Symbolic Integration in Weierstrass-like Extensions *

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Abstract

This paper studies the integration problem in differential fields that may involve quantities reminiscent of the Weierstrass $\wp$ function, which are defined by a first-order nonlinear differential equation. We extend the classical notion of special polynomials to elements of Weierstrass-like extensions and present algorithms for reduction in such extensions. As an application of these results, we derive some new formulae for integrals of powers of $\wp$.

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1 Introduction

Modern algorithms for symbolic integration are primarily governed by two complementary approaches. One approach uses annihilating linear operators to describe functions, and performs computations in operator algebras [33, 13, 17] in order to answer questions about a given integral. The other approach uses differential fields to describe functions, and performs computations in these fields in order to answer questions about a given integral [24, 25, 6]. The latter approach has proved particularly successful for so-called elementary functions, but has also seen extensions to other types of functions, see [23] and the references given there.

Here we are concerned with integrals involving functions that are defined by (possibly nonlinear) first-order differential equations, i.e., functions y which satisfy $Q(y', y) = 0$ for some bivariate polynomial Q. A prototypical example for such a function is the Weierstrass $\wp$ function, which satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{1}$$

for certain constants $g_2, g_3 \in \mathbb{C}$ with $g_2^3 - 27g_3^2 \neq 0$.

In differential algebra aspect, quantities satisfying this differential equation were already considered by Kolchin [18], who called them *Weierstrassian.* Slightly generalizing by also allowing other polynomials Q, we call the extensions generated by these functions *Weierstrass-like.* There has been some recent work related to symbolic integration with Weierstrassian elements. Variants of Liouville's theorem covering this case have been proposed by Kumbhakar and Srinivasan [20] and by Pila and Tsimerman [22].

From a more computational perspective, parallel integration has been applied to integration problems involving functions defined by nonlinear differential equations. Bronstein [7] gives an example involving the Lambert W function, and Böttner [1] gives some examples involving the Weierstrass $\wp$ function. Apart from this, not much is known about integration theory and algorithms when the integrand involves $\wp$ or similar functions. In fact, not too many identities about integrals involving $\wp$ are available in the literature.

Our contribution in this paper consists of three aspects. On the theoretical side, we propose an extension of the classical notion of *special polynomials* to elements of Weierstrass-like extensions. On the algorithmic side, we propose an algorithm for reduction process, thereby continuing an ongoing trend in the development of integration algorithms [2, 3, 11, 10, 15, 4, 8, 30, 9, 14]. Finally, as an application of these results, we obtain some new identities about integrals of powers of $\wp$.

2 An appetizer

We start our study from the integration problem of powers of the Weierstrass $\wp$ function, i.e., the problem of evaluating the integral $I_n(z) := \int \wp(z)^n \, dz$ with $n \in \mathbb{Z}$. We will recall some classical formulae from the book [32]. By differentiating the both sides of the differential equation (1), we obtain that $\wp(z)^2 = \frac{1}{6}\wp''(z) + \frac{1}{12}g_2$, which implies that

$$\int \wp(z)^2\, dz = \frac{1}{6}\wp'(z) + \frac{1}{12}g_2 z + C, \quad C \text{ is a constant.}$$

In the following, we will always omit C if no confusion arises. So the integral $I_2(z)$ is in the field generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}(z)$. After developing the theory of integration in elementary terms in this paper, we will show that the integral of $\wp(z)$ is not elementary over the field $\mathbb{C}(z)(\wp(z), \wp'(z))$. For this reason, we should introduce some new functions, such as the Weierstrass zeta-function $\zeta(z)$ satisfying the equation $\zeta'(z) = -\wp(z)$ and the function $\sigma(z)$ with $\zeta(z) = \sigma'(z)/\sigma(z)$. In terms of these two new functions, we can evaluate the integral

$$\int \frac{1}{\wp(z)}\, dz = \frac{1}{\sqrt{-g_3}}\left(\log\left(\frac{\sigma(z-v)}{\sigma(z+v)}\right) + 2\zeta(v)z\right),$$

where $v \in \mathbb{C}$ be such that $\wp(v) = 0$ and $\wp'(v) = \sqrt{-g_3}$. To evaluate the general integral $I_n(z)$, we need the inverse function of $\wp(z)$, which is defined by the formula [32, p. 438]

$$\wp^{-1}(z) = \int_z^\infty \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt.$$

By using change of variables, we have

$$\int \wp(z)^n\, dz = J_n(\wp(z)), \quad \text{with } J_n(t) := \int \frac{t^n\, dt}{\sqrt{4t^3 - g_2 t - g_3}}.$$

The integral $J_n(t)$ satisfies a linear recurrence equation of the form

$$\begin{aligned} J_n(t) = {} & \frac{g_2(2n-3)}{8n-4} J_{n-2}(t) + \frac{g_3(n-2)}{4n-2} J_{n-3}(t) \\ & + \frac{t^{n-2}\sqrt{4t^3 - g_2 t - g_3}}{4n-2}, \end{aligned}$$

with initial values $J_0(t) = \wp^{-1}(t)$, $J_1(t) = -\zeta(\wp^{-1}(t))$ and

$$J_2(t) = \frac{\sqrt{4t^3 - g_2 t - g_3}}{6} + \frac{g_2}{12}\wp^{-1}(t).$$

The above recurrence can be computed by creative telescoping using the Mathematica package **HolonomicFunctions.m** [19]. From the above recurrence for $J_n(t)$ and changing back the variables, we can obtain the integral

$$\int \wp(z)^3\, dz = \frac{1}{10}\wp(z)\wp'(z) - \frac{3g_2}{20}\zeta(z) + \frac{g_3}{10}z$$

and also the integral

$$\int \wp(z)^4\, dz = \frac{1}{14}\wp(z)^2\wp'(z) + \frac{5g_2}{168}\wp'(z) + \frac{5g_2^2}{336}z - \frac{g_3}{7}\zeta(z).$$

We will revisit these integrals $I_n(z)$ with $n \in \mathbb{N}$ in Section 7.

3 Algebraic Functions

In this section, we recall some terminologies about fields of algebraic functions of one variable from the book [12]. Let k be a field of characteristic 0 and t be transcendental over k. If $m \in k[t, X]$ is an absolutely irreducible polynomial, i.e., m is irreducible over the algebraic closure $\bar{k}$ of k, then k is algebraically closed in $K := k(t)[X]/\langle m \rangle$ by [29, Section 3, Thm 1]. Such K is called a *field of algebraic functions of one variable* over k.

A subring $\mathcal{O} \subsetneq K$ is called a *valuation ring* if $\mathcal{O}$ contains k, and for any nonzero $x \in K$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. A valuation ring is a local ring with a principal maximal ideal. The maximal ideal of a valuation ring is called a *place*. For a given place P, there is a unique valuation ring $\mathcal{O}_P$ of which P is the maximal ideal. The residue field $\mathcal{O}_P/P$ is denoted by Σ_P. It is a finite algebraic extension of k.

Assume $P = u\mathcal{O}_P$. The *order function* of K at P is a map $\nu_P \colon K \to \mathbb{Z} \cup \{+\infty\}$ defined by $\nu_P(x) = \max\{n \mid x \in u^n \mathcal{O}_P\}$. One can prove that ν_P is well-defined and has the following properties:

(i) $\nu_P(x) = +\infty$ if and only if $x = 0$;

(ii) $\nu_P(xy) = \nu_P(x) + \nu_P(y)$ for all $x, y \in K$;

(iii) $\nu_P(x + y) \geq \min\{\nu_P(x), \nu_P(y)\}$ for all $x, y \in K$ and equality holds if $\nu_P(x) \neq \nu_P(y)$

In fact, $\mathcal{O}_P := \{x \in K \mid \nu_P(x) \geq 0\}$, and every elements of P have strictly positive order at P.

Example 1. *Let k be a field of characteristic 0 and $k(t)$ be the field of rational functions over k. A place of $k(t)$ is generated by either t^{-1} or an irreducible polynomial $p \in k[t]$ in their valuation ring $\mathcal{O}_\infty$ or $\mathcal{O}_p$, respectively. We denote the respective order functions by ν_∞ and ν_p. Let $a, b \in k[t]$ with $ab \neq 0$, and $f = a/b \in C(t)$. Then $\nu_\infty(f) = \deg_t b - \deg_t a$. If we write $f = p^n a_0/b_0$ where $n \in \mathbb{Z}$, $\gcd(a_0, b_0) = 1$ and $p \nmid a_0 b_0$, then $\nu_p(f) = n$. We say that $t^{-1}\mathcal{O}_\infty$ is the* infinite place *of $k(t)$ and $p\mathcal{O}_p$ is a* finite place. *In fact,*

$$\mathcal{O}_\infty = \left\{ \frac{a}{b} \mid \deg_t a < \deg_t b \right\} \text{ and } \mathcal{O}_p = \left\{ \frac{a}{b} \mid \ \gcd(a, b) = 1, \ p \nmid b \right\}$$

Let Σ_∞ be the residue field of $\mathcal{O}_\infty$ and Σ_p be that of $\mathcal{O}_p$. It is straightforward to check that $\Sigma_\infty = k$, and Σ_p is isomorphic to $k(\beta)$, where $\beta \in \bar{k}$ is a root of p.

For convenience, we also use $\mathcal{O}_{t^{-1}}$ to refer to $\mathcal{O}_\infty$.

According to [5, page 119], for a place P of K, either there exists a unique irreducible polynomial p such that $p \in P$, in which case P will be called a *finite place*, or $t^{-1} \in P$, in which case P will be called an *infinite place*.

Let q be t^{-1} or an irreducible polynomial of $k[t]$. If $q \in P$, then we say that P *lies above* q, or equivalently, q *lies below* P. In fact, q lies below at least one place of K but not infinitely many places. Now assume that q lies below P. Let

$\mathcal{D}$ be the valuation ring of P. Then $\mathcal{O} := \mathcal{D} \cap k(t)$ is the valuation ring in $k(t)$ of the place $Q := P \cap \mathcal{O}$, which is generated by q. Since Q is the contraction of P in $\mathcal{O}$, one can identify the residue field Σ_Q as a subfield of Σ_P.

Let ν_Q be the order function of $k(t)$ at Q and ν_P be that of K at P. Since $\nu_P\big(k(t) \setminus \{0\}\big)$ forms an additive subgroup of $\mathbb{Z}$ and contains nonzero integers, it is generated by a positive integer r_P, which is called the *ramification index* of P. It is clear that

$$\forall\ x \in k(t) \setminus \{0\},\ \nu_P(x) = r_P \nu_Q(x). \tag{2}$$

In particular, $\nu_P(q) = r_P$.

An element $x \in K$ is said to be *integral* at q if the order of x is nonnegative at each place of K lying above q. The set of all elements integral at q is a free $\mathcal{O}_q$-module of rank $[K : k(t)]$. A basis of this module is called a *local integral basis* at q. If x is integral at each irreducible polynomial in $k[t]$, we say that x is integral over $k[t]$. This is equivalent to saying that the monic minimal polynomial of x belongs to $k[t, X]$. The set of all integral elements is a free $k[t]$-module of rank $[K : k(t)]$, and a basis of this module is called an *integral basis* of K. Several algorithms are known for computing an integral basis for K [29, 26, 31].

The following conventions will be used throughout this paper. Let $v \in k[t]$ be squarefree and let S be the set of all irreducible factors of v. By "integral at v" we mean "integral at each $p \in S$", and similarly, by "local integral basis at v" we mean "local integral basis at each $p \in S$ ". Denote $\mathcal{O}_v = \bigcap_{p \in S} \mathcal{O}_p$. All the elements in K, that are integral at v, form a free $\mathcal{O}_v$-module of rank $[K : k(t)]$.

4 Special Polynomials and Places

In the rest of this paper, we let $(k, ')$ be a differential field of characteristic 0, C_k be its subfield of constants and k' its set of the derivatives in k. Let E be a differential extension of k. Kolchin in [18, page 803] defines that an element $t \in E$ is called *Weierstrassian* over k if t is not a constant and $(t')^2 = \alpha^2(4t^3 - g_2 t - g_3)$, where $\alpha \in k$, $g_2, g_3 \in C_k$ and $27g_3^2 - g_2^3 \neq 0$. In this section, we consider a more general situation.

Definition 2. *Suppose that $t \in E$ is transcendental over k and $t' \in \overline{k(t)}$ is integral over $k[t]$. Let $m(t, X) \in k[t, X]$ be the monic absolutely irreducible minimal polynomial of t'. Then $K := k(t, t')$ is a field of algebraic functions of one variable, and a differential extension of $(k, ')$. We call K a* Weierstrass-like extension *of k.*

If t is transcendental and Weierstrassian over k, then $k(t, t')$ is a Weierstrass-like extension. Definition 2 also generalizes Bronstein's notion of monomial extensions [6, Section 3.4]. A monomial extension is a Weierstrass-like extension with $m(t, X) = X - s(t)$ for some $s \in k[t]$. In a monomial extension, a polynomial $p \in k[t]$ is called special if $p \mid p'$. Being special is closely related to

K having more constants than k, and with differentiation affecting orders in a different way than the usual derivation $\frac{d}{dx}$. That is why special polynomials need special attention.

In our case, we have nontrivial algebraic extensions, so p' need not be a polynomial even if p is. We therefore need to refine the concept of being special for Weierstrass-like extensions.

Definition 3. *Let $\beta \in \bar{k}$. We call β a* special point *of m if $m(\beta, \beta') = 0$. For an irreducible polynomial $p \in k[t]$, we say that p is* special *(w.r.t. m) if it admits a root that is a special point. Otherwise, p is called* normal*.*

By [6, Thm 3.2.4], a k-automorphism of any algebraic extension of k commutes with derivation. Hence conjugation preserves the specialness. Therefore, if one of the roots of a polynomial p is a special point, then all roots of p are special points. The definition of special points corresponds to [6, Thm 3.4.3].

For $q \in k[t]$, we let $\kappa(q)$ be the polynomial obtained by differentiating the coefficients of q, and $\partial_t(q) = \frac{d}{dt}(q)$ be the formal derivative of q with respect to t. The operations κ and ∂_t are both derivations on $k[t]$, and $q' = \kappa(q) + \partial_t(q)t'$. One can see that q' is integral over $k[t]$ since t' is.

Theorem 4. *Let $p \in k[t]$ be irreducible. Then p is special if and only if there exists a place P of K lying above p such that $\nu_P(p') > 0$.*

Proof. Let $\beta \in \bar{k}$ be a root of p. Assume that $p = (t - \beta)\tilde{p}$, where $\tilde{p} \in k(\beta)[t]$ and $\tilde{p}(\beta) \neq 0$. Then:

$$\kappa(p) = -\beta'\tilde{p} + (t - \beta)\kappa(\tilde{p}), \quad \partial_t(p) = \tilde{p} + (t - \beta)\partial_t(\tilde{p}). \tag{3}$$

Recall that t' is integral over $k[t]$. For any place P lying above p, passing to the residue field Σ_P yields

$$\overline{p'} = \overline{\kappa(p)} + \overline{\partial_t(p)}\,\overline{t'}.$$

Since Σ_p can be considered as a subfield of Σ_P through $\Sigma_p \cong k(\beta)$, it follows that

$$\overline{p'} = \tilde{p}(\beta)\big(\overline{t'} - \beta'\big).$$

Hence $\nu_P(p') > 0$ if and only if $\overline{t'} = \beta' \in \Sigma_P$.

At first, assume that $\nu_P(p') > 0$ for some place P lying above p. Then $\overline{t'} = \beta'$. Since $m(t, t') = 0$, taking images into the residue field yields $m(\bar{t}, \overline{t'}) = \bar{0}$, hence $m(\beta, \beta') = 0$, i.e., p is special.

Conversely, we assume that p is special. Then $m(\beta, \beta') = 0$, so β' is a root of $m(\beta, X)$. Assume $m(\beta, X) = (X - \beta')^s\,\tilde{m}(X)$, where $\tilde{m}(X) \in k(\beta)[X]$ and $\tilde{m}(\beta') \neq 0$. By [28, Thm 3.3.7], there is a place P of K lying above p such that $\overline{t'} = \beta' \in \Sigma_P$. Thus $\nu_P(p') > 0$. ∎

This theorem indicates that Definition 3 generalizes the notion of special polynomials in [6, Section 3.4].

Definition 5. *A polynomial $q \in k[t]$ is called* normal *if q is squarefree and all irreducible factors of q are normal.*

Proposition 6. *Let $q \in k[t]$ be a normal polynomial and p be an irreducible factor of q. Then $\nu_P(q') = 0$ for every place P lying above p.*

Proof. Write $q = p\tilde{q}$ with $\gcd(p, \tilde{q}) = 1$. Then $q' = p'\tilde{q} + p\tilde{q}'$. For any place P of K lying above p, $\nu_P(p') = 0$ by Theorem 4. So $\nu_P(p'\tilde{q}) = 0$. Note that $\nu_P(p\tilde{q}') > 0$ since $\tilde{q}'$ is integral over $k[t]$. Hence $\nu_P(q') = 0$. ■

We now introduce the notion of special places in K.

Definition 7. *Let P be a finite place of K lying above $p \in k[t]$. We call P a* special place *if $\nu_P(p') > 0$, and* special of the zeroth kind *if $0 < \nu_P(p') < \nu_P(p)$. We call that P is* normal *if $\nu_P(p') = 0$.*

Extending Bronstein's [6] theory of special polynomials, we encounter places of the zeroth kind, which cannot appear in monomial extensions. The name "the zeroth kind " is in contrast to Bronstein's first-kind case $\nu_P(p') = \nu_P(p)$, because the zeroth-kind case corresponds to $\nu_P(p') < \nu_P(p)$. Places of the zeroth kind allow effective control of orders after differentiation and are the only ones involved in the special reduction to be developed in Section 6. If P is a special place lying above p, then p is also special by Theorem 4.

Proposition 8. *Let P be a finite place of K lying above $p \in k[t]$, and let r be the ramification index of P. Then, for any $f \in K \setminus \{0\}$:*

(i) If $\nu_P(f) = 0$, then $\nu_P(f') \geq \min\{0, \nu_P(p') - r + 1\}$.

(ii) If $\nu_P(f) \neq 0$, then $\nu_P(f') \geq \min\{\nu_P(f), \nu_P(f) + \nu_P(p') - r\}$, and equality holds if P is normal or special of the zeroth kind.

Proof. Note that $\nu_P(p) = r$ by (2). Let $v = \nu_P(f)$. Assume that f admits a local expansion of the form $f = b_0 p^{\frac{v}{r}} + b_1 p^{\frac{v+1}{r}} + \cdots$, where $b_i \in \bar{k}$ and $b_0 \neq 0$. Differentiating f gives

$$f' = b_0' p^{\frac{v}{r}} + b_1' p^{\frac{v+1}{r}} + \cdots + p' \left(\frac{v}{r} b_0 p^{\frac{v-r}{r}} + \frac{v+1}{r} b_1 p^{\frac{v+1-r}{r}} + \cdots \right).$$

(i) If $v = 0$, then the order of f' is no less than the respective orders of p^0 and $p' p^{\frac{1-r}{r}}$.

(ii) If $v \neq 0$, then the order of f' is no less than the respective orders of $p^{\frac{v}{r}}$ and $p' p^{\frac{v-r}{r}}$. Hence $\nu_P(f') \geq \min\{v, v + \nu_P(p') - r\}$. If P is normal or special of the zeroth kind, then $\nu_P(p') < r$. Therefore, $p' p^{\frac{v-r}{r}}$ dominates the order of f', because its order is strictly less than $p^{\frac{v}{r}}$. Thus $\nu_P(f') = v + \nu_P(p') - r$. ■

Corollary 9. *If all special places of K are of the zeroth kind, then the field of constants of K coincides with C_k.*

Proof. Let $c \in K$ be a constant, i.e., $c' = 0$. Assume $c \notin k$, then there is a place P of K such that $\nu_P(c) < 0$ by [12, page 9, Corollary 3]. Note that P is either normal or special of the zeroth kind. It follows from Proposition 8 that $\nu_P(0) = \nu_P(c')$ is a finite number, a contradiction. ∎

We also need an analogue of Proposition 8 at infinite places as a preparation for studying elementary integrablity in Section 6.

Proposition 10. *Let P be an infinite place of K with ramification index r. Then, for any $f \in K \setminus \{0\}$:*

(i) If $\nu_P(f) = 0$, then $\nu_P(f') \geq \min\{0, \nu_P(t') + r + 1\}$.

(ii) If $\nu_P(f) \neq 0$, then $\nu_P(f') \geq \min\{\nu_P(f), \nu_P(f) + \nu_P(t') + r\}$.

Proof. We have $\nu_P(t) = -r$ by (2). Let $v = \nu_P(f)$. Assume f admits a local expansion at infinity: $f = b_0 t^{\frac{-v}{r}} + b_1 t^{\frac{-v-1}{r}} + \cdots$, where $b_i \in \bar{k}$ and $b_0 \neq 0$. Differentiating f yields

$$f' = b_0' t^{\frac{-v}{r}} + b_1' t^{\frac{-v-1}{r}} + \cdots + t' \left(\frac{-v}{r} b_0 t^{\frac{-v-r}{r}} + \frac{-v-1}{r} b_1 t^{\frac{-v-1-r}{r}} + \cdots \right),$$

from which one can derive assertions (i) and (ii) by a similar argument as in the proof of Proposition 8. ∎

5 Hermite Reduction

A core algorithmic technique in symbolic integration is Hermite reduction, which decomposes integrands into an integrable part and a remainder with controlled poles. It is extended from rational functions [21, 16] to transcendental elementary functions by Risch [24, 5, 6], and to algebraic functions by Trager [29, 5] via integral bases. More recently, Hermite reduction has been generalized to D-finite functions [4, 11, 30, 9].

Throughout this section, let $K = k(t, t')$ be a Weierstrass-like extension over k, and let $m \in k[t, X]$ be the monic minimal polynomial of t' with degree n in X. Let $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of K. Then an element $f \in K$ can be written as $f = \sum_{i=1}^n \frac{f_i}{D} \omega_i$ for some $D, f_i \in k[t]$ with $\gcd(D, f_1, \ldots, f_n) = 1$. Such D is unique up to a nonzero multiplicative element of k. We call D the *denominator* of f w.r.t. $\{\omega_1, \ldots, \omega_n\}$. Write $D = D_N D_S$, where all the irreducible factors of D_N are normal and those of D_S are special. Note that D_N and D_S are coprime. See Appendix A for an algorithm to compute D_N and D_S without irreducible factorization.

By the extended Euclidean algorithm, one can uniquely decompose f as $\mathcal{N}(f) + \mathcal{S}(f)$, where

$$\mathcal{N}(f) = \sum_{i=1}^n \frac{a_i}{D_N} \omega_i, \text{ and } \mathcal{S}(f) = \sum_{i=1}^n \frac{b_i}{D_S} \omega_i$$

for some $a_i, b_i \in k[t]$ with $\deg(a_i) < \deg(D_N)$. We call $\mathcal{N}(f)$ and $\mathcal{S}(f)$ the *normal and special parts* of f, respectively. We say that $\mathcal{N}(f) + \mathcal{S}(f)$ is the *canonical representation* of f w.r.t. $\{\omega_1, \ldots, \omega_n\}$. Both $\mathcal{N}$ and $\mathcal{S}$ can be regarded as k-linear operators on K.

The idea of Hermite reduction is to decrease the multiplicity of factors of D_N modulo derivatives. We begin with a technical lemma analogous to a result in [29, Section 4.2]. It will be used later to guarantee the correctness of Hermite reduction.

Lemma 11. *Let $v \in k[t]$ be a normal polynomial and $\mu > 1$ be an integer. Set $\psi_i := v^{\mu}\big(v^{1-\mu}\omega_i\big)'$ with $i = 1, \ldots, n$. Then $\{\psi_1, \ldots, \psi_n\}$ is a local integral basis at v.*

Proof. Let p be an arbitrary irreducible factor of v, and let Q be a place of K lying above p with ramification index r_Q. Then Q is normal. Since each ω_i is integral over $k[t]$, $\nu_Q(\omega_i') > -r_Q$ by Proposition 8. Note that v is normal. Then $\nu_Q(v') = 0$ by Proposition 6. Therefore, $\nu_Q(\psi_i) \geq 0$ because $\psi_i = v\omega_i' - (\mu - 1)v'\omega_i$. Consequently, ψ_i is integral at p. It follows that ψ_i is integral at v.

There are two ways how $\{\psi_1, \ldots, \psi_n\}$ can fail to be a local integral basis at v: (i) $\psi_1, \ldots, \psi_n$ are $\mathcal{O}_v$-linearly dependent; (ii) $\psi_1, \ldots, \psi_n$ are $\mathcal{O}_v$-linearly independent, but there exists an element integral at v which is not a $\mathcal{O}_v$-linear combination of $\psi_1, \ldots, \psi_n$. In both cases, there is an $F \in K$, integral at v, such that $F = \frac{1}{v}\sum_{i=1}^{n} c_i\psi_i$, where $c_1, \ldots, c_n \in k[t]$ are not all zero and $v \nmid c_j$ for some j. We will derive a contradiction from the existence of such element F.

Let $G = \sum_{i=1}^{n} c_i'\omega_i$. Note that c_i' is integral over $k[t]$ since $c_i \in k[t]$, so is G. We have:

$$F + G = v^{\mu-1}\sum_{i=1}^{n}\big(c_i(v^{1-\mu}\omega_i)' + c_i'v^{1-\mu}\omega_i\big) = v^{\mu-1}\sum_{i=1}^{n}(c_iv^{1-\mu}\omega_i)'. \qquad (4)$$

Let $H = \sum_{i=1}^{n} c_iv^{1-\mu}\omega_i = v^{2-\mu}\sum_{i=1}^{n}\frac{c_i}{v}\omega_i$. Then $H \neq 0$. Since $v \nmid c_j$, some irreducible factor p of v appears in the denominator of $\sum_{i=1}^{n}\frac{c_i}{v}\omega_i$. Then there exists a place P of K lying above p such that $\nu_P(H) < \nu_P\big(v^{2-\mu}\big)$, where ν_P denotes the order function at P. Let r_P be the ramification index of P. Then $\nu_P(H) < (2-\mu)r_P \leq 0$ by $\mu > 1$. Then $\nu_P(H') < (1-\mu)r_P = \nu_P\big(v^{1-\mu}\big)$ by Proposition 8.

However, $H' = v^{1-\mu}(F+G)$ by (4), which yields a contradiction since $F+G$ is integral at v. ∎

We now describe the Hermite reduction in K. For convenience, assume that $f = \mathcal{N}(f) = \sum_{i=1}^{n}\frac{f_i}{D}\omega_i \in K$. Then all irreducible factors of D are normal.

Let $D = uv^{\mu}$, where $\mu > 1$ is an integer, v is squarefree, $\gcd(u, v) = 1$ and all irreducible factors of u have multiplicities less than μ. Set $\tilde{f} := fD$, which is integral over $k[t]$, and define $\psi_i := \big(v^{1-\mu}\omega_i\big)'D$. By Lemma 11, $\{\psi_1, \ldots, \psi_n\}$ is a local integral basis at v. Let us decrease the multiplicity of v. We compute $c_1, \ldots, c_n \in k(t)$ such that $\tilde{f} = \sum_{i=1}^{n} c_i\psi_i$. Then $c_i \in \mathcal{O}_v$. For each i, we can find $r_i \in k[t]$ such that $\deg_t r_i < \deg_t v$ and $r_i \equiv c_i \mod v$. Set $\tilde{g} = \sum_{i=1}^n r_i \psi_i$. Then $\tilde{f} - \tilde{g} = vR$ for some $R \in K$ which is integral at v. Hence

$$f = \frac{\tilde{g} + vR}{D} = \sum_{i=1}^n r_i (v^{1-\mu}\omega_i)' + \frac{R}{uv^{\mu-1}}.$$

Using integration by parts,

$$f = \sum_{i=1}^n \Big(\frac{r_i}{v^{\mu-1}}\omega_i\Big)' + \frac{R}{uv^{\mu-1}} - \sum_{i=1}^n \frac{r_i'}{v^{\mu-1}}\omega_i. \tag{5}$$

Let $g = \sum_{i=1}^n \frac{r_i}{v^{\mu-1}}\omega_i$. Since r_i' is integral over $k[t]$, the denominator of $f - g'$ w.r.t. $\{\omega_1, \ldots, \omega_n\}$ has multiplicity less than μ at v by (5).

However, the derivative g' may introduce new factors of the denominator since ω_i' does not need to be integral over $k[t]$. Denote $\vec{\omega} = (\omega_1, \ldots, \omega_n)^\tau$, where τ denotes the transpose of a vector. Let $e \in k[t]$ and $M = (m_{i,j})_{i,j=1}^n \in k[t]^{n \times n}$ be such that

$$e\big(\vec{\omega}\big)' = M\vec{\omega}$$

with $\gcd(e, m_{1,1}, \ldots, m_{n,n}) = 1$. Then e is unique up to a nonzero multiplicative element of k. We call e the *differential denominator* of $\{\omega_1, \ldots, \omega_n\}$. Denote $\vec{r} = (r_1, \ldots, r_n)^\tau$. Then $g = \frac{\vec{r}^\tau \vec{\omega}}{v^{\mu-1}}$. A direct calculation shows that

$$g' = \underbrace{\frac{\big(\vec{r}^\tau\big)'\vec{\omega}}{v^{\mu-1}} + \frac{(1-\mu)v'\vec{r}^\tau\vec{\omega}}{v^\mu}}_{A} + \frac{\vec{r}^\tau M\vec{\omega}}{ev^{\mu-1}}, \tag{6}$$

in which e may introduce new factors of the denominator.

Lemma 12. *The differential denominator $e \in k[t]$ of $\{\omega_1, \ldots, \omega_n\}$ is squarefree.*

Proof. Let p be an irreducible factor of e and write $e = e_0 p$. Then $p\omega_i' = \sum_{j=1}^n \frac{m_{i,j}}{e_0}\omega_j$. By Proposition 8, $\nu_P(p\omega_i') > 0$ for any place P of K lying above p, i.e., $p\omega_i'$ is integral at p. Since $\{\omega_1, \ldots, \omega_n\}$ is an integral basis, $\frac{m_{i,j}}{e_0} \in \mathcal{O}_p$. Then $p \nmid e_0$ by $\gcd(e, m_{1,1}, \ldots, m_{n,n}) = 1$. Hence, e is squarefree. ∎

Remark 13. *The differential denominator e may have special irreducible factors. For example, let $K = \mathbb{Q}(t, t')$ with $(t')^3 = t$. Then 0 is a special point of $X^3 - t$, and hence t is special. Since $t'' = (t')^2/(3t)$, the differential denominator of $\{1, t', t''\}$ is divisible by t.*

The above reduction does not introduce higher multiplicities. The remaining difficulty lies in the possible appearance of new special factors. In general, special poles of integrands are hard to handle. Fortunately, newly-introduced special factors can be removed by an additional modification as follows.

Using notation in (6), we assume $\gcd(e, v) = d$ and $e = e_1 d$. Then e_1 is coprime with v since e is squarefree. By the extended Euclidean algorithm, we can find $\vec{a}^\tau = (a_1, \ldots, a_n)^\tau \in k[t]^n$ such that $\deg_t a_i < \deg_t v$ and $v^{\mu-1}\vec{a}^\tau + \vec{r}^\tau = e_1\vec{b}^\tau$ for some $\vec{b}^\tau \in k[t]^n$. Set $g_a = \vec{a}^\tau\vec{\omega}$. Then

$$g' + g_a' = A + \left(\vec{a}^\tau\right)'\vec{\omega} + \frac{\left(\vec{r}^\tau + v^{\mu-1}\vec{a}^\tau\right)M\vec{\omega}}{e} = B + \frac{\vec{b}^\tau M\vec{\omega}}{d}, \tag{7}$$

where $B = A + \left(\vec{a}^\tau\right)'\vec{\omega}$. Since v is normal, so is d. Then B has no special factor in the denominator. Hence we eliminate the special poles of g' by adding g_a'. Moreover, the denominator of g_a' is a factor of e, which is squarefree.

Then by (5) and (7), the denominator of $f - (g + g_a)'$ is a product of normal irreducible factors, each of which has the multiplicity less than μ at v. Repeating the reduction process until the integrand has a squarefree denominator, we arrive at:

Theorem 14. *Let $f \in K$. Then there exist $g \in K$, $f_0 \in K$ with a normal denominator dividing e, and $h = \sum_{i=1}^n \frac{h_i}{D_*}\omega_i$ where D_* is normal and coprime with e, $\deg h_i < \deg D_*$, such that*

$$\mathcal{N}(f) = g' + f_0 + h.$$

Moreover, h is unique, and $h = 0$ if $f \in K'$.

Proof. By the preceding discussion, one can find $g \in K$ such that $w = \mathcal{N}(f) - g'$ has a normal denominator. Write $w = \sum_{i=1}^n w_i\omega_i$ where $w_i \in k(t)$, then applying the extended Euclidean algorithm to the numerators of w_i yields the desired decomposition $w = f_0 + h$.

It remains to prove the uniqueness of h and verify $h = 0$ if f is a derivative in K. For proving the uniqueness and in-field integrability of h, it suffices to show that $h = 0$ if either $h + f_0$ or $h + f_0 + \mathcal{S}(f)$ belongs to K'.

Let $F \in K$ be such that $F' = h + f_0 + \alpha$, where $\alpha = 0$ or $\alpha = \mathcal{S}(f)$. For any normal irreducible $p \in k[t]$, let P be any place of K lying above p with ramification index r_P. Then $\nu_P(F) \geq 0$. For, otherwise, $\nu_P(F') < -r_P$ by Proposition 8, i.e., $\nu_P(pF') < 0$, which would contradicts to the fact that the multiplicity of p in the denominator of F' is at most one.

The conclusion $\nu_P(F) \geq 0$ implies $\mathcal{N}(F) = 0$. Then F can be written $\frac{\vec{\rho}^\tau\vec{\omega}}{H}$, where $\vec{\rho}^\tau \in k[t]^n$, $\vec{\omega} = (\omega_1, \ldots, \omega_n)^\tau$ and $H \in k[t]$ has only irreducible special factors. Then

$$F' = \frac{\left(\vec{\rho}^\tau\right)'\vec{\omega}}{H} - \frac{H'\vec{\rho}^\tau\vec{\omega}}{H^2} + \frac{\vec{\rho}^\tau M\vec{\omega}}{eH}.$$

Since H' and all entries of $\left(\vec{\rho}^\tau\right)'$ are integral over $k[t]$, the denominator of F' has no irreducible factor, which is normal and coprime with e. Hence, $h = 0$. ■

We call the element h in Theorem 14 the *Hermite remainder* of f (w.r.t. $\{\omega_1, \ldots, \omega_n\}$). The Hermite reduction described above naturally translates into an algorithm for computing f_0, g and h in Theorem 14. The result depends on the choice integral bases.

Example 15. *Let $k = \mathbb{C}(z)$ equipped with $' = d/dz$ and $k(t,t')$ be a Weierstrass-like extension over k, where $(t')^2 = 4t^3 - g_2 t - g_3$, $g_2 = 0$ and $g_3 = -4$. We compute the integral of the function*

$$f = \frac{(t^2 - t - 1)t' - 4 + (2z+2)t^4 + (4z+2)t^3 - 4zt^2 - 4t}{(t+1)t^2}.$$

Let $\{1, t'\}$ be the chosen integral basis of $k(t,t')$. One can check that 0 is not a special point but -1 is. Hence t is normal and $t+1$ is special. By applying the extended Euclidean algorithm, we obtain the canonical representation of f w.r.t. $\{1,t'\}$: $f = \mathcal{N}(f) + \mathcal{S}(f)$, where

$$\mathcal{N}(f) = \frac{-4 - t'}{t^2}, \qquad \mathcal{S}(f) = \frac{2(z+1)t^2 + 2(2z+1)t - 4z + t'}{t+1}.$$

To reduce the multiplicities of t in the denominator of $\mathcal{N}(f)$, we set $\psi_1 = t^2(t^{-1})' = -t'$ and $\psi_2 = t^2(t^{-1}t')' = 2t^3 - 4$. By Lemma 11, $\{\psi_1, \psi_2\}$ is a local integral basis at t. We can find $c_1 = 1$ and $c_2 = -\frac{4}{2t^3-4}$ such that the numerator of $\mathcal{N}(f)$ is $c_1\psi_1 + c_2\psi_2$. Then we can compute $r_1 = 1$ and $r_2 = 1$ such that $\deg_t r_i < \deg_t t$ and $r_i \equiv c_i \mod t$. Set $g = \frac{r_1 + r_2 t'}{t}$, we have $\mathcal{N}(f) = g' - 2t$. Hence the Hermite remainder of f is 0.

The special part $\mathcal{S}(f)$ will be handled in the next section.

6 Special and Polynomial Reductions

In this section, let $K = k(t,t')$ be a Weierstrass-like extension of k and $m \in k[t,X]$ be the monic minimal polynomial of t'. We assume that $m = X^2 - q$, where $q \in k[t]$ is squarefree and $\deg_t q \geq 3$. In such K, one can model extensions generated by transcendental Weierstrassian elements over k.

For $f \in K$, Theorem 14 decomposes its normal part into the sum of a Hermite remainder h, an in-field integrable part g', and an obstacle f_0 that admits no in-field integrability. Since such an obstacle may occur, Hermite reduction serves merely as a preprocessor for the normal part. Moreover, the special part of f is not addressed.

The goal of this section is to control these untreated parts by two further reductions, which will be called *special reduction* and *polynomial reduction*, respectively.

At first, we reduce the special part. In general, it is difficult to determine all special points of m, which is equivalent to finding all algebraic solutions of a first-order differential equation. In order to circumvent this difficulty, we make a technical assumption throughout this section:

Hypothesis 16. *Every special point of m is a constant in $\overline{k}$.*

By the above hypothesis, $q(\beta) = 0$ if β is a special point of m. Thus the special points of m are constant roots of q.

Remark 17. *The hypothesis holds when k is a Liouvillian extension of C_k and $q \in C_k[t]$ by [27, Proposition 3.2].*

Lemma 18. *Let $\beta \in \overline{k}$ and $\beta' = 0$. If $p \in k[t]$ is the monic minimal polynomial of β, then $p \in C_k[t]$. In particular, if $p \in k[t]$ is irreducible and special, then p is a factor of q with constant coefficients.*

Proof. Differentiating both sides of $p(\beta) = 0$, we see that $\kappa(p)(\beta) + \beta'\partial_t(p)(\beta) = 0$. Hence $\kappa(p)(\beta) = 0$, i.e., $p \mid \kappa(p)$. Since p is monic, we have that $\deg_t \kappa(p) < \deg_t p$. So $\kappa(p) = 0$, i.e., all coefficients of p are constants. ∎

Lemma 19. *Let $p \in k[t]$ be irreducible and special. Then there exists exactly one place P of K lying above p. In particular, $\nu_P(p) = 2$, $\nu_P(p') = 1$ and $\nu_P(t') = 1$.*

Proof. Since p is irreducible and special, we have that $p \mid q$ and $p \in C_k[t]$. Let P be a special place lying above p with ramification index r_P and set $v = \nu_P(t')$. As $p' = t'\partial_t(p)$ and $\gcd\big(p, \partial_t(p)\big) = 1$, we have $\nu_P(p') = v$. Since $(t')^2 = q$ and q is squarefree, it follows that $2v = r_P$. Then $r_P \geq 2$. By [12, page 52, Theorem 1], P is the only place lying above p and $r_P = [K : k(t)] = 2$. Hence, $v = 1$. ∎

All special places of K are of the zeroth kind by the above lemma. Then C_k is the field of constants of K by Corollary 9.

By [29, page 31], $\{1, t'\}$ is an integral basis of K. Write $q = q_N q_S$, where $q_N \in k[t]$ is normal, $q_S \in k[t]$ is monic and all its irreducible factors are special. Then $q_S \in C_k[t]$, and

$$t'' = \frac{\kappa(q_N)}{2q_N}t' + \frac{\partial_t(q)}{2}.$$

Hence q_N is the differential denominator of $\{1, t'\}$. Denote by I_K the set of elements in K that are integral over $k[t]$. Under the basis $\{1, t'\}$, Hermite reduction simplifies the normal parts of integrands.

We now describe the special reduction, which decreases the multiplicity of factors of the denominator of special parts modulo derivatives. Assume that $f \in K$ with the special part $\mathcal{S}(f) = \frac{A+Bt'}{D}$, where $A, B, D \in k[t]$, D is monic and $\gcd(A, B, D) = 1$. Assume that $D = uv^{\mu}$ where $\mu > 0$, v is squarefree and coprime with u, and factors of u have multiplicity less than μ. By Hypothesis 16, $u, v \in C_k[t]$ and v divides q. Set $q_v = q/v \in k[t]$. For $a, b \in k[t]$ and $\lambda \in \mathbb{N} \setminus \{0\}$, a direct calculation shows that

$$\left(\frac{a}{v^{\lambda}}\right)' = \frac{\kappa(a) + \partial_t(a)t'}{v^{\lambda}} - \frac{\lambda a \partial_t(v)t'}{v^{\lambda+1}} \tag{8}$$

and

$$\left(\frac{bt'}{v^{\lambda}}\right)' = \frac{b_1 t'}{q_N v^{\lambda}} + \frac{b_2}{v^{\lambda-1}} + \frac{(1-2\lambda)bq_v\partial_t(v)}{2v^{\lambda}} \tag{9}$$

for some $b_1, b_2 \in k[t]$. It proceeds as follows:

(i) If $\mu \geq 1$, we compute $b \in k[t]$ such that $\deg_t b < \deg_t v$ and

$$(1-2\mu)ubq_v\partial_t(v) \equiv 2A \mod v.$$

Such b can be found since q_v, $\partial_t(v)$ and u are coprime with v. It follows from (9) that

$$\tilde{f} := \mathcal{S}(f) - \left(\frac{bt'}{v^\mu}\right)' = \frac{\tilde{A}}{uv^{\mu-1}} + \frac{\tilde{B}t'}{uv^\mu} + R \tag{10}$$

for some $\tilde{A}, \tilde{B} \in k[t]$ and $R \in \frac{I_K}{q_N}$.

(ii) If μ in (10) is greater than or equal to 2, then we compute $a \in k[t]$ such that $\deg_t a < \deg_t v$ and

$$(\mu-1)ua\partial_t(v) \equiv \tilde{B} \mod v.$$

Such a can be found since $u, \partial_t(v)$ are coprime with v. By (8), $\tilde{f} + \left(\frac{a}{v^{\mu-1}}\right)'$ has a denominator, in which the multiplicity of v is at most $\mu - 1$.

Repeating (i) and (ii) to $\mathcal{S}(f)$, we have

Theorem 20. *Let $f \in K$. Then there exist $g \in K$, $f_1 \in \frac{I_K}{q_N}$ and $s = \frac{\theta}{\gamma}t'$, where $\gamma \in k[t]$ is squarefree with only irreducible special factors, $\theta \in k[t]$ and $\deg_t \theta < \deg_t \gamma$, such that*

$$\mathcal{S}(f) = g' + f_1 + s.$$

Moreover, s is unique and $s = 0$ if $f \in K'$.

Proof. By (i) and (ii) given above, there exists $g \in K$ such that $\mathcal{S}(f) - g' = \frac{\theta_0}{\gamma}t' + r$, where $\gamma, \theta_0 \in k[t]$, $r \in \frac{I_K}{q_N}$, and γ is squarefree with merely special factors. Dividing θ_0 by γ yields the desired θ and f_1.

Now we prove the uniqueness and in-field integrability of s. Similar to the proof of Theorem 14, it suffices to prove that $s = 0$ if $f_1 + s + \alpha \in K'$, where $\alpha = 0$ or $\alpha = \mathcal{N}(f)$.

Assume $s \neq 0$ and let $F \in K$ satisfy $F' = f_1 + s + \alpha$. Let p be an irreducible factor of γ and P be the place lying above p. By Lemma 19, $\nu_P(p) = 2$ and $\nu_P(p') = \nu_P(t') = 1$.

We claim that $\nu_P(F) \geq 0$. Otherwise, $\nu_P(F') = \nu_P(F) - 1 \leq -2$ by Proposition 8. On the other hand, γ is squarefree, and thus, $\nu_P(s) \geq \nu_P(\frac{t'}{\gamma}) = -1$. Accordingly, $\nu_P(F') \geq -1$ since f_1 and z are integral at p, a contradiction. The claim holds.

Again by Proposition 8, $\nu_P(F) \geq 0$ implies $\nu_P(F') \geq 0$. Since P is the only place lying above p, F' is integral at p, which contradicts the fact that $\{1, t'\}$ is an integral basis and $F' = f_1 + \frac{\theta}{\gamma}t' + z$. ∎

For a given $f \in K$, the special reduction described above computes f_1, g and s in Theorem 20. We call s the *special remainder* of f (w.r.t. $\{1, t'\}$). Now we show how to use special reduction to integrate the $\mathcal{S}(f)$ in Example 15.

Example 21. *We have* $q = 4(t^3+1)$*. Then* $q_N = 4$ *and* $q_S = t^3+1$*. Recall that* $\mathcal{S}(f) = \frac{A+Bt'}{D}$ *where* $A = 2(z+1)t^2 + 2(2z+1)t - 4z$*,* $B = 1$ *and* $D = t+1$*. Then* $u = 1$*,* $v = t+1$*,* $\mu = 1$ *and* $q_v = 4(t^2-t+1)$*. One can find that* $b = z$ *such that* $\deg_t b < \deg_t v$ *and* $(1-2\mu)\, ub\, q_v\, \partial_t(v) \equiv 2A \mod v$*. Then* $\tilde{A} = 0$*,* $\tilde{B} = 0$ *and* $R = 2t$ *in* (10)*. Therefore,* $\mathcal{S}(f) = \left(\frac{zt'}{t+1}\right)' + 2t$*. The special reminder of* f *is* 0*.*

Combine with the result in Example 15, we see that the obstacles in $\mathcal{N}(f)$ *and* $\mathcal{S}(f)$ *are canceled with each other, hence* $f \in K'$*, i.e.,*

$$\int f\,dz = \frac{1+t'}{t} + \frac{zt'}{t+1}.$$

By Theorems 14 and 20, each element of K is decomposed as the sum of its Hermite remainder, special remainder and an element in $\frac{I_K}{q_N}$. To control the poles at infinity, we develop the polynomial reduction to simplify the elements of $\frac{I_K}{q_N}$.

For $f \in \frac{I_K}{q_N}$, we can write $f = \frac{a+bt'}{q_N} + w + rt'$, where $a, b, w, r \in k[t]$ with $\deg_t a$ and $\deg_t b$ are less than $\deg_t q_N$. Then

$$\mathcal{N}(f) = \frac{a+bt'}{q_N} \text{ and } \mathcal{S}(f) = w + rt'.$$

Moreover, $\mathcal{S}(f)$ is integral over $k[t]$. Write $r = r_n t^n + \cdots + r_0$, and set $\mathbf{\Upsilon}(f) := \frac{r_n}{n+1} t^{n+1} + \cdots + r_0 t \in k[t]$. Then $\partial_t\big(\mathbf{\Upsilon}(f)\big) = r$. By a direct calculation,

$$f - \mathbf{\Upsilon}(f)' = f - \kappa\big(\mathbf{\Upsilon}(f)\big) - rt' = \mathcal{N}(f) + w - \kappa\big(\mathbf{\Upsilon}(f)\big).$$

So the special part of $f - \mathbf{\Upsilon}(f)'$ is $w - \kappa\big(\mathbf{\Upsilon}(f)\big)$, denoted by $\mathcal{S}^*(f)$, which belongs to $k[t]$. Moreover, $\mathcal{S}^*$ is a k-linear operator on $\frac{I_K}{q_N}$.

Lemma 22. *For any* $\delta \in k$ *and* $\lambda \in \mathbb{N}$*, we have* $(\delta t^\lambda t')' \in \frac{I_K}{q_N}$ *and* $\mathcal{S}^*\big((\delta t^\lambda t')'\big)$ *is of degree* $\lambda - 1 + \deg_t q$ *with leading coefficient* $l(\lambda)\delta$*, where*

$$l(\lambda) = \left(\lambda + \frac{\deg_t q}{2}\right) \operatorname{lc}_t(q).$$

Proof. A direct calculation shows that

$$\big(\delta t^\lambda t'\big)' = \underbrace{\lambda\delta q t^{\lambda-1} + \frac{\delta\partial_t(q)t^\lambda}{2}}_{\Gamma_1} + \underbrace{\left(\delta' t^\lambda + \delta\frac{\kappa(q_N)}{2q_N} t^\lambda\right)}_{\Gamma_2} t'.$$

Hence $(\delta t^\lambda t')' \in \frac{I_K}{q_N}$. Furthermore, the polynomial $\Gamma_1 \in k[t]$ is of degree $\lambda - 1 + \deg_t q$ with leading coefficient $l(\lambda)\delta$, $\Gamma_2 \in k(t)$ and $\mathcal{S}(\Gamma_2)$ is either 0 or of degree λ. Set $R := \mathbf{\Upsilon}\big((\delta t^\lambda t')'\big)$. Then $\partial_t(R) = \mathcal{S}(\Gamma_2)$ and $\mathcal{S}^*\big((\delta t^\lambda t')'\big) = \Gamma_1 - \kappa(R)$. In particular, $\deg_t R \le \lambda + 1$. From the assumption that $\deg_t q \ge 3$, we have

$$\deg_t \kappa(R) \le \lambda + 1 < \lambda - 1 + \deg_t q = \deg_t \Gamma_1.$$

Thus $\mathcal{S}^*\big((\delta t^\lambda t')'\big)$ has the same leading term as Γ_1. ∎

Given $f \in \frac{I_K}{q_N}$, we let $d = \deg_t \mathcal{S}^*(f)$ and $\varepsilon = \mathrm{lc}_t\big(\mathcal{S}^*(f)\big)$. If $d \geq \deg_t q - 1$, then we take λ and δ in the above lemma as $d - \deg_t q + 1$ and $\varepsilon / l(\lambda)$, respectively. The same lemma implies the leading term of $\mathcal{S}^*\big((\delta t^\lambda t')'\big)$ is equal $\mathcal{S}^*(f)$.

Let $\tilde{f} = f - (\delta t^\lambda t')'$. Then the degree of $\mathcal{S}^*\big(\tilde{f}\big)$ is less than d. Repeating this process to $\tilde{f}$ until $\deg_t \mathcal{S}^*(\tilde{f}) < \deg_t q - 1$, we find $b \in k[t]$ such that $\mathcal{S}^*(f - b')$ has degree less than $\deg_t q - 1$. With this degree-decreasing process, we have

Theorem 23. *Let $f \in \frac{I_K}{q_N}$. Then there exist $g \in K$, $f_2 \in \frac{I_K}{q_N}$ with $\mathcal{S}(f_2) = 0$ and $\eta \in k[t]$ with $\deg_t \eta < \deg_t q - 1$, such that*

$$f = g' + f_2 + \eta.$$

Moreover, f_2 is unique, and $f \in K'$ if and only if $f_2 = 0$ and $\eta \in k'$.

The proof of Theorem 23 is based on the next lemma.

Lemma 24. *(i) Let P be an infinite place of K and let $f = a + bt'$ be such that $a, b \in k(t)$ are proper fractions. Then $\nu_P(f) \geq \nu_P(t') - \nu_P(t)$.*

(ii) Let $F \in K$ be integral over $k[t]$ and polynomial $\eta \in k[t]$ with $\deg_t \eta < \deg_t q - 1$. If $\nu_P(F' + \eta) \geq \nu_P(t') - \nu_P(t)$ for any infinite place P of K, then $F \in k$.

Proof. (i) Let r_P be the ramification index of P. For any polynomial $w \in k[t]$, we have $\nu_P(w) = -r_P \deg_t w$ since $\nu_P(t) = -r_P < 0$. Then $\nu_P(t') < 0$ by $(t')^2 = q$. As a and b are proper,

$$\nu_P(a) \geq r_P > r_P + \nu_P(t') \text{ and } \nu_P(bt') \geq r_P + \nu_P(t').$$

(ii) Since $\{1, t'\}$ is an integral basis, write $F = A + Bt'$ with $A, B \in k[t]$. Then $F' = A_0 + B_0 t' + \mathcal{N}(F')$, where

$$A_0 = \kappa(A) + \partial_t(B) q + \frac{1}{2} B \partial_t(q), \ B_0 = \partial_t(A) + \kappa(B) + S\left(\frac{\kappa(q_N) B}{2 q_N}\right),$$

and A_0, $B_0 \in k[t]$. By (i), $\nu_P\big(\mathcal{N}(F')\big) \geq \nu_P(t') - \nu_P(t)$. Since

$$A_0 + \eta + B_0 t' = F' + \eta - \mathcal{N}(F'),$$

we have that $\nu_P(A_0 + \eta + B_0 t')$ is no less than $\nu_P(t') - \nu_P(t)$ for any infinite place P. Set

$$C := \frac{t}{t'}(A_0 + \eta + B_0 t') = \frac{(A_0 + \eta) t}{q} t' + B_0 t.$$

Then C is integral at t^{-1}. By [29, page 30, Proposition], $\{1, t'\}$ is normal at t^{-1}. It follows from [10, Lemma 2] that $\frac{(A_0+\eta)t}{q} t'$ and $B_0 t$ are integral at t^{-1}. Accordingly, $B_0 = 0$.

We claim that $B = 0$. Otherwise, $B_0 = 0$ and $B \neq 0$ imply that $\deg_t \partial_t(A) \leq \deg_t B$, i.e., $\deg_t A \leq \deg_t B + 1$. Since $\deg_t q \geq 3$, we have $\deg_t \kappa(A) \leq \deg_t A \leq$

$\deg_t B + 1 < \deg_t B + \deg_t q - 1$. Note that the degree of $\partial_t(B)q + \frac{1}{2}B\partial_t(q)$ is equal to $\deg_t B + \deg_t q - 1$, which is greater than $\deg_t \kappa(A)$. Hence

$$\deg_t A_0 = \deg_t B + \deg_t q - 1 > \deg_t \eta.$$

It follows that

$$\deg_t(A_0 + \eta)t = \deg_t A_0 t = \deg_t B + \deg_t q \geq \deg_t q,$$

hence $\frac{(A_0+\eta)t}{q}$ admits nonpositive order at t^{-1}. As t' is not integral at t^{-1}, neither is $\frac{(A_0+\eta)t}{q}t'$, a contradiction. The claim holds.

Consequently, $A \in k$ by $B_0 = 0$. It follows that $F \in k$. ■

Proof of Theorem 23. Let $b \in k[t]$ be such that the degree of $\mathcal{S}^*(f - b')$ is less than $\deg_t q - 1$. By the definition of $\mathcal{S}^*$, we have

$$f - b' - \big(\mathbf{\Upsilon}(f - b')\big)' = \mathcal{N}(f - b') + \mathcal{S}^*(f - b').$$

Setting $g = b + \mathbf{\Upsilon}(f - b')$, $f_2 = \mathcal{N}(f - b')$ and $\eta = \mathcal{S}^*(f - b')$ gives the desired decomposition.

For the uniqueness of f_2 and in-field integrablity condition for f, it suffices to prove that $f_2 + \eta \in K'$ implies $f_2 = 0$ and $\eta \in k'$. Assume $F' = f_2 + \eta$ for some $F \in K$. By an order comparison similar to those in the proofs of Theorems 14 and 20, F is integral over $k[t]$. Let P be an infinite place of K with ramification index r_P. By Lemma 24 (i), $\nu_P(f_2) \geq \nu_P(t') - \nu_P(t)$. Hence $\nu_P(F' - \eta) \geq \nu_P(t') - \nu_P(t)$. By Lemma 24 (ii), $F \in k$. Thus $f_2 = 0$ and $\eta = F' \in k'$. ■

The process of polynomial reduction naturally translates into an algorithm for computing f_2, g and η in Theorem 23. Combining the Hermite, special and polynomial reductions, we have the following theorem.

Theorem 25. *For $f \in K$, we let h be the Hermite remainder and s be the special remainder of f w.r.t. $\{1, t'\}$ as in Theorems 14 and 20, respectively. Then there exists $g \in K$, a unique element $l \in \frac{I_K}{q_N}$ with no special part, and $\eta \in k[t]$ with $\deg_t \eta < \deg_t q - 1$ such that*

$$f = g' + h + s + l + \eta.$$

Moreover, $f \in K'$ if and only if h, s and l are all zero and $\eta \in k'$.

Proof. The existence of g, l and η follows from Theorems 14, 20 and 23. If $f \in K'$ or $f = 0$, then $l + \eta \in K'$. Hence $l = 0$ and $\eta \in k'$ by Theorem 23. ■

Although η in Theorem 25 is not unique, it is determined up to an element in k' additively. Hence the positive degree terms of η are unique. We call such η a *polynomial remainder* of f (w.r.t. $\{1, t'\}$).

Theorem 25 is not only a criterion for in-field integrablity in Weierstrass-like extensions, but also leads to a necessary condition for elementary integrablity.

Corollary 26. *Assume that C_k is algebraically closed. Let $f \in K$ and η be a polynomial remainder of f. If f has an elementary integral over K, then* $\deg_t \eta \leq \frac{\deg_t q}{2} - 1$.

Proof. Write $f = g' + h + s + l + \eta$ as in Theorem 25 and set $R = f - g'$. If f has an elementary integral over K, so does R. By [6, Thm 5.5.2], there exist $F \in K$, $c_1, \ldots, c_n \in C_k$ and $u_1, \ldots, u_n \in K \setminus \{0\}$ such that

$$R = F' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}.$$

Let P be a place of K with ramification index r_P. If P is normal, then $\nu_P(\frac{u_i'}{u_i}) \geq -r_P$ by Proposition 8. Similar to the proof of Theorem 14, one can show that F has no normal part. If P is special, then $\nu_P(\frac{u_i'}{u_i}) \geq -1$ by Proposition 8 and Lemma 19. Similar to the proof of Theorem 20, one can show that F is integral over $k[t]$.

If P is an infinite place, then $\nu_P(t') = -\frac{r_P \deg_t q}{2}$ by $(t')^2 = q$, which implies $\nu_P(t') + r_P = \frac{r_P(2-\deg_t q)}{2} < 0$. Then

$$\nu_P\left(\frac{u_i'}{u_i}\right) \geq \nu_P(t') + r_P = \nu_P(t') - \nu_P(t)$$

by Proposition 10. Moreover, $\nu_P(h)$, $\nu(s)$ and $\nu_P(l)$ are all greater than or equal to $\nu_P(t') - \nu_P(t)$ by Lemma 24 (i). Hence $\nu_P(F' - \eta) \geq \nu_P(t') - \nu_P(t)$. Then $F \in k$ by of Lemma 24 (ii).

If $\deg_t \eta > 0$, then

$$\nu_P(t') + r_P = \nu_P(t') - \nu_P(t) \leq \nu_P(F' - \eta) = \nu_P(\eta).$$

Hence $\deg_t \eta = -\frac{\nu_P(\eta)}{r_P} \leq -\frac{\nu_P(t')}{r_P} - 1 = \frac{\deg_t q}{2} - 1$. ∎

7 The Appetizer Revisited

We now apply the Hermite reduction and Weierstrass reduction to evaluate the integrals $I_n(z) := \int \wp(z)^n \, dz$ with $n \in \mathbb{N}$ in Section 2. Let $k = \mathbb{C}(z)$ be the field of rational functions equipped with the derivation $' := d/dz$. Then the field of constants of k is $\mathbb{C}$. Let $K = k(t, t')$ be a Weierstrass-like extension and $m = X^2 - q \in k[t, X]$ be the monic minimal polynomial of t', where $q = 4t^3 - g_2 t - g_3$ with $g_2, g_3 \in \mathbb{C}$ and $27g_3^2 - g_2^3 \neq 0$. In this sense, t satisfies the same differential equation as the Weierstrass-$\wp$ function. Then $\{1,\ t'\}$ is an integral basis of K.

Hypothesis 16 holds for our setting by Remark 17, i.e., any special point of m is a constant in $\mathbb{C}$. Then t itself is a polynomial remainder. Since $\frac{\deg_t q}{2} - 1 < 1$, t has no elementary integral over K by Corollary 26. As in Section 2, $\zeta(z)$ stands for the integral of t.

Note that t^n and $(t^n t')'$ lie in $k[t]$ for $n \in \mathbb{N}$. It follows that $\mathcal{S}^*(t^n) = t^n$ and $\mathcal{S}^*\big((t^n t')\big) = (t^n t')'$. Applying the polynomial reduction to t^2 yields that

$$t^2 = \left(\frac{1}{6}t'\right)' + \frac{g_2}{12}.$$

Hence a polynomial remainder of t^2 is $\frac{g_2}{12}$, $t^2 \in K'$, i.e.,

$$\int t^2 \, dz = \frac{1}{6}t' + \frac{g_2}{12}z.$$

Applying the polynomial reduction to t^3, we find that

$$t^3 = \left(\frac{1}{10}tt'\right)' + \frac{3g_2}{20}t + \frac{g_3}{10}.$$

Then t^3 has a polynomial remainder $\frac{3g_2}{20}t + \frac{g_3}{10}$. So t^3 has no elementary integral over K by Corollary 26. Using $\zeta(z)$, we can represent the integral as

$$\int t^3 \, dz = \frac{1}{10}tt' - \frac{3g_2}{20}\zeta + \frac{g_3}{10}z.$$

Similarly, applying the polynomial reduction to t^4 yields that

$$t^4 = \left(\frac{1}{14}t^2 t'\right)' - \frac{g_3}{7}t + \frac{5g_2^2}{336}$$

Then $-\frac{g_3}{7}t + \frac{5g_2^2}{336}$ is a polynomial remainder of t^4, which implies t^4 has no elementary integral over k. With the help of $\zeta(z)$, the integral of t^4 is represented as

$$\int t^4 \, dz = \frac{1}{14}t^2 t' + \frac{5g_2}{168}t' + \frac{5g_2^2}{336}z - \frac{g_3}{7}\zeta.$$

In fact, for any $n \in \mathbb{N}$, t^n admits the following decomposition:

$$t^n = \left(\frac{1}{4n-2}t^{n-2}t'\right)' + \frac{(n-2)g_3}{4n-2}t^{n-3} + \frac{(2n-3)g_2}{8n-4}t^{n-2}.$$

Substituting $t = \wp(z)$ into the identity and integrating w.r.t. z yields the claimed recurrence for $J_n(z)$ in Section 2.

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A Splitting Factorization

Let $K = k(t, t')$ be a Weierstrass-like extension over k and let $m \in k[t, X]$ be the monic minimal polynomial of t'. For any $D \in k[t]$, there exist $D_N, D_S \in k[t]$ such that $D = D_N D_S$, where all irreducible factors of D_N are normal and those of D_S are special. This factorization is unique up to a nonzero multiplicative element in k, and is called the *splitting factorization* of D (w.r.t. m). In this section, an algorithm is presented to compute splitting factorization by gcd-computation and resultants.

Lemma 27. *Let $D \in k[t]$ be squarefree and set $D_0(t, y) := \kappa(D) + \partial_t(D)y$, where y is an indeterminate. Let $R(t) \in k[t]$ be the Sylvester resultant of $D_0(t, y)$ and $m(t, y)$ w.r.t. y. If $\beta \in \bar{k}$ is a root of D, then β is a special point of m if and only if β is a root of R. Consequently, D and $\gcd(R, D)$ have the same irreducible special factors.*

Proof. Let $\beta \in \bar{k}$ be a root of D. Since D is squarefree, we have $\partial_t(D)(\beta) \neq 0$. As $m(t, y)$ is monic in y, the degrees (in y) of $D_0(\beta, y)$ and $m(\beta, y)$ coincide with those of $D_0(t, y)$ and $m(t, y)$, respectively. Then $R(\beta)$ is the resultant of $D_0(\beta, y)$ and $m(\beta, y)$ w.r.t. y.

Write $D = (t - \beta)\tilde{D}$. Then $\tilde{D}(\beta) \neq 0$. A calculation similar to (3) yields $\kappa(D)(\beta) = -\beta'\tilde{D}(\beta)$ and $\partial_t(D)(\beta) = \tilde{D}(\beta)$. Then $D_0(\beta, y) = \tilde{D}(\beta)(y - \beta')$, and $\operatorname{res}_y\big(D_0(\beta, y), m(\beta, y)\big) = 0$ if and only if $m(\beta, \beta') = 0$, i.e., β is a special point of m. ∎

Algorithm 28. SplittingFactorization

Input: *$D \in k[t]$ and m, the monic irreducible polynomial of t'.*

Output: *the splitting factorization of D w.r.t. m.*

1. *Compute the squarefree factorization $D = D_1^{\mu_1} \dots D_n^{\mu_n}$ of D*
2. *$D_N \leftarrow 1$, $D_S \leftarrow 1$*

2. FOR i FROM 1 TO n DO

$$D_0 \leftarrow \kappa(D_i) + \partial_t(D_i)y,\; R \leftarrow \mathrm{resultant}_y\big(D_0(t,y), m(t,y)\big)$$
$$G \leftarrow \gcd(R, D_i)$$
$$D_S \leftarrow D_S\, G^{\mu_i},\; D_N \leftarrow D_N \left(\tfrac{D_i}{G}\right)^{\mu_i}$$

END DO

4. RETURN D_N, D_S

The correctness is guaranteed by Lemma 27. As a by product, we obtain the squarefree factorization of both D_N and D_S.