $|S_\ell| + n$. As a result, the cost of Step 7 is $\tilde{O}((|S_\ell|+n)nd_1 \cdots d_n)$ ops and that of Step 8 is $\tilde{O}((|S_\ell|+n)^\omega)$ ops in each iteration. This implies that the loop costs no more than $\tilde{O}(M)$ ops, where

$$\begin{aligned} M = \sum_{\ell=0}^{m} ((|S_\ell|+n)^\omega + (|S_\ell|+n)nd_1 \cdots d_n) &\leq \left(\sum_{\ell=0}^{m}(|S_\ell|+n)\right)^\omega + \left(\sum_{\ell=0}^{m}(|S_\ell|+n)\right) nd_1 \cdots d_n \\ &= (|S|+mn)^\omega + (|S|+mn)nd_1 \cdots d_n. \end{aligned}$$

Since $|S|$ is no more than $d_1 d_2 \cdots d_n$, the loop needs $\tilde{O}(d_1^\omega \cdots d_n^\omega)$ ops that dominates the whole costs. This completes the proof. ∎

From the above complexity analysis, we can not distinguish the algorithms with **a**-degree cover and **x**-homogeneous cover. In Section 8, we have implemented both algorithms to compare the practical efficiency. The experiments show that our ADC algorithm is more efficient than the DOS algorithm for sparse polynomials.

3.2 Proof of correctness of Theorem 3.7

Before proving Theorem 3.7, we need several lemmas to explore the inner structure of polynomials $c_\alpha(\mathbf{a})$ in S. First we give an explicit expression of the non-constant homogeneous components of $c_{\boldsymbol{\alpha}}(\mathbf{a})$ and find a recurrence relation among the homogeneous components. Then we explain the role of the admissible cover and the magic of linearization in Algorithm 3.6. Finally, we prove Theorem 3.7 by induction on ℓ.

For a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, let $|\boldsymbol{\alpha}| := \sum_{i=1}^n \alpha_i$ and $\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} := \frac{|\boldsymbol{\alpha}|!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$. Let ∂_{x_i} denote the partial derivative with respect to x_i and $\boldsymbol{\partial}^{\boldsymbol{\alpha}}$ denote $(\partial_{x_1})^{\alpha_1}(\partial_{x_2})^{\alpha_2}\cdots(\partial_{x_n})^{\alpha_n}$. For n variables $\mathbf{a} = \{a_1, a_2, \ldots, a_n\}$, we use $D_{\mathbf{a}}$ to denote the directional derivative in the direction of $\mathbf{a}$, i.e. $D_{\mathbf{a}} := \sum_{i=1}^n a_i \partial_{x_i}$. For $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{F}^n$, the notation $D_{\mathbf{s}}$ means $D_{\mathbf{a}}|_{\mathbf{a}=\mathbf{s}}$. Then for any $k \in \mathbb{N}^+$,

$$D_{\mathbf{a}}^k := (D_{\mathbf{a}})^k = \sum_{|\boldsymbol{\alpha}|=k} \binom{k}{\boldsymbol{\alpha}} \mathbf{a}^{\boldsymbol{\alpha}} \boldsymbol{\partial}^{\boldsymbol{\alpha}}$$

by the multinomial theorem since ∂_{x_i} and ∂_{x_j} commute.

By the directional derivative and Taylor's expansion, the homogeneous components of polynomials in S_ℓ^H can be expressed as follows.

Lemma 3.9. *(See [32, Lemma 3.5]) Let* $d := \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\}$. *For any* $k \in \mathbb{N}$ *and* $\ell \in \{0, 1, \ldots, d\}$, *we have*

$$H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x}+\mathbf{a})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})) = \sum_{i=0}^{\ell} \frac{1}{i!} D_{\mathbf{a}}^i \left(H_{\mathbf{x}}^{d-\ell+i}(p(\mathbf{x}))\right) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})) \tag{3.2}$$

and

$$H_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x}+\mathbf{a})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x}))\right) = \begin{cases} \dfrac{1}{k!} D_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{d-\ell+k}(p(\mathbf{x}))\right), & \text{if } k \geq 1, \\ H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})), & \text{if } k = 0. \end{cases} \tag{3.3}$$

Moreover, for any $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$ *and* $k \geq 1$, $H_{\mathbf{a}}^k(c_{\boldsymbol{\alpha}}(\mathbf{a}))$ *is the coefficient of* $\mathbf{x}^{\boldsymbol{\alpha}}$ *in* $\frac{1}{k!} D_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{|\boldsymbol{\alpha}|+k}(p)\right)$.

Proof. Note that $c_{\boldsymbol{\alpha}}(\mathbf{a})$ is exactly the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}}$ in $H_{\mathbf{x}}^{|\boldsymbol{\alpha}|}(p(\mathbf{x}+\mathbf{a})) - H_{\mathbf{x}}^{|\boldsymbol{\alpha}|}(q(\mathbf{x}))$, so it is sufficient to prove Equations (3.2) and (3.3). By Taylor's expansion, we have

$$p(\mathbf{x}+\mathbf{a}) = \sum_{i=0}^{d} \frac{1}{i!} D_{\mathbf{a}}^{i}(p)(\mathbf{x}) = \sum_{i=0}^{d}\sum_{j=0}^{d} \frac{1}{i!} D_{\mathbf{a}}^{i}(H_{\mathbf{x}}^{j}(p))(\mathbf{x}).$$

Note that if $D_{\mathbf{a}}^{i}(H_{\mathbf{x}}^{j}(p))$ is not equal to zero, then it is homogeneous of degree $j-i$ in $\mathbf{x}$. Consequently, we obtain Equation (3.2). Moreover, note that $D_{\mathbf{a}}^{i}(H_{\mathbf{x}}^{d-\ell+i}(p))(\mathbf{x})$ is homogeneous of degree i with respect to $\mathbf{a}$. So we get Equation (3.3), which completes the proof. ∎

Since

$$\frac{1}{k!} D_{\mathbf{a}}^{k}\left(H_{\mathbf{x}}^{|\boldsymbol{\alpha}|+k}(p(\mathbf{x}))\right) = \frac{1}{k!}\sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^{\boldsymbol{\beta}} \boldsymbol{\partial}^{\boldsymbol{\beta}}\left(H_{\mathbf{x}}^{|\boldsymbol{\alpha}|+k}(p(\mathbf{x}))\right),$$

dropping the terms except $H^{k}(c_{\boldsymbol{\alpha}}(\mathbf{a}))\cdot \mathbf{x}^{\boldsymbol{\alpha}}$ in the above polynomial, we can get

$$H^{k}(c_{\boldsymbol{\alpha}}(\mathbf{a}))\cdot \mathbf{x}^{\boldsymbol{\alpha}} = \frac{1}{k!}\sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^{\boldsymbol{\beta}} \boldsymbol{\partial}^{\boldsymbol{\beta}}\left([\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}](p(\mathbf{x}))\cdot \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}\right), \tag{3.4}$$

where $[\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}](p(\mathbf{x}))$ denotes the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ in $p(\mathbf{x})$. Therefore, for any $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$, we can write $D_{\mathbf{a},\boldsymbol{\alpha}}^{k}(f(\mathbf{x})) := \sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^{\boldsymbol{\beta}} \boldsymbol{\partial}^{\boldsymbol{\beta}}\left([\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}](f(\mathbf{x}))\cdot \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}\right)$ and use $D_{\mathbf{s},\boldsymbol{\alpha}}^{k}$ to denote $D_{\mathbf{a},\boldsymbol{\alpha}}^{k}\big|_{\mathbf{a}=\mathbf{s}}$ for $\mathbf{s} \in \mathbb{F}^{n}$. The following lemma is derived straightforward.

Lemma 3.10. *Let $k \in \mathbb{N}^{+}$ and $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$. Then we have $H^{k}(c_{\boldsymbol{\alpha}}(\mathbf{a}))\cdot \mathbf{x}^{\boldsymbol{\alpha}} = \frac{1}{k!} D_{\mathbf{a},\boldsymbol{\alpha}}^{k}(p(\mathbf{x}))$.*

For the directional derivative, we know $D_{\mathbf{a}}^{k}(f(\mathbf{x})) = (D_{\mathbf{a}}^{1})^{k}(f(\mathbf{x}))$. However $D_{\mathbf{a},\boldsymbol{\alpha}}^{k}(f(\mathbf{x}))$ may be different from $(D_{\mathbf{a},\boldsymbol{\alpha}}^{1})^{k}(f(\mathbf{x}))$, as the following example shows.

Example 3.11. *Let $\mathbb{F} = \mathbb{Q}$, $p(x,y), q(x,y) \in \mathbb{Q}[x,y]$ with $p(x,y) = x^{3}+y^{3}$ and $q(x,y) = p(x,y)+1$. Expanding $p(x+a,y+b) - q(x,y)$, we have $p(x+a,y+b)-q(x,y) = 3a\cdot x^{2} + 3b\cdot y^{2} + 3a^{2}\cdot x + 3b^{2}\cdot y + (a^{3}+b^{3}-1)$. Then we have $c_{(1,0)}(a,b) = 3a^{2}$,*

$$\begin{aligned}
D_{(a,b),(1,0)}^{1}(p(\mathbf{x})) &= \sum_{i+j=1} \binom{1}{(i,j)} a^{i}b^{j}\cdot \partial_{x}^{i}\partial_{y}^{j}\left([x^{1+i}y^{0+j}](p(x,y))\cdot x^{1+i}y^{0+j}\right)\\
&= \binom{1}{(1,0)} a\cdot \partial_{x}\left([x^{2}](p(x,y))\cdot x^{2}\right) + \binom{1}{(0,1)} b\cdot \partial_{y}\left([xy](p(x,y))\cdot xy\right) = 0,\\
D_{(a,b),(1,0)}^{2}(p(\mathbf{x})) &= \sum_{i+j=2} \binom{2}{(i,j)} a^{i}b^{j}\cdot \partial_{x}^{i}\partial_{y}^{j}\left([x^{1+i}y^{0+j}](p(x,y))\cdot x^{1+i}y^{0+j}\right)\\
&= \binom{2}{(2,0)} a^{2}\cdot \partial_{x}^{2}\left([x^{3}](p(x,y))\cdot x^{3}\right) + \binom{2}{(1,1)} ab\cdot \partial_{x}\partial_{y}\left([x^{2}y](p(x,y))\cdot x^{2}y\right)\\
&\quad + \binom{2}{(0,2)} b^{2}\cdot \partial_{y}^{2}\left([xy^{2}](p(x,y))\cdot xy^{2}\right)\\
&= \frac{2!}{0!2!} a^{2}\cdot \partial_{x}^{2}(x^{3}) = 6a^{2}x
\end{aligned}$$

and $\left(D_{(a,b),(1,0)}^{1}\right)^{2}(p(\mathbf{x})) = D_{(a,b),(1,0)}^{1}(0) = 0$. Therefore, we can check that $H^{k}\left(c_{(1,0)}(a,b)\right)\cdot x$ is equal to $\frac{1}{k!} D_{(a,b),(1,0)}^{k}(p(x,y))$ for $k = 1,2$, but $D_{(a,b),(1,0)}^{2}(p(\mathbf{x}))$ is not equal to $\left(D_{(a,b),(1,0)}^{1})^{2}(p(\mathbf{x})\right)$.

Now we rewrite the expression of $D_{\mathbf{a},\boldsymbol{\alpha}}^{k}(f)$ and derive a recurrence relation for $D_{\mathbf{a},\boldsymbol{\alpha}}^{k}(f)$.

Lemma 3.12. *Let $\boldsymbol{\alpha} \in \mathbb{N}^n$, $k, \ell \in \mathbb{N}^+$ and $f \in \mathbb{F}[\mathbf{x}]$. Let $\mathbf{e}_i \in \mathbb{N}^n$ denote a unit vector with the i-th component being one and others being zero. Then we have:*

(1) $D^k_{\mathbf{a},\boldsymbol{\alpha}}(f(\mathbf{x})) = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \left(\left[\mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}}\right](f(\mathbf{x})) \cdot \mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}} \right)$.

(2) $D^{k+\ell}_{\mathbf{a},\boldsymbol{\alpha}}(f(\mathbf{x})) = \sum_{|\boldsymbol{\beta}|=\ell} \binom{\ell}{\boldsymbol{\beta}} \mathbf{a}^{\boldsymbol{\beta}} \boldsymbol{\partial}^{\boldsymbol{\beta}} \left(D^k_{\mathbf{a},\boldsymbol{\alpha}+\boldsymbol{\beta}}(f(\mathbf{x})) \right)$.

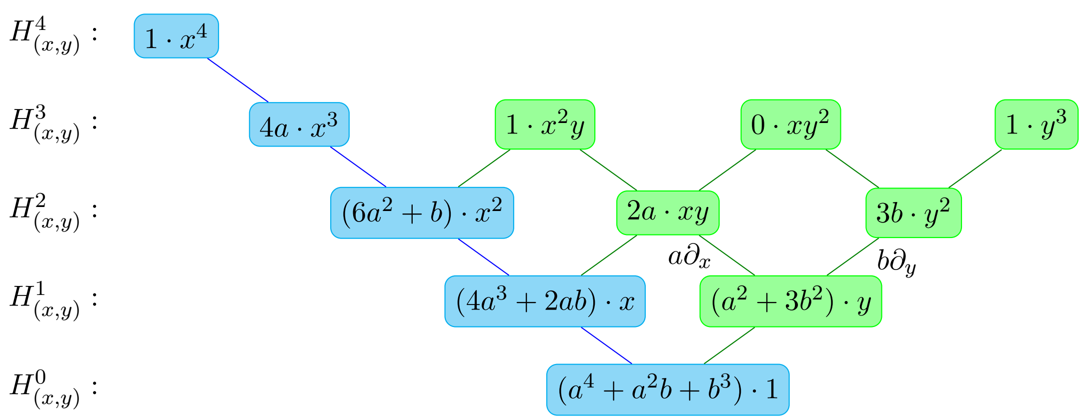
Proof. (1): note that for any $\boldsymbol{\beta} \in \mathbb{N}^n$ with $|\boldsymbol{\beta}| = k$, $\boldsymbol{\beta}$ can be expressed as a sum of k unit vectors. Moreover, there are $\binom{k}{\boldsymbol{\beta}}$ different k-tuples $(j_1, j_2, \ldots, j_k)$ such that $\boldsymbol{\beta} = \sum_{i=1}^k \mathbf{e}_{j_i}$. Then combining the definition of $D^k_{\mathbf{a},\boldsymbol{\alpha}}(f(\mathbf{x}))$, we can get (1).

(2): applying (1) twice, we have

$$
\begin{aligned}
& D^{k+\ell}_{\mathbf{a},\boldsymbol{\alpha}}(f(\mathbf{x})) \\
&= \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \sum_{j_{k+1}=1}^{n} \cdots \sum_{j_{k+\ell}=1}^{n} \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \\
&\qquad \left(\left[\mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}+\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}}\right](f(\mathbf{x})) \cdot \mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}+\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \right) \\
&= \sum_{j_{k+1}=1}^{n} \cdots \sum_{j_{k+\ell}=1}^{n} \mathbf{a}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \\
&\quad \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \left(\left[\mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}+\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}}\right](f(\mathbf{x})) \cdot \mathbf{x}^{\boldsymbol{\alpha}+\sum_{i=1}^k \mathbf{e}_{j_i}+\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \right) \right) \\
&= \sum_{j_{k+1}=1}^{n} \cdots \sum_{j_{k+\ell}=1}^{n} \mathbf{a}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \left(D^k_{\mathbf{a},\boldsymbol{\alpha}+\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}}(f(\mathbf{x})) \right).
\end{aligned}
$$

Then as the proof of (1), we can finally obtain (2) by set $\boldsymbol{\beta} = \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}$. ∎

Example 3.13. *Let $\mathbb{F} = \mathbb{Q}$ and $p = x^4 + x^2y + y^3 \in \mathbb{Q}[x, y]$. After expanding, we get $p(x+a, y+b) = x^4 + 4a \cdot x^3 + x^2y + y^3 + (6a^2+b) \cdot x^2 + 2a \cdot xy + 3b \cdot y^2 + (4a^3+2ab) \cdot x + (a^2+3b^2) \cdot y + (a^4+a^2b+b^3)$. All terms of $p(x+a, y+b)$ are listed in the following figure.*



Taking $q(x, y) = 0$ in Lemma 3.10, we get $D^k_{(a,b),(i,j)}(p(x, y)) = k! H^k_{(a,b)}([x^iy^j](p(x+a, y+b))) \cdot x^iy^j$ for all $k \geq 1$. So we can read off $D^k_{(a,b),(i,j)}(p(x, y))$ from the above figure. For instance,

$D^2_{(a,b),(0,1)}(p(x, y)) = 2! \cdot (a^2+3b^2) \cdot y$, $D^1_{(a,b),(1,1)}(p(x, y)) = 2a \cdot xy$ and $D^1_{(a,b),(0,2)}(p(x, y)) = 3b \cdot y^2$.

Taking $k=\ell=1$ in Lemma 3.12 (2), we obtain a recurrence relation among these three terms:

$$D^2_{(a,b),(0,1)}(p(x,y)) = \sum_{i+j=1} \binom{1}{(i,j)} a^i b^j \partial_x^i \partial_y^j \left(D^1_{(a,b),(i,1+j)}(p(x,y))\right)$$
$$= \binom{1}{(1,0)} a\partial_x \left(D^1_{(a,b),(1,1)}(p(x,y))\right) + \binom{1}{(0,1)} b\partial_y \left(D^1_{(a,b),(0,2)}(p(x,y))\right).$$

This implies

$$2(a^2+3b^2)y = a\partial_x(2axy) + b\partial_y\left(3by^2\right). \tag{3.5}$$

By the definition of $D^k_{\mathbf{a},\boldsymbol{\alpha}}(p)$ and Lemma 3.10, we get

$$2(a^2+3b^2)y = a^2\partial_x^2(x^2y) + 2ab\partial_x\partial_y(0\cdot xy^2) + b^2\partial_y^2(y^3). \tag{3.6}$$

Note that the term x^4 does not involve in the above two equations (3.5) and (3.6) because $y \nmid x^4$. In this example, the term x^4 only affects all terms in the blue branch, such as $3!\cdot 4a^3x = a^3\partial_x^3(x^4)$.

Without introducing the notation $D^k_{\mathbf{a},\boldsymbol{\alpha}}$, by Lemma 3.9 (or Lemma 3.5 in [32]) we only get "global" relations, such as

$$2!\cdot H^2_{(a,b)}\left(H^1_{(x,y)}(p(x+a,y+b))\right) = D^2_{(a,b)}\left(H^{1+2}_{(x,y)}(p(x,y))\right).$$

This implies two relations among the rows (instead of the points) in the figure:

$$2(2abx+a^2y+3b^2y) = (a\partial_x+b\partial_y)^2(x^2y+0\cdot xy^2+y^3)$$
$$= (a\partial_x+b\partial_y)(bx^2+2axy+3by^2).$$

From Observation 3.4 in [32], we know if $D^1_{\mathbf{a}}(f(\mathbf{x})) = D^1_{\mathbf{b}}(f(\mathbf{x}))$, then $D^k_{\mathbf{a}}(f(\mathbf{x})) = D^k_{\mathbf{b}}(f(\mathbf{x}))$ for all $k\geq 1$. Now we show that $D^k_{\mathbf{a},\boldsymbol{\alpha}}(f(\mathbf{x}))$ can be determined by $D^1_{\mathbf{a},\boldsymbol{\beta}}(f(\mathbf{x}))$ for all $\boldsymbol{\beta}\in\mathbb{N}^n$ with $\boldsymbol{\beta}\geq\boldsymbol{\alpha}$ and $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|+k-1$. This is why we introduce the second condition in the definition of an admissible cover.

Lemma 3.14. *Let $\mathbf{r},\mathbf{s}\in\mathbb{F}^n$, $\boldsymbol{\alpha}\in\mathbb{N}^n$, $k\in\mathbb{N}^+$ and $f(\mathbf{x})\in\mathbb{F}[\mathbf{x}]$. If $D^1_{\mathbf{r},\boldsymbol{\beta}}(f(\mathbf{x})) = D^1_{\mathbf{s},\boldsymbol{\beta}}(f(\mathbf{x}))$ for all $\boldsymbol{\beta}\in\mathbb{N}^n$ with $\boldsymbol{\beta}\geq\boldsymbol{\alpha}$ and $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|+k-1$, then we have $D^k_{\mathbf{r},\boldsymbol{\alpha}}(f(\mathbf{x})) = D^k_{\mathbf{s},\boldsymbol{\alpha}}(f(\mathbf{x}))$.*

Proof. The proof is by induction on k. It is clear to see that the lemma is true for $k=1$. Now assume the equality holds for k. For $k+1$, assume that $D^1_{\mathbf{r},\boldsymbol{\beta}}(f(\mathbf{x})) = D^1_{\mathbf{s},\boldsymbol{\beta}}(f(\mathbf{x}))$ for all $\boldsymbol{\beta}\in\mathbb{N}^n$ with $\boldsymbol{\beta}\geq\boldsymbol{\alpha}$ and $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|+(k+1)-1$. We have $D^{k+1}_{\mathbf{r},\boldsymbol{\alpha}}(f(\mathbf{x})) = \sum_{i=1}^n \mathbf{r}^{\mathbf{e}_i}\boldsymbol{\partial}^{\mathbf{e}_i}(D^k_{\mathbf{r},\boldsymbol{\alpha}+\mathbf{e}_i}(f(\mathbf{x})))$ by Lemma 3.12 (2). Note that for all $\boldsymbol{\gamma}\in\mathbb{N}^n$ with $\boldsymbol{\gamma}\geq\boldsymbol{\alpha}+\mathbf{e}_i$ and $|\boldsymbol{\gamma}| = |\boldsymbol{\alpha}+\mathbf{e}_i|+k-1$, we have $\boldsymbol{\gamma}\geq\boldsymbol{\alpha}$ and $|\boldsymbol{\gamma}| = |\boldsymbol{\alpha}|+(k+1)-1$. Thus by assumption we have $D^1_{\mathbf{r},\boldsymbol{\gamma}}(f(\mathbf{x})) = D^1_{\mathbf{s},\boldsymbol{\gamma}}(f(\mathbf{x}))$. It follows from the inductive hypothesis that $D^k_{\mathbf{r},\boldsymbol{\alpha}+\mathbf{e}_i}(f(\mathbf{x})) = D^k_{\mathbf{s},\boldsymbol{\alpha}+\mathbf{e}_i}(f(\mathbf{x}))$. So

$$D^{k+1}_{\mathbf{r},\boldsymbol{\alpha}}(f(\mathbf{x})) = \sum_{i=1}^n \mathbf{r}^{\mathbf{e}_i}\boldsymbol{\partial}^{\mathbf{e}_i}\left(D^k_{\mathbf{s},\boldsymbol{\alpha}+\mathbf{e}_i}(f(\mathbf{x}))\right)$$
$$= \sum_{i=1}^n \mathbf{r}^{\mathbf{e}_i}\boldsymbol{\partial}^{\mathbf{e}_i}\left(\sum_{|\boldsymbol{\gamma}|=k}\binom{k}{\boldsymbol{\gamma}}\mathbf{s}^{\boldsymbol{\gamma}}\boldsymbol{\partial}^{\boldsymbol{\gamma}}\left([\mathbf{x}^{\boldsymbol{\alpha}+\mathbf{e}_i+\boldsymbol{\gamma}}](f(\mathbf{x}))\cdot\mathbf{x}^{\boldsymbol{\alpha}+\mathbf{e}_i+\boldsymbol{\gamma}}\right)\right)$$
$$= \sum_{|\boldsymbol{\gamma}|=k}\binom{k}{\boldsymbol{\gamma}}\mathbf{s}^{\boldsymbol{\gamma}}\boldsymbol{\partial}^{\boldsymbol{\gamma}}\left(\sum_{i=1}^n \mathbf{r}^{\mathbf{e}_i}\boldsymbol{\partial}^{\mathbf{e}_i}([\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}+\mathbf{e}_i}](f(\mathbf{x}))\cdot\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}+\mathbf{e}_i})\right)$$
$$= \sum_{|\boldsymbol{\gamma}|=k}\binom{k}{\boldsymbol{\gamma}}\mathbf{s}^{\boldsymbol{\gamma}}\boldsymbol{\partial}^{\boldsymbol{\gamma}} D^1_{\mathbf{r},\boldsymbol{\alpha}+\boldsymbol{\gamma}}(f(\mathbf{x})).$$

Because $\boldsymbol{\alpha}+\boldsymbol{\gamma} \geq \boldsymbol{\alpha}$ and $|\boldsymbol{\alpha}+\boldsymbol{\gamma}| = |\boldsymbol{\alpha}|+|\boldsymbol{\gamma}| = |\boldsymbol{\alpha}|+(k+1)-1$, we have $D^1_{\mathbf{r},\boldsymbol{\alpha}+\boldsymbol{\gamma}}(f(\mathbf{x})) = D^1_{\mathbf{s},\boldsymbol{\alpha}+\boldsymbol{\gamma}}(f(\mathbf{x}))$ by assumption. Applying Lemma 3.12 (2) again, the proof is completed. ∎

Combining Lemma 3.10 and the above lemma, we get the following lemma immediately.

Lemma 3.15. *Let $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$, $k \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r}, \mathbf{s} \in \mathbb{F}^n$. If*

$$H^1(c_{\boldsymbol{\beta}})(\mathbf{r}) = H^1(c_{\boldsymbol{\beta}})(\mathbf{s}) \quad \text{for all } \boldsymbol{\beta} \in \mathbb{N}^n \text{ with } \boldsymbol{\beta} \geq \boldsymbol{\alpha} \text{ and } |\boldsymbol{\beta}| = |\boldsymbol{\alpha}| + k - 1,$$

then we have

$$H^k(c_{\boldsymbol{\alpha}})(\mathbf{r}) = H^k(c_{\boldsymbol{\alpha}})(\mathbf{s}).$$

Now we are ready to show that for an admissible cover, the linearization does not change the zero set of the polynomial system S.

Lemma 3.16. *Let $\{S_0, S_1, \ldots, S_m\}$ be an admissible cover of S and $\ell \in \{0, 1, \ldots, m\}$. If there exist $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$, then for all $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in \cup_{i=0}^{\ell} S_i$ we have*

$$L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\alpha}})(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\alpha}})(\mathbf{a}). \tag{3.7}$$

Furthermore, we have $c_{\boldsymbol{\alpha}}(\mathbf{r}) = L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\alpha}})(\mathbf{r})$.

Proof. Since $c_{\boldsymbol{\alpha}}(\mathbf{r}) = L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\alpha}})(\mathbf{r})$, it is sufficient to prove Equation (3.7) by induction on ℓ.

For $\ell = 0$, we have $\deg(c_{\boldsymbol{\alpha}}(\mathbf{a})) \leq 1$ by Definition 3.5, so $L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\alpha}})(\mathbf{a}) = c_{\boldsymbol{\alpha}}(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\alpha}})(\mathbf{a})$.

For $\ell > 0$, suppose the lemma holds for smaller ℓ. Then it is sufficient to show that $H^k(c_{\boldsymbol{\alpha}})(\mathbf{r}) = H^k(c_{\boldsymbol{\alpha}})(\mathbf{s})$ for all $k \in \mathbb{N}$ with $k \geq 2$. By Lemma 3.15, we know the proof is completed by showing that $H^1(c_{\boldsymbol{\beta}})(\mathbf{r}) = H^1(c_{\boldsymbol{\beta}})(\mathbf{s})$ for $\boldsymbol{\beta} \in \mathbb{N}^n$ with $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$ and $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}| + k - 1$. Because $|\boldsymbol{\beta}| \geq |\boldsymbol{\alpha}|+2-1 > |\boldsymbol{\alpha}|$, we have $\boldsymbol{\beta} > \boldsymbol{\alpha}$. Then we get either $c_{\boldsymbol{\beta}}(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$ or $c_{\boldsymbol{\beta}}(\mathbf{a}) = 0$ by Definition 3.5. For $\ell - k + 1 \leq \ell - 2 + 1 \leq \ell - 1$, we have $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i) \subseteq \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i)$. This means $\mathbf{r}$ and $\mathbf{s}$ are zeros of all polynomials $c_{\boldsymbol{\beta}}(\mathbf{a}) \in S_{\ell-k+1}$. Then we have

$$L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}})(\mathbf{r}) = c_{\boldsymbol{\beta}}(\mathbf{r}) = 0 \quad \text{and} \quad L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}})(\mathbf{s}) = 0. \tag{3.8}$$

On the other hand, $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{(\ell-k+1)-1} S_i)$ because $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i) \subseteq \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{(\ell-k+1)-1} S_i)$. By the inductive hypothesis with $\ell - k + 1$, we get $L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}})(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}})(\mathbf{a})$. So

$$H^0(L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}})) = H^0(L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}})). \tag{3.9}$$

Note that $H^1(c_{\boldsymbol{\beta}})(\mathbf{a}) = H^1(L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}}))(\mathbf{a}) = H^1(L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}}))(\mathbf{a})$. Combining the equations (3.8) and (3.9), we have

$$\begin{aligned} H^1(c_{\boldsymbol{\beta}})(\mathbf{r}) &= H^1(L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}}))(\mathbf{r}) = -H^0(L_{\mathbf{a}=\mathbf{r}}(c_{\boldsymbol{\beta}})) \\ &= -H^0(L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}})) = H^1(L_{\mathbf{a}=\mathbf{s}}(c_{\boldsymbol{\beta}}))(\mathbf{s}) = H^1(c_{\boldsymbol{\beta}})(\mathbf{s}), \end{aligned}$$

which completes the proof. ∎

Proof of Theorem 3.7. We shall prove the theorem by induction on ℓ.

For $\ell = 0$, we know that any $c_{\boldsymbol{\alpha}}(\mathbf{a})$ in S_0 satisfies $\deg(c_{\boldsymbol{\alpha}}(\mathbf{a})) \leq 1$ by Definition 3.5. Thus we have $L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_0) = S_0$ and $\mathbb{V}_{\mathbb{F}}(S_0) = \mathbb{V}_{\mathbb{F}}(L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_0))$.

For $\ell > 0$, assume the theorem holds for $\ell - 1$, i.e., $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i) = \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i))$. Taking $\mathbf{r}, \mathbf{s} = \mathbf{s}^{(\ell-1)}, \mathbf{s}^{(\ell)} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-2} S_i)$ in Lemma 3.16, we know that the linearizations of $c_{\boldsymbol{\beta}}(\mathbf{a})$ at $\mathbf{s}^{(\ell)}$ and

$\mathbf{s}^{(\ell-1)}$ are equal for all $c_{\boldsymbol{\beta}}(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$. This means $\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i) = \cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i)$. Then we have

$$\mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) \subseteq \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) = \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i)\right) = \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} S_i\right),$$

where the last equality follows from the inductive hypothesis. Note that $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell} S_i)$ is also a subset of $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$. So we only need to prove that for all $\mathbf{r} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$,

$$\mathbf{r} \in \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} S_i\right) \quad \text{if and only if} \quad \mathbf{r} \in \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right). \tag{3.10}$$

Because $s^{(\ell)} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$, we have $c_{\boldsymbol{\alpha}}(\mathbf{r}) = L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(c_{\boldsymbol{\alpha}})(\mathbf{r})$ for all $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in \cup_{i=0}^{\ell} S_i$ by Lemma 3.16. Then the claim (3.10) follows immediately. ∎

The main distinction between the reasoning of our general scheme and the DOS algorithm can be shown in Lemma 3.15. In [32], Lemma 3.5 proves Equation (3.2) which we extend to Lemmas 3.9 and 3.10. Observation 3.4 in [32] is generalized by Lemma 3.14. The subsequent proof for the correctness of the DOS algorithm can be summarized by the lemma below. Then the remaining steps for proving Theorem 3.7 and the correctness of the DOS algorithm are similar.

Lemma 3.17. *Let* $d := \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\}$, $i \in \{0, 1, \ldots, d\}$, $k \in \mathbb{N}$ *with* $k \geq 2$ *and* $\mathbf{r}, \mathbf{s} \in \mathbb{F}^n$. *If*

$$H^1(c_{\boldsymbol{\beta}})(\mathbf{r}) = H^1(c_{\boldsymbol{\beta}})(\mathbf{s}) \quad \textit{for all } c_{\boldsymbol{\beta}}(\mathbf{a}) \in S_{i-k+1}^H,$$

then we have

$$H^k(c_{\boldsymbol{\alpha}})(\mathbf{r}) = H^k(c_{\boldsymbol{\alpha}})(\mathbf{s}) \quad \textit{for all } c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_i^H.$$

Given two points $\mathbf{r}$ and $\mathbf{s}$ in $\mathbb{F}^n$, we use diagrams to explain how Lemma 3.15 makes the statement more precise than this lemma for the case where $d = \deg_{\mathbf{x}}(p) = \deg_{\mathbf{x}}(q) = 4$ and $n = 2$. Let $p(x+a, y+b) - q(x,y) = \sum_{(\alpha,\beta)\in\Lambda} c_{(\alpha,\beta)}(a,b) x^{\alpha} y^{\beta}$. For the k-th homogeneous component of $c_{(\alpha,\beta)}$, if its values at $\mathbf{r}$ and $\mathbf{s}$ are equal, we draw a point at position (α, β, k) in the space. Furthermore, if $H^k(c_{(\alpha,\beta)})(\mathbf{r}) = H^k(c_{(\alpha,\beta)})(\mathbf{s})$ for all $(\alpha, \beta) \in \mathbb{N}^2$ such that the sum of α and β is a fixed constant i, which means there are points (α, β, k) on the same line, then we draw a segment to connect them to each other. Note that the degree of any polynomial in S_{d-i}^H is no more than i, which will be proved exactly in Lemma 3.19. Lemma 3.17 implies that the dark green segment on one triangle face can conclude all the segments on this face with $k \geq 2$ in Figure 3.1. More precisely, Lemma 3.15 tells us that in Figure 3.2, on one triangle face, every point with $k \geq 2$ can be deduced from the part of dark green segment which is cut out by two dotted line from this point. For example, Point A can be inferred from Segment ℓ.

3.3 Two special admissible covers

By Theorem 3.7, we see that any admissible cover of the polynomial system S corresponds to an algorithm for solving the SET problem via linear system solving. We now present two special admissible covers. The first admissible cover defined below is classifying the polynomial system S according to their degree in $\mathbf{a}$, which is called the $\mathbf{a}$-*degree cover*.

Theorem 3.18 ($\mathbf{a}$-degree cover, Theorem 1.3, restated)**.** *Let* d' *be the degree of* $p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x})$ *with respect to* $\mathbf{a}$. *Let* $S_i^D := \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S \mid \deg(c_{\boldsymbol{\alpha}}(\mathbf{a})) = i\}$. *Then the cover* $\{S_0^D, S_1^D, \ldots, S_{d'}^D\}$ *of* S *is admissible.*

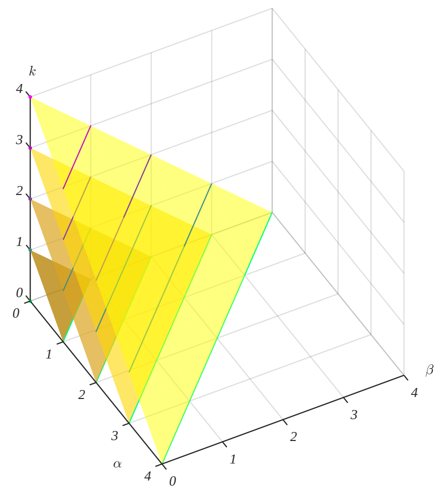


Figure 3.1: Graph for Lemma 3.17

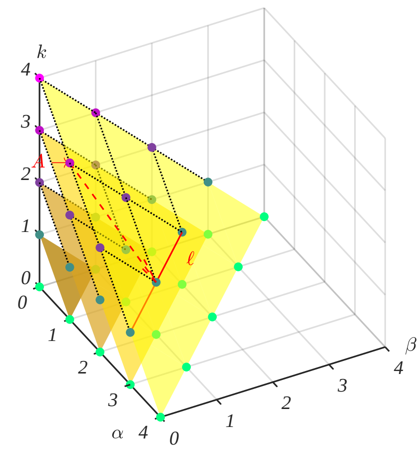


Figure 3.2: Graph for Lemma 3.15

Proof. We will check that this cover satisfies the two conditions mentioned in Definition 3.5. The condition (1) can be checked directly by definition. As for (2), assume that $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_{\ell}^{D}$ and $\boldsymbol{\beta}$ is an arbitrary vector in $\mathbb{N}^n$ with $\boldsymbol{\beta} > \boldsymbol{\alpha}$ and $\mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))$. We will argue by contradiction that $\deg(c_{\boldsymbol{\alpha}}(\mathbf{a})) > \deg(c_{\boldsymbol{\beta}}(\mathbf{a}))$. By assumption, we know that $\deg(c_{\boldsymbol{\alpha}}(\mathbf{a})) = \ell$. If there is a monomial $\mathbf{a}^{\boldsymbol{\gamma}} \in \mathrm{Supp}(c_{\boldsymbol{\beta}}(\mathbf{a}))$ with $|\boldsymbol{\gamma}| \geq \ell$, then by Equation (3.4), we have $\mathbf{x}^{\boldsymbol{\gamma}+\boldsymbol{\beta}} \in \mathrm{Supp}(p(\mathbf{x}))$. Since $|\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}| \geq |\boldsymbol{\beta}| - |\boldsymbol{\alpha}| > 0$, we obtain $\mathbf{a}^{\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}} \in \mathrm{Supp}(H^{|\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}|}(c_{\boldsymbol{\alpha}}(\mathbf{a}))) \in \mathrm{Supp}(c_{\boldsymbol{\alpha}}(\mathbf{a}))$ by Equation (3.4). However, note that $\boldsymbol{\beta} > \boldsymbol{\alpha}$ implies $|\boldsymbol{\beta}| > |\boldsymbol{\alpha}|$, so $|\boldsymbol{\gamma}+\boldsymbol{\beta}-\boldsymbol{\alpha}| = |\boldsymbol{\gamma}| + |\boldsymbol{\beta}| - |\boldsymbol{\alpha}| > \ell$, which leads to a contradiction to the fact that ℓ is the degree of $c_{\boldsymbol{\alpha}}(\mathbf{a})$. ∎

Note that $\boldsymbol{\beta} > \boldsymbol{\alpha}$ implies $|\boldsymbol{\beta}| > |\boldsymbol{\alpha}|$. This inspires the second admissible cover called the $\mathbf{x}$-*homogeneous cover*. Before we prove it, we first present a useful lemma.

Lemma 3.19. *Let* $d := \deg_{\mathbf{x}}(p(\mathbf{x}))$. *For any* $\boldsymbol{\alpha} \in \mathbb{N}^n$ *with* $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$, *we have* $\deg_{\mathbf{a}}(c_{\boldsymbol{\alpha}}(\mathbf{a})) \leq d - |\boldsymbol{\alpha}|$.

Proof. Note that $[\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}](p(\mathbf{x})) \neq 0$ yields $|\boldsymbol{\alpha}+\boldsymbol{\beta}| \leq d$, so $k = |\boldsymbol{\beta}| \leq d - |\boldsymbol{\alpha}|$ in Equation (3.4). That is to say, $H^k(c_{\boldsymbol{\alpha}}(\mathbf{a})) \cdot \mathbf{x}^{\boldsymbol{\alpha}} = 0$ if $k > d - |\boldsymbol{\alpha}|$. Then the conclusion follows. ∎

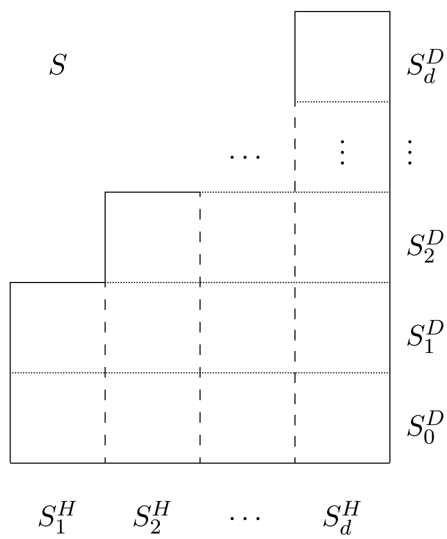
Theorem 3.20 ($\mathbf{x}$-homogeneous cover, Theorem 1.3, restated)**.** *Let d be the maximal degree of $p(\mathbf{x})$ and $q(\mathbf{x})$. Let $S_i^H := \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S \mid |\boldsymbol{\alpha}| = d - i\}$ for $i = 0, 1, \ldots, d$. Then the cover $\{S_0^H, S_1^H, \ldots, S_d^H\}$ of S is admissible.*

Proof. We first show that $\{S_0^H, S_1^H, \ldots, S_d^H\}$ is exactly a cover of S. It is sufficient to show that S is a subset of $\cup_{i=0}^{d} S_i^H$. This is true because we have $\mathbf{x}^{\boldsymbol{\alpha}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x})) \subseteq \mathrm{Supp}(p(\mathbf{x})) \cup \mathrm{Supp}(q(\mathbf{x}))$ for arbitrary $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$ and then $0 \leq |\boldsymbol{\alpha}| \leq d$.

Then we check this cover is admissible. If there exists $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_0^H$ with a nonlinear monomial $\mathbf{a}^{\boldsymbol{\beta}}$, then we have $\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in \mathrm{Supp}(p(\mathbf{x}))$ by Equation (3.4). For $|\boldsymbol{\beta}| \geq 2$, there is a unit vector $\mathbf{e}_j \in \mathbb{N}^n$ such that $\mathbf{e}_j < \boldsymbol{\alpha}+\boldsymbol{\beta}$. Thus $\mathbf{a}^{\mathbf{e}_j}\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{e}_j} \in \mathrm{Supp}_{\mathbf{x}\cup\mathbf{a}}(D^1_{\mathbf{a},\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{e}_j}(p(\mathbf{x})))$. By Lemma 3.10, $\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{e}_j} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))$. However, $|\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{e}_j| = |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| - |\mathbf{e}_j| \geq d + 2 - 1 > d$, which leads to a contradiction to the fact that $\deg_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x})) \leq \max\{\deg_{\mathbf{x}}(p(\mathbf{x}))), \deg_{\mathbf{x}}(q(\mathbf{x}))\} = d$. So the condition (1) in Definition 3.5 is satisfied. Finally, for any $\ell = 1, 2, \ldots, d$, let $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S_{\ell}^H$. Then for all $\boldsymbol{\beta} \in \mathbb{N}^n$ with $\boldsymbol{\beta} > \boldsymbol{\alpha}$ and $\mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{Supp}_{\mathbf{x}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x}))$, we have $|\boldsymbol{\beta}| > |\boldsymbol{\alpha}| = d - \ell$, so $c_{\boldsymbol{\beta}}(\mathbf{a}) \in S^H_{d-|\boldsymbol{\beta}|} \subseteq \cup_{i=0}^{\ell-1} S_i^H$. Therefore, $S_0^H, S_1^H, \ldots, S_d^H$ also satisfy the condition (2) in Definition 3.5 and the proof is complete. ∎

This is the cover defined in the DOS algorithm and we call it **x**-homogeneous cover. As a consequence, we reproved the correctness of the DOS algorithm in our general framework of admissible covers.

After introducing two special admissible covers, we would like to compare them and explain their connection. For simplicity, we always assume $H_{\mathbf{x}}^d(p(\mathbf{x}+\mathbf{a})) = H_{\mathbf{x}}^d(q(\mathbf{x}))$ with $\deg(p(\mathbf{x})) = \deg(q(\mathbf{x})) = d$ in the following discussion. For this special case, $S_0^H = \varnothing$. We get $\deg_{\mathbf{a}}(p(\mathbf{x}+\mathbf{a}) - q(\mathbf{x})) = d$ by Equation (3.4). Lemma 3.19 yields $S_\ell^H \subseteq \cup_{i=0}^{\ell} S_i^D$. Hence the connection between two different covers is like the following figure.



Now we give some examples to illustrate our two algorithms induced by our two elaborated admissible covers.

Example 3.21. *Let* $\mathbb{F} = \mathbb{Q}$, $p_i(x,y,z), q_i(x,y,z) \in \mathbb{Q}[x,y,z]$, $i = 1,2$ *with* $p_1(x,y,z) = x^4+x^2y+y^2$, $q_1(x,y,z) = p_2(x,y+1,z+2)+z$, $p_2(x,y,z) = x^4+x^3y+xy^2+z^2$ *and* $q_2(x,y,z) = p_2(x,y+1,z+2)+xy$.

(1) Compute F_{p_1,q_1}. *We expand* $p_1(x+a,y+b,z+c) - q_1(x,y,z)$ *and get that*

$$\begin{aligned} & p_1(x+a,y+b,z+c) - q_1(x,y,z) \\ =&(4a\cdot x^3) + ((6a^2+b-1)\cdot x^2 + 2a\cdot xy) \\ &+ ((4a^3+2ab)\cdot x + (a^2+2b-2)\cdot y - z) + (a^4+a^2b+b^2-1). \end{aligned}$$

Then we can separate the coefficients of $p_1(x+a,y+b,z+c) - q_1(x,y,z)$ *with respect to* x, y *and* z *in two different methods as following.*

S	S_1^H	S_2^H	S_3^H	S_4^H
S_4^D				$a^4+a^2b+b^2-1$
S_3^D			$4a^3+2ab$	
S_2^D		$6a^2+b-1$	a^2+2b-2	
S_1^D	$4a$	$2a$		
S_0^D			-1	

So we can get $F_{p_1,q_1}=\varnothing$ at once if we use the **a**-*degree cover, while by the* **x**-*homogeneous cover, we will calculate until we get S_3^H.*

(2) *Compute F_{p_2,q_2}. We expand $p_2(x+a,y+b,z+c)-q_2(x,y,z)$ and get that*

$$\begin{aligned}
&p_2(x+a,y+b,z+c)-q_2(x,y,z)\\
=&((4a+b-1)\cdot x^3+3a\cdot x^2y)\\
&+((6a^2+3ab)\cdot x^2+(3a^2+2b-3)\cdot xy+a\cdot y^2)\\
&+((4a^3+3a^2b+b^2-1)\cdot x+(a^3+2ab)\cdot y+(2c-4)\cdot z)\\
&+(a^4+a^3b+ab^2+c^2-4).
\end{aligned}$$

Then we can separate the coefficients of $p_2(x+a,y+b,z+c)-q_2(x,y,z)$ with respect to x, y and z in two different methods as following.

S	S_1^H	S_2^H	S_3^H	S_4^H
S_4^D				$a^4+a^3b+ab^2+c^2-4$
S_3^D			$4a^3+3a^2b+b^2-1$ a^3+2ab	
S_2^D		$6a^2+3ab$ $3a^2+2b-3$		
S_1^D	$4a+b-1$ $3a$	a	$2c-4$	
S_0^D				

So we can get $F_{p_2,q_2}=\varnothing$ if we use **x**-*homogeneous cover and calculate S_2^H, while by* **a**-*degree cover, we have to solve $2c-4=0$ needlessly.*

4 Isotropy groups and orbital decompositions

In this section, we first recall the notion of isotropy groups under shifts, which plays a central role in the summability criteria and existence criteria of telescopers. Then we present different types of partial fraction decompositions of $\mathbb{F}(\mathbf{x})$ with respect to different orbital factorizations as in [21]. These decompositions can be computed via algorithms for the SET problem over integers and will be used in the next sections for reducing the rational summability problem and the existence problem of telescopers to simpler cases.

4.1 Isotropy groups

Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ be the free abelian group generated by shift operators $\sigma_{x_1}, \ldots, \sigma_{x_n}$ and A be a subgroup of G. Let p be a multivariate polynomial in $\mathbb{F}[\mathbf{x}]$. The set

$$[p]_A := \{\sigma(p) \mid \sigma \in A\}$$

is called the A-*orbit* of p. Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are said to be A-*shift equivalent* or A-*equivalent* if $[p]_A = [q]_A$, denoted by $p \sim_A q$. The relation $\sim_A$ is an equivalence relation.

Definition 4.1 (Sato's Isotropy Group [66])**.** *Let A and p be defined as above. The set*

$$A_p := \{\sigma \in A \mid \sigma(p) = p\}.$$

is a subgroup of A, called the isotropy *group of p in A.*

If two polynomials p, q in $\mathbb{F}[\mathbf{x}]$ are A-shift equivalent, then $A_p = A_q$. The following remark says that we can test the A-equivalence of polynomials and compute a basis of A_p by algorithms for the SET problem over integers in Section 3.

Remark 4.2. *(1) Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are G-equivalent if and only if there exists a $\sigma \in G$ such that $\sigma(p) = q$. Therefore, the G-equivalence relation of p, q can be obtained via the computation of $Z_{p,q}$ in Section 3. When $p = q$, the group G_p is isomorphic to $Z_{p,p}$. Both of them are free abelian groups and a basis of G_p can be given by a basis of $Z_{p,p}$.*

(2) When $A = \langle \sigma_{x_1}, \ldots, \sigma_{x_r} \rangle$ with $1 \le r \le n$, we can view p, q as polynomials in $x_1, \ldots, x_r$ and the other variables are parameters. Then the A-equivalence relation of p, q and a basis of the isotropy group A_p are also available by algorithms in Section 3.

(3) In general, let $A = \langle \tau_1, \ldots, \tau_r \rangle$, where $\{\tau_1, \ldots, \tau_r\} (1 \le r \le n)$ are $\mathbb{Z}$-linearly independent. We will utilize Proposition 5.12 below to construct a difference isomorphism between $(\mathbb{F}(\mathbf{x}), \tau_i)$ and $(\mathbb{F}(\mathbf{x}), \sigma_{x_i})$ such that $\phi \circ \tau_i = \sigma_{x_i} \circ \phi$ for $1 \le i \le r$. Let $B = \langle \sigma_{x_1}, \ldots, \sigma_{x_r} \rangle$. Then p and q are A-equivalent if and only if $\phi(p)$ and $\phi(q)$ are B-equivalent. Furthermore, we have $\tau_1^{a_1} \cdots \tau_r^{a_r} \in A_p$ if and only if $\sigma_{x_1}^{a_r} \cdots \sigma_{x_r}^{a_r} \in B_{\phi(p)}$ for any $a_1, \ldots, a_r \in \mathbb{Z}$.

A structure property of the quotient group G/G_p is given by Sato [66, Lemma A-3] as follows.

Lemma 4.3. *G/G_p is a free abelian group.*

If $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ is a non-constant polynomial, then G_p is a proper subgroup of G. By Lemma 4.3, we have $\operatorname{rank}(G_p) < \operatorname{rank}(G)$, where $\operatorname{rank}(G)$ denotes the rank of the free abelian group G. This property about the rank of isotropy groups plays a key rule in the reduction method of solving rational summability problem and the existence problem of telescopers.

If $n > 1$, let $H = \langle \sigma_{x_1}, \dots, \sigma_{x_{n-1}} \rangle$ be the subgroup of G generated by $\sigma_{x_1}, \dots, \sigma_{x_{n-1}}$. The isotropy group of p in H is $H_p = \{\tau \in H \mid \tau(p) = p\}$. By Lemma 4.3, both G/G_p and H/H_p are free abelian groups. So the rank of G_p and H_p are strictly less than that of G and H respectively if p has positive degree in x_1.

Lemma 4.4. *G_p/H_p is a free abelian group of* $\operatorname{rank}(G_p/H_p) \leq 1$.

Proof. Define a group homomorphism $\varphi : G_p/H_p \to \mathbb{Z}$ by

$$\sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n} H_p \mapsto k_n.$$

It can be verified that φ is well-defined. For any $\tau_1, \tau_2 \in G_p$, if they are in the same coset of H_p in G_p, then $\tau_1 \tau_2^{-1} \in H_p$. This implies $\tau_1 \tau_2^{-1} \in H$ and hence $\varphi(\tau_1 H_p) = \varphi(\tau_2 H_p)$. Moreover, the converse is true since $G_p \cap H = H_p$. So φ is injective. Then we have $G_p/H_p \cong \operatorname{im} \varphi = k\mathbb{Z}$ for some integer $k \in \mathbb{Z}$. So G_p/H_p is a free abelian group generated by $\varphi^{-1}(k)$. ∎

Remark 4.5. *Let p be a polynomial in $\mathbb{F}[\mathbf{x}]$. By Remark 4.2.(1), one can compute a basis $\{\tau_1, \tau_2, \dots, \tau_r\}$ of G_p. If $\tau_i \in H$ for all $i = 1, \dots, r$, then $G_p = H_p$ and $G_p/H_p = \{\mathbf{1}\}$. So* $\operatorname{rank}(G_p/H_p) = 0$ *and $\{\tau_1, \tau_2, \dots, \tau_r\}$ is a basis of H_p. If $\tau_\ell \notin H$ for some $\ell \in \{1, \dots, r\}$, then* $\operatorname{rank}(G_p/H_p) = 1$. *Write $\tau_i = \sigma_{x_1}^{b_{i,1}} \cdots \sigma_{x_n}^{b_{i,n}}$ with $b_{i,j} \in \mathbb{Z}$ for each $i = 1, \dots, r$. Let $B = (b_{i,j}) \in \mathbb{Z}^{r \times n}$. Since $\tau_\ell \notin H$, we have $b_{\ell,n} \neq 0$. Using unimodular row reduction, one can compute a unimodular matrix $U \in \mathbb{Z}^{r \times r}$ such that $C = UB$, where $C = (c_{i,j}) \in \mathbb{Z}^{r \times n}$ satisfies $c_{1,n} = \gcd(b_{1,n}, b_{2,n}, \dots, b_{r,n}) \neq 0$ and $c_{i,n} = 0$ for all $i = 2, \dots, r$. Let $\sigma_i = \sigma_{x_1}^{c_{i,1}} \cdots \sigma_{x_n}^{c_{i,n}}$ for each $i = 1, \dots, r$. Then $\{\sigma_1, \dots, \sigma_r\}$ is another basis of G_p because U is an invertible matrix over $\mathbb{Z}$. Moreover, $G_p/H_p = \langle \bar{\sigma}_1 \rangle$ and $\{\sigma_2, \dots, \sigma_r\}$ is a basis of H_p.*

Example 4.6. *Consider polynomials in $\mathbb{Q}[x, y, z]$. Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $H = \langle \sigma_x, \sigma_y \rangle$.*

(1) For $p = x^2 + 2xy + z^2$, we have $G_p = H_p = \{\mathbf{1}\}$.

(2) For $p = (x - 3y)^2(y + z) + 1$, we have $G_p = \langle \tau \rangle$ and $H_p = \{\mathbf{1}\}$, where $\tau = \sigma_x^3 \sigma_y \sigma_z^{-1}$. So $G_p/H_p = \langle \bar{\tau} \rangle$, where $\bar{\tau} = \tau H_p$ denotes the coset in G_p/H_p represented by $\tau \in G_p$.

(3) Let $p = x + 2y + z$, we have $G_p = \langle \tau_1, \tau_2 \rangle$ and $H_p = \langle \tau_2 \rangle$, where $\tau_1 = \sigma_x \sigma_y^{-1} \sigma_z$ and $\tau_2 = \sigma_x^2 \sigma_y^{-1}$. So $G_p/H_p = \langle \bar{\tau}_1 \rangle$.

4.2 Orbital decompositions

A polynomial $p \in \mathbb{F}[\mathbf{x}]$ is said to be *monic* if the leading coefficient of p is 1 under a fix monomial order. Let $\hat{\mathbf{x}}_1$ denote the $m - 1$ variables $x_2, \dots, x_m$. For any subgroup A of $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ and any polynomial Q in $\mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, one can classify all of the monic irreducible factors in x_1 of Q into distinct A-orbits which leads to a factorization

$$Q = c \cdot \prod_{i=1}^{I} \prod_{j=1}^{J_i} \tau_{i,j}(d_i)^{e_{i,j}},$$

where $c \in \mathbb{F}(\hat{\mathbf{x}}_1)$, $I, J_i, e_{i,j} \in \mathbb{N}$, $\tau_{i,j} \in A$, $d_i \in \mathbb{F}[\mathbf{x}]$ being monic irreducible polynomials in distinct A-orbits, and for each i, $\tau_{i,j}(d_i) \neq \tau_{i,j'}(d_i)$ if $1 \leq j \neq j' \leq J_i$. With respect to this fixed representation, we have the unique irreducible partial fraction decomposition for a rational function $f = P/Q \in \mathbb{F}(\mathbf{x})$ of the form

$$f = p + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{\ell=1}^{e_{i,j}} \frac{a_{i,j,\ell}}{\tau_{i,j}(d_i)^\ell}, \tag{4.1}$$

where $p, a_{i,j,\ell} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(a_{i,j,\ell}) < \deg_{x_1}(d_i)$ for all i, j, ℓ. Note that the representation in (4.1) depends on the choice of representatives d_i in distinct A-orbits. However, the sum $\sum_{j=1}^{J_i} \frac{a_{i,j,\ell}}{\tau_{i,j}(d_i)^\ell}$ only depends on the multiplicity ℓ and the orbit $[d_i]_A$ instead of its representative d_i. Based on this fact, we shall formulate a unique decomposition of a rational function with respect to the group A. In this sense, we can decompose $\mathbb{F}(\mathbf{x})$ as a vector space over $\mathbb{E} = \mathbb{F}(\hat{\mathbf{x}}_1)$.

Given an irreducible polynomial $d \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(d) > 0$ and $j \in \mathbb{N}^+$, we define a subspace of $\mathbb{F}(\mathbf{x})$

$$V_{[d]_A,j} = \operatorname{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \,\middle|\, a \in \mathbb{E}[x_1], \tau \in A, \deg_{x_1}(a) < \deg_{x_1}(d) \right\}. \tag{4.2}$$

For any fraction in $V_{[d]_A,j}$, the irreducible factors of its denominator are in the same A-orbit as d. Let $V_0 = \mathbb{E}[x_1]$ denote the set of all polynomials in x_1. By the irreducible partial fraction decomposition, any rational function $f \in \mathbb{F}(\mathbf{x})$ can be uniquely written in the form

$$f = f_0 + \sum_j \sum_{[d]_A} f_{[d]_A,j}, \tag{4.3}$$

where $f_0 \in V_0$ and $f_{[d]_A,j}$ are in distinct $V_{[d]_A,j}$ spaces. Let T_A be the set of all distinct A-orbits of monic irreducible polynomials in $\mathbb{F}[\mathbf{x}]$, whose degrees with respect to x_1 are positive. Then $\mathbb{F}(\mathbf{x})$ has the following direct sum decomposition

$$\mathbb{F}(\mathbf{x}) = V_0 \bigoplus \left(\bigoplus_{j \in \mathbb{N}^+} \bigoplus_{[d]_A \in T_A} V_{[d]_A,j} \right). \tag{4.4}$$

Definition 4.7. *The decomposition (4.4) of $\mathbb{F}(\mathbf{x})$ is called the* orbital decomposition *of $\mathbb{F}(\mathbf{x})$ with respect to the variable x_1 and the group A. If f is written in the form (4.3), we call f_0 and $f_{[d]_A,j}$* orbital components *of f with respect to x_1 and A.*

A key feature of subspaces $V_{[d]_A,j}$ is the A-invariant property. In the field of univariate rational functions, the orbital decomposition of $\mathbb{F}(x_1)$ with respect to the group $A = \langle \sigma_{x_1} \rangle$ was first given in [46] by Karr.

Lemma 4.8. *If $f \in V_{[d]_A,j}$ and $P \in \mathbb{F}(\hat{\mathbf{x}}_1)[A]$, then $P(f) \in V_{[d]_A,j}$.*

Proof. Let $f = \sum a_i/\tau_i(d)^j$ and $P = \sum p_\sigma \sigma$ with $p_\sigma \in \mathbb{F}(\hat{\mathbf{x}}_1)$ and $\sigma \in A$. For any $\sigma \in A$, we have that $\sigma\tau_i$ is still in A, because A is a group. Since the shift operators do not change the degree and multiplicity of a polynomial, we have $\deg_{x_1}(\sigma(a_i)) < \deg_{x_1}(d)$ and then $\frac{p_\sigma \sigma(a_i)}{\sigma(\tau_i(d))^j}$ is in $V_{[d]_A,j}$. So $P(f) \in V_{[d]_A,j}$ by the linearity of the space. ∎

Example 4.9. *Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{E} = \mathbb{Q}(y, z)$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$. Consider the rational function f_1 in $\mathbb{Q}(x, y, z)$ of the form*

$$f_1 = \frac{x - z^2}{\underbrace{x^2 + 2xy + z^2}_{d_1 := d_{1,1}}} + \frac{x - y - 2z}{\underbrace{x^2 + 2xy + 2x + z^2}_{d_{1,2}}} + \frac{y + z^2}{\underbrace{x^2 + 2xy + 8x + 2y + z^2 - 2z + 8}_{d_{1,3}}}.$$

If $A = \langle \sigma_x \rangle$, then the orbital partial fraction decomposition of f_1 is

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} \text{ with } f_{1,1} = \frac{x - z^2}{d_{1,1}},\ f_{1,2} = \frac{x - y - 2z}{d_{1,2}} \text{ and } f_{1,3} = \frac{y + z^2}{d_{1,3}},$$

where $f_{1,i} \in V_{[d_{1,i}]_A,1}$ *for* $i = 1, 2, 3$ *and* $d_1, d_{1,2}, d_{1,3}$ *are in distinct* $\langle \sigma_x \rangle$*-orbits. If* $A = \langle \sigma_x, \sigma_y \rangle$*, then the orbital partial fraction decomposition of* f_1 *is*

$$f_1 = f_{1,1} + f_{1,2} \text{ with } f_{1,1} = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} \text{ and } f_{1,2} = \frac{y + z^2}{d_{1,3}},$$

where $f_{1,1} \in V_{[d_1]_A,1}$*,* $f_{1,2} \in V_{[d_{1,3}]_A,1}$ *and* $d_1, d_{1,3}$ *are in distinct* $\langle \sigma_x, \sigma_y \rangle$*-orbits. If* $A = \langle \sigma_x, \sigma_y, \sigma_z \rangle$*, then* $f_1 \in V_{[d_1]_A,1}$ *is one component in the orbital decomposition because*

$$f_1 = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} + \frac{y + z^2}{\sigma_x \sigma_y^3 \sigma_z^{-1}(d_1)}.$$

Example 4.10. *Let* $\mathbb{F} = \mathbb{Q}$ *and* $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$*. Consider the rational function* $f = f_1 + f_2 + f_3$ *in* $\mathbb{Q}(x, y, z)$ *with* f_1 *given in Example 4.9,*

$$f_2 = \frac{x + z}{\underbrace{(x - 3y)^2(y + z) + 1}_{d_2}} \text{ and } f_3 = \left(y + \frac{z}{y^2 + z - 1} - \frac{1}{y^2 + z} \right) \frac{1}{(\underbrace{x + 2y + z}_{d_3})^2}.$$

If $A = G$*, then the orbital partial fraction decomposition of* f *is*

$$f = f_1 + f_2 + f_3 \quad \text{with} \quad f_i \in V_{[d_i]_G,1} \text{ for } i = 1, 2 \text{ and } f_3 \in V_{[d_3]_G,2},$$

where d_1, d_2, d_3 *are in distinct* $\langle \sigma_x, \sigma_y, \sigma_z \rangle$*-orbits.*

5 The rational summability problem

In this section, we solve the rational summability problem for multivariate rational functions and design an algorithm for rational summability testing. In Section 5.1 we use a special orbital decomposition in Section 4.2 to reduce the summability problem of a general rational function to its orbital components and then further to simple fractions by Abramov's reduction. In Section 5.2, we use the structure of isotropy groups to reduce the number of variables in the summability problem inductively.

5.1 Orbital reduction for summability

Let f be a rational function in $\mathbb{F}(\mathbf{x})$, where $\mathbf{x} = \{x_1, \ldots, x_m\}$. Recall that $\hat{\mathbf{x}}_1 = \{x_2, \ldots, x_m\}$. Let n be an integer such that $1 \leq n \leq m$. We consider the $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summability problem of f in $\mathbb{F}(\mathbf{x})$. Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$. Taking $\mathbb{E} = \mathbb{F}(\hat{\mathbf{x}}_1)$ and $A = G$ in equality (4.2), we get the subspace $V_{[d]_G,j}$ of $\mathbb{F}(\mathbf{x})$

$$V_{[d]_G,j} = \operatorname{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \,\middle|\, a \in \mathbb{E}[x_1], \tau \in G, \deg_{x_1}(a) < \deg_{x_1}(d) \right\}.$$

where $j \in \mathbb{N}^+$ and $d \in \mathbb{E}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(d) > 0$. According to Equation (4.3), f can be decomposed into the form

$$f = f_0 + \sum_j \sum_{[d]_G} f_{[d]_G,j}, \tag{5.1}$$

where $f_0 \in V_0 = \mathbb{E}[x_1]$ and $f_{[d]_G,j}$ are in distinct $V_{[d]_G,j}$ spaces. Then we obtain the orbital decomposition (4.4) of $\mathbb{F}(\mathbf{x})$ with respect to the group $A = G$.

Lemma 5.1. *Let $f \in \mathbb{F}(\mathbf{x})$. Then f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable if and only if f_0 and each $f_{[d]_G,j}$ are $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable for all $[d]_G \in T_G$ and $j \in \mathbb{N}^+$.*

Proof. The sufficiency is due to the linearity of difference operators Δ_{x_i}. For the necessity, suppose $f = \sum_{i=1}^n \Delta_{x_i}(g^{(i)})$ with $g^{(i)} \in \mathbb{F}(\mathbf{x})$. By the orbital decomposition of rational functions (5.1), we can write $f, g^{(i)}$ in the form

$$f = f_0 + \sum_j \sum_{[d]_G} f_{[d]_G,j} \quad \text{and} \quad g^{(i)} = g_0^{(i)} + \sum_j \sum_{[d]_G} g_{[d]_G,j}^{(i)} \quad \text{for } 1 \le i \le n.$$

By the linearity of Δ_{x_i}, we see

$$f = \sum_{i=1}^n \Delta_{x_i}\left(g_0^{(i)}\right) + \sum_j \sum_{[d]_G} \left(\sum_{i=1}^n \Delta_{x_i}\left(g_{[d]_G,j}^{(i)}\right)\right).$$

By Lemma 4.8, it is another expression of f with respect to $V_{[d]_G,j}$. Such a decomposition is unique, so $f_0 = \sum_{i=1}^n \Delta_{x_i}(g_0^{(i)})$ and $f_{[d]_G,j} = \sum_{i=1}^n \Delta_{x_i}(g_{[d]_G,j}^{(i)})$, which are $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable. ∎

Using Lemma 5.1, we can reduce the summability problem of a rational function to its orbital components. Note that polynomials in x_1 are always (σ_{x_1})-summable. Thus Problem 2.4 can be reduced to that for rational functions in $V_{[d]_G,j}$, which are of the form

$$f = \sum_\tau \frac{a_\tau}{\tau(d)^j}, \tag{5.2}$$

where $\tau \in G$, $a_\tau \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(a_\tau) < \deg_{x_1}(d)$ and d is irreducible in x_1 over $\mathbb{F}(\hat{\mathbf{x}}_1)$.

Let σ be an automorphism on $\mathbb{F}(\mathbf{x})$ and $a, b \in \mathbb{F}(\mathbf{x})$. Then for any integer $k \in \mathbb{Z}$, we have the reduction formula

$$\frac{a}{\sigma^k(b)} = \sigma(h) - h + \frac{\sigma^{-k}(a)}{b}, \tag{5.3}$$

where $h = 0$ if $k = 0$, $h = \sum_{i=0}^{k-1} \frac{\sigma^{i-k}(a)}{\sigma^i(b)}$ if $k > 0$ and $h = -\sum_{i=0}^{-k-1} \frac{\sigma^i(a)}{\sigma^{i+k}(b)}$ if $k < 0$. For any $\tau = \sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n} \in G$, applying the reduction formula (5.3) with $\sigma = \sigma_{x_i}$ for $i = 1, \ldots, n$, we get

$$\frac{a}{\sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n}(b)} = \sum_{i=1}^n (\sigma_{x_i}(h_i) - h_i) + \frac{\sigma_{x_1}^{-k_1} \cdots \sigma_{x_n}^{-k_n}(a)}{b}, \tag{5.4}$$

where

$$h_i = \begin{cases} 0, & \text{if } k_i = 0, \\ \displaystyle\sum_{\ell=0}^{k_i-1} \frac{\sigma_{x_i}^{\ell-k_i} \sigma_{x_{i-1}}^{-k_{i-1}} \cdots \sigma_{x_1}^{-k_1}(a)}{\sigma_{x_i}^{\ell} \sigma_{x_{i+1}}^{k_{i+1}} \cdots \sigma_{x_n}^{k_n}(b)}, & \text{if } k_i > 0, \\ \displaystyle -\sum_{\ell=0}^{-k_i-1} \frac{\sigma_{x_i}^{\ell} \sigma_{x_{i-1}}^{-k_{i-1}} \cdots \sigma_{x_1}^{-k_1}(a)}{\sigma_{x_i}^{\ell+k_i} \sigma_{x_{i+1}}^{k_{i+1}} \cdots \sigma_{x_n}^{k_n}(b)}, & \text{if } k_i < 0. \end{cases}$$

for $i = 1, \ldots, n$. The equation (5.4) is called the $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-*reduction* formula. Rewriting every fraction of f in (5.2) by the reduction formula (5.4), we get the following lemma.

Lemma 5.2. *Let $f \in V_{[d]_G,j}$ be in the form (5.2). Then we can decompose it into the form*

$$f = \sum_{i=1}^{n} \Delta_{x_i}(g_i) + r \text{ with } r = \frac{a}{d^j}, \tag{5.5}$$

where $g_i \in \mathbb{F}(\mathbf{x})$, $a = \sum_\tau \tau^{-1}(a_\tau)$ with $\deg_{x_1}(a) < \deg_{x_1}(d)$. In particular, f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable if and only if r is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable.

Example 5.3. *Consider the rational function $f_1 \in \mathbb{Q}(x, y, z)$ given in Example 4.9. Then $f_1 \in V_{[d_1]_G,1}$ and it can be written as*

$$f_1 = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} + \frac{y + z^2}{\sigma_x\sigma_y^3\sigma_z^{-1}(d_1)},$$

where $d_1 = x^2 + 2xy + z^2$. By applying the $(\sigma_x, \sigma_y, \sigma_z)$-reduction formula, we have

$$f_1 = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1) + r_1 \text{ with } r_1 = \frac{2x - 1}{d_1},$$

where

$$u_1 = \frac{y + z^2}{\sigma_y^3\sigma_z^{-1}(d_1)},\ v_1 = \frac{x - y + 1 - 2z}{d_1} + \sum_{\ell=0}^{2} \frac{y + \ell - 3 + z^2}{\sigma_y^\ell\sigma_z^{-1}(d_1)},\ w_1 = -\frac{y - 3 + z^2}{\sigma_z^{-1}(d_1)}.$$

Then f_1 is $(\sigma_x, \sigma_y, \sigma_z)$-summable if and only if r_1 is $(\sigma_x, \sigma_y, \sigma_z)$-summable.

The results in Lemmas 5.1 and 5.2 are summarized as follows. The following lemma reduces the rational summability problem from general rational functions to simple fractions.

Corollary 5.4 (Lemma 1.5, restated)**.** *Let $f \in \mathbb{F}(\mathbf{x})$. Then we can decompose f into the form*

$$f = \sum_{i=1}^{n} \Delta_{\sigma_{x_1}}(g_i) + r \text{ with } r = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{5.6}$$

where $g_i \in \mathbb{F}(\mathbf{x})$, $a_{i,j} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $d_i \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$ and the d_i's are monic irreducible polynomials in distinct G-orbits. Furthermore, f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable if and only if each $a_{i,j}/d_i^j$ is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable for all i, j with $1 \le i \le I$ and $1 \le j \le J_i$.

5.2 Summability criteria

By Corollary 5.4, we reduce the rational summability problem to that for simple fractions

$$f = \frac{a}{d^j}, \tag{5.7}$$

where $j \in \mathbb{N}^+$, $a \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $d \in \mathbb{F}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(a) < \deg_{x_1}(d)$. In this section, we shall present a criterion on the summability for such simple fractions.

For the univariate summability problem, we recall the following well known result in [2, 4, 10, 56, 60, 65]. Since the univariate case is the base of our induction method, we give a proof for the sake of completeness.

Lemma 5.5. *Let $f \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). Then f is (σ_{x_1})-summable in $\mathbb{F}(\mathbf{x})$ if and only if $a = 0$.*

Proof. The sufficiency is trivial since $f = \Delta_{x_1}(0)$. To show the necessity, suppose f is (σ_{x_1})-summable but $a \neq 0$. Since $f = a/d^j \in V_{[d]_G,j}$, by the proof of Lemma 5.1 we can further assume $f = \Delta_{x_1}(g)$ for some $g \in V_{[d]_G,j}$. Write g in the form $g = \sum_{i=\ell_0}^{\ell_1} a_i/\sigma_{x_1}^i(d)^j$ with $a_{\ell_0}a_{\ell_1} \neq 0$. Then

$$f = \Delta_{x_1}(g) = \sum_{i=\ell_0}^{\ell_1+1} \frac{\tilde{a}_i}{\sigma_{x_1}^i(d)^j},$$

where $\tilde{a}_i = \sigma_{x_1}(a_{i-1}) - a_i$ for $\ell_0 + 1 \leq i \leq \ell_1$, $\tilde{a}_{\ell_0} = -a_{\ell_0}$ and $\tilde{a}_{\ell_1+1} = \sigma_{x_1}(a_{\ell_1})$. Note that $\tilde{a}_{\ell_0}$ and $\tilde{a}_{\ell_1+1}$ are nonzero. For any integer $i \in \mathbb{Z}$, $\sigma_{x_1}^i(d)$ is still an irreducible polynomial. However, there is only one irreducible factor in the denominator of $f = a/d^j$. So we must have $\sigma_{x_1}^i(d) = d$ for some nonzero integer i. It implies that d is free of x_1. This is a contradiction because d has positive degree in x_1. ∎

For the multivariate summability problem with $n > 1$, let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ and $H = \langle \sigma_{x_1}, \ldots, \sigma_{x_{n-1}} \rangle$. The isotropy group of the polynomial d in G and H are denoted by G_d and H_d, respectively, i.e.,

$$G_d = \{\tau \in G \mid \tau(d) = d\} \quad \text{and} \quad H_d = \{\tau \in H \mid \tau(d) = d\}.$$

By Lemma 4.4, we know either $\operatorname{rank}(G_d/H_d) = 0$ or $\operatorname{rank}(G_d/H_d) = 1$.

When $\operatorname{rank}(G_d/H_d) = 0$, the summability problem in n variables can be reduced to that in $n-1$ variables.

Lemma 5.6. *Let $f = a/d^j \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). If $n > 1$ and $\operatorname{rank}(G_d/H_d) = 0$, then f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable in $\mathbb{F}(\mathbf{x})$ if and only if f is $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable in $\mathbb{F}(\mathbf{x})$.*

Proof. The sufficiency is obvious by definition. For the necessity, suppose f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable but not $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable. By the orbital decomposition of f in (5.1) and Lemma 5.1, we get

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) \tag{5.8}$$

with $g_1, \ldots, g_n$ in the same subspace $V_{[d]_G,j}$ as f. As an analogue to (5.5) in $n-1$ variables $x_1, \ldots, x_{n-1}$, we can decompose g_n as

$$g_n = \sum_{i=1}^{n-1} \Delta_{x_i}(u_i) + \sum_{\ell=0}^{\rho} \frac{\lambda_\ell}{\sigma_{x_n}^\ell(\mu)^j}, \tag{5.9}$$

where $u_i \in \mathbb{F}(\mathbf{x})$, $\rho \in \mathbb{N}$, $\lambda_\ell \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $\mu \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(\lambda_\ell) < \deg_{x_1}(d)$ and μ is in the same G-orbit as d.

Furthermore, we can assume $\lambda_0\lambda_\rho \neq 0$ and each nonzero $\lambda_\ell/\sigma_{x_n}^\ell(\mu)^j$ is not $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable. Substituting g_n in (5.9) into (5.8), we get

$$f + \sum_{\ell=0}^{\rho+1} \frac{\tilde{\lambda}_\ell}{\sigma_{x_n}^\ell(\mu)^j} = \sum_{i=1}^{n-1} \Delta_{x_i}(h_i), \tag{5.10}$$

where $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_{\rho+1} = -\sigma_{x_n}(\lambda_\rho)$, $\tilde{\lambda}_\ell = \lambda_\ell - \sigma_{x_n}(\lambda_{\ell-1})$ for all $1 \leq \ell \leq \rho$ and $h_i = g_i + \Delta_{x_n}(u_i)$ for all $1 \leq i \leq n-1$.

Since $\operatorname{rank}(G_d/H_d) = 0$ and $G_d = G_\mu$, it follows that all $\sigma_{x_n}^{\ell}(\mu)$ with $\ell \in \mathbb{Z}$ are in distinct H-orbits. In particular, $[\mu]_H, [\sigma_{x_n}(\mu)]_H \ldots, [\sigma_{x_n}^{\rho+1}(\mu)]_H$ are distinct H-orbits. On the other hand, the left hand side of (5.10) is $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable, but $\tilde{\lambda}_0/\mu^j$ is not $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable according to the assumption. By Lemma 5.1 (in $n-1$ variables), the only choice is that $\mu \sim_H d$. Similarly, $\sigma_{x_n}^{\rho+1}(\mu) \sim_H d$ and hence $\mu \sim_H \sigma_{x_n}^{\rho+1}(\mu)$. This leads to a contradiction since ρ is a non-negative integer. ∎

Lemma 5.7. *Let $f \in \mathbb{F}(\mathbf{x})$ and K be a subgroup of $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ with rank r $(1 \leq r \leq n)$. If $\{\sigma_i\}_{i=1}^r$ and $\{\tau_i\}_{i=1}^r$ are two bases of K, then f is $(\sigma_1, \ldots, \sigma_r)$-summable if and only if f is $(\tau_1, \ldots, \tau_r)$-summable.*

To prove the basis change property of the summability problem in Lemma 5.7, we first show the following lemma. It can be seen as a variant of the reduction formula (5.4). Since it is useful in computation, we give a detailed proof by induction.

Lemma 5.8. *Let $\sigma_1, \ldots, \sigma_r$ be elements in G and $K = \langle \sigma_1, \ldots, \sigma_r \rangle$ be the subgroup of G generated by $\sigma_1, \ldots, \sigma_r$. Then for every $\tau \in K$,*

$$\tau - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_1 + \cdots + (\sigma_r - \mathbf{1})\tilde{\sigma}_r,$$

for some $\tilde{\sigma}_i \in \mathbb{F}[K]$.

Proof. We prove this lemma by induction on the number of σ_i. If r=1, then $\tau = \sigma_1^{k_1}$ for some $k_1 \in \mathbb{Z}$. We have $\sigma_1^{k_1} - \mathbf{1} = (\sigma_1 - \mathbf{1})\mu$, where $\mu = 0$ if $k_1 = 0$, $\mu = \sum_{i=0}^{k_1-1} \sigma_1^i$ if $k_1 > 0$ and $\mu = -\sum_{i=0}^{-k_1-1} \sigma_1^{i+k_1}$ if $k_1 < 0$. If $r \geq 2$, assume that the conclusion holds for $r-1$. Write $\tau = \sigma_1^{k_1} \cdots \sigma_r^{k_r}$ for some $k_1, \ldots, k_r \in \mathbb{Z}$. Then

$$\tau - \mathbf{1} = \left(\sigma_1^{k_1} - \mathbf{1}\right) \sigma_2^{k_2} \cdots \sigma_r^{k_r} + \left(\sigma_2^{k_2} \cdots \sigma_r^{k_r} - \mathbf{1}\right).$$

If $\sigma_2^{k_2} \cdots \sigma_r^{k_r} = \mathbf{1}$, then we are done. Otherwise, by the inductive hypothesis, we get $\tau - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_1 + \cdots + (\sigma_r - \mathbf{1})\tilde{\sigma}_r$ for some $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r \in \mathbb{F}[K]$. In fact, the above argument gives the following explicit expression

$$\tilde{\sigma}_i = \begin{cases} 0 & \text{if } k_i = 0, \\ \displaystyle\sum_{\ell=0}^{k_i-1} \sigma_i^{\ell} \sigma_{i+1}^{k_{i+1}} \cdots \sigma_r^{k_r} & \text{if } k_i > 0, \\ \displaystyle -\sum_{\ell=0}^{-k_i-1} \sigma_i^{\ell+k_i} \sigma_{i+1}^{k_{i+1}} \cdots \sigma_r^{k_r} & \text{if } k_i < 0, \end{cases}$$

for $i = 1, \ldots, r$. ∎

Proof of Lemma 5.7. Suppose f is $(\tau_1, \ldots, \tau_r)$-summable. This means

$$f = \Delta_{\tau_1}(h_1) + \cdots + \Delta_{\tau_r}(h_r), \tag{5.11}$$

for some $h_1, \ldots, h_r \in \mathbb{F}(\mathbf{x})$. For each $i = 1, \ldots, r$, since $\tau_i \in \langle \sigma_1, \ldots, \sigma_r \rangle$, it follows from Lemma 5.8 that $\tau_i - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_{i,1} + \cdots + (\sigma_r - \mathbf{1})\tilde{\sigma}_{i,r}$ for some $\tilde{\sigma}_{i,j} \in \mathbb{F}[K]$ with K being the subgroup generated by $\sigma_1, \ldots, \sigma_r$. Applying this operator to h_i yields that

$$\Delta_{\tau_i}(h_i) = \Delta_{\sigma_1}(h_{i,1}) + \cdots + \Delta_{\sigma_r}(h_{i,r}), \tag{5.12}$$

where $h_{i,j} = \tilde{\sigma}_{i,j}(h_i)$ for $j = 1, \ldots, r$. Combining Equations (5.11) and (5.12), we have

$$f = \sum_{i=1}^{r} \Delta_{\tau_i}(h_i) = \sum_{i=1}^{r}\sum_{j=1}^{r} \Delta_{\sigma_j}(h_{i,j}) = \sum_{j=1}^{r} \Delta_{\sigma_j}\left(\sum_{i=1}^{r} h_{i,j}\right),$$

where the last equality follows from the linearity of Δ_{σ_j}. Thus f is $(\sigma_1, \ldots, \sigma_r)$-summable. Similarly, the other direction is also true. ∎

Theorem 5.9 (Theorem 1.6, restated). *Let $f = a/d^j \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). Let $\{\tau_i\}_{i=1}^{r}(1 \leq r < n)$ be a basis of G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Then f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable if and only if*

$$a = \Delta_{\tau_1}(b_1) + \cdots + \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for all $1 \leq i \leq r$.

Proof. The sufficiency follows from the fact that $f = \sum_{i=1}^{r} \Delta_{\tau_i}(b_i/d^j)$ and Lemma 5.8. For the necessity, we proceed by induction on n. If $n = 1$, then G_d is a trivial group and the univariate case follows from Lemma 5.5. If $n > 1$, suppose the inductive hypothesis is true for $n-1$ as follows.

If $\{\theta_i\}_{i=1}^{s}$ is a basis of H_d, then f is $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable if and only if $a = \sum_{i=1}^{s} \Delta_{\theta_i}(b_i)$ for some $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for all $1 \leq i \leq s$.

Now we proceed by a case distinction according to the rank of G_d/H_d which is either 0 or 1 by Lemma 4.4. If $\operatorname{rank}(G_d/H_d) = 0$, then $H_d = G_d$. The conclusion follows from Lemma 5.6 and the inductive hypothesis. If $\operatorname{rank}(G_d/H_d) = 1$, by Lemma 5.7, we may assume that $\{\tau_i\}_{i=1}^{r}$ is a basis of G_d such that $H_d = \langle \tau_1, \ldots, \tau_{r-1}\rangle$ and $G_d/H_d = \langle \bar{\tau}_r \rangle$. Here $\bar{\tau}_r$ represents the element $\tau_r H_d$ with $\tau_r \in G_d$. Then we can choose $\tau_r = \sigma_{x_1}^{-k_1} \cdots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}^{k_n}$ such that k_n is a positive integer. Otherwise, replace τ_r by τ_r^{-1}. Since $\bar{\tau}_r$ is a generator of G_d/H_d, we have that k_n is the smallest positive integer such that $\sigma_{x_n}^{k_n}(d) \sim_H d$.

By the decomposition (4.4), we can assume $f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n)$ with $g_i \in V_{[d]_G, j}$. In here, using Lemma 5.2, g_n can be decomposed as

$$g_n = \sum_{i=1}^{n-1} \Delta_{x_i}(u_i) + \sum_{\ell=0}^{k_n-1} \frac{\lambda_\ell}{\sigma_{x_n}^{\ell}(d)^j},$$

where $u_i \in \mathbb{F}(\mathbf{x})$ and $\lambda_\ell \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(\lambda_\ell) < \deg_{x_1}(d)$. Then we have

$$f - \Delta_{x_n}\left(\sum_{\ell=0}^{k_n-1} \frac{\lambda_\ell}{\sigma_{x_n}^{\ell}(d)^j}\right) = \sum_{i=1}^{n-1} \Delta_{x_i}(h_i), \tag{5.13}$$

where $h_i = g_i + \Delta_{x_n}(u_i)$. Note that $\sigma_{x_n}^{k_n}(d) = \sigma_{x_1}^{k_1} \cdots \sigma_{x_{n-1}}^{k_{n-1}}(d)$ and apply the reduction formula (5.4) to simplify (5.13). We get

$$\tilde{f} := \sum_{\ell=0}^{k_n-1} \frac{\tilde{\lambda}_\ell}{\sigma_{x_n}^{\ell}(d)^j} = \sum_{i=1}^{n-1} \Delta_{x_i}(\tilde{h}_i), \tag{5.14}$$

where $\tilde{h}_i \in \mathbb{F}(\mathbf{x})$, $\tilde{\lambda}_0 = a + \lambda_0 - \sigma_{x_1}^{-k_1} \cdots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}(\lambda_{k_n-1})$ and $\tilde{\lambda}_\ell = \lambda_\ell - \sigma_{x_n}(\lambda_{\ell-1})$ for $1 \leq \ell \leq k_n - 1$.

Note that $[d]_H, [\sigma_{x_n}(d)]_H, \ldots, [\sigma_{x_n}^{k_n-1}(d)]_H$ are distinct H-orbits due to the minimality of k_n. From the equation (5.14), $\tilde{f}$ is $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable. So by Lemma 5.1, each $\frac{\tilde{\lambda}_\ell}{\sigma_{x_n}^{\ell}(d)^j}$ is $(\sigma_{x_1}, \ldots, \sigma_{x_{n-1}})$-summable for $0 \leq \ell \leq k_n - 1$. Let W denote the vector subspace of $\mathbb{F}(\mathbf{x})$ over $\mathbb{F}$ consisting of all elements in the form of $\sum_{i=1}^{r-1} \Delta_{\tau_i}(b_i)$ with $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $\deg_{x_1}(b_i) < \deg_{x_1}(d)$.

(If $r = 1$, take $W = \{0\}$.) If two rational functions $g, h \in \mathbb{F}(\mathbf{x})$ satisfy the property that $g - h \in W$, we say g, h are congruent modulo W, denoted by $g \equiv h \pmod W$. Since $H_d = H_{\sigma_{x_n}^{\ell}(d)}$, we apply the inductive hypothesis to conclude that

$$\begin{cases} 0 \equiv a + \lambda_0 - \sigma_{x_1}^{-k_1} \cdots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}(\lambda_{k_n - 1}) & \pmod W \\ 0 \equiv \lambda_1 - \sigma_{x_n}(\lambda_0) & \pmod W \\ \qquad \vdots & \\ 0 \equiv \lambda_{k_n-1} - \sigma_{x_n}(\lambda_{k_n-2}) & \pmod W. \end{cases}$$

Since W is G-invariant, it follows from the equations that

$$a \equiv \sigma_{x_1}^{-k_1} \cdots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}^{k_n}(\lambda_0) - \lambda_0 \equiv \Delta_{\tau_r}(\lambda_0) \pmod W.$$

This completes the proof. ∎

Remark 5.10. *For the bivariate case with $n = 2$, Theorem 5.9 coincides with the known criterion in [44, Theorem 3.3] and [28, Theorem 3.7]. In this case,* $\operatorname{rank}(G_d) \le 1$ *and* $H_d = \{\mathbf{1}\}$. *If* $\operatorname{rank}(G_d) = 0$, *then a/d^j is $(\sigma_{x_1}, \sigma_{x_2})$-summable in $\mathbb{F}(\mathbf{x})$ if and only if $a = 0$. If* $\operatorname{rank}(G_d) = 1$ *and G_d is generated by $\tau = \sigma_{x_1}^{\ell_1}\sigma_{x_2}^{-\ell_2} \in G$ for some $\ell_2 \neq 0$, then a/d^j is $(\sigma_{x_1}, \sigma_{x_2})$-summable if and only if $a = \sigma_{x_1}^{\ell_1}\sigma_{x_2}^{-\ell_2}(b) - b$ for some $b \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b) < \deg_{x_1}(d)$.*

Example 5.11. *Let $f = 1/(x_1^s + \cdots + x_n^s) \in \mathbb{Q}(x_1, \ldots, x_n)$ with $s, n \in \mathbb{N} \setminus \{0\}$. Let G_d be the isotropy group of $d = x_1^s + \cdots + x_n^s$ in $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$. Then we can decide for all cases the $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summability of f in $\mathbb{Q}(x_1, \ldots, x_n)$.*

(1) If $s = 1$ and $n > 1$, then d is irreducible. The rank of G_d is $n - 1$ and one basis is given by $\tau_1, \ldots, \tau_{n-1}$ with $\tau_i = \sigma_{x_i}\sigma_{x_{i+1}}^{-1}$ for $i = 1, \ldots, n-1$. Since $1 = \tau_1(x_1) - x_1$, it follows that f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable. In fact, we have

$$\frac{1}{x_1 + \cdots + x_n} = \Delta_{x_1}\left(\frac{x_1}{x_1 + \cdots + x_n}\right) + \Delta_{x_2}\left(\frac{-x_1 - 1}{x_1 + \cdots + x_n}\right).$$

This means f is $(\sigma_{x_1}, \sigma_{x_2})$-summable, so is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable.

(2) If $s \ge 1$ and $n = 1$, then $f = 1/x_1^s$. Since the isotropy group of x_1 in $\langle \sigma_{x_1} \rangle$ is $\{\mathbf{1}\}$, by Theorem 5.9, we get that f is not (σ_{x_1})-summable.

(3) If $s > 1$ and $n = 2$, then $f = 1/(x_1^s + x_2^s) = \sum_{j=1}^{s} a_j/(x_1 - \beta_j x_2)$, where β_j's are distinct roots of $z^s = -1$ and $a_j = 1/s(\beta_j x_2)^{s-1}$. There exists $j \in \{1, \ldots, s\}$ such that $\beta_j \notin \mathbb{Z}$. Then for $d_j = x_1 - \beta_j x_2$, we have $G_{d_j} = \{\mathbf{1}\}$. So a_j/d_j is not $(\sigma_{x_1}, \sigma_{x_2})$-summable in $\mathbb{C}(x_1, x_2)$ and by Lemma 5.1, neither is f. Hence f is not $(\sigma_{x_1}, \sigma_{x_2})$-summable in $\mathbb{Q}(x_1, x_2)$. This result has appeared in [28, Example 3.8].

(4) If $s > 1$ and $n > 2$, then d is irreducible. Since $G_d = \{\mathbf{1}\}$, by Theorem 5.9, it follows that f is not $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable.

Now we transfer the $(\tau_1, \ldots, \tau_r)$-summability problem to the $(\sigma_{x_1}, \ldots, \sigma_{x_r})$-summability problem.

Proposition 5.12. *Let $\{\tau_i\}_{i=1}^{r} (1 \le r \le n)$ be a family of linearly independent elements in $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$. Then there exists an $\mathbb{F}$-automorphism ϕ of $\mathbb{F}(\mathbf{x})$ such that ϕ is a difference isomorphism between the difference fields $(\mathbb{F}(\mathbf{x}), \tau_i)$ and $(\mathbb{F}(\mathbf{x}), \sigma_{x_i})$ for all $i = 1, \ldots, r$. Therefore, for any $f \in \mathbb{F}(\mathbf{x})$, f is $(\tau_1, \ldots, \tau_r)$-summable in $\mathbb{F}(\mathbf{x})$ if and only if $\phi(f)$ is $(\sigma_{x_1}, \ldots, \sigma_{x_r})$-summable in $\mathbb{F}(\mathbf{x})$.*

Proof. Assume $\tau_i = \sigma_{x_1}^{a_{i,1}} \cdots \sigma_{x_m}^{a_{i,m}}$ with $a_{i,j} = 0$ if $j > n$ and write $\alpha_i = (a_{i,1}, \ldots, a_{i,m}) \in \mathbb{Z}^m$ viewed as a vector in $\mathbb{Q}^m$ for $i = 1, \ldots, r$. Then $\alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$. So we can find the other vectors $\alpha_{r+1}, \ldots, \alpha_m$ such that $\{\alpha_1, \ldots, \alpha_m\}$ forms a basis of $\mathbb{Q}^m$. Let $\alpha_i = (a_{i,1}, \ldots, a_{i,m})$ for $i = r+1, \ldots, m$ and $A = (a_{i,j}) \in \mathbb{Q}^{m\times m}$. Then A is an invertible matrix. Thus we define an $\mathbb{F}$-automorphism $\phi : \mathbb{F}(\mathbf{x}) \to \mathbb{F}(\mathbf{x})$ by

$$(\phi(x_1), \ldots, \phi(x_m)) := (x_1, \ldots, x_m)A.$$

Let $u_j := \phi(x_j) = \sum_{i=1}^m a_{i,j}x_i$ for all $1 \le j \le m$. Then ϕ satisfies the relation $\phi \circ \tau_i = \sigma_{x_i} \circ \phi$ for all $i = 1, \ldots, r$, which means the following diagrams

$$\begin{array}{ccc} \mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \\ {\scriptstyle\tau_1}\downarrow & & \downarrow{\scriptstyle\sigma_1} \\ \mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \end{array} \qquad \cdots \qquad \begin{array}{ccc} \mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \\ {\scriptstyle\tau_r}\downarrow & & \downarrow{\scriptstyle\sigma_r} \\ \mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \end{array}$$

are commutative. This is true since for any $f \in \mathbb{F}(x_1, \ldots, x_m)$, we have

$$\begin{aligned} \phi\left(\tau_i(f(x_1, \ldots, x_m)\right) &= \phi\left(f(x_1 + a_{i,1}, \ldots, x_m + a_{i,m})\right) \\ &= f(u_1 + a_{i,1}, \ldots, u_m + a_{i,m}) \end{aligned}$$

and

$$\begin{aligned} \sigma_{x_i}\left(\phi(f(x_1, \ldots, x_m)\right) &= \sigma_{x_i}\left(f\left(u_1, \ldots, u_m\right)\right) \\ &= f\left(u_1 + a_{i,1}, \ldots, u_m + a_{i,m}\right). \end{aligned}$$

It follows that

$$f = \sum_{i=1}^r \Delta_{\tau_i}(g_i) \quad \Longleftrightarrow \quad \phi(f) = \sum_{i=1}^r \Delta_{x_i}(\phi(g_i))$$

whenever $f, g_1, \ldots, g_r \in \mathbb{F}(\mathbf{x})$. This proves our assertion. ∎

Combining Theorem 5.9 and Proposition 5.12, the summability problem 2.4 in n variables can be reduced to that in fewer variables. So we can design the following recursive algorithm for testing $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summability of multivariate rational functions. Furthermore, the $(\tau_1, \ldots, \tau_n)$-summability problem can also be solved via the transformation in Proposition 5.12.

Algorithm 5.13 (Constructive Testing of the Rational Summability)**.**
IsSummable(f**,** $[x_1, \ldots, x_n]$**).**
INPUT: a multivariate rational function $f \in \mathbb{F}(\mathbf{x})$ and a list $[x_1, \ldots, x_n]$ of variable names;
OUTPUT: unnormalised certificates $g_1, \ldots, g_n$ for f if f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable in $\mathbb{F}(\mathbf{x})$; false *otherwise.*

1 *using shift equivalence testing and partial fraction decomposition, decompose f into $f = f_0 + \sum_{j\in\mathbb{N}^+} \sum_{[d]_G} f_{[d]_G,j}$ as in Equation (5.1).*

2 *apply the reduction to f_0 and each nonzero component $f_{[d]_G,j}$ such that*

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + r \text{ with } r = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j},$$

where $a_{i,j}/d_i^j$ is the remainder of $f_{[d_i]_G,j}$ described in Lemma 5.2.

if $r=0$, *then* **return** $g_1, \ldots, g_n$.
for $i=1,\ldots,I$ **do**
  by Remark 4.2, one can compute a basis $\tau_{i,1},\ldots,\tau_{i,r_i}$ *for the isotropy group* G_{d_i} *of* d_i.
  for $j=1,\ldots,J_i$ **do**
    if $n=1$ *or* $G_{d_i}=\{\mathbf{1}\}$ *then*
      return false *if* $a_{i,j}\neq 0$.
    else
      find an $\mathbb{F}$*-automorphism* ϕ_i *of* $\mathbb{F}(\mathbf{x})$ *given in Proposition 5.12 such that* $\phi_i\circ\tau_{i,\ell}=\sigma_{x_\ell}\circ\phi_i$ *for* $\ell=1,\ldots,r_i$.
      set $\tilde{a}_{i,j}=\phi_i(a_{i,j})$.
      execute **IsSummable**$(\tilde{a}_{i,j},\ [x_1,\ldots,x_{r_i}])$.
      if $\tilde{a}_{i,j}$ *is* $(\sigma_{x_1},\ldots,\sigma_{x_{r_i}})$*-summable in* $\mathbb{F}(\mathbf{x})$, *let*

$$\tilde{a}_{i,j}=\Delta_{x_1}\left(\tilde{b}_{i,j}^{(1)}\right)+\cdots+\Delta_{x_{r_i}}\left(\tilde{b}_{i,j}^{(r_i)}\right);$$

      return false *otherwise.*
      applying ϕ_i^{-1} *to the previous equation yields that*

$$a_{i,j}=\Delta_{\tau_{i,1}}\left(b_{i,j}^{(1)}\right)+\cdots+\Delta_{\tau_{i,r_i}}\left(b_{i,j}^{(r_i)}\right),$$

      where $(b_{i,j}^{(1)},\ldots,b_{i,j}^{(r_i)})=(\phi_i^{-1}(\tilde{b}_{i,j}^{(1)}),\ldots,\phi_i^{-1}(\tilde{b}_{i,j}^{(r_i)}))$.
      using Lemma 5.8, compute $h_{i,j}^{(1)},\ldots,h_{i,j}^{(n)}\in\mathbb{F}(\mathbf{x})$ *such that*

$$\frac{a_{i,j}}{d_i^j}=\sum_{\ell=1}^{r_i}\Delta_{\tau_{i,\ell}}\left(\frac{b_{i,j}^{(\ell)}}{d_i^j}\right)=\sum_{\ell=1}^{n}\Delta_{x_\ell}\left(h_{i,j}^{(\ell)}\right)$$

      update $g_\ell=g_\ell+h_{i,j}^{(\ell)}$ *for* $\ell=1,\ldots,n$.
return $g_1,\ldots,g_n$.

Recall that a map $\phi:\mathbb{F}(\mathbf{x})\to\mathbb{F}(\mathbf{x})$ is called a $\mathbb{Q}$-*affine* map if $\phi(f(\mathbf{x}))=f(\mathbf{x}\cdot A+\mathbf{b})$, where A is an invertible matrix in $\mathrm{GL}_m(\mathbb{Q})$ and $\mathbf{b}$ is a vector in $\mathbb{Q}^m$. Note that the identity map, all shift operations and all difference isomorphisms constructed in Proposition 5.12 are $\mathbb{Q}$-affine maps. The composition of two $\mathbb{Q}$-affine maps is still a $\mathbb{Q}$-affine map.

If f is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable in $\mathbb{F}(\mathbf{x})$, Algorithm 5.13 will output unnormalised certificates for f in the form

$$g=\sum_{\ell=1}^{\rho}\prod_{k=1}^{K_\ell}\psi_{\ell,k}(u_{\ell,k}),$$

where $u_{\ell,k}\in\mathbb{F}(\mathbf{x})$ and the $\psi_{\ell,k}$'s are $\mathbb{Q}$-affine maps. For each $1\le\ell\le\rho$, the product $\prod_{k=1}^{K_\ell}u_{\ell,k}$ is called a *kernel* of g.

For convenience, we analyse the complexity of Algorithm 5.13 for $\mathbb{F}=\mathbb{Q}$. The following theorem shows that the rational summability problem can be solved in polynomial time.

Theorem 5.14. *Let* δ *be an integer in* $\mathbb{N}$ *and* $f(\mathbf{x})$ *be a multivariate rational function in* $\mathbb{Q}(\mathbf{x})_\delta$. *Then*

(1) If f is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable in $\mathbb{Q}(\mathbf{x})$, then Algorithm 5.13 will output unnormalised certificates for f, whose kernels are in $\mathbb{Q}(\mathbf{x})_{O(D(\delta,n))}$ with

$$D(\delta,n)=\left(\prod_{i=1}^{n} i^{2^{i-1}}\right)\delta^{2^n}; \tag{5.15}$$

(2) the total runtime of Algorithm 5.13 is $\tilde{O}(C(m,\delta,n))$ ops in $\mathbb{Q}$, where

$$C(m,\delta,n)=\left(\prod_{i=1}^{n} i^{7\cdot 2^{i-2}m-1}\right)\delta^{7\cdot 2^{n-1}m-1}.$$

Proof. (1) Note that f_0 and all the fractions obtained by partial fraction decomposition at Step 1 are in $\mathbb{Q}(\mathbf{x})_{(O(\delta),O(\delta^2),\ldots,O(\delta^2))}$, and so are the g_i and $a_{i,j}$'s obtained at Step 2. Furthermore, for each iteration of the loops in Steps 4 and 6, we get $\tilde{a}_{i,j}$ at Step 11, whose degree in x_i is no more than $O(n\delta^2)$ if $i=1,\ldots,n$ and no more than $O(\delta^2)$ if $i=n+1,\ldots,m$.

We prove the result by induction on n. If $n=1$, the degree bound is correct because the algorithm outputs the g_i's obtained at Step 2. If $n>1$, assume that the degree bound is true for $n-1$. Since the degree in x_k of the product-term $1/d_i^j$ at Step 15 is no more than δ, to estimate the degree bound of the kernels of the output g_i, it is sufficient to estimate that of $\tilde{b}_{i,j}^{(k)}$ at Step 13, which is $O(D(n\delta^2,n-1))=O((\prod_{i=1}^{n-1} i^{2^{i-1}})(n\delta^2)^{2^{n-1}})=O(D(\delta,n))$.

(2) We first estimate the total cost of the first two steps. Step 1 includes three basic operations: factorization of the denominator of f, shift equivalence testing for each pair of irreducible factors $(d_i,d_{i'})$ and partial fraction decomposition. By Fact 2.8 and Theorem 3.8, the first two operations cost $\tilde{O}(\delta^{m-1}\delta^{6m}+\delta^{\omega m}(m\delta)^2)$ ops. Since the partial fraction decomposition is performed in $\mathbb{Q}(\hat{\mathbf{x}}_1)(x_1)$, outputting f_0 and at most δ fractions in $\mathbb{Q}(\mathbf{x})_{(O(\delta),O(\delta^2),\ldots,O(\delta^2))}$, it takes $\tilde{O}(\delta(\delta((m-1)\delta^{2(m-1)}))+\delta\cdot\delta^{2(m-1)}\cdot\delta^2)$ ops in $\mathbb{Q}$ by multipoint evaluation and interpolation. At Step 2, each $a_{i,j}$ is obtained by shift and expansion and the number of $a_{i,j}$'s is no more than δ, so the cost of Step 2 is $\tilde{O}(\delta(m\delta^{2(m-1)+1}))$. So the total cost of Step 1 and Step 2 is $\tilde{O}(\delta^{7m-1})$ ops.

The remaining part of the proof is completed by induction on n. For $n=1$, the most expensive steps are the first two steps, so the total cost of the algorithm is $\tilde{O}(\delta^{7m-1})$ ops. If $n>1$, suppose the inductive hypothesis is true for $n-1$. For each iteration of the loops in Steps 4 and 6, recall that the degree of $\tilde{a}_{i,j}$ at Step 11 in x_i is in $O(n\delta^2)$ for $i=1,\ldots,n$ and in $O(\delta^2)$ for $i=n+1,\ldots,m$. Thus Step 11 costs $\tilde{O}(m(n\delta^2)^n\delta^{2(m-n)})$ ops and Step 12 takes $\tilde{O}(C(m,n\delta^2,n-1))$ ops. Note that the number of the iteration is $\sum_{i=1}^{I} J_i\le\delta$, so the total cost of the algorithm is

$$\begin{aligned}
&\tilde{O}\left(\delta^{7m-1}+\delta\left(mn^n\delta^{2m}+C(m,n\delta^2,n-1)\right)\right)\\
=&\tilde{O}\left(\delta^{7m-1}+\delta\left(mn^n\delta^{2m}+\left(\prod_{i=1}^{n-1} i^{7\cdot 2^{i-2}m-1}\right)n^{7\cdot 2^{n-2}m-1}\delta^{7\cdot 2^{n-1}m-2}\right)\right)\\
=&\tilde{O}\left(\delta^{7m-1}+mn^n\delta^{2m+1}+\left(\prod_{i=1}^{n} i^{7\cdot 2^{i-2}m-1}\right)\delta^{7\cdot 2^{n-1}m-1}\right)\\
=&\tilde{O}(C(m,\delta,n))
\end{aligned}$$

ops. This completes the proof. ∎

Example 5.15. *Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $f = f_1 + f_2 + f_3 \in \mathbb{Q}(x, y, z)$ be the same as in Example 4.10.*

(1) *After the $(\sigma_x, \sigma_y, \sigma_z)$-reduction for f_1, see Example 5.3, we get*

$$f_1 = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1) + r_1 \text{ with } r_1 = \frac{2x-1}{d_1}, \tag{5.16}$$

where $u_1, v_1, w_1 \in \mathbb{Q}(x, y, z)$ and $d_1 = x^2 + 2xy + z^2$. By Example 4.6 (1), the isotropy group $G_{d_1} = \{\mathbf{1}\}$ is trivial. By Theorem 5.9, we see r_1 is not $(\sigma_x, \sigma_y, \sigma_z)$-summable because its numerator $a_1 = 2x - 1$ is not zero. Hence f_1 is not $(\sigma_x, \sigma_y, \sigma_z)$-summable.

(2) *For $f_2 = a_2/d_2$ with $a_2 = x + z$ and $d_2 = (x - 3y)^2(y + z) + 1$, we know from Example 4.6 (2) that a basis of G_{d_2} is $\{\sigma_x^3 \sigma_y \sigma_z^{-1}\}$. For any $\{\mu, \nu\} \subseteq \{x, y, z\}$, since the isotropy group of d_2 in $\langle \sigma_\mu, \sigma_\nu \rangle$ is trivial, we get that f_2 is not (σ_μ, σ_ν)-summable in $\mathbb{Q}(x, y, z)$. By Theorem 5.9, we see that f_2 is $(\sigma_x, \sigma_y, \sigma_z)$-summable in $\mathbb{Q}(x, y, z)$ if and only if a_2 is (τ)-summable in $\mathbb{Q}(x, y, z)$ with $\tau = \sigma_x^3 \sigma_y \sigma_z^{-1}$. Choose one $\mathbb{Q}$-automorphism ϕ_2 of $\mathbb{Q}(x, y, z)$ given in Proposition 5.12 as follows*

$$\phi_2(h(x, y, z)) = h(3x, x + y, -x + z),$$

for any $h \in \mathbb{Q}(x, y, z)$. Then $\phi_2 \circ \tau = \sigma_x \circ \phi_2$. Hence a_2 is (τ)-summable in $\mathbb{Q}(x, y, z)$ if and only if $\phi_2(a_2)$ is (σ_x)-summable in $\mathbb{Q}(x, y, z)$. Since

$$\phi_2(a_2) = 2x + z = \Delta_x((x - 1)(x + z)) \tag{5.17}$$

is (σ_x)-summable, it follows that f_2 is $(\sigma_x, \sigma_y, \sigma_z)$-summable. In fact, applying ϕ_2^{-1} to Equation (5.17) yields that

$$a_2 = x + z = \Delta_\tau(b) \text{ with } b = \frac{1}{9}(x - 3)(2x + 3z).$$

By Lemma 5.8, we have

$$f_2 = \Delta_\tau\left(\frac{b}{d_2}\right) = \Delta_x(u_2) + \Delta_y(v_2) + \Delta_z(w_2), \tag{5.18}$$

where $u_2 = \sum_{\ell=0}^{2} \sigma_x^\ell \sigma_y \sigma_z^{-1}\left(\frac{b}{d_2}\right)$, $v_2 = \sigma_z^{-1}\left(\frac{b}{d_2}\right)$ and $w_2 = -\sigma_z^{-1}\left(\frac{b}{d_2}\right)$.

(3) *For $f_3 = a_3/d_3^2$ with $a_3 = y + z/(y^2 + z - 1) - 1/(y^2 + z)$ and $d_3 = x + 2y + z$, we know from Example 4.6 (2) that a basis of G_{d_3} is $\{\tau_1, \tau_2\}$, where $\tau_1 = \sigma_x^2 \sigma_y^{-1}$, $\tau_2 = \sigma_x \sigma_z^{-1}$. To decide the $(\sigma_x, \sigma_y, \sigma_z)$-summability of f_3, we construct a $\mathbb{Q}$-automorphism ϕ_3 of $\mathbb{Q}(x, y, z)$ such that $\phi_3 \circ \tau_1 = \sigma_x \circ \phi_3$ and $\phi_3 \circ \tau_2 = \sigma_y \circ \phi_3$ as follows*

$$\phi_3(h(x, y, z)) = h(2x + y, -x, -y + z),$$

for any $h \in \mathbb{Q}(x, y, z)$. Then it remains to decide the (σ_x, σ_y)-summability of

$$\phi_3(a_3) = -x + \underbrace{\frac{z - y}{x^2 - y + z - 1}}_{\sigma_y(\tilde{d})} - \underbrace{\frac{1}{x^2 - y + z}}_{\tilde{d}}$$

in $\mathbb{Q}(x, y, z)$. So we use the (σ_x, σ_y)-reduction to reduce $\phi_3(a_3)$ and obtain

$$\phi_3(a_3) = \Delta_x\left(\tilde{b}_1\right) + \Delta_y\left(\tilde{b}_2\right) + \frac{z - y}{x^2 - y + z}, \tag{5.19}$$

where $\tilde{b}_1 = -\frac{1}{2}x(x-1)$ *and* $\tilde{b}_2 = \frac{z-y+1}{x^2-y+z}$. *Since the isotropy group of* $\tilde{d}$ *in* $\langle \sigma_x, \sigma_y \rangle$ *is trivial,* $\phi_3(a_3)$ *is not* (σ_x, σ_y)*-summable. Hence* f_3 *is not* $(\sigma_x, \sigma_y, \sigma_z)$*-summable. Even so, in this case, using the above calculation, we can further decompose* f_3 *into a summable part and a remainder. Let us see how to do this. Starting from the decomposition* (5.19) *of* $\phi_3(a_3)$ *with respect to the* (σ_x, σ_y)*-summability problem, we apply* ϕ_3^{-1} *to both sides of this decomposition to obtain that*

$$a_3 = \Delta_{\tau_1}(b_1) + \Delta_{\tau_2}(b_2) + \frac{z}{y^2+z}.$$

where $b_1 = \phi_3^{-1}(\tilde{b}_2) = -\frac{1}{2}y(y+1)$ *and* $b_2 = \phi_3^{-1}(\tilde{b}_2) = \frac{z+1}{y^2+z}$. *By Lemma* 5.8 *with* $\tau = \tau_1, \tau_2$, *we have*

$$\begin{aligned} f_3 = \frac{a_3}{d_3^2} &= \Delta_{\tau_1}\left(\frac{b_1}{d_3^2}\right) + \Delta_{\tau_2}\left(\frac{b_2}{d_3^2}\right) + \underbrace{\frac{z}{(y^2+z)d_3^2}}_{r_3} \\ &= \Delta_x(u_3) + \Delta_y(v_3) + \Delta_z(w_3) + r_3, \end{aligned} \tag{5.20}$$

where $u_3 = \sum_{\ell=0}^{1} \sigma_x^{\ell}\sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right) + \sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$, $v_3 = -\sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right)$ *and* $w_3 = -\sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$.

(4) *For* $f = f_1 + f_2 + f_3$, *from Example* 4.10 *we know that* f_1, f_2, f_3 *are in distinct* $V_{[d]_G, j}$ *spaces. Since* f_1 *is not* $(\sigma_x, \sigma_y, \sigma_z)$*-summable, it follows from Lemma* 5.1 *that* f *is not* $(\sigma_x, \sigma_y, \sigma_z)$*-summable. Moreover, combining Equations* (5.16), (5.18) *and* (5.20), *we decompose* f *into*

$$f = \Delta_x(u) + \Delta_y(v) + \Delta_z(w) + r \text{ with } r = \frac{2x-1}{d_1} + \frac{z}{(y^2+z)d_3^2},$$

where $u = \sum_{i=1}^{2} u_i$, $v = \sum_{i=1}^{2} v_i$ *and* $w = \sum_{i=1}^{2} w_i$ *are rational functions in* $\mathbb{Q}(x, y, z)$.

As we discussed in the above example, given a rational function $f \in \mathbb{F}(\mathbf{x})$, we can compute rational functions $g_1, \ldots, g_n, r \in \mathbb{F}(\mathbf{x})$ such that

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + r \tag{5.21}$$

satisfying the property that f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable if and only if $r = 0$. This process can be achieved by induction on n. However, this remainder r is not unique, which depends on the choice of the difference isomorphisms ϕ_i. So how to choose a minimal remainder r is still an open problem. Moreover, different choices of the isomorphism might also lead to more efficient reductions.

6 The existence problem of telescopers

Similar to the summability problem, there are mainly two steps of solving the existence problem 2.2 of telescopers. First we use the orbital decomposition and Abramov's reduction to simplify the existence problem in Section 6.1. Then in Section 6.2, we use the exponent separation introduced in [24] to further reduce the existence problem to simple fractions and use the summability criteria in Section 5.2 to derive the existence criteria.

6.1 Orbital reduction for existence of telescopers

Let f be a rational function in $\mathbb{K}(t, \mathbf{x})$, where $\mathbf{x} = \{x_1, \ldots, x_m\}$. Let n be an integer such that $1 \leq n \leq m$. We consider the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for the rational function f in $\mathbb{K}(t,\mathbf{x})$. Let $G_t = \langle \sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ be the free abelian group generated by the shift operators $\sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n}$. Taking $\mathbb{E} = \mathbb{K}(t, \hat{\mathbf{x}}_1)$ and $A = G_t$ in Equality (4.2), we get

$$V_{[d]_{G_t},j} = \operatorname{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \,\middle|\, a \in \mathbb{E}[x_1], \tau \in G_t, \deg_{x_1}(a) < \deg_{x_1}(d) \right\},$$

where $j \in \mathbb{N}^+$ and $d \in \mathbb{E}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(d) > 0$. Then f can be decomposed as

$$f = f_0 + \sum_j \sum_{[d]_{G_t}} f_{[d]_{G_t},j}, \tag{6.1}$$

where $f_0 \in V_0 = \mathbb{E}[x_1]$ and $f_{[d]_{G_t},j}$ are in distinct $V_{[d]_{G_t},j}$ spaces. It induces the following orbital decomposition of $\mathbb{K}(t,\mathbf{x})$ with respect to the group G_t

$$\mathbb{K}(t,\mathbf{x}) = V_0 \bigoplus \left(\bigoplus_{j\in\mathbb{N}^+} \bigoplus_{[d]_{G_t}\in T_{G_t}} V_{[d]_{G_t},j} \right)$$

as a vector space over $\mathbb{K}(t, \hat{\mathbf{x}}_1)$. This orbital decomposition is G_t-invariant. Moreover for any L in $\mathbb{K}(t)\langle S_t \rangle$, if $f \in V_{[d]_{G_t},j}$, then $L(f) \in V_{[d]_{G_t},j}$. Note that such an operator L commutes with the difference operator Δ_{x_i} for $i = 1, \ldots, n$. So by Remark 2.5 and the similar argument as in the proof of Lemma 5.1, we arrive at the following lemma.

Lemma 6.1. *Let $f \in \mathbb{K}(t,\mathbf{x})$. Then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ if and only if f_0 and each $f_{[d]_{G_t},j}$ have a telescoper of the same type for all $[d]_{G_t} \in T_{G_t}$ and $j \in \mathbb{N}^+$.*

Since $f_0 \in V_0 = \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ is always (σ_{x_1})-summable, it follows that $L = 1$ is a telescoper for f_0. For $f \in V_{[d]_{G_t},j}$, it can be written as

$$f = \sum_\tau \frac{a_\tau}{\tau(d)^j}, \tag{6.2}$$

where $\tau \in G_t$, $a_\tau \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t,\mathbf{x}]$ with $\deg_{x_1}(a_\tau) < \deg_{x_1}(d)$ and d is irreducible in x_1 over $\mathbb{K}(t, \hat{\mathbf{x}}_1)$. Each $\tau \in G_t$ is in the form of $\tau = \sigma_t^{k_0} \sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n}$ for some $k_0, k_1, \ldots, k_n \in \mathbb{Z}$. Using the $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-reduction formula (5.4), we get the following decomposition.

Lemma 6.2. *Let $f \in V_{[d]_{G_t},j}$ be in the form (6.2). Then we can decompose it into the form*

$$f = \sum_{i=1}^n \Delta_{x_i}(g_i) + r \ \text{ with } r = \sum_{\ell=0}^{\rho} \frac{a_\ell}{\sigma_t^\ell(\mu)^j},$$

where $\rho \in \mathbb{N}$, $g_i \in \mathbb{K}(t,\mathbf{x})$, $a_\ell \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $\mu \in \mathbb{K}[t,\mathbf{x}]$, $\deg_{x_1}(a_\ell) < \deg_{x_1}(d)$, μ is in the same G_t-orbit as d, and $\sigma_t^\ell(\mu)$, $\sigma_t^{\ell'}(\mu)$ are not G-equivalent for $0 \leq \ell \neq \ell' \leq \rho$. Therefore f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ if and only if r has a telescoper of the same type.

Example 6.3. *Let $\mathbb{K} = \mathbb{Q}$, $G_t = \langle \sigma_t, \sigma_x, \sigma_y, \sigma_z \rangle$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$.*

(1) Consider the rational function f in $\mathbb{Q}(t,x,y,z)$ of the form

$$f = \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3 \sigma_x \sigma_y \sigma_z(d)}$$

where $d = x^2 + 2xy + z^2 + t$. *Then* $f \in V_{[d]_{G_t},1}$ *and applying* $(\sigma_x, \sigma_y, \sigma_z)$*-reduction formula to* f *yields*

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3(d)}, \tag{6.3}$$

where

$$u_0 = \frac{1}{\sigma_t^3\sigma_y\sigma_z(d)}, \ v_0 = \frac{1}{\sigma_t^3\sigma_z(d)} \ \text{and } w_0 = \frac{1}{\sigma_t^3(d)}.$$

Since there is no nonzero integer s *such that* $\sigma_t^s(d)$ *and* d *are* G*-equivalent, the equation* (6.3) *gives a required decomposition for* f *in Lemma* 6.2*.*

(2) Consider the rational function f *in* $\mathbb{Q}(t, x, y, z)$ *of the form*

$$f = \frac{1}{t(t+y+2z)d} + \frac{y+z-1}{(t+3z)\sigma_t(d)} - \frac{y+z}{(t+3z)\sigma_t\sigma_x^3\sigma_y^2(d)},$$

where $d = 3y + (x+z)^2 + t$. *Then* $f \in V_{[d]_{G_t},1}$ *and applying* $(\sigma_x, \sigma_y, \sigma_z)$*-reduction formula to* f *yields that*

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + \frac{1}{t(t+y+2z)d} + \frac{1}{(t+3z)\sigma_t(d)}, \tag{6.4}$$

where

$$u_0 = -\sum_{\ell=0}^{2} \frac{y+z}{(t+3z)\sigma_t\sigma_x^\ell\sigma_y^2(d)}, \ v_0 = -\sum_{\ell=0}^{1} \frac{y+\ell-2+z}{(t+3z)\sigma_t\sigma_y^\ell(d)} \ \text{and} \ \ w_0 = 0.$$

Since the isotropy group of d *in* G_t *is* $G_{t,d} = \langle \sigma_t^3\sigma_y^{-1}, \sigma_x\sigma_z^{-1}\rangle$*, the minimal positive integer* s *such that* $\sigma_t^s(d)$ *and* d *are* G*-equivalent is* $s = 3$*. So* d *and* $\sigma_t(d)$ *are not* G*-equivalent. Thus the equation* (6.4) *gives a required decomposition for* f *in Lemma* 6.2*.*

6.2 Criteria on the existence of telescopers

Combining Lemmas 6.1 and 6.2, we reduce the existence problem (2.2) to that for rational functions in the form

$$f = \sum_{i=0}^{I} \frac{a_i}{\sigma_t^i(d)^j}, \tag{6.5}$$

where $j \in \mathbb{N}^+$, $a_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$,$d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a_i) < \deg_{x_1}(d)$ and d is irreducible such that $\sigma_t^i(d)$ and $\sigma_t^{i'}(d)$ are not G-equivalent for $0 \leq i \neq i' \leq I$.

Let $G_t = \langle \sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n}\rangle$ and $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n}\rangle$ be a subgroup of G_t. Let G_d and $G_{t,d}$ be the isotropy groups of the polynomial d in G and G_t, respectively. By Lemma 4.4, the quotient group $G_{t,d}/G_d$ is free and of rank 0 or 1.

In the case of $\text{rank}(G_{t,d}/G_d) = 0$, the existence problem of telescopers is equivalent to the summability problem.

Lemma 6.4. *Let* $f \in \mathbb{K}(t, \mathbf{x})$ *be in the form* (6.5)*. If* $\text{rank}(G_{t,d}/G_d) = 0$*, then* f *has a telescoper of type* $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ *if and only if each* $a_i/\sigma_t^i(d)^j$ *is* $(\sigma_{x_1}, \ldots, \sigma_{x_n})$*-summable in* $\mathbb{K}(t, \mathbf{x})$ *for* $0 \leq i \leq I$*.*

Proof. Suppose that each $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable for $0 \le i \le I$. By the linearity of the difference operators Δ_{x_i}, we see that $L = \mathbf{1}$ is a telescoper for f. Conversely, assume that $L = \sum_{\ell=0}^{\rho} e_\ell S_t^\ell$ with $e_\ell \in \mathbb{K}(t)$ is a telescoper of type $(\sigma_t;\sigma_{x_1},\ldots,\sigma_{x_n})$ for f. Without loss of generality, we may suppose that $e_0 \neq 0$. Then we have

$$L(f) = \sum_{\ell=0}^{\rho}\sum_{i=0}^{I} e_\ell \sigma_t^\ell\left(\frac{a_i}{\sigma_t^i(d)^j}\right) = \sum_{\ell=0}^{I+\rho}\left(\frac{\sum_{i=0}^{\ell} e_i\sigma_t^i(a_{\ell-i})}{\sigma_t^\ell(d)^j}\right)$$

is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable where $e_\ell = 0$ if $\ell > \rho$ and $a_i = 0$ if $i > I$. Since $\operatorname{rank}(G_{t,d}/G_d) = 0$, all $\sigma_t^\ell(d)$ with $\ell \in \mathbb{Z}$ are in distinct G-orbits. By Lemma 5.1, for any ℓ with $0 \le \ell \le \rho$, there exist $g_{\ell,1},\ldots,g_{\ell,n} \in \mathbb{K}(t,\mathbf{x})$ such that

$$\frac{\sum_{i=0}^{\ell} e_i\sigma_t^i(a_{\ell-i})}{\sigma_t^\ell(d)^j} = \Delta_{x_1}(g_{\ell,1}) + \cdots + \Delta_{x_n}(g_{\ell,n}). \tag{6.6}$$

To show that each $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable for $0 \le i \le I$, we proceed by induction. For $i = 0$, substituting $\ell = 0$ into (6.6), we get $a_0/d^j = \Delta_{x_1}(g_{0,1}/e_0) + \cdots + \Delta_{x_n}(g_{0,n}/e_0)$. Suppose we have $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable for $i = 0,\ldots,s-1$ with $s \le I$. Taking $\ell = s$ in Equation (6.6) yields that

$$\frac{a_s}{\sigma_t^s(d)^j} = \Delta_{x_1}\left(\frac{g_{s,1}}{e_0}\right) + \cdots + \Delta_{x_n}\left(\frac{g_{s,n}}{e_0}\right) - \frac{1}{e_0}\sum_{i=1}^{s} e_i\sigma_t^i\left(\frac{a_{s-i}}{\sigma_t^{s-i}(d)^j}\right).$$

By the inductive hypothesis, we have $a_{s-i}/\sigma_t^{s-i}(d)^j$ is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable for $1 \le i \le s$. Note that $e_i \in \mathbb{K}(t)$ is free of $\mathbf{x}$. Due to the commutativity between σ_t and σ_{x_i} for $i = 1,\ldots,n$, we get $\frac{1}{e_0}\sum_{i=1}^{s} e_i\sigma_t^i\left(\frac{a_{s-i}}{\sigma_t^{s-i}(d)^j}\right)$ is $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable. Hence $a_s/\sigma_t^s(d)^j$ is also $(\sigma_{x_1},\ldots,\sigma_{x_n})$-summable. ∎

Example 6.5. *We continue the Example 6.3 (1) and write $f \in \mathbb{Q}(t,x,y,z)$ as*

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + r \text{ with } r = \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3(d)},$$

where $u_0, v_0, w_0 \in \mathbb{Q}(t,x,y,z)$ and $d = x^2+2xy+z^2+t$. Note that the isotropy groups $G_{t,d} = G_d = \{\mathbf{1}\}$ are trivial. The first term $(2x-1)/d$ of r is not $(\sigma_x,\sigma_y,\sigma_z)$-summable in $\mathbb{Q}(t,x,y,z)$ by the similar reason as in Example 5.15 (1). Since $\operatorname{rank}(G_{t,d}/G_d) = 0$, *we know from Lemma 6.4 that r does not have any telescoper of type $(\sigma_t;\sigma_x,\sigma_y,\sigma_z)$ and neither does f.*

Lemma 6.6. *Let $f = \sum_{i=1}^{I} a_i/\sigma_t^i(d)^j \in \mathbb{K}(t,\mathbf{x})$ be in the form (6.5). If* $\operatorname{rank}(G_{t,d}/G_d) = 1$, *then f has a telescoper of type $(\sigma_t;\sigma_{x_1},\ldots,\sigma_{x_n})$ if and only if each $a_i/\sigma_t^i(d)^j$ has a telescoper of the same type for $0 \le i \le I$.*

Proof. Sufficiency follows from Remark 2.5. The proof of necessity is a natural generalization from the trivariate case [24, lemma 5.3] to the multivariate case. Suppose $L = \sum_{i=0}^{\ell} e_iS_t^i \in \mathbb{K}(t)\langle S_t\rangle$ is a telescoper for f. Since $\operatorname{rank}(G_{t,d}/G_d) = 1$, there is a minimal positive integer k_0 such that $\sigma_t^{k_0}(d) = \sigma_{x_1}^{k_1}\cdots\sigma_{x_n}^{k_n}(d)$ for some integers $k_1,\ldots,k_n$. In the expression (6.5), we require that $\sigma_t^i(d)$ and $\sigma_t^{i'}(d)$ are not G-equivalent for any $0 \le i \neq i' \le I$. By the minimality of k_0, we may assume $f = \sum_{i=0}^{k_0-1} a_i/\sigma_t^i(d)^j$. The k_0-exponent separation of L (see [24, Section 4]) is defined as follows

$$L = L_0 + L_1 + \cdots + L_{k_0-1},$$

where $L_i = \sum_{j=0}^{\ell} e_{jk_0+i} S_t^{jk_0+i}$ and $e_i = 0$ if $i > \ell$. Since $L(f)$ is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable, by Lemma 5.1 each orbital component of $L(f)$ is summable. So we have

$$\begin{cases} L_0 \dfrac{a_0}{d^j} + L_{k_0-1} \dfrac{a_1}{\sigma_t(d)^j} + \cdots + L_1 \dfrac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} & \equiv 0 \\ L_1 \dfrac{a_0}{d^j} + \quad L_0 \dfrac{a_1}{\sigma_t(d)^j} + \cdots + L_2 \dfrac{a_{k_0-1}}{\sigma_x^{k_0-1}(d)^j} & \equiv 0 \\ \qquad\qquad \cdots & \\ L_{k_0-1} \dfrac{a_0}{d^j} + L_{k_0-2} \dfrac{a_1}{\sigma_t(d)^j} + \cdots + L_0 \dfrac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} & \equiv 0, \end{cases} \tag{6.7}$$

where $f \equiv 0$ means that f is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable in $\mathbb{K}(t, \mathbf{x})$. Taking

$$\mathcal{V} = \left[\frac{a_0}{d^j}, \frac{a_1}{\sigma_t(d)^j}, \ldots, \frac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} \right]^T,$$

then Equation (6.7) can be written as

$$\mathcal{L}_{k_0} \cdot \mathcal{V} \equiv 0,$$

where

$$\mathcal{L}_{k_0} = \begin{bmatrix} L_0 & L_{k_0-1} & L_{k_0-2} & \cdots & L_1 \\ L_1 & L_0 & L_{k_0-1} & \cdots & L_2 \\ L_2 & L_1 & L_0 & \cdots & L_3 \\ \vdots & \vdots & \vdots & & \vdots \\ L_{k_0-1} & L_{k_0-2} & L_{k_0-3} & \cdots & L_0 \end{bmatrix}.$$

According to [24, Proposition 4.3], there exist nonzero operators $T_0, \ldots, T_{k_0-1} \in \mathbb{K}(t)\langle S_t \rangle$ and a matrix $\mathcal{M}$ over $\mathbb{K}(t)\langle S_t \rangle$ such that

$$\mathcal{M} \cdot \mathcal{L}_{k_0} = \operatorname{diag}(T_0, \ldots, T_{k_0-1}).$$

For each $0 \leq i \leq k_0 - 1$, we know that T_i is a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for $a_i/\sigma_t^i(d)^j$, because the operators in $T_i \in \mathbb{K}(t)\langle S_t \rangle$ commute with the difference operators $\Delta_{x_1}, \ldots, \Delta_{x_n}$.

∎

Now we consider the existence problem of telescopers for simple fractions in the form

$$f = \frac{a}{d^j} \tag{6.8}$$

where $j \in \mathbb{N}^+$, $a \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a) < \deg_{x_1}(d)$ and d is irreducible such that $\operatorname{rank}(G_{t,d}/G_d) = 1$.

Theorem 6.7 (Theorem 1.7, restated). *Let $f \in \mathbb{K}(t, \mathbf{x})$ be as in (6.8). Let $\{\tau_0, \tau_1, \ldots, \tau_r\}(1 \leq r < n)$ be a basis of $G_{t,d}$ such that $G_{t,d}/G_d = \langle \bar{\tau}_0 \rangle$ and $\{\tau_1, \ldots, \tau_r\}$ is a basis of G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Let $\tau_0 = \sigma_t^{k_0} \sigma_{x_1}^{-k_1} \cdots \sigma_{x_n}^{-k_n}$ for some $k_i \in \mathbb{Z}$ and set $T_0 = S_t^{k_0} S_{x_1}^{-k_1} \cdots S_{x_n}^{-k_n}$. Then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ if and only if there exists a nonzero operator $L \in \mathbb{K}(t)\langle T_0 \rangle$ such that*

$$L(a) = \Delta_{\tau_1}(b_1) + \cdots \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for $1 \leq i \leq r$.

Proof. Firstly, suppose that $L_0 = \sum_{\ell=0}^{\rho} e_\ell T_0^\ell \in \mathbb{K}(t)\langle T_0\rangle$ is a nonzero operator such that $L_0(a) = \sum_{i=1}^{r} \Delta_{\tau_i}(b_i)$ for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$. Set $L = \sum_{\ell=0}^{\rho} e_\ell S_t^{\ell k_0}$. Then

$$\begin{aligned} L(f) &= \sum_{\ell=0}^{\rho} \frac{e_\ell \sigma_t^{\ell k_0}(a)}{\sigma_t^{\ell k_0}(d)^j} = \sum_{\ell=0}^{\rho} \frac{e_\ell \sigma_t^{\ell k_0}(a)}{\sigma_{x_1}^{\ell k_1} \cdots \sigma_{x_n}^{\ell k_n}(d)^j} \\ &= \sum_{i=1}^{n} \Delta_{x_i}(g_i) + \frac{\sum_{\ell=0}^{\rho} e_\ell \sigma_t^{\ell k_0} \sigma_{x_1}^{-\ell k_1} \cdots \sigma_{x_n}^{-\ell k_n}(a)}{d^j} \quad \text{for some } g_i \in \mathbb{K}(t, \mathbf{x}) \\ &= \sum_{i=1}^{n} \Delta_{x_i}(g_i) + \frac{L_0(a)}{d^j} \qquad (6.9) \\ &= \sum_{i=1}^{n} \Delta_{x_i}(g_i) + \frac{1}{d^j} \sum_{i=1}^{r} (\tau_i(b_i) - b_i)) \\ &= \sum_{i=1}^{n} \Delta_{x_i}(g_i) + \sum_{i=1}^{r} \left(\tau_i \left(\frac{b_i}{d^j} \right) - \frac{b_i}{d^j} \right) \\ &= \sum_{i=1}^{n} \Delta_{x_i}(g_i + h_i) \quad \text{for some } h_i \in \mathbb{K}(t, \mathbf{x}). \qquad (6.10) \end{aligned}$$

The last equality follows from Lemma 5.8.

Conversely, let L be a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ for f. By the k_0-exponent separation (see [24, Section 4]) of L and Lemma 5.1, without loss of generality, we may assume $L = \sum_{\ell=0}^{\rho} e_\ell S_t^{\ell k_0} \in \mathbb{K}(t)\langle S_t\rangle$ is a telescoper for f. Then

$$L\left(\frac{a}{d^j}\right) = \sum_{\ell=0}^{\rho} \frac{e_\ell \sigma_t^{\ell k_0}(a)}{\sigma_{x_1}^{\ell k_1} \cdots \sigma_{x_n}^{\ell k_n}(d)^j} = \sum_{i=1}^{n} \Delta_{x_i}(h_i) + \frac{1}{d^j} h$$

for some $h_1, \ldots, h_n, h \in \mathbb{K}(t, \mathbf{x})$ with

$$h = \sum_{\ell=0}^{\rho} e_\ell \sigma_t^{\ell k_0} \sigma_{x_1}^{-\ell k_1} \cdots \sigma_{x_n}^{-\ell k_n}(a) = \sum_{\ell=0}^{\rho} e_\ell \tau_0^\ell(a). \qquad (6.11)$$

Since $L(a/d^j)$ is $(\sigma_{x_1}, \ldots, \sigma_{x_n})$-summable and $\{\tau_1, \ldots, \tau_r\}$ is a basis of G_d, by Theorem 5.9 with $\mathbb{F} = \mathbb{K}(t)$ we get

$$h = \Delta_{\tau_1}(b_1) + \cdots + \Delta_{\tau_r}(b_r) \qquad (6.12)$$

for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for $1 \le i \le r$. Combining Equations (6.11) and (6.12) yields that a has a telescoper $L_0 = \sum_{\ell=0}^{\rho} e_\ell T_0^\ell$ of type $(\tau_0; \tau_1, \ldots, \tau_r)$. ∎

Proposition 6.8. *Let $\tau \in G_t \setminus G$ and $f = a/b$ with $a, b \in \mathbb{K}[t, \mathbf{x}]$ and $\gcd(a, b) = 1$. Then there exist $e_0, \ldots, e_r \in \mathbb{K}(t)$, not all zero, such that $\sum_{i=0}^{r} e_i \tau^i(f) = 0$ if and only if $b = b_1 b_2$ with $b_1 \in \mathbb{K}[t]$ and $b_2 \in \mathbb{K}[t, \mathbf{x}]$ satisfying that $\tau(b_2) = b_2$.*

Proof. First we suppose $b = b_1 b_2$ with b_1, b_2 satisfying the above conditions. Then for any $i \in \mathbb{N}$,

$$\tau^i(f) = \frac{\tau^i(a)}{\tau^i(b_1 b_2)} = \frac{\tau^i(a)}{\tau^i(b_1) b_2} = \frac{\tau^i(a/b_1)}{b_2}. \qquad (6.13)$$

Note that $b_1 \in \mathbb{K}[t]$ and the total degrees of the polynomials $\tau^i(a)$ in $\mathbf{x}$ are the same as that of a. Thus all shifts of a/b_1 lie in a finite dimensional linear space over $\mathbb{K}(t)$. So there exist $e_0, e_1, \ldots, e_r \in \mathbb{K}(t)$, not all zero, such that $\sum_{i=0}^{r} e_i \tau^i(a/b_1) = 0$. This implies $\sum_{i=0}^{r} e_i \tau^i(f) = 0$.

Conversely, suppose $\sum_{i=0}^{r} e_i\tau^i(f) = 0$. Let b_1 and b_2 be the content and primitive part of b as a polynomial in $\mathbf{x}$ over $\mathbb{K}(t)$. If $b_2 \in \mathbb{K}$, then we are done. Now we assume that $b_2 \notin \mathbb{K}$. Then all of its irreducible factors have positive total degree in $\mathbf{x}$. Assume that there exists an irreducible polynomial p such that $\tau(p) \neq p$. By Lemma 4.3, the quotient group $G_t/G_{t,p}$ is free, so is torsion free. So for any integer $i \neq 0$, $\tau^i(p) \neq p$. Among all of such irreducible factors of b_2, we can find one factor p such that $\tau^i(p) \nmid b_2$ for any integer $i < 0$. Let s be the largest integer such that $\tau^s(p) \mid b_2$. Then the irreducible polynomial $\tau^{r+s}(p)$ divides $\tau^r(b_2)$, but $\tau^{r+s}(p) \nmid \tau^i(b_2)$ for any $0 \leq i \leq r-1$. Otherwise $\tau^{r+s-i}(p) \mid b_2$, which contradicts the choice of s. Therefore we have $\sum_{i=0}^{r} e_i\tau^i(f) \neq 0$, since p depends on $\mathbf{x}$ and the coefficients e_i are in $\mathbb{K}(t)$. This leads to a contradiction. So every irreducible factor p of b_2 satisfies the property that $\tau(p) = p$. This implies that $\tau(b_2) = b_2$. ∎

Lemma 6.9. *Let $\tau \in G_t \setminus G$ and $f = a/(b_1b_2)$ with $b_1 \in \mathbb{K}[t]$, $a, b_2 \in \mathbb{K}[t, \mathbf{x}]$ and $\tau(b_2) = b_2$. Then we can compute $e_0, \ldots, e_r \in \mathbb{K}[t]$, not all zero, such that $\sum_{i=0}^{r} e_i\tau^i(f) = 0$. Furthermore, if $f \in \mathbb{K}(t, \hat{\mathbf{x}}_1)_d(x_1)_{d_0}$ with two positive integers d and d_0, then the computation takes $\tilde{O}(md^{m-n}(d_0 + nd)^{n+1})$ ops in $\mathbb{K}$ with r being no more than $d_0 + nd + 1$ and $e_i \in \mathbb{K}[t]_{2(d_0+nd)(d_0+nd+1)^2}$ for $i = 0, 1, \ldots, r$.*

Proof. By Proposition 6.12 below, we can construct a difference isomorphism between $(\mathbb{K}(t, \mathbf{x}), \tau)$ and $(\mathbb{K}(t, \mathbf{x}), \sigma_t)$ such that $\varphi \circ \tau = \sigma_t \circ \varphi$ and $\varphi(\mathbb{K}[t]) \subseteq \mathbb{K}[t]$. Then $\sigma_t(\varphi(b_2)) = \varphi(\tau(b_2)) = \varphi(b_2)$ and for all $e_i(t) \in \mathbb{K}[t]$,

$$\sum_{i=0}^{r} e_i(t)\tau^i(f) = 0 \iff \sum_{i=0}^{r} e_i(\varphi(t))\sigma_t^i(\varphi(f)) = 0.$$

So we only need to consider the case $\tau = \sigma_t$. Now suppose that $f = a/(b_1b_2)$ with $b_1 \in \mathbb{K}[t]$, $a, b_2 \in \mathbb{K}[t, \mathbf{x}]$ and $\sigma_t(b_2) = b_2$. It suffices to find a nonzero operator $L \in \mathbb{K}[t]\langle S_t\rangle$ such that $L(f) = 0$. We write $a = \sum_{i=0}^{s} a_it^i$ with $a_i \in \mathbb{K}[\mathbf{x}]$. For each $0 \leq i \leq s$, let $L_i = t^i\sigma_t(b_1)S_t - (t+1)^ib_1$. Then L_i is an operator in $\mathbb{K}[t]\langle S_t\rangle$ such that $L_i(t^i/b_1) = 0$. Since $\sigma_t(b_2) = b_2$ and $\sigma_t(a_i) = a_i$, we have $L_i(a_it^i/(b_1b_2)) = (a_i/b_2)L_i(t^i/b_1) = 0$. Let $L \in \mathbb{K}[t]\langle S_t\rangle$ be the LCLM of L_i for all $i = 0, \ldots, s$ and write $L = R_iL_i$ with $R_i \in \mathbb{K}[t]\langle S_t\rangle$. Then L is a nonzero operator and

$$L(f) = L\left(\sum_{i=0}^{s} \frac{a_it^i}{b_1b_2}\right) = \sum_{i=0}^{s} L\left(\frac{a_it^i}{b_1b_2}\right) = \sum_{i=0}^{s} R_iL_i\left(\frac{a_it^i}{b_1b_2}\right) = 0.$$

Suppose $f \in \mathbb{K}(t, \hat{\mathbf{x}}_1)_d(x_1)_{d_0}$. Then the degree of $\varphi(f)$ in t is no more than $d_0 + nd$, and that in x_i is no more than $(n-1)d + d_0$ if $1 \leq i \leq n$ and no more than d if $n+1 \leq i \leq m$. The expansion of $\varphi(f)$ takes $\tilde{O}((m+1)(d_0+nd)(d_0+(n-1)d)^nd^{m-n}) = \tilde{O}(md^{m-n}(d_0+nd)^{n+1})$ ops by multipoint evaluation and interpolation. In addition, s is no more than $d_0 + nd$ and $L_i \in \mathbb{K}[t]_{2(d_0+nd)}\langle S_t\rangle_1$ for $0 \leq i \leq s$. Applying Fact 2.10, we get

$$L \in \mathbb{K}[t]_{2(d_0+nd)(d_0+nd+1)^2}\langle S_t\rangle_{d_0+nd+1}$$

to be the LCLM of the L_i's using $\tilde{O}((d_0 + nd)^{2\omega+1})$ ops. ∎

Remark 6.10. *For the bivariate case with $m = n = 1$, our existence criterion coincides with the known result in [6, Theorem 1] and [27, Theorem 4.11]. Let $f = a/d^j \in \mathbb{K}(t, \mathbf{x})$, where $j \in \mathbb{N}^+$, $a \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a) < \deg_{x_1}(d)$ and d is irreducible. Let $G_t = \langle\sigma_t; \sigma_{x_1}\rangle$ and $G = \langle\sigma_{x_1}\rangle$. Since the degree of d in x_1 is positive, we have $G_d = \{\mathbf{1}\}$. If $\operatorname{rank}(G_{t,d}/G_d) = 0$, then by Lemma 6.4 and Theorem 5.9, f has a telescoper of type (σ_t, σ_{x_1}) if and only if $a = 0$. If*

$\operatorname{rank}(G_{t,d}/G_d) = 1$, *then there exists* $\tau = \sigma_t^s \sigma_{x_1}^k \in G_{t,d}$ *with* $s > 0$ *such that* $G_{t,d}/G_d = \langle \bar{\tau} \rangle$. *By Theorem 6.7 and Proposition 6.8,* f *has a telescoper of type* $(\sigma_t; \sigma_{x_1})$ *if and only if* $a = c/b$ *with* $c \in \mathbb{K}[t, \mathbf{x}]$, $b \in \mathbb{K}[t, \hat{\mathbf{x}}_1]$, $\gcd(b, c) = 1$, *where* b *can be written as* $b = b_1 b_2$ *with* $b_1 \in \mathbb{K}[t]$ *and* $b_2 \in \mathbb{K}[t, \hat{\mathbf{x}}_1]$ *such that* $\tau(b_2) = b_2$. *Since* $m = n = 1$, *we have* $b \in \mathbb{K}[t]$ *and hence* f *always has a telescoper of type* $(\sigma_t; \sigma_{x_1})$.

Example 6.11. *Let* $f = 1/(t^s + x_1^s + \cdots + x_n^s) \in \mathbb{Q}(t, x_1, \ldots, x_n)$ *with* $s, n \in \mathbb{N} \setminus \{0\}$. *Then* $d = t^s + x_1^s + \cdots + x_n^s$ *is irreducible over* $\mathbb{Q}$ *if* $n > 1$. *Let* $G_{t,d}$ *and* G_d *be the isotropy group of* d *in* $G_t = \langle \sigma_t, \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$ *and* $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$, *respectively. Then we can decide the existence of telescopers of type* $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ *for all cases of* f.

(1) *If* $s = 1$, *then* d *is irreducible. Since* $G_{t,d} = \langle \tau \rangle$ *with* $\tau = \sigma_t \sigma_{x_1}^{-1}$ *and* $G_d = \{\mathbf{1}\}$, *we have* $G_{t,d}/G_d = \langle \bar{\tau} \rangle$ *and* $\operatorname{rank}(G_{t,d}/G_d) = 1$. *Observing that* $\tau - 1$ *is an annihilator of the numerator of* f, *by Theorem 6.7 we get* f *has a telescoper. Indeed* $L(f) = \Delta_{x_1}(f) + \Delta_{x_2}(0) + \cdots + \Delta_{x_n}(0)$, *where* $L = S_t - 1$.

(2) *If* $s > 1$ *and* $n = 1$, *then* $f = 1/(t^s + x_1^s) = \sum_{j=1}^s a_j/(t - \beta_j x_1)$, *where* β_j*'s are distinct roots of* $z^s = -1$ *and* $a_j = 1/s(\beta_j x_2)^{s-1}$. *There exists* $j \in \{1, \ldots, s\}$ *such that* $\beta^j \notin \mathbb{Z}$. *Then for* $d_j = t - \beta_j x_1$, *we have* $G_{t,d_j} = G_{d_j} = \{\mathbf{1}\}$. *So* a_j/d_j *is not* σ_{x_1}*-summable in* $\mathbb{C}(t, x_1)$ *and neither is* f. *By Lemma 6.4, we get that* f *does not have any telescoper of type* (σ_t, σ_{x_1}) *in* $\mathbb{C}(t)\langle S_t \rangle$. *Hence* f *does not have any telescoper of the same type in* $\mathbb{Q}(t)\langle S_t \rangle$.

(3) *If* $s > 1$ *and* $n > 1$, *then* d *is irreducible. Since* $G_d = \{\mathbf{1}\}$, *by Theorem 5.9, it follows that* f *is not* $(\sigma_{x_1}, \ldots, \sigma_{x_n})$*-summable. Since* $G_{t,d} = \{\mathbf{1}\}$ *and* $\operatorname{rank}(G_{t,d}/G_d) = 0$, *by Lemma 6.4, we conclude that* f *does not have any telescoper.*

Proposition 6.12. *Let* $\{\tau_0, \tau_1, \ldots, \tau_r\}(1 \leq r \leq n)$ *be a family of* $\mathbb{Z}$*-linearly independent elements in* G_t *such that* $\tau_0 \in G_t \setminus G$ *and* $\{\tau_1, \ldots, \tau_r\} \subseteq G$. *Then there exists a* $\mathbb{K}$*-automorphism* φ *of* $\mathbb{K}(t, \mathbf{x})$ *such that* φ *is a difference isomorphism between the difference fields* $(\mathbb{K}(t, \mathbf{x}), \tau_0)$ *and* $(\mathbb{K}(t, \mathbf{x}), \sigma_t)$, *and simultaneously a difference isomorphism between* $(\mathbb{K}(t, \mathbf{x}), \tau_i)$ *and* $(\mathbb{K}(t, \mathbf{x}), \sigma_{x_i})$ *for all* $i = 1, \ldots, r$. *Furthermore,* $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$ *and hence for any* $f \in \mathbb{K}(t, \mathbf{x})$, f *has a telescoper of type* $(\tau_0; \tau_1, \ldots, \tau_r)$ *if and only if* $\varphi(f)$ *has a telescoper of type* $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_r})$.

Proof. Let $\tau_i = \sigma_t^{a_{i,0}} \sigma_{x_1}^{a_{i,1}} \cdots \sigma_{x_m}^{a_{i,m}}$, where $a_{i,j} = 0$ if $j > n$. Define $\alpha_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m}) \in \mathbb{Z}^{m+1}$ for $i = 0, 1, \ldots, r$. Since $\alpha_0, \alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$, we can find vectors $\alpha_{r+1}, \ldots, \alpha_m \in \mathbb{Q}^{m+1}$ such that $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ is a basis of $\mathbb{Q}^{m+1}$ over $\mathbb{Q}$. Write $\alpha_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m})$ for $i = r+1, \ldots, m$. Since $\tau_0 \in G_t \setminus G$ and $\{\tau_1, \ldots, \tau_r\} \subseteq G$, we have $a_{0,0} \neq 0$ and $a_{i,0} = 0$ for $i = 1, \ldots, r$. So we can further assume that $a_{i,0} = 0$ for $i = r+1, \ldots, m$. Let $A = (a_{i,j}) \in \mathbb{Q}^{(m+1)\times(m+1)}$ which is invertible. Let φ be a $\mathbb{K}$-automorphism of $\mathbb{K}(t, \mathbf{x})$ defined by

$$(\varphi(t), \varphi(x_1), \ldots, \varphi(x_m)) := (t, x_1, \ldots, x_m)A.$$

Then $\varphi(t) = a_{0,0} \cdot t$ and $\varphi(x_j) = a_{0,j} \cdot t + \sum_{i=1}^m a_{i,j} \cdot x_i$ for $j = 1, \ldots, m$. It can be checked that $\varphi \circ \tau_0 = \sigma_t \circ \varphi$ and $\varphi \circ \tau_i = \sigma_{x_i} \circ \varphi$ for $1 \leq i \leq r$. This means the following diagrams are commutative.

$$\begin{array}{ccc} \mathbb{K}(t,\mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t,\mathbf{x}) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \sigma_t} \\ \mathbb{K}(t,\mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t,\mathbf{x}) \end{array} \qquad \cdots \qquad \begin{array}{ccc} \mathbb{K}(t,\mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t,\mathbf{x}) \\ \downarrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \sigma_{x_i}} \\ \mathbb{K}(t,\mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t,\mathbf{x}) \end{array}$$

Note that $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$. It follows that

$$\sum_{\ell=0}^{\rho} e_\ell(t)\tau_0^\ell(f) = \sum_{i=1}^{r} \Delta_{\tau_i}(g_i) \quad \Longleftrightarrow \quad \sum_{\ell=0}^{\rho} e_\ell(a_{0,0}t)\sigma_t^\ell(\varphi(f)) = \sum_{i=1}^{r} \Delta_{x_i}(\varphi(g_i)),$$

whenever $e_\ell(t) \in \mathbb{K}(t)$ and $f, g_i \in \mathbb{K}(t, \mathbf{x})$. This completes our proof. ∎

The way of constructing a difference isomorphism in Proposition 6.12 is almost the same as that in Proposition 5.12. The only difference is that in Proposition 6.12, we require $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$.

Let $f = a/d^j$ be in the form (6.8) with $\operatorname{rank}(G_{t,d}/G_d) = 1$. By Theorem 6.7, there are two cases according to whether G_d is trivial or not. If $G_d = \{\mathbf{1}\}$, then a/d^j has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$ if and only if there exists a nonzero operator $L \in \mathbb{K}(t)\langle T_0\rangle$ such that $L(a) = 0$. This problem is solved by Proposition 6.8. If G_d is nontrivial, we can apply the transformation in Proposition 6.12 to reduce the existence problem of telescopers to that of fewer variables. Moreover, the general existence of telescopers of type $(\tau_0; \tau_1, \ldots, \tau_n)$ for rational functions has also been solved.

Algorithm 6.13 (Constructive Testing of the Existence of Telescopers)**.**
IsTelescoperable(f**,** $[x_1, \ldots, x_n]$**,** t**).**
INPUT: a multivariate rational function $f \in \mathbb{K}(t, \mathbf{x})$*, a set* $\{x_1, \ldots, x_n\}$ *of variable names and a variable name* t *for telescoping;*
OUTPUT: a telescoper L *and its unnormalised certificates* $g_1, \ldots, g_n$ *if* f *has a telescoper of type* $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$*;* false *otherwise.*

using shift equivalence testing and irreducible partial fraction decomposition, decompose f *into* $f = f_0 + \sum_{j\in\mathbb{N}^+} \sum_{[d]_{G_t}} f_{[d]_{G_t},j}$ *as in Equation (6.1).*

apply the reduction to f_0 *and each nonzero component* $f_{[d]_{G_t},j}$ *such that*

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + r \ \text{ with } r = \sum_{i=1}^{I}\sum_{j=1}^{J_i}\sum_{\ell=0}^{s_{i,j}} \frac{a_{i,j,\ell}}{\sigma_t^\ell(d_i)^j},$$

where $\sum_{\ell=0}^{s_{i,j}} \frac{a_{i,j,\ell}}{\sigma_t^\ell(d_i)^j}$ *is the remainder of* $f_{[d_i]_{G_t},j}$ *described in Lemma 6.2.*

if $r = 0$*, then* **return** $L = \mathbf{1}$ *and* $g_1, \ldots, g_n$*.*

for $i = 1, \ldots, I$ **do**

  using Remark 4.5, one can find elements $\tau_{i,0}, \tau_{i,1}, \ldots, \tau_{i,r_i} \in G_{t,d_i}$ *such that* $G_{t,d_i}/G_{d_i} = \langle\bar{\tau}_{i,0}\rangle$ *and* $\{\tau_{i,1}, \ldots, \tau_{i,r_i}\}$ *forms a basis for* G_{d_i}*.*

  for $j = 1, \ldots, J_i$*,* $\ell = 1, \ldots, s_{i,j}$ **do**

    if $\operatorname{rank}(G_{t,d_i}/G_{d_i}) = 0$ *then*

      execute Algorithm 5.13 with **IsSummable(**$r_{i,j,\ell}$**,** $[x_1, \ldots, x_n]$**)***, where* $r_{i,j,\ell} := \frac{a_{i,j,\ell}}{\sigma_t^\ell(d_i)^j}$*.*

      if $r_{i,j,\ell}$ *is* $(\sigma_{x_1}, \ldots, \sigma_{x_n})$*-summable in* $\mathbb{F}(\mathbf{x})$*, let*

$$r_{i,j,\ell} = \Delta_{x_1}\left(h_{i,j,\ell}^{(1)}\right) + \cdots + \Delta_{x_n}\left(h_{i,j,\ell}^{(n)}\right)$$

      and set $L_{i,j,\ell} = \mathbf{1}$*;* **return** false *otherwise.*

    if $\operatorname{rank}(G_{t,d_i}/G_{d_i}) = 1$ *then*

      choose $\tau_{i,0} = \sigma_t^{k_{i,0}}\sigma_{x_1}^{-k_{i,1}}\cdots\sigma_{x_n}^{-k_{i,n}}$ *with* $k_{i,0} > 0$*.*

      set $T_{i,0} = S_t^{k_{i,0}} S_{x_1}^{-k_{i,1}} \cdots S_{x_n}^{k_{i,n}}$*.*

      if $n = 1$ *or* $G_{d_i} = \{\mathbf{1}\}$ *then* *using Proposition 6.8, check whether there exists a nonzero operator* $\bar{L}_{i,j,\ell}(t, T_{i,0}) \in \mathbb{K}(t)\langle T_{i,0}\rangle$ *such that* $\bar{L}_{i,j,\ell}(t, T_{i,0})(a_{i,j,\ell}) = 0$. *If so, use Lemma 6.9 to find such an operator* $\bar{L}_{i,j,\ell}(t, T_{i,0})$ *and set* $L_{i,j,\ell}(t, S_t) = \bar{L}_{i,j,\ell}(t, S_t^{k_{i,0}})$. *By Equation (6.9) we obtain*

$$L_{i,j,\ell}\left(\frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j}\right) = \sum_{\lambda=1}^{n} \Delta_{x_\lambda}\left(h_{i,j,\ell}^{(\lambda)}\right) + \underbrace{\frac{\bar{L}_{i,j,\ell}(a_{i,j,\ell})}{\sigma_t^{\ell}(d_i)^j}}_{=0};$$

return false *otherwise.*

else

find a $\mathbb{K}$*-automorphism* φ_i *of* $\mathbb{K}(t, \mathbf{x})$ *given in Proposition 6.12 such that* $\varphi_i \circ \tau_{i,0} = \sigma_t \circ \varphi_i$ *and* $\varphi_i \circ \tau_{i,\ell} = \sigma_{x_i} \circ \varphi_i$ *for* $\ell = 1, \ldots, r_i$.

set $\tilde{a}_{i,j,\ell} = \varphi_i(a_{i,j,\ell})$.

execute **IsTelescoperable(**$\tilde{a}_{i,j,\ell}$**,** $[x_1, \ldots, x_{r_i}], t$**)**.

if $\tilde{a}_{i,j,\ell}$ *has a telescoper of type* $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_{r_i}})$, *let*

$$\tilde{L}_{i,j,\ell}(t, S_t)(\tilde{a}_{i,j,\ell}) = \sum_{\lambda=1}^{r_i} \Delta_{x_\lambda}\left(\tilde{b}_{i,j,\ell}^{(\lambda)}\right);$$

return false *otherwise.*

apply φ_i^{-1} *to both sides of the previous equation to get*

$$\bar{L}_{i,j,\ell}(t, T_{i,0})(a_{i,j,\ell}) = \sum_{\lambda=1}^{r_i} \Delta_{\tau_{i,\lambda}}\left(b_{i,j,\ell}^{(\lambda)}\right),$$

where $\bar{L}_{i,j,\ell}(t, T_{i,0}) = \tilde{L}_{i,j,\ell}(t/k_{i,0}, T_{i,0})$ *and* $b_{i,j,\ell}^{(\lambda)} = \varphi_i^{-1}(\tilde{b}_{i,j,\ell}^{(\lambda)})$ *for all* $\lambda = 1, \ldots, r_i$.

set $L_{i,j,\ell}(t, S_t) = \bar{L}_{i,j,\ell}(t, S_t^{k_{i,0}})$ *and by Equations (6.9) and (6.10) we obtain*

$$\begin{aligned} L_{i,j,\ell}\left(\frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j}\right) &= \sum_{\lambda=1}^{n} \Delta_{x_\lambda}\left(u_{i,j,\ell}^{(\lambda)}\right) + \frac{\bar{L}_{i,j,\ell}(a_{i,j,\ell})}{\sigma_t^{\ell}(d_i)^j} \\ &= \sum_{\lambda=1}^{n} \Delta_{x_\lambda}\left(h_{i,j,\ell}^{(\lambda)}\right) \end{aligned}$$

for some $u_{i,j,\ell}^{(\lambda)}, h_{i,j,\ell}^{(\lambda)} \in \mathbb{K}(t, \mathbf{x})$.

let $L \in \mathbb{K}(t)\langle S_t\rangle$ *be the LCLM of* $L_{i,j,\ell}$ *for all* i, j, ℓ *and write*

$$L = R_{i,j,\ell} L_{i,j,\ell}$$

for some $R_{i,j,\ell} \in \mathbb{K}(t)\langle S_t\rangle$.

update $g_\lambda = L(g_\lambda) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{\ell=1}^{s_{i,j}} R_{i,j,\ell}(h_{i,j,\ell}^{(\lambda)})$ *for all* $\lambda = 1, \ldots, n$.

return L *and* $g_1, \ldots, g_n$.

If f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$, Algorithm 6.13 will output unnormalised certificates for f in the form

$$g = \sum_{\ell=1}^{\rho} \prod_{k=1}^{K_\ell} \psi_{\ell,k}(v_{\ell,k}),$$

where $v_{\ell,k} \in \mathbb{F}(t, \mathbf{x})$ and the $\psi_{\ell,k}$'s are $\mathbb{Q}$-affine maps. For each $1 \leq \ell \leq \rho$, the product $\prod_{k=1}^{K_\ell} v_{\ell,k}$ is called a *kernel* of g.

Now we give the complexity of Algorithm 6.13 for $\mathbb{K} = \mathbb{Q}$, which implies that the existence problem of telescopers can be solved in polynomial time.

Theorem 6.14. *Let δ be an integer in $\mathbb{N}$ and $f(t, \mathbf{x})$ be a multivariate rational function in $\mathbb{Q}(t, \mathbf{x})_\delta$ with coefficients whose denominators and numerators are in $O(1)$.*

(1) If f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \ldots, \sigma_{x_n})$, Algorithm 6.13 will output a telescoper in $\mathbb{Q}[t]_{O(D_1(\delta,n))}\langle S_t\rangle_{O(D_2(\delta,n))}$ for f and a tuple of unnormalised certificates whose kernels are in $\mathbb{Q}(t)_{O(D_1(\delta,n))}(\mathbf{x})_{O(D_3(\delta,n))}$, where

$$D_1(\delta, n) = \left(\prod_{i=1}^{n} i^{(2n+3)2^{i-1}+i-n-1}\right) \delta^{(2n+3)2^n},$$

$$D_2(\delta, n) = \left(\prod_{i=1}^{n} i^{2^i-1}\right) \delta^{2^{n+1}-1} \quad \text{and} \quad D_3(\delta, n) = \left(\prod_{i=1}^{n} i^{2^i}\right) \delta^{2^{n+1}}.$$

(2) The total runtime of Algorithm 5.13 is $\tilde{O}(C'(m, \delta, n))$ ops in $\mathbb{Q}$, where

$$C'(m, \delta, n) = \left(\prod_{i=1}^{n} i^{7\cdot 2^{i-1}m-1}\right) \delta^{7\cdot 2^n m-1}.$$

Proof. (1) Let $D(\delta, n)$ be the function defined as in (5.15). By the similar argument in the proof of Theorem 5.14, the g_i and $a_{i,j,\ell}$'s obtained at Step 2 are in $\mathbb{Q}(t, \mathbf{x})_{(O(\delta^2),O(\delta),O(\delta^2),\ldots,O(\delta^2))}$, and the number of iterations of the loops in Steps 4 and 6 is $\sum_{i=1}^{I}\sum_{j=1}^{J_i} s_{i,j} \leq \delta$. For each iteration of the loops in Steps 4 and 6, we get the kernels of $h_{i,j,\ell}^{(\lambda)}$'s in $\mathbb{Q}(t, \mathbf{x})_{O(D(\delta^2,n))}$ if we execute Step 9, and by Lemma 6.9, the kernels of $h_{i,j,\ell}^{(\lambda)}$'s in $\mathbb{Q}(t)_{O(n^3\delta^6)}(\mathbf{x})_{O(\delta^2)}$ with

$$L_{i,j,\ell} \in \mathbb{Q}[t]_{O(n^3\delta^6)}\langle S_t\rangle_{O(n\delta^2)}$$

if we execute Step 14. The rational function $\tilde{a}_{i,j,\ell}$ obtained at Step 17 is of degree no more than $O(n\delta^2)$ in t or x_i if $i = 1, \ldots, n$, and no more than $O(\delta^2)$ in x_i if $i = n+1, \ldots, m$. The proof will be completed by induction on n.

If $n = 1$, we execute either Step 9 or Step 14 for each iteration. Therefore, the algorithm outputs L whose bidegree in (t, S_t) is no more than $(O(\delta^{10}), O(\delta^3))$ and g_i whose kernels are in $\mathbb{Q}(t)_{O(\max\{D(\delta^2,1),\delta^{10}\})}(\mathbf{x})_{O(D(\delta^2,1))}$. So the base case is true.

If $n > 1$, assume the degree bounds are true for $n-1$. We first estimate the bidegree bound of the telescoper L. Note that in each iteration, the bidegrees of $\tilde{L}_{i,j,\ell}$'s in (t, S_t) at Step 19 are $(O(D_1(n\delta^2, n-1)), O(D_2(n\delta^2, n-1)))$. After executing Step 22, by Fact 2.10, we obtain that the bidegree bound of L in (t, S_t) is $(O(D_1(n\delta^2, n-1)\delta^2 D_2(n\delta^2, n-1)), O(\delta D_2(n\delta^2, n-1)))$. It is straightforward to compute that

$$\begin{aligned}\delta D_2(n\delta^2, n-1) &= \delta\left(\prod_{i=1}^{n-1} i^{2^i-1}\right)\left(n\delta^2\right)^{2^n-1} \\ &= \left(\prod_{i=1}^{n-1} i^{2^i-1}\right) n^{2^n-1}\delta^{2^{n+1}-1} \\ &= D_2(\delta, n)\end{aligned}$$

and

$$\begin{aligned}
D_1(n\delta^2, n-1)\delta^2 D_2(n\delta^2, n-1) &= \delta D_1(n\delta^2, n-1) D_2(\delta, n) \\
&= \left(\prod_{i=1}^{n-1} i^{((2n+1)2^{i-1}+i-n)+(2^i-1)}\right) n^{2^n-1} \left(n\delta^2\right)^{(2n+1)2^{n-1}} \delta^{2^{n+1}} \\
&= \left(\prod_{i=1}^{n-1} i^{(2n+3)2^{i-1}+i-n-1}\right) n^{(2n+3)2^{n-1}-1} \delta^{(2n+3)2^n} \\
&= D_1(\delta, n).
\end{aligned}$$

To estimate the degree bound of the kernels of g_i's, we first estimate that of $h_{i,j,\ell}$'s at Step 9 and that of $\tilde{b}_{i,j,\ell}^{(\lambda)}$ at Step 19. The kernels of $h_{i,j,\ell}$'s at Step 9 are in $\mathbb{K}(t, \mathbf{x})_{D(\delta^2,n)}$ and the kernels of $\tilde{b}_{i,j,\ell}^{(\lambda)}$ at Step 19 are in $\mathbb{K}(t)_{O(D_1(n\delta^2,n-1))}(\mathbf{x})_{O(D_3(n\delta^2,n-1))}$. A direct computation yields that $D(\delta^2, n) = (\prod_{i=1}^{n} i^{2^{i-1}})\delta^{2^{n+1}}$ is smaller than $D_3(n\delta^2, n-1) = (\prod_{i=1}^{n-1} i^{2^i})(n\delta^2)^{2^n} = D_3(\delta, n)$. As a result, the kernels of g_λ's at Step 23 are in $\mathbb{K}(t)_{O(D_1(\delta,n)+D_1(n\delta^2,n-1))}(\mathbf{x})_{O(D_3(\delta,n))} = \mathbb{K}(t)_{O(D_1(\delta,n))}(\mathbf{x})_{O(D_3(\delta,n))}$.

(2) By the proof similar to that of Theorem 5.14, the total cost of the first three steps is $\tilde{O}(\delta^{7m+6})$ ops. For each iteration of the loops in Steps 4 and 6, Step 8 costs $\tilde{O}(C(m, \delta^2, n))$ ops by Theorem 5.14 and Step 14 takes $\tilde{O}(n^{n+1} m \delta^{2(m+1)})$ ops by Lemma 6.9. By Fact 2.10, the cost of the computation of the LCLM at Step 22 is $\tilde{O}(\delta^{4\omega+6})$ ops for $n = 1$ and $\tilde{O}(\delta^{2\omega} D_1(n\delta^2, n-1)(D_2(n\delta^2, n-1))^\omega)$ ops for $n > 1$. The proof will be completed by induction on n.

If $n = 1$, executing either Step 8 or Step 14 in each iteration, the most expensive steps are the first three steps, Step 8 or Step 14 in each iteration, as well as Step 22. Thus the cost of the algorithm is $\tilde{O}(\delta^{7m+6} + \delta \max\{C(m, \delta^2, 1), m\delta^{2(m+1)}\} + \delta^{4\omega+6}) = \tilde{O}(\delta^{14m-1})$.

If $n > 1$, suppose the result is true for $n - 1$. For each iteration of the loops in Steps 4 and 6, Step 8 costs $\tilde{O}(C(m, \delta^2, n))$ ops and Step 18 takes $\tilde{O}(C'(m, n\delta^2, n-1))$ ops. As a result, the total cost of the algorithm is

$$\begin{aligned}
&\tilde{O}\left(\delta\left(C(m, \delta^2, n) + C'(m, n\delta^2, n-1)\right) + \delta^{2\omega} D_1(n\delta^2, n-1)(D_2(n\delta^2, n-1))^\omega\right) \\
=&\tilde{O}\left(\left(\prod_{i=1}^{n} i^{7\cdot 2^{i-2}m-1}\right)\delta^{7\cdot 2^n m-1} + \left(\prod_{i=1}^{n-1} i^{7\cdot 2^{i-1}m-1}\right)\delta(n\delta^2)^{7\cdot 2^{n-1}m-1}\right. \\
&\left.+\delta^{2\omega}\left(\left(\prod_{i=1}^{n-1} i^{(2n+1)2^{i-1}+i-n}\right)(n\delta^2)^{(2n+1)2^{n-1}}\right)\left(\left(\prod_{i=1}^{n-1} i^{2^i-1}\right)(n\delta^2)^{2^n-1}\right)^\omega\right) \\
=&\tilde{O}\left(\left(\prod_{i=1}^{n} i^{7\cdot 2^{i-2}m-1}\right)\delta^{7\cdot 2^n m-1} + \left(\prod_{i=1}^{n} i^{7\cdot 2^{i-1}m-1}\right)\delta^{7\cdot 2^n m-1}\right. \\
&\left.+\left(\prod_{i=1}^{n} i^{(2n+1+2\omega)2^{i-1}+i-n-\omega}\right)\delta^{(2n+1+2\omega)2^n}\right) \\
=&\tilde{O}(C'(m, \delta, n))
\end{aligned}$$

ops in $\mathbb{Q}$.

∎

Example 6.15. *Let $\mathbb{K} = \mathbb{Q}$ and $f \in \mathbb{Q}(t, x, y, z)$. We will decide constructively the existence of telescopers of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for various cases of f. Let $G_t = \langle \sigma_t, \sigma_x, \sigma_y, \sigma_z \rangle$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$.*

(1) For $f = \frac{1}{(t+1)(t+2z)d}$ with $d = (t - 3y + x)^2(t + y)(t + z) + 1$, we find a basis of the isotropy group $G_{t,d}$ is $\{\tau_0\}$, where $\tau_0 = \sigma_t\sigma_x^{-4}\sigma_y^{-1}\sigma_z^{-1}$. Then $G_{t,d}/G_d = \langle\bar{\tau}_0\rangle$. Since $\operatorname{rank}(G_{t,d}/G_d) = 1$ and $G_d = \{\mathbf{1}\}$ is a trivial group, we know from Theorem 6.7 that f has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ if and only if there exists a nonzero operator $L_0 \in \mathbb{Q}(t)\langle T_0\rangle$ with $T_0 = S_tS_x^{-4}S_y^{-1}S_z^{-1}$ such that

$$L_0(a) = 0, \ \text{where } a = fd = \frac{1}{(t+1)(t+2z)}.$$

Note that the prime part of the denominator $b = (t + 1)(t + 2z)$ of a with respect to the variables $\{x, y, z\}$ is $b_2 = t + 2z$ and $\tau_0(b_2) \neq b_2$. By Proposition 6.8, there does not exist any operator $L_0 \in \mathbb{Q}(t)\langle T_0\rangle$ such that $L_0(a) = 0$. So f does not have any telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$.

(2) For $f = \frac{1}{(t+1)d}$ with d being the same as in Example 6.15 (1), it is easy to check that for $a = \frac{1}{t+1}$,

$$L_0(a) = 0 \ \text{with } L_0 = T_0 - \frac{t+1}{t+2},$$

where $T_0 = S_tS_x^{-4}S_y^{-1}S_z^{-1}$. So by Theorem 6.7, f has a telescoper L of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$. In fact, we can take $L = S_t - \frac{t+1}{t+2}$. Then

$$\begin{aligned} L(f) &= \frac{\sigma_t(a)}{\sigma_t(d)} - \frac{t+1}{t+2}\cdot\frac{a}{d} = \frac{\sigma_t(a)}{\sigma_x^4\sigma_y\sigma_z(d)} - \frac{t+1}{t+2}\cdot\frac{a}{d} \\ &= \Delta_x(u) + \Delta_y(v) + \Delta_z(w) + \underbrace{\frac{\tau_0(a) - ((t+1)/(t+2))a}{d}}_{=L_0(a)/d=0} \\ &= \Delta_x(u) + \Delta_y(v) + \Delta_z(w), \end{aligned}$$

where $u = \sum_{\ell=0}^{3} \frac{\sigma_t(a)}{\sigma_x^\ell\sigma_y\sigma_z(d)}$, $v = \frac{\sigma_t(a)}{\sigma_z(d)}$, and $w = \frac{\sigma_t(a)}{d}$. Additionally, this is a non-trivial example in two respects. Firstly, since $G_d = \{\mathbf{1}\}$, this rational function f is not $(\sigma_x, \sigma_y, \sigma_z)$-summable in $\mathbb{Q}(t, x, y, z)$. Secondly, for any $\{\mu, \nu\} \subseteq \{x, y, z\}$, since the isotropy group of d in $\langle\sigma_t, \sigma_\mu, \sigma_\nu\rangle$ is trivial and f is not (σ_μ, σ_ν)-summable, by Lemma 6.4, f does not have any telescoper in $\mathbb{Q}(t)\langle S_t\rangle$ of type $(\sigma_t; \sigma_\mu, \sigma_\nu)$.

(3) We continue the Example 6.3 (2) and write f in the form

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + r_1 + r_2,$$

where $u_0, v_0, w_0 \in \mathbb{Q}(t, x, y, z)$ and $r_1 = \frac{1}{t(t+y+2z)d}$, $r_2 = \frac{1}{(t+3z)\sigma_t(d)}$ with $d = 3y + (x + z)^2 + t$.

(a) For $r_1 = a_1/d$ with $a_1 = 1/(t(t + y + 2z))$, we find that a basis of $G_{t,d}$ is $\{\tau_0, \tau_1\}$, where $\tau_0 = \sigma_t^3\sigma_y^{-1}$, $\tau_1 = \sigma_x\sigma_z^{-1}$. Then by Theorem 6.7, r_1 has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ if and only if a_1 has a telescoper of type $(\tau_0; \tau_1)$. Choose one $\mathbb{Q}$-automorphism ϕ_1 of $\mathbb{Q}(t, x, y, z)$ given in Proposition 6.12 as follows

$$\phi_1(h(t, x, y, z)) = h(3t, x, -t + y, -x + z),$$

for any $h \in \mathbb{Q}(x,y,z)$. Then $\phi_1 \circ \tau_0 = \sigma_t \circ \phi_1$ and $\phi_1 \circ \tau_1 = \sigma_x \circ \phi_1$. So a_1 has a telescoper of type $(\tau_0; \tau_1)$ if and only if $\phi_1(a_1)$ has a telescoper of type $(\sigma_t; \sigma_x)$. A direct calculation yields that

$$\phi_1(a_1) = \frac{1}{3t(\underbrace{2t + y - 2x + 2z}_{\tilde{d}})}.$$

Again consider the isotropy group of $\tilde{d}$ in $\langle \sigma_t, \sigma_x \rangle$, which is generated by $\tilde{\tau}_0 = \sigma_t \sigma_x^2$. Since $(\tilde{\tau}_0 - \frac{t}{t+1})(\frac{1}{3t}) = 0$, by similar arguments used in Example 6.15 (2), we see that $\phi_1(a_1)$ has a telescoper $\tilde{L}_1 \in \mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_x)$ and in particular we find

$$\tilde{L}_1(t, S_t)(\phi_1(a_1)) = \Delta_x(\tilde{b}_1) \tag{6.14}$$

with $\tilde{L}_1 = S_t - \frac{t}{t+1}$, $\tilde{b}_1 = -\frac{1}{3(t+1)(2t+y-2x+2+2z)}$. So by Theorem 6.7, r_1 has a telescoper $L \in \mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$. In fact, we can find an explicit expression for L. Applying ϕ_1^{-1} to Equation (6.14) yields that

$$\bar{L}_1(t, T_0)(a_1) = \Delta_{\tau_1}(b_1),$$

where $T_0 = S_t^3 S_y^{-1}$, $\bar{L}_1(t, T_0) = \tilde{L}_1(\frac{t}{3}, T_0) = T_0 - \frac{t}{t+3}$, $b_1 = \phi_1^{-1}(\tilde{b}_1) = -\frac{1}{(t+3)(t+y+2z+2)}$. Let $L_1(t, S_t) = \bar{L}_1(t, S_t^3) = S_t^3 - \frac{t}{t+3}$. Then we have

$$\begin{aligned} L_1(r_1) &= \frac{\sigma_t^3(a_1)}{\sigma_t^3(d)} - \frac{t}{t+3} \cdot \frac{a_1}{d} = \frac{\sigma_t^3(a_1)}{\sigma_y(d)} - \frac{t}{t+3} \cdot \frac{a_1}{d} \\ &= \Delta_x(0) + \Delta_y(v_1) + \Delta_z(0) + \frac{\bar{L}_1(a_1)}{d} \text{ with } v_1 = \frac{\sigma_t^3 \sigma_y^{-1}(a_1)}{d} \end{aligned}$$

and using Lemma 5.8 with $\tau = \tau_1$, we get

$$\frac{\bar{L}_1(a_1)}{d} = \Delta_{\tau_1}\left(\frac{b_1}{d}\right) = \Delta_x(u_1) + \Delta_y(0) + \Delta_z(w_1)$$

with $u_1 = \sigma_z^{-1}\left(\frac{b_1}{d}\right)$ and $w_1 = -\sigma_z^{-1}\left(\frac{b_1}{d}\right)$. Hence L_1 is a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for r_1 and

$$L_1(r_1) = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1).$$

(b) *Similarly, for $r_2 = a_2/\sigma_t(d)$ with $a_2 = 1/(t+3z)$, we apply the algorithm* IsTelescoperable *to r_2. The result is true and we obtain*

$$L_2(r_2) = \Delta_x(u_2) + \Delta_y(v_2) + \Delta_z(w_2),$$

where $L_2 = S_t^3 - 1$, $u_2 = \sigma_z^{-1}\left(\frac{b_2}{\sigma_t(d)}\right)$, $v_2 = \frac{\sigma_t^3 \sigma_y^{-1}(a_2)}{\sigma_t(d)}$ and $w_2 = -\sigma_z^{-1}\left(\frac{b_2}{\sigma_t(d)}\right)$ with $b_2 = -\frac{1}{t+3z+3}$.

(c) *For $r = r_1 + r_2$, using the LCLM algorithm to compute the least common multiple L of L_1, L_2 in $\mathbb{Q}(t)\langle S_t \rangle$, we obtain*

$$L = R_1 L_1 = R_2 L_2 = S_t^6 - \frac{2(t+3)}{t+6} S_t^3 + \frac{t}{t+6}$$

with $R_1 = S_t^3 - \frac{t+3}{t+6}$ and $R_2 = S_t^3 - \frac{t}{t+6}$. Then

$$L(r) = \Delta_x(u) + \Delta_y(v) + \Delta_z(w),$$

where $u = \sum_{i=1}^{2} R_i(u_i)$, $v = \sum_{i=1}^{2} R_i(v_i)$ *and* $u = \sum_{i=1}^{2} R_i(w_i)$ *are rational functions in* $\mathbb{Q}(t, x, y, z)$. *Updating* $u = u + L(u_0)$, $v = v + L(v_0)$ *and* $w = w + L(w_0)$, *we get*

$$L(f) = \Delta_x(u) + \Delta_y(v) + \Delta_z(w).$$

So L *is a telescoper of type* $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ *for* f.

6.3 Examples and applications

Creative telescoping is a powerful tool for proving combinatorial identities algorithmically [61]. The following example shows an application of telescopers for multivariate rational functions.

Example 6.16. *We show that*

$$F(t) = \sum_{x=0}^{t}\sum_{y=0}^{t}\sum_{z=0}^{t} f(t, x, y, z) = 0,$$

where

$$f(t, x, y, z) = \frac{(2y-t)(2x-t)(2z-t)}{(y+t+1)(-2t+y-1)(x+t+1)(-2t+x-1)(z+t+1)(-2t+z-1)}.$$

Applying Algorithm 6.13 to f, *we find that* f *has a telescoper* $L = S_t - 1$ *of type* $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ *with certificates* (u, v, w), *where*

$$u = \frac{(-2y+t+1)(-2z+t+1)(8t^2-2tx-x^2+19t-2x+11)}{(x+t+1)(2t-x+2)(2t-x+3)(y+t+2)(2t-y+3)(z+t+2)(2t-z+3)},$$

$$v = \frac{(-2x+t)(-2z+t+1)(8t^2-2ty-y^2+19t-2y+11)}{(x+t+1)(2t-x+1)(y+t+1)(2t-y+2)(2t-y+3)(z+t+2)(2t-z+3)}$$

and

$$w = \frac{(-2x+t)(-2y+t)(8t^2-2tz-z^2+19t-2z+11)}{(x+t+1)(2t-x+1)(y+t+1)(2t-y+1)(z+t+1)(2t-z+2)(2t-z+3)}.$$

Thus we have

$$\begin{aligned}
&\sum_{x=0}^{t}\sum_{y=0}^{t}\sum_{z=0}^{t} (f(t+1, x, y, z) - f(t, x, y, z)) \\
=&\sum_{x=0}^{t}\sum_{y=0}^{t}\sum_{z=0}^{t} (\Delta_x(u) + \Delta_y(v) + \Delta_z(w)) \\
=&\sum_{y=0}^{t}\sum_{z=0}^{t}(u(t, t+1, y, z) - u(t, 0, y, z)) + \sum_{x=0}^{t}\sum_{z=0}^{t}(v(t, x, t+1, z) - v(t, x, 0, z)) \\
&+ \sum_{x=0}^{t}\sum_{y=0}^{t}(w(t, x, y, t+1) - w(t, x, y, 0)).
\end{aligned}$$

Then applying L to $F(t)$ yields

$$\begin{aligned}
&F(t+1)-F(t)\\
=&\sum_{y=0}^{t}\sum_{z=0}^{t}\underbrace{\left(u(t,t+1,y,z)-u(t,0,y,z)+f(t+1,t+1,y,z)\right)}_{g_1}\\
&+\sum_{x=0}^{t}\sum_{z=0}^{t}\underbrace{\left(v(t,x,t+1,z)-v(t,x,0,z)+f(t+1,x,t+1,z)\right)}_{g_2}\\
&+\sum_{x=0}^{t}\sum_{y=0}^{t}\underbrace{\left(w(t,x,y,t+1)-w(t,x,y,0)+f(t,x,y,t+1)\right)}_{g_3}\\
&+\sum_{x=0}^{t}f(t+1,x,t+1,t+1)+\sum_{y=0}^{t}f(t+1,t+1,y,t+1)+\sum_{z=0}^{t}f(t+1,t+1,t+1,z)\\
&+f(t+1,t+1,t+1,t+1).
\end{aligned}$$

So we reduce the triple sum to the double sums and the single sums. One can check that $g_1 = 0$. By Algorithm 5.13, one can find that g_2 is σ_x-summable and g_3 is (σ_x, σ_y)-summable. Similarly, we further reduce the double sums to the single sums. Applying the Algorithm 5.13 (specialized to the univariate case) again, we simplify the single sums and finally obtain that $F(t+1) - F(t) = 0$. Since the initial value of $F(t)$ is $F(0) = f(0,0,0,0) = 0$, we conclude that $F(t) = 0$. This completes the proof.

Under some assumptions, there are several packages to compute the creative telescoping in more than two variables. The Mathematica package "HolonomicFunctions" developed by Koutschan contains two functions "CreativeTelescoping" and "FindCreativeTelescoping" to construct telescopers for holonomic functions in different ways [52]. Another Mathematica function "FindRecurrence", the core of the Mathematica package "MultiSum" by Wegschaider, is designed to find telescopers for proper hypergeometric functions [79]. For rational functions in three variables, an effective algorithm has been presented in [23] to compute their minimal telescopers.

Experiments show that Algorithm 6.13 is more efficient to test the existence of telescopers and construct one telescoper. For example, for the rational function

$$f(t,x,y)=\frac{4t+2}{(45t+5x+10y+47)(45t+5x+10y+2)(63t-5x+2y+58)(63t-5x+2y-5)},$$

Algorithm 6.13 tells us that it has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y)$ and outputs a telescoper and its corresponding certificates within two seconds in Maple, while the algorithm in [23] takes about three minutes and the other three functions in Mathematica packages use much more timings as shown in [23]. Given a rational function, one could use our algorithm to pre-check the existence of its telescopers and find a telescoper if such a telescoper exists. After that one may apply the other efficient methods to find a telescoper with lower degree in S_t.

7 Conclusion and future work

In this paper, we constructively solve the summability problem and the existence problem of telescopers for multivariate rational functions, and present a new efficient algorithm for solving the shift equivalence testing (SET) problem of multivariate polynomials.

Our algorithm can compute a telescoper for a given multivariate rational function if the existence of telescopers is guaranteed, but the computed telescoper may not be of minimal order. So a natural question is how to compute the minimal telescoper (which is unique if it is monic) for multivariate rational function if it exists. Similar to the trivariate case [23], we may first need an additive decomposition to decompose a rational function f as a sum of a summable function and a remainder r, as shown in Example 5.15, such that f is summable if and only if the remainder r is zero. Then we need to deal with the problem that the sum of two remainders in the additive decomposition may not be a remainder. The similar problem appears and has been solved in the case for trivariate rational functions [23, Section 4] and the case for bivariate hypergeometric terms [25, Section 5].

For the efficiency, we may need to consider how to choose a "minimal" remainder, which may depend on the choices of difference isomorphisms. Choosing a "good" admissible cover may help us to discover a more efficient SET algorithm. In theory, we use an irreducible partial fraction decomposition in summation algorithms, but in practice, an incomplete partial fraction decomposition would be enough, like in the univariate case [9, 60].

In the future research, we hope to explore more summation algorithms for other classes of functions, like multivariate hypergeometric terms [79]. This would be an extension of Gosper's algorithm [36] which only works for the summation of univariate hypergeometric terms and has many applications in proving combinatorial identities [61]. Some algorithms have been developed for special bivariate hypergeometric terms [30] and for multiple binomial sums [15].

In the differential case, telescopers always exist for D-finite functions [82]. One interesting problem is how to find a telescoper [15, 26], especially the minimal one. Another problem is the integrability problem proposed by Picard [62–64], which is a continuous analogue of the summability problem. Given a rational function $f \in \mathbb{F}(x_1, \ldots, x_m)$, the integrability problem is deciding whether there exist rational functions $g_i \in \mathbb{F}(x_1, \ldots, x_m)$ such that

$$f = \partial_{x_1}(g_1) + \cdots + \partial_{x_m}(g_m),$$

where ∂_{x_i} is the usual partial derivative with respect to x_i. When $m = 1$, it can be solved by Ostrogradsky-Hermite reduction [43, 57]. When $m = 2$, it was solved by Picard [63, p. 475–479]. In more than two variables, there is no complete algorithm for deciding the integrability of rational functions. Under a regularity assumption, some related results are listed in [15].

8 Appendix: implementations and timings

We have implemented Algorithms 3.6, 5.13 and 6.13 in the computer algebra system Maple 2020. In this section, we compare the efficiency of the algorithms for solving the SET problem and illustrate the usage of our package "RationalWZ" by several examples. Our maple code and more examples are available for download at

http://www.mmrc.iss.ac.cn/~schen/RationalWZ-2022.html

We have implemented the G algorithm, the KS algorithm, the DOS algorithm, and the algorithm applying **a**-degree cover to Algorithm 3.6 (ADC) in Maple 2020 with $\mathbb{F} = \mathbb{Q}$.

Fixing one admissible cover, there are two methods to calculate it and then to implement Algorithm 3.6. A direct method is expanding $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})$ with $2n$ variables to get the set of its coefficients in $\mathbf{x}$ and then the admissible cover, while another is obtaining the members of the admissible cover successively by computing partial derivatives dynamically. For a more efficient implementation, the DOS algorithm and the ADC algorithm is realized by partial derivatives and expansion respectively.

The test suite was generated as follows.

Let $n, d, t, d' \in \mathbb{N}$ and $d' < d$. Let $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$. We first generated randomly a t-term polynomial $p(\mathbf{x})$ of degree d, as well as a polynomial $dis(\mathbf{x})$ of degree d'. Then we generated randomly a vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ and let $q(\mathbf{x}) = p(\mathbf{x} + \mathbf{a}) + dis(\mathbf{x})$. Therefore, the calculation is most likely to terminate after computing $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{d-1-d'} S_i^H)$. By setting $0 \le a_i \le 99$, we can avoid the case where the memory is not enough to complete the computation.

Note that, in all the tests, the algorithms take the expanded forms of examples given above as input. All timings are measured in seconds on a macOS Monterey (Version 12.0.1) MacBook Pro with 32GB Memory and Apple M1 Pro Chip.

For a selection of random polynomials and vectors for different choices of n, t, d, d' as above, we first tabulate the timings of the G algorithm, the KS algorithm, the DOS algorithm and the ADC algorithm. Note that $d' = -\infty$ means $dis = 0$, implying that p is shift equivalent to q.

n	t	d	d'	G	KS	DOS	ADC
3	10	15	13	5.476	2.090	0.014	0.008
3	10	15	10	0.243	1.124	0.023	0.020
3	10	15	5	21.719	1.809	0.050	0.032
3	10	15	0	573.178	2.576	0.068	0.039
3	10	15	$-\infty$	18.491	0.714	0.043	0.036
3	100	15	13	0.205	10.025	0.044	0.028
3	100	15	10	0.482	9.997	0.046	0.046
3	100	15	5	22.114	11.317	0.061	0.062
3	100	15	0	2152.378	19.470	0.083	0.069
3	100	15	$-\infty$	1200.473	13.640	0.085	0.068

The experimental results illustrate that the DOS algorithm and the ADC algorithm outperform the other two algorithms. Furthermore, we conducted experiments in more complicated cases.

n	t	d	d'	DOS	ADC
5	100	40	35	199.177	59.889
5	100	40	30	24.684	90.159
5	100	40	20	379.835	95.761
5	100	40	10	681.189	665.885
5	100	40	0	182.671	67.261
5	100	40	$-\infty$	709.223	77.880
5	10000	20	18	2.724	122.744
5	10000	20	15	3.088	163.258
5	10000	20	10	5.290	139.685
5	10000	20	5	10.755	125.359
5	10000	20	0	23.949	151.010
5	10000	20	$-\infty$	24.562	136.187

The experimental results indicate that the ADC algorithm outperforms the other for most of non-dense testing examples, while the DOS algorithm has a clear advantage for dense ones. It may be because the timing of expansion grows fast with the number of terms in the input polynomial. In conclusion, we present an algorithm, the ADC algorithm, which is complementary to the DOS algorithm for solving the SET problem.

From the runtime comparison, we decided to use the ADC algorithm in the package RationalWZ. In our setting, the base field $\mathbb{F}$ can be $\mathbb{Q}$ or the field of rational functions $\mathbb{Q}(u_1, \dots, u_s)$. The following instructions show how to load the modules.

```
> read "RationalWZ.mm";
> with(ShiftEquivalenceTesting):
> with(OrbitalDecomposition):
> with(RationalReduction):
> with(RationalSummation):
> with(RationalTelescoping):
```

Example 8.1. *Compute the dispersion set (over $\mathbb{Z}$) of two polynomials.*

(1) *For $p, q \in \mathbb{Q}[x, y]$ in Example 3.4, we type*

```
> ShiftEquivalent(x^2 + 2*x*y + y^2 + 2*x + 6*y, x^2 + 2*x*y + y^2 + 4*x +
   8*y + 11, [x, y])
```

$$[-1, 2]$$

which implies $Z_{p,q} = \{(-1, 2)\}$. So $p(x-1, y+2) = q(x, y)$.

(2) *For $p, q \in \mathbb{Q}[x, y, z]$ in Example 3.21 (1) , we type*

```
> ShiftEquivalent(x^4 + x^3*y + x*y^2 + z^2, x^4 + x^3*(y + 1) + x*(y + 1)^2
    + (z + 2)^2 + x*y, [x, y, z])
```

$$[\,]$$

which implies $Z_{p,q} = \varnothing$. So p, q are not shift equivalent.

Example 8.2. *Decide the $(\sigma_x, \sigma_y, \sigma_z)$-summability of a rational function $f \in \mathbb{Q}(x, y, z)$. Let f_3, r_3 be the same as in Example 5.15 (3).*

(1) *Applying the function "IsSummable" to $f = f_3$, we see that f is not $(\sigma_x, \sigma_y, \sigma_z)$-summable.*

```
> IsSummable((y + z/(y^2 + z - 1) - 1/(y^2 + z))/(x + 2*y + z)^2, [x, y, z])
```

false

(2) *Applying the function "IsSummable" to $f = f_3 - r_3$, we see that f is $(\sigma_x, \sigma_y, \sigma_z)$-summable and its certificates are as follows:*

```
> IsSummable((y + z/(y^2 + z - 1) - 1/(y^2 + z))/(x + 2*y + z)^2 - z/((y^2 +
    z)*(x + 2*y + z)^2), [x, y, z])
```

$$true, \left[-\frac{y(y-1)}{2\,(x-2+2\,y+z)^2} - \frac{y(y-1)}{2\,(x-1+2\,y+z)^2} + \frac{z}{(y^2+z-1)(x-1+2\,y+z)^2}, \frac{y(y-1)}{2\,(x-2+2\,y+z)^2}, -\frac{z}{(y^2+z-1)(x-1+2\,y+z)^2}\right]$$

Example 8.3. *Decide the existence of telescopers of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for a rational function $f \in \mathbb{Q}(t, x, y, z)$.*

(1) *Applying the function "IsTelescoperable" to f in Example 6.15 (1), we see that f does not have a telescoper in $\mathbb{Q}(t)\langle S_t\rangle$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$.*

```
> IsTelescoperable(1/((t + 1)*(t + 2*z)*((t - 3*y + x)^2*(t + y)*(t + z) +
   1)), [x, y, z], t, 'St')
```

false

(2) *Applying the function "IsTelescoperable" to $f = r_1$ in Example 6.15 (3), we see that f has a telescoper $L = -\frac{t}{t+3} + S_t^3$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ and its certificates are as follows:*

```
> IsTelescoperable(1/(t*(t + y + 2*z)*(3*y + (x + z)^2 + t)), [x, y, z], t,
  'St')
```

$$true, -\frac{t}{t+3} + St^3, \left[-\frac{1}{6\left(\frac{t}{2}+\frac{y}{2}+z\right)\left(\frac{t}{3}+1\right)\left(x^2+2\,x(z-1)+(z-1)^2+t+3\,y\right)},\right.$$
$$\left.\frac{1}{(t+3)(t+2+y+2\,z)(x^2+2\,xz+z^2+t+3\,y)}, \frac{1}{6\left(\frac{t}{3}+1\right)\left(\frac{t}{2}+\frac{y}{2}+z\right)\left(x^2+2\,x(z-1)+(z-1)^2+t+3\,y\right)}\right]$$

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