# <span id="page-0-1"></span>**BIJECTIONS AROUND SPRINGER NUMBERS**

SHAOSHI CHEN, YANG LI, ZHICONG LIN, AND SHERRY H.F. YAN

Abstract. Arnol'd proved in 1992 that Springer numbers enumerate the Snakes, which are type *B* analogs of alternating permutations. Chen, Fan and Jia in 2011 introduced the labeled ballot paths and established a "hard" bijection with snakes. Callan conjectured in 2012 and Han–Kitaev–Zhang proved recently that rc-invariant alternating permutations are counted by Springer numbers. Very recently, Chen–Fang–Kitaev–Zhang investigated multi-dimensional permutations and proved that weakly increasing 3-dimensional permutations are also counted by Springer numbers. In this work, we construct a sequence of "natural" bijections linking the above four combinatorial objects.

### <span id="page-0-0"></span>1. INTRODUCTION

The *Euler numbers*  $E_n$  are the coefficients of the Taylor expansion:

$$
\tan(x) + \sec(x) = \sum_{n\geqslant 0} E_n \frac{x^n}{n!} = 1 + x + 1 \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} + 61 \frac{x^6}{6!} + \cdots
$$

Let  $\mathfrak{S}_n$  denote the set of permutations of  $[n] := \{1, 2, \ldots, n\}$ . A permutation  $\pi \in \mathfrak{S}_n$ possessing the down-up (or alternating) property

(1.1) 
$$
\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots
$$

is called an *alternating permutation*. It is a classical result in Enumerative Combinatorics  $[10, Prop. 1.6.1]$  that  $E_n$  counts alternating permutations of length *n*.

The *Springer numbers S<sup>n</sup>* defined by

$$
\frac{1}{\cos(x) - \sin(x)} = \sum_{n \ge 0} S_n \frac{x^n}{n!} = 1 + x + 3\frac{x^2}{2!} + 11\frac{x^3}{3!} + 57\frac{x^4}{4!} + 361\frac{x^5}{5!} + 2763\frac{x^6}{6!} + \cdots
$$

can be considered as type *B* analogs of the Euler numbers. Let  $\mathfrak{B}_n$  denote the set of *signed permutations* of length *n*, whose elements are those of the form  $\pm \pi_1 \pm \pi_2 \cdots \pm \pi_n$  with  $\pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$ . For convenience, we write  $-n$  by  $\bar{n}$  for each positive integer *n*. A signed permutation  $\pi \in \mathfrak{B}_n$  satisfying  $\pi_1 > 0$  and the down-up property [\(1.1\)](#page-0-0) is called a *snake* (of type  $B_n$ ). In his study of connections between combinatorics of the Coxeter groups and singularities of smooth functions, Arnol'd  $[1]$  proved that  $S_n$  enumerates snakes of length *n*. For example, *S*<sup>3</sup> counts the following 11 snakes of length 3

1¯23*,* 1¯32*,* 1¯3¯2*,* 213*,* 2¯13*,* 2¯31*,* 2¯3¯1*,* 312*,* 3¯12*,* 3¯21*,* 3¯2¯1*.*

Besides snakes of length *n*, there are various interesting combinatorial interpretations for the Springer number *Sn*, among which are

' labeled ballot paths of *n* steps, introduced and proved by Chen, Fan and Jia [\[4\]](#page-6-2);

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<span id="page-1-3"></span>

<span id="page-1-0"></span>FIGURE 1. Bijections around Springer numbers

- ' rc-invariant alternating permutations of length 2*n*, proved recently by Han, Kitaev and Zhang [\[7\]](#page-6-3), verifying an observation by Callan in 2012;
- ' weakly increasing 3-dimensional permutations, found and proved very recently by Chen, Fang, Kitaev and Zhang [\[2\]](#page-6-4).

Chen, Fan and Jia  $[4]$  established a bijection between the labeled ballot paths and snakes, which is "hard" in the sense that the proof of its bijectivity is intricate. On the other hand, the proofs of the latter two assertions in [\[7\]](#page-6-3) and [\[2\]](#page-6-4) are not bijective. In this note, we construct a sequence of "natural" bijections linking the above four combinatorial objects counted by Spring numbers (see Fig. [1\)](#page-1-0).

## 2. Bijecting weakly increasing 3-dimensional permutations with snakes

A pair  $(\sigma, \pi) \in \mathfrak{S}_n^2$  is called a *weakly increasing* 3-dimensional permutation<sup>[1](#page-1-1)</sup> (3-WIP for short) in  $[2]$  if

$$
\max(\sigma_1, \pi_1) \le \max(\sigma_2, \pi_2) \le \cdots \le \max(\sigma_n, \pi_n).
$$

Let  $\mathfrak{W}_n$  be the set of all 3-WIPs of length *n*. It is convenience to write  $(\sigma, \pi) \in \mathfrak{W}_n$  in 2-array as

<span id="page-1-2"></span>
$$
\begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}.
$$

For example,

(2.1) 
$$
\begin{bmatrix} 1 & 5 & 2 & 6 & 7 & 3 & 8 & 9 & 4 \\ 2 & 5 & 6 & 3 & 1 & 7 & 8 & 4 & 9 \end{bmatrix} \in \mathfrak{W}_9.
$$

For  $\sigma \in \mathfrak{S}_n$ , denote by  $\sigma^{-1}$  its inverse in  $\mathfrak{S}_n$ . A letter  $k, 2 \leq k \leq n$ , is called a *cycle peak* of  $\sigma$  if  $\sigma_k^{-1} < k > \sigma_k$ . For example, with  $\sigma = 267953184$  written in its cycle form as  $(1, 2, 6, 3, 7)(5)(4, 9)(8)$ , its cycle peaks are 6, 7 and 9. For our purpose, we need to

<span id="page-1-1"></span><sup>1</sup>Since the pair  $(\sigma, \pi)$  can be written as 3-dimensional array

$$
\begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix},
$$

it is referred to as 3-dimensional permutation in [\[2\]](#page-6-4).

<span id="page-2-0"></span>recall Foata's "transformation fondamentale"  $f : \mathfrak{S}_n \to \mathfrak{S}_n$  (see [\[10,](#page-6-0) Page 23]), which is constructed in two steps:

- Write the permutation  $\sigma$  in its *standard cycle form* by requiring: (a) each cycle has its largest letter in the leftmost position; (b) the cycles are listed from left to right in increasing order of their largest letters.
- Erasing all the parentheses results in the permutation  $f(\sigma)$ .

Continuing with the running example  $\sigma$ , its standard cycle form is  $(5)(7, 1, 2, 6, 3)(8)(9, 4)$ and we get  $o(\sigma) = 571263894$  after erasing all the parentheses. It is plain to see that Foata's bijection *f* transforms cycle peaks of  $\sigma$  to left peaks of  $\pi = f(\sigma)$ , where a letter  $\pi_i$ of  $\pi$  is a *left peak* if  $\pi_{i-1} < \pi_i > \pi_{i+1}$  with the convention  $\pi_0 = 0$  and  $\pi_{n+1} = +\infty$ .

We are ready to construct the bijection  $\Phi$  from  $\mathfrak{W}_n$  to  $\mathcal{S}_n$ , the set of snakes of length *n*. Given  $(\sigma, \pi) \in \mathfrak{W}_n$ , we perform the following three steps to get the snake  $\Phi(\sigma, \pi)$ :

- (1) Construct a permutation  $\tau$  such that  $\tau(\sigma_i) = \pi_i$  for each *i* with its cycle peak *k* is hatted if and only if  $k = \sigma_{\ell} = \pi_{\ell+1}$  for some  $\ell$ . Take  $(\sigma, \pi)$  in  $(2.1)$  as example, we have  $\tau = (5)(\hat{7}, 1, 2, 6, 3)(8)(\hat{9}, 4)$ , written in its standard cycle form.
- (2) Write  $\tau$  in its standard cycle form and erase the parentheses to get  $\tilde{\tau} = f(\tau)$ , which is a permutation in  $\mathfrak{S}_n$  with some of its left peaks hatted. Continuing with the running example, we have  $\tilde{\tau} = f(\tau) = 5\tilde{7}1263894$ .
- (3) A letter  $\tilde{\tau}_i$  of  $\tilde{\tau}$  is a *right valley* if  $\tilde{\tau}_{i-1} > \tilde{\tau}_i < \tilde{\tau}_{i+1}$  with the convention  $\tilde{\tau}_0 = 0$ and  $\tilde{\tau}_{n+1} = +\infty$ . We associate each right valley of  $\tilde{\tau}$  with the closest left peak to its left in  $\tilde{\tau}$ . Now define the snake  $\Phi(\sigma, \pi)$  from  $\tilde{\tau}$  by first removing hats and then putting bars over the letters in two situations: a non-right-valley in even position or a right valley whose associated left peak in  $\tilde{\tau}$  is hatted. Continuing with the running example, we have  $\Phi(\sigma, \pi) = 57\overline{1} \overline{2} 638\overline{9} \overline{4}$ .

**Theorem 2.1.** *The mapping*  $\Phi : \mathfrak{W}_n \to \mathcal{S}_n$  *is a bijection.* 

*Proof.* We need to verify that  $\Phi(\sigma, \pi) = p \in \mathfrak{B}_n$  is a snake. As  $\tilde{\tau}_1$  is not a right valley,  $p_1 > 0$ . We distinguish two cases.

- If *i* is odd and  $i < n$ , then we need to show that  $p_i > p_{i+1}$ . If  $\tilde{\tau}_i < \tilde{\tau}_{i+1}$ , then  $p_{i+1} = -\tilde{\tau}_{i+1}$  and so  $p_i > p_{i+1}$ . If  $\tilde{\tau}_i > \tilde{\tau}_{i+1}$ , then  $p_i = \tilde{\tau}_i$  is positive and so  $p_i > p_{i+1}$ .
- If *i* is even and  $i < n$ , then we need to show that  $p_i < p_{i+1}$ . If  $\tilde{\tau}_i < \tilde{\tau}_{i+1}$ , then  $p_{i+1} = \tilde{\tau}_{i+1}$  and so  $p_i < p_{i+1}$ . If  $\tilde{\tau}_i > \tilde{\tau}_{i+1}$ , then  $p_i = -\tilde{\tau}_i$  and so  $p_i < p_{i+1}$ .

It follows that  $p$  is a snake and  $\Phi$  is well-defined.

It is plain to see that steps (2) and (3) in the construction of  $\Phi$  are reversible. To see that step (1) is reversible, we define its inverse explicitly. Given  $\tau \in \mathfrak{S}_n$  with some of its cycle peaks hatted, form the 3-array

$$
T = \begin{bmatrix} 1 & 2 & \cdots & n \\ \tau_1 & \tau_2 & \cdots & \tau_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix},
$$

where  $c_i = \max(i, \tau_i)$  for each *i*. Observe that *k* appears twice in the bottom row of *T* iff *k* is a cycle peak in  $\tau$ . Rearrange the columns of T in weakly increasing order of their bottom values by requiring that whenever two columns  $(j, \tau_j, c_j)$  and  $(k, \tau_k, c_k)$  with  $j < k$ and  $c_j = c_k = k$ , then put  $(k, \tau_k, c_k)$  before  $(j, \tau_j, c_j)$  iff *k* is a cycle peak in  $\tau$  with hat.

<span id="page-3-2"></span>Finally, removing the bottom row to retrieve the desired 3-WIP  $(\sigma, \pi)$ . This shows that step (1) is reversible and so  $\Phi$  is a bijection.

#### 3. Bijecting snakes with labeled ballot paths

For  $\pi \in \mathfrak{S}_n$ , the *complement* of  $\pi$  is

$$
\pi^c := (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n),
$$

while the *reverse* of *π* is

$$
\pi^r := \pi_n \pi_{n-1} \cdots \pi_1.
$$

A permutation  $\pi$  is said to be *rc-invariant* if  $\pi^{rc} = \pi$ . For instance, the permutation  $\pi = 41352$  is rc-invariant. The set of rc-invariant alternating permutations of length 2*n* is denoted by  $\mathfrak{R}_n$ . This class of permutations was considered by Callan in the OEIS  $[9]$ , where he suspected that  $|\mathfrak{R}_n| = S_n$ . His conjecture has been confirmed recently by Han, Kitaev and Zhang [\[7\]](#page-6-3). With rc-invariant alternating permutations as intermediate structure, we construct a "new" bijection from snakes to labeled ballot paths.

<span id="page-3-0"></span>First we construct a bijection from  $S_n$  to  $\mathfrak{R}_n$ , which is based on the following simple fact.

**Lemma 3.1.** Given  $\pi = \pi_1 \pi_2 \cdots \pi_{2n} \in \mathfrak{R}_n$ , then  $\pi_n > n > \pi_{n+1}$  (resp.,  $\pi_n < n < \pi_{n+1}$ ) if *n is odd (resp., even).*

*Proof.* Since  $\pi$  is rc-invariant, we have  $\pi_i + \pi_{2n+1-i} = 2n+1$ . The result then follows from the fact that  $\pi$  is alternating.

<span id="page-3-1"></span>**Theorem 3.2.** *There exists a bijection*  $\psi : \mathcal{S}_n \to \mathfrak{R}_n$ *.* 

*Proof.* Given a snake  $\pi \in S_n$ , define  $\tilde{\pi} = \tilde{\pi}_1 \cdots \tilde{\pi}_n$  with

$$
\tilde{\pi}_i := \begin{cases} n + \pi_i, & \text{if } \pi_i > 0; \\ n + 1 + \pi_i, & \text{if } \pi_i < 0. \end{cases}
$$

Note that  $\tilde{\pi}_k + \tilde{\pi}_l \neq 2n + 1$  for any  $k \neq l$ . Thus, we can define  $\psi(\pi)$  to be the unique rc-invariant permutation whose first half (resp., later half) is ˜*π r* (resp., ˜*π*) if *n* is odd (resp., even). It is clear that  $\tilde{\pi}$  is alternating. As  $\pi_1 > 0$ ,  $\tilde{\pi}_1 > n$  and so  $\psi(\pi)$  is alternating by Lemma [3.1.](#page-3-0) For example, if  $\pi = 215\overline{43}$ , then  $\tilde{\pi} = 761023$  and  $\psi(\pi) = 32106745198$ ; if  $\pi = 153624$ , then  $\tilde{\pi} = 724183$  and  $\psi(\pi) = 105129116724183$ .

It is plain to see that  $\psi$  is reversible and so is a bijection.

A *partial Motzkin path* is a lattice path in the quarter plane  $\mathbb{N}^2$  starting at  $(0,0)$  and using three possible steps:

$$
(1, 1) = U
$$
 (up step),  $(1, -1) = D$  (down step) and  $(1, 0) = H$  (horizontal step).

For a partial Motzkin path  $P = p_1 p_2 \cdots p_n$  with *n* steps, let  $h_i(P)$  be the *height* of the *i*-th step of *P*:

$$
h_i(P) := |\{j \mid j < i, p_j = U\}| - |\{j \mid j < i, p_j = D\}|.
$$

A partial Motzkin path without horizontal steps is called a *ballot path*. As introduced in [\[4\]](#page-6-2), a *labeled ballot path* of length *n* is a pair  $(P, w)$ , where  $P = p_1 p_2 \cdots p_n$  is a ballot path of *n* 

<span id="page-4-0"></span>steps and  $w = w_1 w_2 \cdots w_n \in \mathbb{N}^n$  is a weight function satisfying

$$
0 \leqslant w_i \leqslant \begin{cases} h_i(P), & \text{if } p_i = U; \\ h_i(P) - 1, & \text{if } p_i = D. \end{cases}
$$

Let  $\mathcal{P}_n$  be the set of labeled ballot paths of length *n*.

Partial Motzkin paths ending at *x*-axis are the usual Motzkin paths. A *two-colored Motzkin path* is a Motzkin path whose horizontal steps are colored by  $H$  or  $H$ . Denote by  $\mathcal{M}_n^{(2)}$  the set of all two-colored Motzkin paths of length *n*. A *restricted Laguerre history* of length *n* is a pair  $(M, w)$ , where  $M = m_1 m_2 \cdots m_n \in \mathcal{M}_n^{(2)}$  and  $w = w_1 w_2 \cdots w_n \in \mathbb{N}^n$  is a weight function satisfying

$$
0 \leqslant w_i \leqslant \begin{cases} h_i(M), & \text{if } m_i = U, H; \\ h_i(M) - 1, & \text{if } m_i = D, \tilde{H}. \end{cases}
$$

Let  $\mathcal{L}_n$  denote the set of all restricted Laguerre histories of length *n*. In the following, we show that the classical Foata–Zeilberger bijection  $[6]$  (see also  $[5]$ )  $\Psi_{FZ}$  from  $\mathfrak{S}_n$  to  $\mathcal{L}_n$ restricted to a bijection from  $\mathfrak{R}_n$  to  $\mathcal{P}_n$ .

We need to recall the Foata–Zeilberger bijection  $\Psi_{FZ}$ . Given a permutation  $\pi \in \mathfrak{S}_n$ , we use the convention  $\pi_0 = 0$  and  $\pi_{n+1} = +\infty$ . For each  $i \in [n]$ , let  $(312)_i(\pi)$  be the number of  $312$ -patterns in  $\pi$  with *i* representing the 2, i.e.,

$$
(\underline{31}2)_i(\pi) := |\{k : k < j \text{ and } \pi_k < \pi_j = i < \pi_{k-1}\}|.
$$

Define  $\Psi_{FZ}(\pi) = (M, w)$ , where for  $i \in [n]$  with  $\pi_j = i$ :

$$
m_i = \begin{cases} U & \text{if } \pi_{j-1} > \pi_j < \pi_{j+1}, \\ D & \text{if } \pi_{j-1} < \pi_j > \pi_{j+1}, \\ H & \text{if } \pi_{j-1} < \pi_j < \pi_{j+1}, \\ \tilde{H} & \text{if } \pi_{j-1} > \pi_j > \pi_{j+1}, \end{cases}
$$

and  $w_i = \left(\frac{312}{\pi}\right)$ , For example, if  $\pi = 431296857 \in \mathfrak{S}_9$ , then

$$
\Psi_{FZ}(\pi) = (UH\tilde{H}DUUHDD, 010000210).
$$

The inverse algorithm  $\Psi_{FZ}^{-1}$  building a permutation  $\pi$  (in  $n$  steps) from a Laguerre history  $(M, w) \in \mathcal{L}_n$  can be described iteratively as:

- Initialization:  $\pi = \diamond$ ;
- At the *i*-th  $(1 \leq i \leq n)$  step of the algorithm, replace the  $(w_i + 1)$ -th  $\diamond$  (from left to right) of  $\pi$  by

$$
\begin{cases}\n\diamond i \diamond & \text{if } m_i = U, \\
i \diamond & \text{if } m_i = H, \\
i & \text{if } m_i = D, \\
\diamond i & \text{if } m_i = \tilde{H};\n\end{cases}
$$

• The final permutation is obtained by removing the last remaining  $\diamond$ . For example, if  $(M, w) = (UH\tilde{H}DUUHDD, 010000210) \in \mathcal{L}_9$ , then

$$
\pi = \diamond \rightarrow \diamond 1 \diamond \rightarrow \diamond 12 \diamond \rightarrow \diamond 312 \diamond \rightarrow 4312 \diamond \rightarrow 4312 \diamond 5 \diamond
$$
  

$$
\rightarrow 4312 \diamond 6 \diamond 5 \diamond \rightarrow 4312 \diamond 6 \diamond 57 \diamond \rightarrow 4312 \diamond 6857 \diamond \rightarrow 431296857.
$$

<span id="page-5-3"></span>From the inverse algorithm  $\Psi_{FZ}^{-1}$ , one could check easily the following lemma that was known in  $[5]$ .

<span id="page-5-0"></span>**Lemma 3.3.** *Suppose that*  $(M, w) \in \mathcal{L}_n$  *and*  $\pi = \Psi_{FZ}^{-1}(M, w)$ *. Then for any*  $1 \leq i \leq n$ *,* 

$$
(\underline{31}2)_i(\pi) + (2\underline{31})_i(\pi) = \begin{cases} h_i(M), & \text{if } m_i = U, H; \\ h_i(M) - 1, & \text{if } m_i = D, \tilde{H}. \end{cases}
$$

*Here*  $(231)_i(\pi)$  *denotes the number of*  $231$ *-patterns in*  $\pi$  *with i representing the* 2*, i.e.,* 

$$
(231)_i(\pi) = |\{k : k > j \text{ and } \pi_{k+1} < \pi_j = i < \pi_k\}|.
$$

Given  $(M, w) \in \mathcal{L}_n$ , let  $(M, w)^{rc} = (M', w')$ , where

$$
m'_{n+1-i} = \begin{cases} m_i & \text{if } m_i = H, \tilde{H}, \\ U & \text{if } m_i = D, \\ D & \text{if } m_i = U; \end{cases}
$$

and

$$
w'_{n+1-i} = \begin{cases} h_i(M) - w_i, & \text{if } m_i = U, H; \\ h_i(M) - 1 - w_i, & \text{if } m_i = D, \tilde{H}. \end{cases}
$$

<span id="page-5-1"></span>**Lemma 3.4.** For any  $\pi \in \mathfrak{S}_n$ , if  $\Psi_{FZ}(\pi) = (M, w)$ , then  $\Psi_{FZ}(\pi^{rc}) = (M, w)^{rc}$ .

*Proof.* As  $(312)_{n+1-i}(\pi^{rc}) = (231)_i(\pi)$ , the result then follows from the construction of  $\Psi_{FZ}$ and Lemma [3.3.](#page-5-0)  $\Box$ 

For any  $\pi \in \mathfrak{R}_n$ , since  $\pi$  is rc-invariant and alternating, it follows from Lemma [3.4](#page-5-1) that  $\Psi_{FZ}(\pi) = (D, w)$ , where *D* is a Dyck path (i.e., ballot path ending at *x*-axis) and  $(D, w)^{rc} = (D, w)$ . If we denote  $\Psi(\pi)$  the labeled ballot path by keeping the first half of  $(D, w)$ , then  $\Psi$  establishes a bijection between  $\mathfrak{R}_n$  and  $\mathcal{P}_n$ . Composing  $\psi$  in Theorem [3.2](#page-3-1) with  $\Psi$  leads to the following result.

<span id="page-5-2"></span>**Theorem 3.5.** *The functional composition*  $\Psi \circ \psi$  *establishes a bijection between*  $S_n$  *and*  $\mathcal{P}_n$ *.* 

For example, if  $\pi = 2\overline{1}547\overline{63}$ , then  $\psi(\pi) = 5214111279683411310$ . After applying  $\Psi$ we get the labeled ballot path (*UUUDDUU*, 0012000).

# 4. Concluding remarks

It should be mentioned that a bijection different with  $\psi \circ \Phi$  between  $\mathfrak{W}_n$  and  $\mathfrak{R}_n$  was found in [\[3\]](#page-6-8). Josuat-Vergès [\[8\]](#page-6-9) established another bijection between  $S_n$  and  $\mathcal{P}_n$  which does not seem to be directly related with that one in  $|4|$ . However, it turns out accidentally that our bijection  $\Psi \circ \psi$  in Theorem [3.5](#page-5-2) is closely related with that one in [\[4\]](#page-6-2), though constructed quite differently. In fact, under the one-to-one correspondence  $(P, w) \mapsto (P, \bar{w})$  on  $\mathcal{P}_n$  with

$$
\bar{w}_i := \begin{cases} h_i(P) - w_i, & \text{if } p_i = U, \\ h_i(P) - 1 - w_i, & \text{if } p_i = D, \end{cases}
$$

our bijection  $\Psi \circ \psi$  is the same as that one between  $\mathcal{S}_n$  and  $\mathcal{P}_n$  introduced in [\[4\]](#page-6-2).

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