Additive Decompositions in Primitive Extensions*

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ABSTRACT

This paper extends the classical Hermite-Ostrogradsky reduction for rational functions to more general functions in primitive extensions of certain types. For an element f in such an extension K, the extended reduction decomposes f as the sum of a derivative in K and another element r such that f has an antiderivative in K if and only if $r = 0$; and f has an elementary antiderivative over K if and only if r is a linear combination of logarithmic derivatives over the constants when K is a logarithmic extension. Moreover, r is minimal in some sense. Additive decompositions may lead to reduction-based creative-telescoping methods for nested logarithmic functions, which are not necessarily D-finite.

KEYWORDS

Additive decompositions, Creative telescoping, Elementary functions, Symbolic integration

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1 INTRODUCTION

Symbolic integration, together with its discrete counterpart symbolic summation, nowadays has played a crucial role in building the infrastructure for applying computer algebra tools to solve problems in combinatorics and mathematical physics [17, 18, 30]. The early history of symbolic integration starts from the first tries of developing programs in LISP to evaluate integrals in freshman calculus symbolically in the 1960s. Two representative packages at the time were Slagle's SAINT [31] and Moses's SIN [22], which were both based on integral transformation rules and pattern recognition. The algebraic approach for symbolic integration is initialized by Ritt [28] in terms of differential algebra [16], which eventually leads to the Risch algorithm for the integration of elementary functions [26, 27]. The efficiency of the Risch algorithm is further improved by Rothstein [29], Davenport [13], Trager [32], Bronstein [7, 8] etc. Some standard references on this topic are Bronstein's book [9] and Raab's survey [25] that gives an overview of the Risch algorithm and its recent developments.

The central problem in symbolic integration is whether the integral of a given function can be written in "closed form". Its algebraic formulation is given in terms of differential fields and their extensions [9, 16]. A differential field F is a field together with a derivation $'$ that is an additive map on F satisfying the product rule $(fg)' = f'g + fg'$ for all $f, g \in F$. A given element f in F is said to be *integrable* in F if $f = g'$ for some $g \in F$. The problem of deciding whether a given element is integrable or not in F is called the *integrability problem* in F. For example, if F is the field of rational functions, then for $f = 1/x^2$ we can find $g = -1/x$, while for $f = 1/x$ no suitable g exists in F. When f is not integrable in F, there are several other questions we may ask. One possibility is to ask whether there is a pair (g, r) in $F \times F$ such that $f = g' + r$, where r is minimal in some sense and $r = 0$ if f is integrable. This problem is called the *decomposition problem* in F. Extensive work has been done to solve the integrability and decomposition problems in differential fields of various kinds.

Abel and Liouville pioneered the early work on the integrability problem in the 19th century [28]. In 1833, Liouville provided a first decision procedure for solving the integrability problem on algebraic functions [20]. For an overview of Liouville's work on integration in finite terms, we refer to Lützen's book [21, pp. 351–422]. For other classes of functions, complete algorithms for solving the integrability problem are much more recent: 1) the Risch algorithm [26, 27] in the case of elementary functions was presented in 1969; 2) the Almkvist–Zeilberger algorithm [2] (also known as the differential Gosper algorithm) in the case of hyperexponential functions was given in 1990; 3) Abramov and van Hoeij's algorithm [1] generalized the previous algorithm to the general D-finite functions of arbitrary order in 1997.

The decomposition problem was first considered by Ostrogradsky [23] in 1845 and later by Hermite [15] for rational functions. The idea of Ostrogradsky and Hermite is crucial for algorithmic treatments of the problem, since it avoids the root-finding of polynomials and only uses the extended Euclidean algorithm and squarefree factorization to obtain the additive decomposition of a rational function. This reduction is a basic tool for the integration of rational functions and also plays an important role in the base case of our work. We will refer to this reduction as the *rational reduction* in this paper. The rational reduction has been extended to more general classes of functions including algebraic functions [10, 32], hyperexponential functions [4, 14], multivariate rational functions [5, 19], and more recently including D-finite functions [6, 12, 33]. Blending reductions with creative telescoping [2, 34] leads to the fourth and

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most recent generation of creative telescoping algorithms, which are called reduction-based algorithms [3–5, 10, 12].

The telescoping problem can also be formulated for elementary functions [11, 24]. Two related problems are how to decide the existence of telescopers for elementary functions and how to compute one if telescopers exist. Reduction algorithms have been shown to be crucial for solving these two problems. This naturally motivates us to design reduction algorithms for elementary functions.

In this paper, we extend the rational reduction to elements in straight and flat towers of primitive extensions (see Definition 3.5). Our extended reductions solve the decomposition problems in such towers without solving any Risch equations (Theorems 4.8 and 5.15), and determine elementary integrability in such towers when primitive extensions are logarithmic (Theorem 6.1).

The remainder of this paper is organized as follows. We present basic notions and terminologies on differential fields, and collect some useful facts about integrability in primitive extensions in Section 2. We define the notions of straight and flat towers, and describe some straightforward reduction processes in Section 3. Additive decompositions in straight and flat towers are given in Sections 4 and 5, respectively. The two decompositions are used to determine elementary integrability in Section 6. Examples are given in Section 7 to illustrate that the decompositions may be useful to study the telescoping problem for elementary functions that are not D-finite.

2 PRELIMINARIES

Let $(F, ')$ be a differential field of characteristic zero. An element c of F is called a constant if $c' = 0$. Let C_F denote the set of constants in F, which is a subfield of F. Let (E, D) be a differential field containing F. We say that E is a differential field extension of F if the restriction of D on F is equal to the derivation $'$. The derivation D is also denoted by $'$ when there is no confusion.

Let $(E, ')$ be a differential field extension of F. For $S \subset E$, we use S' to denote the set $\{f' \mid f \in S\}$. If S is a C_E-linear subspace, so is S'. For $a, b \in E$, we write $a \equiv b \mod S$ if $a - b \in S$.

An element z of E is said to be *primitive* over F if $z' \in F$. If z is primitive and transcendental over F with $C_{F(z)} = C_F$, then it is called a *primitive monomial* over F, which is a special instance of Liouvillian monomials according to Definition 5.1.2 in [9].

Let z be a primitive monomial over F in the rest of this section. For $p \in F[z]$, the degree and leading coefficient of p are denoted by $\deg_z(p)$ and $\operatorname{lc}_z(p)$, respectively. By Theorem 5.1.1 in [9], p is squarefree if and only if $\gcd(p, p') = 1$. For $m \in \mathbb{N}$, $F[z]^{(m)}$ stands for $\{p \in F[z] \mid \deg_z(p) < m\}$.

An element $f \in F(z)$ is said to be *z-proper* if the degree of its numerator in z is lower than that of its denominator. In particular, zero is z-proper. It is well-known that f can be uniquely written as the sum of a z-proper element and a polynomial in z. They are called the fractional and polynomial parts of f, and denoted by $\operatorname{fp}_z(f)$ and $\operatorname{pp}_z(f)$, respectively.

An element of $F(z)$ is simple if its denominator is squarefree. By Theorem 5.3.1 in [9], for $f \in F(z)$, there exists a simple element h of $F(z)$ such that $f \equiv h \mod F(z)' + F[z]$. It follows that $f \equiv \operatorname{fp}_z(h) \mod F(z)' + F[z]$, which allows us to focus on simple and z-proper elements. So we say that an element of $F(z)$ is *z-simple* if it is both simple and z-proper. For $f \in F(z)$, Algorithm `HermiteReduce` in [9, page 139] computes a z-simple element g in $F(z)$ such that $f \equiv g \mod F(z)' + F[z]$. This algorithm is fundamental for our additive decompositions in primitive extensions.

Lemma 2.1. *Let $g \in F(z)' + F[z]$. Then $g = 0$ if it is z-simple.*

Proof. Suppose that $g \neq 0$. Since g is z-proper, there exists a nontrivial irreducible polynomial $p \in F[z]$ dividing the denominator of g. Since $g \in F(z)' + F[z]$, there exist $a \in F(z)$ and $b \in F[z]$ such that $g = a' + b$. The order of g at p is equal to -1. But the order of a' at p is either nonnegative or less than -1 by Lemma 4.4.2 (i) in [9], and the order of b at p is nonnegative, a contradiction. ∎

Every element $f \in F(z)$ is congruent to a unique z-simple element g modulo $F(z)' + F[z]$ by Theorem 5.3.1 in [9] and Lemma 2.1. We call g the *Hermitian part* of f with respect to z, denoted by $\operatorname{hp}_z(f)$. The map hp_z is C_F-linear on $F(z)$. Its kernel is $F(z)' + F[z]$. Thus, two elements have the same Hermitian parts if they are congruent modulo $F(z)' + F[z]$. This fact is frequently used later.

Example 2.2. *Let $F = \mathbb{C}(x)$ with $x' = 1$ and $z = \log(x)$. Then z is a primitive monomial over F. By Theorem 5.1.1 in [9], $C_{F(z)} = C_F$. Applying Algorithm* `HermiteReduce`*, we have*

$$f := \frac{z^3 + 2xz^2 + z - x^3 - 1}{z^2 + 2xz + x^2} = \left(\frac{x}{z+x}\right)' - \frac{x^2}{z+x} + z.$$

Then $f \equiv -x^2/(z+x) \mod F(z)' + F[z]$ and $\operatorname{hp}_z(f) = -x^2/(z+x)$.

Now, we collect some basic facts about primitive monomials. They are either straightforward or scattered in [9]. We list them below for the reader's convenience.

Lemma 2.3. *If p belongs to both $F[z]$ and $F(z)'$, then there exists c in C_F such that $\operatorname{lc}_z(p) \equiv cz' \mod F'$.*

Proof. Assume $p = r'$ for some $r \in F(z)$. Then $r \in F[z]$ by Lemma 4.4.2.(i) in [9]. Set $d = \deg_z(p)$ and $\ell = \operatorname{lc}_z(p)$. Then $\deg_z(r) \leq d+1$ by Lemma 5.1.2 in [9]. Assume that $r \equiv az^{d+1} + bz^d \mod F[z]^{(d)}$ for some $a, b \in F$. Then

$$r' \equiv a'z^{d+1} + ((d+1)az' + b')z^d \mod F[z]^{(d)}.$$

Since $p = r'$, we have that $a' = 0$ and $\ell = (d+1)az' + b'$. It follows that $\ell \equiv cz' \mod F'$ with $c = (d+1)a$. ∎

The next lemma will be used to decrease the degree of a polynomial modulo $F(z)'$. Its proof is a straightforward application of integration by parts.

Lemma 2.4. *For all $f \in F$ and $d \in \mathbb{N}$, we have*

$$f'z^d \equiv 0 \mod F(z)' + F[z]^{(d)}.$$

Recall that an element f in F is said to be a *logarithmic derivative in F* if $f = a'/a$ for some nonzero element $a \in F$.

Lemma 2.5. *Let f be a logarithmic derivative in $F(z)$. Then $\operatorname{hp}_z(f)$ is a logarithmic derivative in $F(z)$, and $f = \operatorname{hp}_z(f) + r$, where r is a logarithmic derivative in F.*

Proof. If $f = 0$, then we choose $r = 0$, which equals $1'/1$. Otherwise, there exist two monic polynomials $u, v \in F[z]$ and $w \in F$ such that $f = u'/u - v'/v + w'/w$ by the logarithmic derivative identity on page 104 of [9]. Note that $u'/u - v'/v$ is z-simple by Lemma 5.1.2 in [9] and w'/w is in F. Thus, $\mathrm{hp}_z(f) = u'/u - v'/v$ and $r = w'/w$. ∎

3 PRIMITIVE EXTENSIONS

Let $(K_0, ')$ be a differential field of characteristic zero. Set $C = C_{K_0}$. Consider a tower of differential fields

$$K_0 \subset K_1 \subset \cdots \subset K_n, \tag{3.1}$$

where $K_i = K_{i-1}(t_i)$ for all i with $1 \le i \le n$. The tower given in (3.1) is said to be *primitive over* K_0 if t_i is a primitive monomial over K_{i-1} for all i with $1 \le i \le n$. The notation introduced in (3.1) will be used in the rest of the paper.

The assumption $C_{K_n} = C$ has a useful consequence.

Lemma 3.1. *Let the tower* (3.1) *be primitive.*

(i) *$t_1', \ldots, t_n'$ are linearly independent over C;*

(ii) *If $K_0 = C(t_0)$ with $t_0' = 1$ and $t_i' \in K_0$ for some i with $1 \le i \le n$, then $\mathrm{hp}_{t_0}(t_i')$ is nonzero.*

Proof. (i) If $c_1 t_1' + \cdots + c_n t_n' = 0$ for some $c_1, \ldots, c_n \in C$, then $c_1 t_1 + \cdots + c_n t_n \in C$, which implies that $c_1 = \cdots = c_n = 0$, because $t_1, \ldots, t_n$ are algebraically independent over K_0.

(ii) By the rational reduction, $t_i' = u' + v$ for some $u, v \in K_0$ with v being t_0-simple. Then v is nonzero. For otherwise, $t_i - u$ would be a constant outside C. ∎

The following lemma tells us a way to modify the leading coefficient of a polynomial in $K_{n-1}[t_n]$ via integration by parts and Algorithm HermiteReduce.

Lemma 3.2. *Let the tower* (3.1) *be primitive with $n \ge 1$. Then, for all $\ell \in K_{n-1}$ and $d \in \mathbb{N}$, there exist a t_{n-1}-simple element $g \in K_{n-1}$ and a polynomial $h \in K_{n-2}[t_{n-1}]$ such that*

$$\ell t_n^d \equiv (g+h)t_n^d \mod K_n' + K_{n-1}[t_n]^{(d)}.$$

Proof. By Algorithm HermiteReduce, there are $f, g \in K_{n-1}$ with g being t_{n-1}-simple, and $h \in K_{n-2}[t_{n-1}]$ such that $\ell = f' + g + h$. Then $\ell t_n^d = f' t_n^d + (g+h)t_n^d$. Applying Lemma 2.4 to the term $f' t_n^d$, we prove the lemma. ∎

Let $\prec$ be the purely lexicographic ordering on the set of monomials in $t_1, t_2, \ldots, t_n$ with $t_1 \prec t_2 \prec \ldots \prec t_n$. For $i \in \{0, 1, \ldots, n-1\}$ and $p \in K_i[t_{i+1}, \ldots, t_n]$ with $p \ne 0$, the head monomial of p, denoted by $\mathrm{hm}_i(p)$, is defined to be the highest monomial in $t_{i+1}, \ldots, t_n$ appearing in p with respect to $\prec$. The head coefficient of p, denoted by $\mathrm{hc}_i(p)$, is defined to be the coefficient of $\mathrm{hm}_i(p)$, which belongs to K_i. The head coefficient of zero is set to be zero.

Example 3.3. *Let $\xi = t_1^2 t_2 t_3 + t_2 t_3$. Viewing ξ as an element of $K_0[t_1, t_2, t_3]$, we have $\mathrm{hm}_0(\xi) = t_1^2 t_2 t_3$ and $\mathrm{hc}_0(\xi) = 1$, while, viewing ξ in $K_1[t_2, t_3]$, we have $\mathrm{hm}_1(\xi) = t_2 t_3$ and $\mathrm{hc}_1(\xi) = t_1^2 + 1$.*

The next lemma will be used in Section 5. We present it below because it holds for primitive towers.

Lemma 3.4. *Let $n \ge 1$. For a polynomial $p \in K_{n-1}[t_n]$, there are polynomials $p_i \in K_i[t_{i+1}, \ldots, t_n]$ such that $p \equiv \sum_{i=0}^{n-1} p_i \mod K_n'$, and that $\mathrm{hc}_i(p_i)$ is t_i-simple for all i with $1 \le i \le n-1$. Moreover, $\deg_{t_n}(p_i) \le \deg_{t_n}(p)$ for all i with $0 \le i \le n-1$.*

Proof. We proceed by induction on n. If $n = 1$, then it suffices to set $p_0 = p$. Assume that $n > 1$ and that the lemma holds for $n-1$.

Let $p \in K_{n-1}[t_n]$ and $d = \deg_{t_n}(p)$. By Lemma 3.2,

$$p \equiv (g+h)t_n^d \mod K_n' + K_{n-1}[t_n]^{(d)},$$

where $g \in K_{n-1}$ is t_{n-1}-simple and $h \in K_{n-2}[t_{n-1}]$. By the induction hypothesis, there exist $h_j \in K_j[t_{j+1}, \ldots, t_{n-1}]$ such that $h = \sum_{j=0}^{n-2} h_j + u'$ for some u in K_{n-1} and that $\mathrm{hc}_j(h_j)$ is t_j-simple for all j with $1 \le j \le n-2$. Moreover, set $h_{n-1} = g$. By Lemma 2.4,

$$p \equiv \sum_{j=0}^{n-1} h_j t_n^d \mod K_n' + K_{n-1}[t_n]^{(d)}. \tag{3.2}$$

We need to argue inductively on d. If $d = 0$, then it is sufficient to set $p_j = h_j$ for all j with $0 \le j \le n-1$, as $K_{n-1}[t_n]^{(0)} = \{0\}$. Assume that $d > 0$ and that the lemma holds for all polynomials in $K_{n-1}[t_n]^{(d)}$. By (3.2) and the induction hypothesis on d, we have

$$p \equiv \sum_{j=0}^{n-1} h_j t_n^d + \sum_{j=0}^{n-1} \tilde{p}_j \mod K_n',$$

where $\tilde{p}_j$ is in $K_j[t_{j+1}, \ldots, t_n]$, $\mathrm{hc}_j(\tilde{p}_j)$ is t_j-simple when $j \ge 1$, and $\deg_{t_n}(\tilde{p}_j) < d$. Set $p_j = h_j t_n^d + \tilde{p}_j$. Then $p \equiv \sum_{j=0}^{n-1} p_j \mod K_n'$. Since $\mathrm{hc}_j(p_j)$ is $\mathrm{hc}_j(h_j)$ if $h_j \ne 0$ and $\mathrm{hc}_j(p_j)$ is $\mathrm{hc}_j(\tilde{p}_j)$ if $h_j = 0$, the requirements on each $\mathrm{hc}_j(p_j)$ with $j \ge 1$ are fulfilled. The induction on d is completed, and so is the induction on n. ∎

Definition 3.5. *Let the tower* (3.1) *be primitive. Then it is said to be* straight *if $\mathrm{hp}_{t_{i-1}}(t_i') \ne 0$ for all i with $2 \le i \le n$. The tower is said to be* flat *if $t_i' \in K_0$ for all i with $1 \le i \le n$.*

Example 3.6. *Let $K_0 = \mathbb{C}(x)$ with $x' = 1$. Let*

$$\log(x) = \int x^{-1} dx \quad \text{and} \quad \mathrm{Li}(x) = \int \log(x)^{-1} dx.$$

Then the tower $K_0 \subset K_0(\log(x)) \subset K_0(\log(x), \mathrm{Li}(x))$ is straight, while the tower $K_0 \subset K_0(\log(x)) \subset K_0(\log(x), \log(x+1))$ is flat. They contain no new constants by Lemma 5.1.1 in [9].

In this paper, we consider additive decompositions for elements in either straight or flat towers, where $K_0 = C(t_0)$ with $t_0' = 1$.

4 STRAIGHT TOWERS

In this section, we assume that the tower (3.1) is straight and that $K_0 = C(t_0)$ with $t_0' = 1$. The subfield C of constants is denoted by K_{-1} in recursive definitions and induction proofs to be carried out.

Our idea is to reduce a polynomial in $K_{n-1}[t_n]$ to another one of lower degree via integration by parts, whenever it is possible. The notion of t_n-rigid elements describes $r \in K_{n-1}$ such that $r t_n^d$ cannot be congruent to a polynomial of degree lower than d modulo K_n'.

Definition 4.1. *An element $r \in K_{-1}$ is said to be* t_0-rigid *if $r = 0$. Let $r \in K_{n-1}$, $f = \mathrm{fp}_{t_{n-1}}(r)$ and $p = \mathrm{pp}_{t_{n-1}}(r)$. We say that r is* t_n-rigid *if f is t_{n-1}-simple, $f \ne c\,\mathrm{hp}_{t_{n-1}}(t_n')$ for any nonzero $c \in C$, and $\mathrm{lc}_{t_{n-1}}(p)$ is t_{n-1}-rigid.*

Zero is t_n-rigid because $\mathrm{hp}_{t_{n-1}}(t_n')$ is nonzero. Furthermore, let r be t_{n-1}-simple. Then $r t_n^d$ cannot be congruent to a polynomial of a lower degree if and only if r is t_n-rigid by Lemma 2.3.

Example 4.2. *Let $t_0 = x$, $t_1 = \log(x)$ and $t_2 = \mathrm{Li}(x)$. Let*

$$\ell_1 = \frac{1}{x + k_1} \quad \text{and} \quad \ell_2 = \frac{1}{t_1 + k_2} + \ell_1 t_1^2 + x t_1 + x^2.$$

Then ℓ_1 is t_1-rigid if $k_1 \neq 0$ and ℓ_2 is t_2-rigid if $k_1 k_2 \neq 0$.

The next lemma, together with Lemma 2.3, reveals that a nonzero polynomial p in $K_{n-1}[t_n]$ with a t_n-rigid leading coefficient has no antiderivative in K_n.

Lemma 4.3. *Let $r \in K_{n-1}$ be t_n-rigid. If*

$$r \equiv c t_n' \mod K_{n-1}' \tag{4.1}$$

for some $c \in C$, then both r and c are zero.

Proof. We proceed by induction on n. If $n = 0$, then $r = 0$ by Definition 4.1. Thus, $c t_0' \equiv 0 \mod K_{-1}'$. Consequently, $c = 0$ because $K_{-1}' = \{0\}$ and $t_0' = 1$.

Assume that $n > 0$ and that the lemma holds for $n - 1$.

Set $f = \mathrm{fp}_{t_{n-1}}(r)$. Then $f = \mathrm{hp}_{t_{n-1}}(r)$, since f is t_{n-1}-simple by Definition 4.1. Applying the map $\mathrm{hp}_{t_{n-1}}$ to (4.1), we have $f = c\,\mathrm{hp}_{t_{n-1}}(t_n')$ by Lemma 2.1. Hence, $c = 0$ and $f = 0$ by Definition 4.1.

Set $p = \mathrm{pp}_{t_{n-1}}(r)$. Then (4.1) becomes $p \equiv 0 \mod K_{n-1}'$, which, together with Lemma 2.3, implies that $\mathrm{lc}_{t_{n-1}}(p) \equiv \tilde{c} t_{n-1}' \mod K_{n-2}'$ for some $\tilde{c} \in C$. It follows from the induction hypothesis that $\mathrm{lc}_{t_{n-1}}(p)$ is zero, and so is p. Thus, r is zero. ∎

In $K_{n-1}[t_n]$, we define a class of polynomials that have no antiderivatives in K_n.

Definition 4.4. *For $n \geq 0$, a polynomial in $K_{n-1}[t_n]$ is said to be t_n-*straight *if its leading coefficient is t_n-rigid.*

Zero is a t_n-straight polynomial, because its leading coefficient is zero, which is t_n-rigid.

Proposition 4.5. *Let $p \in K_{n-1}[t_n]$ be a t_n-straight polynomial. Then $p = 0$ if $p \in K_n'$.*

Proof. By Lemma 2.3, $\mathrm{lc}_{t_n}(p) \equiv c t_n' \mod K_{n-1}'$ for some $c \in C$. Then $\mathrm{lc}_{t_n}(p) = 0$ by Lemma 4.3. Consequently, $p = 0$. ∎

Next, we reduce a polynomial to a t_n-straight one.

Lemma 4.6. *For $p \in K_{n-1}[t_n]$, there exists a t_n-straight polynomial $q \in K_{n-1}[t_n]$ with $\deg_{t_n}(q) \leq \deg_{t_n}(p)$ such that $p \equiv q \mod K_n'$.*

Proof. If $p = 0$, then we choose $q = 0$. Assume that p is nonzero. We proceed by induction on n.

If $n = 0$, then $p \equiv 0 \mod K_0'$, as every element of $K_{-1}[t_0]$ has an antiderivative in the same ring.

Assume that $n > 0$ and that the lemma holds for $n - 1$.

Let $p \in K_{n-1}[t_n]$ with degree d and leading coefficient ℓ. We are going to concoct a t_n-rigid element r such that

$$\ell \equiv r \mod K_n'. \tag{4.2}$$

This congruence helps us decrease degrees.

By Algorithm `HermiteReduce`, there are t_{n-1}-simple elements g, u in K_{n-1} and polynomials h, v in $K_{n-2}[t_{n-1}]$ such that

$$\ell \equiv g + h \mod K_{n-1}' \quad \text{and} \quad t_n' \equiv u + v \mod K_{n-1}'.$$

By the induction hypothesis, for any $c \in C$, $h - cv \equiv \tilde{h}_c \mod K_{n-1}'$, where $\tilde{h}_c$ is a t_{n-1}-straight polynomial in $K_{n-2}[t_{n-1}]$. It follows that $\ell \equiv g - cu + \tilde{h}_c \mod K_n'$. If there exists $\tilde{c} \in C$ such that $g = \tilde{c}u$, then let $r = \tilde{h}_{\tilde{c}}$. Otherwise, let $c = 0$ and $r = g + \tilde{h}_0$. Then $r \in K_{n-1}$ is t_n-rigid and (4.2) holds.

If $d = 0$, then $p = \ell$. By (4.2), we have $p \equiv r \mod K_n'$. Let $q = r$, which is t_n-straight by Definition 4.4.

Assume that $d > 0$ and each polynomial in $K_{n-1}[t_n]^{(d)}$ is congruent to a t_n-straight polynomial modulo K_n'. It follows from (4.2) and Lemma 2.3 that $\ell \equiv r + c t_n' \mod K_{n-1}'$. By Lemma 2.4 and the equality $c t_n' t_n^d = \left(\frac{c}{d+1} t_n^{d+1}\right)'$, we have $p \equiv r t_n^d + \tilde{q} \mod K_n'$ for some $\tilde{q} \in K_{n-1}[t_n]^{(d)}$. If $r \neq 0$, then set $q = r t_n^d + \tilde{q}$. Otherwise, applying the induction hypothesis on d to $\tilde{q}$ yields a t_n-straight polynomial q with $p \equiv q \mod K_n'$. The above reduction clearly implies that $\deg_{t_n}(q) < \deg_{t_n}(p)$. ∎

Example 4.7. *Let us consider $\int \log(x)\,\mathrm{Li}(x)^2\,dx$. Set $t_1 = \log(x)$ and $t_2 = \mathrm{Li}(x)$. Then we reduce the integrand $t_1 t_2^2$. We have that $\mathrm{lc}_{t_2}(t_1 t_2^2) = t_1$. Since t_1 is not t_2-rigid, $t_1 t_2^2$ can be reduced. In fact, $t_1 t_2^2 = x' t_1 t_2^2$. By Lemma 2.4 and a straightforward calculation, we get that $t_1 t_2^2 \equiv (2x/t_1) t_2 + (x^2/t_1) \mod C(x, t_1, t_2)'$. Since $2x/t_1$ is t_2-rigid, we have $(2x/t_1)t_2 + (x^2/t_1)$ is t_2-straight. Hence, $t_1 t_2^2$ has no antiderivative in $C(x, t_1, t_2)$ by Proposition 4.5.*

Below is an additive decomposition in a straight tower.

Theorem 4.8. *For $f \in K_n$, the following assertions hold.*

(i) *There exist a t_n-simple element $g \in K_n$ and a t_n-straight polynomial $p \in K_{n-1}[t_n]$ such that*

$$f \equiv g + p \mod K_n'. \tag{4.3}$$

(ii) *$f \in K_n'$ if and only if both g and p in (4.3) are zero.*

(iii) *If $f \equiv \tilde{g} + \tilde{p} \mod K_n'$, where $\tilde{g} \in K_n$ is a t_n-simple element and $\tilde{p} \in K_{n-1}[t_n]$, then $g = \tilde{g}$ and $\deg_{t_n}(p) \leq \deg_{t_n}(\tilde{p})$.*

Proof. (i) By Algorithm `HermiteReduce`, there exist a t_n-simple element $g \in K_n$ and a polynomial $h \in K_{n-1}[t_n]$ such that

$$f \equiv g + h \mod K_n'.$$

By Lemma 4.6, h can be replaced by a t_n-straight polynomial p.

(ii) Since $f \in K_n'$, the congruence (4.3) becomes $g + p \equiv 0 \mod K_n'$. Applying the map hp_{t_n} to the new congruence, we have that $g = 0$, because $g = \mathrm{hp}_{t_n}(g + p)$. Thus, $p = 0$ by Proposition 4.5.

(iii) Since $g - \tilde{g} \equiv \tilde{p} - p \mod K_n'$, we have $g = \tilde{g}$ by Lemma 2.1. If $\deg_{t_n}(\tilde{p}) < \deg_{t_n}(p)$, then $p - \tilde{p}$ is t_n-straight, because $\mathrm{lc}_{t_n}(p - \tilde{p})$ equals $\mathrm{lc}_{t_n}(p)$. So $p - \tilde{p} = 0$ by Proposition 4.5, a contradiction. ∎

Example 4.9. *Consider the integral*

$$\int \frac{1}{\mathrm{Li}(x)^2} + \log(x)\,\mathrm{Li}(x)^2\,dx.$$

The integrand is equal to $(-\log(x)/\mathrm{Li}(x))' + 1/(x\,\mathrm{Li}(x)) + \log(x)\,\mathrm{Li}(x)^2$ by Algorithm `HermiteReduce`. Therefore, it has no antiderivative in $C(x, \log(x), \mathrm{Li}(x))$ by Theorem 4.8 and Example 4.7.

5 FLAT TOWERS

In this section, we let the tower (3.1) be flat. The ground field K_0 will be specialized to $C(t_0)$ later in this section. We are not able to fully carry out the same idea as in Section 4, because $\mathrm{hp}_{t_{i-1}}(t_i') = 0$

for all $i = 2, \ldots, n$. This spoils Lemma 4.3 and Proposition 4.5. So we need to study integrability in a flat tower differently.

This section is divided into two parts. First, we extend Lemma 2.4 to the differential ring $K_0[t_1, \ldots, t_n]$. Second, we present a flat counterpart of the results in Section 4.

5.1 Scales

Let us denote $K_0[t_1, \ldots, t_n]$ by R_n. For a monomial ξ in $t_1, \ldots, t_n$, the C-linear subspace $\{p \in R_n \mid p \prec \xi\}$ is denoted by $R_n^{(\xi)}$. The notion of scales is motivated by the following example.

Example 5.1. *Let $n = 2$, and $\xi_0 = 1, \xi_1 = t_1$ and $\xi_2 = t_2$. And let $\ell = t_1' + t_2'$. Using integration by parts, we find three congruences*

$$\ell\xi_0 \equiv 0 \mod K_2', \quad \ell\xi_1 \equiv -t_1' t_2 \mod K_2', \quad \ell\xi_2 \equiv -t_2' t_1 \mod K_2'.$$

The first and third congruences lead to monomials lower than ξ_0 and ξ_2, respectively. But the second one leads to t_2, which is higher than ξ_1. The notion of scales aims to prevent the second congruence from the reduction to be carried out.

Definition 5.2. *For $p \in R_n \setminus K_0$ with $\operatorname{hm}_0(p) = t_1^{e_1} \cdots t_n^{e_n}$, the* scale *of p with respect to n is defined to be s if $e_1 = 0, \ldots, e_{s-1} = 0$ and $e_s > 0$. For $p \in K_0$, the* scale *of p with respect to n is defined to be n. The scale of p with respect to n is denoted by* $\operatorname{scale}_n(p)$.

Example 5.3. *Let $\xi_0 = 1$, $\xi_1 = t_1 t_2$ and $\xi_2 = t_3^2$. Regarding ξ_0, ξ_1 and ξ_2 as elements in R_3, we have that $\operatorname{scale}_3(\xi_0) = 3$, $\operatorname{scale}_3(\xi_1) = 1$ and $\operatorname{scale}_3(\xi_2) = 3$; while, regarding them as elements in R_4, we have that $\operatorname{scale}_4(\xi_0) = 4$, $\operatorname{scale}_4(\xi_1) = 1$ and $\operatorname{scale}_4(\xi_2) = 3$.*

Notably, if $p \in K_0$, then $\operatorname{scale}_n(p) = n$, which varies as n does. On the other hand, $\operatorname{scale}_n(p) = \operatorname{scale}_m(p)$ if $p \in R_m \setminus K_0$ with $m \leq n$.

The next lemma extends Lemma 2.4 and indicates what kind of integration by parts will be used for reduction.

Lemma 5.4. *Let ξ be a monomial in $t_1, \ldots, t_n$. Then the following assertions hold.*

(i) *For all $f \in K_0$, $f'\xi \equiv 0 \mod K_n' + R_n^{(\xi)}$.*
(ii) *Let $s = \operatorname{scale}_n(\xi)$. Then, for all $c_1, \ldots, c_s \in C$,*

$$(c_1 t_1' + \cdots + c_s t_s')\xi \equiv 0 \mod K_n' + R_n^{(\xi)}.$$

Proof. (i) It follows from integration by parts and the fact that ξ' belongs to $R_n^{(\xi)}$.

(ii) Set $L_0 = 0$ and $L_i = \sum_{j=1}^{i} c_j t_j$ for $i = 1, \ldots, n$.

If $\xi = 1$, then $s = n$ and $L_n'\xi \in K_n'$. The assertion clearly holds. Assume that $\xi = t_s^{e_s} \cdots t_n^{e_n}$ with $e_s > 0$. Then $L_s'\xi = L_{s-1}'\xi + c_s t_s'\xi$. Note that $L_{s-1}'\xi$ belongs to $K_n' + R_n^{(\xi)}$ by a direct use of integration by parts. Set $\eta = \xi / t_s^{e_s}$. Then the term $c_s t_s' \xi$ is equal to $\frac{c_s}{e_s+1}\left(t_s^{e_s+1}\right)'\eta$. Integration by parts leads to

$$c_s t_s' \xi \equiv \frac{-c_s}{e_s + 1} t_s^{e_s+1} \eta' \mod K_n'. \tag{5.1}$$

If $\eta = 1$, then $c_s t_s' \xi \in K_n'$ by (5.1). Otherwise, we have $e_j > 0$ for some j with $s < j \leq n$. Then each monomial in $t_s^{e_s+1}\eta'$ is of total degree $\sum_{j=s}^{n} e_j$ and is of degree $e_s + 1$ in t_s. So $t_s^{e_s+1}\eta' \prec \xi$. Consequently, $c_s t_s' \xi \in K_n' + R_n^{(\xi)}$ by (5.1). ∎

In the rest of this section, we let $K_0 = C(t_0)$ with $t_0' = 1$. By Lemma 3.1 (ii), we may further assume that t_i' is nonzero and t_0-simple for all i with $1 \leq i \leq n$.

Definition 5.5. *For every k with $1 \leq k \leq n$, an element of K_0 is said to be* k-rigid *if either it is zero or it is t_0-simple and not a C-linear combination of $t_1', \ldots, t_k'$.*

Proposition 5.6. *For $p \in R_n$, there exists $q \in R_n$ such that $p \equiv q \mod K_n'$ and that $\operatorname{hc}_0(q)$ is s-rigid, where $s = \operatorname{scale}_n(q)$. Moreover, we have $\operatorname{hm}_0(q) \preceq \operatorname{hm}_0(p)$.*

Proof. Set $q = 0$ if $p = 0$. Assume $p \neq 0$ and $\xi = \operatorname{hm}_0(p)$. By the rational reduction, $\operatorname{hc}_0(p) = f' + g$ for some $f, g \in K_0$ with g being t_0-simple. Then $p = f'\xi + g\xi \mod R_n^{(\xi)}$. By Lemma 5.4 (i), $p \equiv g\xi + r \mod K_n'$ for some $r \in R_n^{(\xi)}$. Set $s = \operatorname{scale}_n(\xi)$. If g is nonzero and s-rigid, then set $q = g\xi + r$. Otherwise, $p \equiv \tilde{r} \mod K_n'$ for some $\tilde{r} \in R_n^{(\xi)}$ by Lemma 5.4 (ii). The proposition follows from a direct Noetherian induction on $\operatorname{hm}_0(\tilde{r})$ with respect to $\prec$. ∎

Example 5.7. *Let $K_0 = \mathbb{C}(x)$, $t_1 = \log(x)$, $t_2 = \log(x + 1)$ and*

$$p = t_1^2 t_2 + (2/x) t_1 t_2 + ((2/(x + 1)) t_1.$$

Then $\operatorname{hc}_0(p) = 1$, which is not 1-rigid. Since $t_1^2 t_2 = x' t_1^2 t_2$, we have that $p = \left(x t_1^2 t_2\right)' + q$, where $q = \left(\frac{2}{x} - 2\right) t_1 t_2 - \frac{x}{x+1} t_1^2 + \frac{2}{x+1} t_1$. We can then reduce q further, because $\operatorname{hc}_0(q) = (2t_1 - 2x)'$, which is not 1-rigid either. Repeating this reduction a finite number of times, we see that $\int p \, dx = (x + 1) t_1^2 t_2 - 2x t_1 t_2 - x t_1^2 + (2x + 2) t_2 + 4x t_1 - 6x$.

5.2 Reduction

A flat analogue of straight polynomials is given below.

Definition 5.8. *A polynomial in $C[t_0]$ is said to be t_0-flat if it is zero. For $n \geq 1$, $p \in K_{n-1}[t_n]$ is called a t_n-*flat polynomial *if there exist $p_i \in K_i[t_{i+1}, \ldots, t_n]$ for all i with $0 \leq i \leq n - 1$ such that $p = \sum_{i=0}^{n-1} p_i$, $\operatorname{hc}_i(p_i)$ is t_i-simple for all $i \geq 1$, and $\operatorname{hc}_0(p_0)$ is s-rigid, where $s = \operatorname{scale}_n(p_0)$. The sequence $\{p_i\}_{i=0,1,\ldots,n-1}$ is called a* sequence associated to p.

Example 5.9. *Let $n = 3$ and $t_0 = x$, $t_1 = \log(x)$, $t_2 = \log(x + 1)$ and $t_3 = \log(x + 2)$. Consider $p \in K_2[t_3]$*

$$p = \underbrace{\frac{1}{t_2} t_3^2}_{p_2} + \underbrace{\frac{1}{t_1} t_2 t_3}_{p_1} + \underbrace{\frac{1}{x + k} t_3^3 + x t_2 t_3}_{p_0},$$

where $k \in \mathbb{Z}$. Obviously, $\operatorname{hc}_2(p_2)$ is t_2-simple and $\operatorname{hc}_1(p_1)$ is t_1-simple. Moreover, $\operatorname{scale}_3(p_0) = 3$ and $\operatorname{hc}_0(p_0)$ is 3-rigid if $k \notin \{0, 1, 2\}$. So p is t_3-flat if $k \notin \{0, 1, 2\}$.

We are going to extend the results in Section 4 to the flat case, based on the following technical lemma.

Lemma 5.10. *Let $n \geq 1$ and p be a t_n-flat polynomial in $K_{n-1}[t_n]$ with $d = \deg_{t_n}(p)$ and $\ell = \operatorname{lc}_{t_n}(p)$. Let $\{p_i\}_{i=0,1,\ldots,n-1}$ be a sequence associated to p, and ℓ_i be the coefficient of t_n^d in p_i. Then*

(i) *$\operatorname{fp}_{t_{n-1}}(\ell)$ is t_{n-1}-simple.*
(ii) *If $\ell_m \neq 0$ for some $m > 0$, then $\ell \notin K_{m-1}[t_m, t_{m+1}, \ldots, t_{n-1}]$.*
(iii) *If $n > 1$, then $\operatorname{pp}_{t_{n-1}}(\ell) - c t_n'$ is t_{n-1}-flat for all $c \in C$.*

Proof. The lemma is trivial if $p = 0$. Assume that p is nonzero. Then $\ell = \sum_{i=0}^{n-1} \ell_i$, $\mathrm{fp}_{t_{n-1}}(\ell) = \ell_{n-1}$ and $\mathrm{pp}_{t_{n-1}}(\ell) = \sum_{i=0}^{n-2} \ell_i$.

(i) Note that $\ell_i = 0$ if $\deg_{t_n}(p_i) < d$, and $\mathrm{hc}_i(\ell_i) = \mathrm{hc}_i(p_i)$ otherwise, because $\prec$ is a purely lexicographic with $t_{i+1} \prec \cdots \prec t_n$. Then $\mathrm{fp}_{t_{n-1}}(\ell)$ is t_{n-1}-simple by Definition 5.8.

(ii) Without loss of generality, assume that $\ell_{n-1} = \cdots = \ell_{m+1} = 0$ and $\ell_m \neq 0$ for some $m > 0$. Then we have $f_m := \mathrm{hc}_m(p_m) \neq 0$. By Definition 5.8, f_m is t_m-simple. So f_m is not in $K_{m-1}[t_m]$, which implies $\ell_m \notin K_{m-1}[t_m, \ldots, t_{n-1}]$. Since $\ell_i \in K_{m-1}[t_m, \ldots, t_{n-1}]$ for all i with $0 \le i \le m-1$, we see that $\ell = \ell_m + \sum_{i=0}^{m-1} \ell_i$ does not belong to $K_{m-1}[t_m, \ldots, t_{n-1}]$ either.

(iii) Assume that $n > 1$. Then

$$\mathrm{pp}_{t_{n-1}}(\ell) - ct'_n = \ell_{n-2} + \cdots + \ell_1 + \tilde{\ell}_0, \tag{5.2}$$

where $\tilde{\ell}_0 = \ell_0 - ct'_n$ and $\mathrm{hc}_i(\ell_i)$ is t_i-simple, $i = 1, \ldots, n-2$.

Set $s = \mathrm{scale}_n(p_0)$ and $\tilde{s} = \mathrm{scale}_{n-1}(\tilde{\ell}_0)$. It suffices to prove that $\mathrm{hc}_0(\tilde{\ell}_0)$ is $\tilde{s}$-rigid by (5.2) and Definition 5.8.

Case 1. $\ell_0 \notin K_0$. Then $s < n$.

$$\mathrm{hm}_0(p_0) = t_s^{e_s} \cdots t_{n-1}^{e_{n-1}} t_n^d \quad \text{and} \quad \mathrm{hm}_0(\ell_0) = t_s^{e_s} \cdots t_{n-1}^{e_{n-1}},$$

where $e_s > 0$. Moreover, $s = \mathrm{scale}_{n-1}(\ell_0)$, $\mathrm{hm}_0(\ell_0) = \mathrm{hm}_0(\tilde{\ell}_0)$ and $\mathrm{hc}_0(p_0) = \mathrm{hc}_0(\ell_0) = \mathrm{hc}_0(\tilde{\ell}_0)$. In particular, $\tilde{s} = s$. Hence, $\mathrm{hc}_0(\tilde{\ell}_0)$ is $\tilde{s}$-rigid, because $\mathrm{hc}_0(p_0)$ is s-rigid.

Case 2. $\ell_0 \in K_0$ with $\ell_0 \neq 0$. Then $\mathrm{hm}_0(p_0) = t_n^d$ and $s = n$. Moreover, $\tilde{s} = n-1$, since $\tilde{\ell}_0 \in K_0$. Note that p is t_n-flat. So $\mathrm{hc}_0(p_0)$ is not a C-linear combination of $\{t'_1, \ldots, t'_{n-1}, t'_n\}$, and neither is ℓ_0 because $\ell_0 = \mathrm{hc}_0(p_0)$. Consequently, $\tilde{\ell}_0$ is not a C-linear combination of $\{t'_1, \ldots, t'_{n-1}\}$, and neither is $\mathrm{hc}_0(\tilde{\ell}_0)$, because $\mathrm{hc}_0(\tilde{\ell}_0) = \tilde{\ell}_0$. Thus, $\mathrm{hc}_0(\tilde{\ell}_0)$ is $(n-1)$-rigid.

Case 3. $\ell_0 = 0$. Then $\tilde{s} = n-1$ and $\mathrm{hc}_0(\tilde{\ell}_0) = \tilde{\ell}_0 = -ct'_n$, which is $\tilde{s}$-rigid by Lemma 3.1 (i). ∎

The next lemma is a flat-analogue of Lemma 4.3

Lemma 5.11. *Let $n \ge 1$ and $p \in K_{n-1}[t_n]$ be t_n-flat. If*

$$\mathrm{lc}_{t_n}(p) \equiv ct'_n \mod K'_{n-1} \tag{5.3}$$

for some $c \in C$, then both p and c are zero.

Proof. If $n = 1$, then the tower $K_0 \subset K_1$ is also straight, and p is t_1-straight by Definition 4.4 and Lemma 3.1 (ii). Both p and c are zero by Lemma 4.3.

Assume $n > 1$ and the lemma holds for $n-1$. Set $\ell = \mathrm{lc}_{t_n}(p)$. Applying the map $\mathrm{hp}_{t_{n-1}}$ to (5.3), we have $\mathrm{hp}_{t_{n-1}}(\ell) = 0$. Then $\mathrm{fp}_{t_{n-1}}(\ell) = 0$ by Lemma 5.10 (i) and Lemma 2.1. Consequently, we have $\ell \in K_{n-2}[t_{n-1}]$. Let $q = \ell - ct'_n$. Then q is t_{n-1}-flat by Lemma 5.10 (iii). On the other hand, $q \in K'_{n-1}$ by (5.3). Then $\mathrm{lc}_{t_{n-1}}(q) \equiv \tilde{c}t'_{n-1} \mod K'_{n-2}$ for some $\tilde{c} \in C$ by Lemma 2.3. So $q = 0$ by the induction hypothesis. Accordingly,

$$\ell = ct'_n \in K_0. \tag{5.4}$$

Let $\{p_i\}_{i=0,1,\ldots,n-1}$ be a sequence associated to p. Let $d = \deg_{t_n}(p)$ and ℓ_0 be the coefficient of t_n^d in p_0. By (5.4) and Lemma 5.10 (ii), we have $\ell = \ell_0$. Then $\ell_0 \in K_0$, which implies that $\mathrm{hm}_0(p_0) = t_n^d$. Therefore, $\mathrm{scale}_n(p_0) = n$. Accordingly, ct'_n is n-rigid by Definition 5.8. It follows form Definition 5.5 that $c = 0$. By (5.4), we conclude that ℓ is zero, and so is p. ∎

The following proposition corresponds to Proposition 4.5.

Proposition 5.12. *Let $n \ge 1$ and p be a t_n-flat polynomial in $K_{n-1}[t_n]$. If $p \in K'_n$, then $p = 0$.*

Proof. Since $p \in K'_n$, we have $\mathrm{lc}_{t_n}(p) \equiv ct'_n \mod K'_{n-1}$ for some $c \in C$ by Lemma 2.3. Then $p = 0$ by Lemma 5.11. ∎

The next lemma corresponds to Lemma 4.6.

Lemma 5.13. *For $p \in K_{n-1}[t_n]$, there exists a t_n-flat polynomial $q \in K_{n-1}[t_n]$ such that $p \equiv q \mod K'_n$. Moreover, $\deg_{t_n}(q)$ is no more than $\deg_{t_n}(p)$.*

Proof. By Lemma 3.4, there exist $p_i \in K_i[t_{i+1}, \ldots, t_n]$ for all i with $0 \le i \le n-1$ such that $p \equiv \sum_{i=0}^{n-1} p_i \mod K'_n$. Moreover, $\mathrm{hc}_i(p_i) \in K_i$ is t_i-simple for all $i \ge 1$, and $\deg_{t_n}(p_i) \le \deg_{t_n}(p)$ for all $i \ge 0$. By Proposition 5.6, there exists an element $r \in R_n$ such that $p_0 \equiv r \mod K'_n$ and that $\mathrm{hc}_0(r)$ is s-rigid, where s equals $\mathrm{scale}_n(r)$. Furthermore, $\mathrm{hm}_0(r) \preceq \mathrm{hm}_0(p_0)$ implies that $\deg_{t_n}(r) \le \deg_{t_n}(p_0)$. Set q to be $\sum_{i=1}^{n-1} p_i + r$. Then q is t_n-flat, $p \equiv q \mod K'_n$, and $\deg_{t_n}(q) \le \deg_{t_n}(p)$. ∎

Example 5.14. *Let p be given in Example 5.9, where we set $k = 1$. By integration by parts, we have*

$$p \equiv p_2 + p_1 + \underbrace{-3t'_3 t_2 t_3^2 + x t_2 t_3}_{q_0} \mod K'_3.$$

Then $\mathrm{scale}_3(q_0) = 2$ and $\mathrm{hc}_0(q_0) = -3t'_3 = -3/(x+2)$, which is 2-rigid. Hence, $p_2 + p_1 + q_0$ is t_3-flat.

We are ready to present the main result of this section.

Theorem 5.15. *For $f \in K_n$, the following assertions hold.*

(i) *There exist a t_n-simple element $g \in K_n$ and a t_n-flat polynomial $p \in K_{n-1}[t_n]$ such that*

$$f \equiv g + p \mod K'_n. \tag{5.5}$$

(ii) *$f \equiv 0 \mod K'_n$ if and only if both g and p are zero.*

(iii) *If $f \equiv \tilde{g} + \tilde{p} \mod K'_n$, where $\tilde{g} \in K_n$ is t_n-simple and $\tilde{p} \in K_{n-1}[t_n]$, then $g = \tilde{g}$ and $\deg_{t_n}(p) \le \deg_{t_n}(\tilde{p})$.*

Proof. (i) Applying Algorithm `HermiteReduce` to f with respect to t_n, we get a t_n-simple element g of K_n and an element h of $K_{n-1}[t_n]$ such that $f \equiv g + h \mod K'_n$. We can replace h with a t_n-flat polynomial p by Lemma 5.13.

(ii) Assume $f \in K'_n$. Then (5.5) becomes $g + p \equiv 0 \mod K'_n$. Applying the map hp_{t_n} to the above congruence yields $g = 0$ by Lemma 2.1. Thus, $p \equiv 0 \mod K'_n$. Consequently, $p = 0$ by Proposition 5.12.

(iii) Since $(g - \tilde{g}) + (p - \tilde{p}) \equiv 0 \mod K'_n$ and $g - \tilde{g}$ is t_n-simple, we have $g = \tilde{g}$ by Lemma 2.1. So $p - \tilde{p} \equiv 0 \mod K'_n$. By Lemma 2.3, we have $\mathrm{lc}_{t_n}(p - \tilde{p}) \equiv ct'_n \mod K'_{n-1}$ for some $c \in C$. If $\deg_{t_n}(\tilde{p})$ is smaller than $\deg_{t_n}(p)$, then $\mathrm{lc}_{t_n}(p) = \mathrm{lc}_{t_n}(p - \tilde{p}) \equiv ct'_n \mod K'_{n-1}$. By Lemma 5.11, we conclude $p = 0$, a contradiction. ∎

6 ELEMENTARY INTEGRABILITY

Let $(F, ')$ be a differential field. An element $f \in F$ is said to be *elementarily integrable over* F if there exist an elementary extension E of F and an element g of E such that $f = g'$ [9, Definition 5.1.4]. We study elementary integrability of elements in K_n given in (3.1) built up by a straight or flat tower using Theorems 4.8 and 5.15.

Denote by $\mathbb{L}_i$ the C-linear subspace spanned by the logarithmic derivatives in K_i for all i with $0 \le i \le n$.

THEOREM 6.1. *Let the tower given in* (3.1) *be either straight or flat, in which* C *is algebraically closed,* $K_0 = C(t_0)$, $t_0' = 1$ *and* t_i' *belongs to* $\mathbb{L}_{i-1}$ *for all* i *with* $1 \le i \le n$. *Assume that, for* $f \in K_n$,

$$f \equiv g + p \mod K_n', \tag{6.1}$$

where $g \in K_n$ *is* t_n*-simple and* $p \in K_{n-1}[t_n]$ *is either* t_n*-straight if* (3.1) *is straight or* t_n*-flat if* (3.1) *is flat. Then* f *is elementarily integrable over* K_n *if and only if* $g + p \in \mathbb{L}_n$

PROOF. Clearly, f is elementarily integrable over K_n if $g+p \in \mathbb{L}_n$.

Conversely, there exists $r \in \mathbb{L}_n$ such that $f \equiv r \mod K_n'$ by Liouville's theorem [9, Theorem 5.5.1]. By (6.1),

$$g + p \equiv r \mod K_n', \tag{6.2}$$

Since hp_{t_n} is C-linear, $r = \mathrm{hp}_{t_n}(r) + \tilde{r}$ for some $\tilde{r} \in \mathbb{L}_{n-1}$ by Lemma 2.5. On the other hand, $\mathrm{hp}_{t_n}(g + p) = g$, as g is t_n-simple. So $g = \mathrm{hp}_{t_n}(r)$ by (6.2) and Lemma 2.1. Hence, $g \in \mathbb{L}_n$ and

$$p \equiv \tilde{r} \mod K_n'. \tag{6.3}$$

Let $d = \deg_{t_n}(p)$ and $\ell = \mathrm{lc}_{t_n}(p)$. If $d > 0$, then $\ell = \mathrm{lc}_{t_n}(p - \tilde{r})$, which, together with (6.3) and Lemma 2.3, implies that $\ell \equiv ct_n' \mod K_{n-1}'$ for some $c \in C$. Thus $\ell = 0$ by Lemma 4.3 in the straight case and by Lemma 5.11 in the flat case, a contradiction. So $d = 0$, and, consequently, $\ell = p$.

We show that (6.2) implies $g + p \in \mathbb{L}_n$ by induction. If $n = 0$, then p is zero. The assertion holds. Assume that the assertion holds for $n-1$. By the equality $\ell = p$, the congruence (6.3) and Lemma 2.3, $\ell \equiv \tilde{r} + ct_n' \mod K_{n-1}'$ for some c in C. It follows that

$$\mathrm{fp}_{t_{n-1}}(\ell) + \mathrm{pp}_{t_{n-1}}(\ell) \equiv \tilde{r} + ct_n' \mod K_{n-1}' \tag{6.4}$$

Note that $\mathrm{fp}_{t_{n-1}}(\ell)$ is t_{n-1}-simple, and that $\mathrm{pp}_{t_{n-1}}(\ell)$ is t_{n-1}-straight (resp. flat) by Definition 4.4 (resp. Lemma 5.10). Moreover, $\tilde{r} + ct_n'$ belongs to $\mathbb{L}_{n-1}$. By (6.4) and the induction hypothesis, we see that ℓ belongs to $\mathbb{L}_{n-1}$, and so does p. Accordingly, $g + p \in \mathbb{L}_n$. ∎

To determine whether an element r of K_n belongs to $\mathbb{L}_n$, we proceed as follows. First, we verify whether $\mathrm{fp}_{t_n}(r)$ is t_n-simple and $\mathrm{pp}_{t_n}(r)$ belongs to K_{n-1}. If so, we check whether the residues of $\mathrm{fp}_{t_n}(r)$ with respect to t_n are constants by the Rothstein–Trager resultants (see Theorem 4.4.3 in [9]). Then we repeat the above steps with $\mathrm{pp}_{t_n}(r)$ recursively.

EXAMPLE 6.2. *Let* K_0, t_1 *and* t_2 *be given in Example 5.7. We compute an additive decomposition for*

$$f = \frac{1}{xt_1} + \frac{1}{xt_2 + t_2} + t_1^2 t_2 + \frac{2}{x} t_1 t_2 + \frac{2}{x+1} t_1 + \frac{1}{x+2}.$$

By Theorem 5.15 and Example 5.7, we have

$$f = a' + \underbrace{\frac{1}{xt_2 + t_2}}_{g} + \underbrace{\frac{1}{xt_1} + \frac{1}{x+2}}_{p},$$

where $a = (x+1)t_1^2 t_2 - 2xt_1t_2 - xt_1^2 + (2x+2)t_2 + 4xt_1 - 6x$. *As the Rothstein–Trager resultant of each fraction in* $g + p$ *has only constant roots,* $g + p$ *is a* C*-linear combination of logarithmic derivatives in* K_2. *So* f *is elementarily integrable over* K_2 *by Theorem 6.1. Indeed,*

$$\int f\, dx = a + \log(t_2) + \log(t_1) + \log(x+2).$$

7 TELESCOPERS FOR ELEMENTARY FUNCTIONS

The problem of creative telescoping is classically formulated for D-finite functions in terms of linear differential operators [2, 34]. Raab in his thesis [24] has studied the telescoping problem viewed as a special case of the parametric integration problem in differential fields. However, there are no theoretical results concerning the existence of telescopers for elementary functions. To be more precise, let F be a differential field with two derivations D_x and D_y that commute with each other and let F_∂ be the set $\{f \in F \mid \partial(f) = 0\}$ for $\partial \in \{D_x, D_y\}$. For a given element $f \in F$, the telescoping problem asks whether there exists a nonzero linear differential operator $L = \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in F_{D_y}$ such that $L(f) = D_y(g)$ for some g in a specific differential extension E of F. We call L a *telescoper* for f and g the corresponding *certificate* for L in E. Usually, we take E to be the field F itself or an elementary extension of F. In contrast to D-finite functions, telescopers may not exist for elementary functions as shown in the following example.

EXAMPLE 7.1. *Let* $F = \mathbb{C}(x, y)$ *and* $E = F(t_1, t_2)$ *be a differential field extension of* F *with* $t_1 = \log(x^2 + y^2)$ *and* $t_2 = \log(1 + t_1)$. *We first show that* $f = 1/t_1 \in F(t_1)$ *has no telescoper with certificate in any elementary extension of* $F(t_1)$. *Since* t_1 *is a primitive monomial over* F, *we have* $F(t_1)_{D_y} = \mathbb{C}(x)$. *We claim that for any* $i \in \mathbb{N}$, $D_x^i(f)$ *can be decomposed as* $D_x^i(f) = D_y(g_i) + a_i/t_1$, *where* $g_i \in F(t_1)$, *and* $a_i \in F$ *satisfies the recurrence relation*

$$a_{i+1} = D_x(a_i) - D_y(xa_i/y) \quad \text{with } a_0 = 1.$$

For $n = 0$, *the claim holds by taking* $g_0 = 0$. *Assume that the claim holds for all* $i < k$. *Applying the induction hypothesis and Algorithm* `HermiteReduce` *to* $D_x^k(f)$ *yields*

$$D_x^k(f) = D_x(D_x^{k-1}(f)) = D_x\left(D_y(g_{k-1}) + \frac{a_{k-1}}{t_1}\right)$$
$$= D_y\left(D_x(g_{k-1}) + \frac{a_{k-1}x}{yt_1}\right) + \frac{D_x(a_{k-1}) - D_y(\frac{xa_{k-1}}{y})}{t_1}.$$

This completes the induction. A straightforward calculation shows that $a_i = A_i/y^{2i}$ *for some* $A_i \in \mathbb{C}[x, y] \setminus \{0\}$ *with* $\deg_y(A_i) < 2i$. *Using the notion of residues in [9, page 118], we have*

$$\mathrm{residue}_{t_1}\left(\frac{a_i}{t_1}\right) = \frac{a_i}{D_y(t_1)} = \frac{(x^2 + y^2)A_i}{2y^{2i+1}},$$

which is not in $\mathbb{C}(x)$. *Then* $D_x^i(f)$ *is not elementarily integrable over* $F(t_1)$ *for any* $i \in \mathbb{N}$ *by the residues criterion in [9, Theorem 5.6.1]. Assume that* f *has a telescoper* $L := \sum_{i=0}^d \ell_i D_x^i$ *with* $\ell_i \in \mathbb{C}(x)$ *not all zero. Then* $L(f)$ *is elementarily integrable over* $F(t_1)$. *However,*

$$L(f) = D_y\left(\sum_{i=0}^d \ell_i g_i\right) + \frac{\sum_{i=0}^d \ell_i a_i}{t_1}.$$

Since all of the ℓ_i's are in $\mathbb{C}(x)$ and $\gcd(x^2+y^2, y^m) = 1$ for any $m \in \mathbb{N}$, the residue of $\sum_{i=0}^d \ell_i a_i/t_1$ is not in $\mathbb{C}(x)$, which implies that $L(f)$ is not elementarily integrable over $F(t_1)$, a contradiction.

We now show that $p = ft_2$ has no telescoper with certificate in any elementary extension of $F(t_1, t_2)$. Since t_2 is also a primitive monomial over $F(t_1)$, we have $E_{D_y} = \mathbb{C}(x)$. Assume that $L := \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in \mathbb{C}(x)$ not all zero is a telescoper for p. Then $L(p)$ is elementarily integrable over E. By a direct calculation, we get $L(p) = L(f)t_2 + r$ with $r \in F(t_1)$. The elementary integrability of $L(p)$ implies that $L(f) = cD_y(t_2) + D_y(b)$ for some $c \in \mathbb{C}(x)$ and $b \in F(t_1)$ by the formula (5.13) in the proof of Theorem 5.8.1 in [9, page 157]. We claim that $c = 0$. Since $D_x^i(f) = u_i/t_1^{i+1}$ with $u_i \in F[t_1]$ and $\deg_{t_1}(u_i) < i+1$ and $D_y(t_2) = D_y(t_1)/(1+t_1)$, the orders of $D_x^i(f)$ and $D_y(t_2)$ at $1+t_1$ are equal to 0 and 1, respectively. If c is nonzero, the order of $cD_y(t_2)$ at $1+t_1$ is equal to 1, which does not match with that of $L(f) - D_y(b)$ by Lemma 4.4.2 (i) in [9], a contradiction. Then $L(f) = D_y(b)$, i.e., L is a telescoper for f, which contradicts with the first assertion.

The next example shows that additive decompositions in Theorems 4.8 and 5.15 are useful for detecting the existence of telescopers for elementary functions that are not D-finite.

Example 7.2. *Let $F = \mathbb{C}(x, y)$ and $E = F(t)$ be a differential field extension of F with $t = \log(x^2+y^2)$. Consider the function $f = t + 1 - \frac{2y}{(x^2+y^2)t^2}$. Since the derivatives $D_x^i(1/t^2) = a_i/t^{i+2}$ with $a_i \in F \setminus \{0\}$ are linearly independent over F, we see that $1/t^2$ is not D-finite over F, and neither is f. Note that f can be decomposed as*

$$f = D_y(1/t) + t + 1.$$

Since $t+1$ is D-finite, it has a telescoper, and so does f.

8 CONCLUSION

In this paper, we developed additive decompositions in straight and flat towers, which enable us to determine in-field integrability and elementary integrability in a straightforward manner. It is natural to ask whether one can develop an additive decomposition in a general primitive tower. Moreover, we plan to investigate about the existence and the construction of telescopers for elementary functions using additive decompositions.

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