Existence Problem of Telescopers: Beyond the Bivariate Case *

Shaoshi Chen[1,2], Qinghu Hou[3], George Labahn[2], Ronghua Wang[4]

[1]KLMM, AMSS, Chinese Academy of Sciences,
Beijing, 100190, China
[2]Symbolic Computation Group, University of Waterloo,
Ontario, N2L3G1, Canada
[3]Center for Applied Mathematics, Tianjin University,
Tianjin, 300072, China
[4]Center for Combinatorics, Nankai University,
Tianjin, 300071, China

schen@amss.ac.cn, qh_hou@tju.edu.cn
glabahn@uwaterloo.ca
wangwang@mail.nankai.edu.cn

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Abstract

In this paper, we solve the existence problem of telescopers for rational functions in three discrete variables. We reduce the problem to that of deciding the summability of bivariate rational functions, which has been solved recently. The existence criteria we present is needed for detecting the termination of Zeilberger's algorithm to the function classes studied in this paper.

1 Introduction

The method of creative telescoping is an algorithmic tool in the symbolic evaluation of parameterized definite sums and integrals. In order to evaluate a

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multiple sum of a given summand $f(x, y_1, \ldots, y_n)$ with respect to $y_1, \ldots, y_n$ with x a discrete parameter, the key step of creative telescoping is to find a nonzero linear recurrence operator L in x such that

$$L(f) = \Delta_{y_1}(g_1) + \cdots + \Delta_{y_n}(g_n),$$

where Δ_{y_i} denotes the difference operator in y_i and the g_i's belong to the same class of functions as f. The operator L is then called a *telescoper* for f. In order to be useful in applications one needs to address two problems: (1) determine whether such an operator L exists for a given f and, (2) if a telescoper exists then design an algorithm for computing it. In this paper we focus on the problem of existence of a telescoper for a given f.

The existence of telescopers is closely related to the termination of Zeilberger's algorithm for computing telescopers. Since the 1990's, extensive work has been done around the existence problem. A sufficient condition was first given by Zeilberger [29] where it was shown that telescopers exist for all holonomic functions. Later Wilf and Zeilberger in [27], using a linear algebra approach proved that telescopers always exist for proper hypergeometric terms. However, holonomicity and properness are only sufficient conditions, that is, there are cases in which the input functions are not holonomic (proper) but telescopers still exist, see [16]. The first necessary and sufficient conditions for the existence of telescopers was given by Abramov and Le [5] for rational functions in two discrete variables. This was later extended to the hypergeometric case by Abramov [3] and to the q-hypergeometric case by Chen et al. in [14]. Recently, the remaining six cases of the existence problem of telescopers for bivariate mixed hypergeometric terms are solved in [12]. To our knowledge, all of the previous works were only focusing on the problem for bivariate functions of a special class. Our long-term goal is to determine necessary and sufficient conditions for the existence problem for general multivariate functions. In this paper, we solve the problem for the starting case, that is, the case of rational functions in three discrete variables.

The previous existence criteria are all based on reduction algorithms which decompose an input function into the sum of a summable function and a non-summable one. The existence is then detected by checking whether the non-summable part is of a special form (so-called proper terms). The reduction algorithms can also be used to decide the summabilty of univariate functions. Recently, the reduction algorithms for univariate rational functions were extended to the bivariate case in [13, 21]. The generalized reduction is also the main ingredient for the existence problem for rational functions of three variables. However, the existence problem in the trivariate case is considerably more involved. As an example the rational function $1/(x + y + z^2)$ is not proper (even after the reduction), but it does have a telescoper (see Example 6.3), a phenomenon which does not happen in the bivariate case.

The remainder of this paper is organized as follows. The basic notation and concepts on telescopers are given in Section 2. In Sections 3 and 4, we review the previous work on solving the summability problem for bivariate rational functions and present special properties of linear recurrence operators. The existence problem for general rational functions are reduced to one with simpler rational functions in Section 5 with the existence criteria for these special rational functions presented in Section 6. The paper ends with a conclusion along with topics for future research.

2 preliminaries

Let $\mathbb{K}$ be a field of characteristic zero and let $\mathbb{E} = \mathbb{K}(x, y, z)$ be the field of rational functions in x, y, z over $\mathbb{K}$. For $f \in \mathbb{E}$ define the shift operators $\sigma_x, \sigma_y, \sigma_z$ on $\mathbb{E}$ by $\sigma_x(f) = f(x+1, y, z)$, $\sigma_y(f) = f(x, y+1, z)$, and $\sigma_z(f) = f(x, y, z+1)$, respectively. Let $\mathcal{R} := \mathbb{E}[S_x, S_y, S_z]$ denote the ring of linear recurrence operators over $\mathbb{E}$, in which S_x, S_y, S_z commute and $S_v \cdot f = \sigma_v(f) \cdot S_v$ for any $f \in \mathbb{E}$ and $v \in \{x, y, z\}$. The action of an operator $P = \sum_{i,j,k} p_{i,j,k} S_x^i S_y^j S_z^k$ in $\mathcal{R}$ on a rational function $f \in \mathbb{E}$ is then given by

$$P(f) = \sum_{i,j,k} p_{i,j,k} f(x + i, y + j, z + k).$$

The *difference operators* Δ_x, Δ_y and Δ_z with respect to x, y and z are defined by

$$\Delta_x = S_x - 1,\ \Delta_y = S_y - 1,\ \text{ and }\ \Delta_z = S_z - 1.$$

A rational function $f \in \mathbb{E}$ is said to be *(σ_y, σ_z)-summable* in $\mathbb{E}$ if $f = \Delta_y(g) + \Delta_z(h)$ for some $g, h \in \mathbb{E}$. We also just say summable if the meaning is clear. For brevity, we sometimes just write $f \equiv_{y,z} 0$ if f is (σ_y, σ_z)-summable.

Definition 2.1. *A nonzero linear recurrence operator $L \in \mathbb{K}(x)[S_x]$ is called a* telescoper *for a rational function $f \in \mathbb{E}$ if $L(f)$ is (σ_y, σ_z)-summable in $\mathbb{E}$, that is, there exists $g, h \in \mathbb{E}$ such that*

$$L(f) = \Delta_y(g) + \Delta_z(h).$$

Then the central problem to be solved in this paper is:

Problem 2.2. *Given $f \in \mathbb{E}$, decide whether f has a telescoper in $\mathbb{K}(x)[S_x]$.*

An operator $L \in \mathbb{K}(x)[S_x]$ is called a *common left multiple* of $L_1, \ldots, L_m \in \mathbb{K}(x)[S_x]$ if there exist operators $L_1', \ldots, L_m' \in \mathbb{K}(x)[S_x]$ such that

$$L = L_1' L_1 = \cdots = L_m' L_m.$$

Since $\mathbb{K}(x)[S_x]$ is a left Euclidean domain, such an L always exists. Amongst all of them, the one of smallest degree in S_x is called the least common left multiple (LCLM). When the field $\mathbb{K}$ is computable, e.g., $\mathbb{K} = \mathbb{Q}$, then many efficient algorithms for computing LCLM have been developed [11, 6].

Remark 2.3. *Let $f = f_1 + \cdots + f_m$ with all $f_i \in \mathbb{E}$. If each f_i has a telescoper L_i for $i = 1, \ldots, m$, then the LCLM of the L_i is a telescoper for f. This fact follows from the definition of LCLM along with the commutativity between operators in $\mathbb{K}(x)[S_x]$ and the difference operators Δ_y, Δ_z.*

Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ be the free Abelian group generated by $\sigma_x, \sigma_y, \sigma_z$. Let $f \in \mathbb{E}$ and H be a subgroup of G. We call

$$[f]_H := \{\sigma(f) \mid \sigma \in H\}$$

the H-orbit at f. Two elements $f, g \in \mathbb{E}$ are said to be H-equivalent if $[f]_H = [g]_H$, denoted by $f \sim_H g$. The relation $\sim_H$ is an equivalence relation. Typically, we will take $H = G$ or $H = \langle \sigma_y, \sigma_z \rangle$ in the rest of this paper.

Example 2.4. *Let $f = y^2+x+2z$ and $g = y^2+x-4y+2z+7$. Then f and g are G-equivalent since $g = \sigma_x \sigma_y^{-2} \sigma_z(f)$. However they are not $\langle \sigma_y, \sigma_z \rangle$-equivalent. Indeed, if $g = \sigma_y^n \sigma_z^k(f)$ for some $n, k \in \mathbb{Z}$ then equating the coefficients leads to the linear system $\{2n = -4, n^2 + 2k = 7\}$. But this implies that $n = -1$ and $k = 3/2$, a contradiction.*

3 Summability

The first necessary step for solving the existence problem of telescopers is to decide whether a given multivariate function $f(x_1, \ldots, x_n)$ in a specific class of functions is equal to $\Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n)$ for some $g_1, \ldots, g_n$ in the same class as f. For univariate rational functions the summability problem was first solved by Abramov [1, 2], with alternative methods later presented in [24, 25]. The Gosper algorithm [18] solves the problem for univariate hypergeometric terms. This was then used by Zeilberger [28] to design a fast algorithm to construct telescopers for bivariate hypergeometric terms. The Gosper algorithm was extended further to the D-finite case by Abramov and van Hoeij in [8, 4], and to more general difference-field setting by Karr [22, 23] and Schneider [26]. A significant step in the path towards the multivariate case was taken by Chen et al. in [15], which gave some necessary conditions for the summability of bivariate hypergeometric terms. Chen and Singer in [13] then presented the first necessary and sufficient condition for the summability of bivariate rational functions. Based on the theoretical criterion in [13], Hou and Wang [21] then gave a practical algorithm for deciding the summability in the bivariate rational case.

In this section, we will recall the summability criterion for bivariate rational functions from [21]. Let $\mathbb{F} := \mathbb{K}(x)$ and $f \in \mathbb{F}(y, z)$. The key idea is to decompose f into the following form

$$f = \Delta_y(g) + \Delta_z(h) + r,$$

where $g, h \in \mathbb{F}(y, z)$ and r is of the form

$$r = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{a_{i,j}}{d_i^j} \tag{3.1}$$

with $a_{i,j} \in \mathbb{F}(y)[z]$, $\deg_z(a_{i,j}) < \deg_z(d_i)$, $d_i \in \mathbb{F}[y,z]$ are irreducible polynomials, and d_i, $d_{i'}$ are not $\langle \sigma_y, \sigma_z \rangle$-equivalent for any $i \neq i'$. The existence of such decompositions has been shown in [21, Lemma 3.1]. Then f is (σ_y, σ_z)-summable if and only if r is (σ_y, σ_z)-summable. Since shift operators preserve the multiplicities of the fractions $a_{i,j}/d_i^j$, we have r is (σ_y, σ_z)-summable if and only if $\sum_{i=1}^m a_{i,j}/d_i^j$ is (σ_y, σ_z)-summable for each j. Furthermore, Lemma 3.2 in [21] shows that $\sum_{i=1}^n a_{i,j}/d_i^j$ is (σ_y, σ_z)-summable if and only if $a_{i,j}/d_i^j$ is (σ_y, σ_z)-summable for all i with $1 \leq i \leq n$. After this, the summability problem for general rational functions in $\mathbb{F}(y,z)$ is reduced to the summable problem for simple fractions of the special form a/d^j. The following theorem [21, Theorem 3.3] then gives a criterion for deciding the summability of such special fractions.

Theorem 3.1. *Let $f = a/d^j \in \mathbb{F}(y,z)$ with $d \in \mathbb{F}[y,z]$ being irreducible, $a \in \mathbb{F}(y)[z] \setminus \{0\}$ and $\deg_z(a) < \deg_z(d)$. Then f is (σ_y, σ_z)-summable if and only if*

(1) *there exist integers t, ℓ with $t \neq 0$ such that*

$$\sigma_y^t(d) = \sigma_z^\ell(d), \tag{3.2}$$

(2) *for the smallest positive integer t such that (3.2) holds, we have $a = \sigma_y^t \sigma_z^{-\ell}(p) - p$ for some $p \in \mathbb{F}(y)[z]$ with $\deg_z(p) < \deg_z(d)$.*

Example 3.2. *Let $f = 1/(y^n + z^n)$ for $n \in \mathbb{N}$. When $n = 1$, Theorem 3.1 implies that f must be (σ_y, σ_z)-summable. In fact, we have*

$$\frac{1}{y+z} = \Delta_y\left(\frac{y}{y+z}\right) + \Delta_z\left(\frac{-y-1}{y+z}\right).$$

However, when $n > 1$ there exists no $(t, \ell) \in \mathbb{Z}^2$ such that $t \neq 0$ and $\sigma_y^t(y^n + z^n) = \sigma_z^\ell(y^n + z^n)$. Thus in this case f is not (σ_y, σ_z)-summable.

Definition 3.3. *For a rational function $f \in \mathbb{F}(y,z)$, we call the triple $(g, h, r) \in \mathbb{F}(y,z)^3$ an* additive decomposition *of f with respect to y and z if $f = \Delta_y(g) + \Delta_z(h) + r$, where r is of the form (3.1) and all fractions $a_{i,j}/d_i^j$ are not (σ_y, σ_z)-summable.*

Remark 3.4. *From the decision procedure for summability given above, additive decompositions always exist for rational functions in $\mathbb{F}(y,z)$. However, we remark that such decompositions may not be unique.*

4 Exponent Separation

In this section, we will present some special properties of linear recurrence operators having to do with separating exponents. This separation of exponents of an operator will be used in next section for separating orbits of shift operators and will help in simplifying the existence problem.

Let $m \in \mathbb{N}$ and L be a nonzero operator in $\mathbb{K}(x)[S_x]$. We can always decompose L into the form

$$L = L_0 + L_1 + \cdots + L_{m-1}, \tag{4.1}$$

where $L_i = \sum_{j=0}^{r_i} \ell_{i,j} S_x^{jm+i}$ for $i = 0, 1, \ldots, m-1$. We call such a decomposition an *m-exponent separation* of L. It is clear that $L = 0$ if and only if $L_i = 0$ for all i. Denote

$$\mathcal{L}_m = \begin{bmatrix} L_0 & L_{m-1} & L_{m-2} & \ldots & L_1 \\ L_1 & L_0 & L_{m-1} & \ldots & L_2 \\ L_2 & L_1 & L_0 & \ldots & L_3 \\ \vdots & \vdots & \vdots & & \vdots \\ L_{m-1} & L_{m-2} & L_{m-3} & \ldots & L_0 \end{bmatrix}. \tag{4.2}$$

The next lemma and proposition will show that the m rows of $\mathcal{L}$ are linearly independent over the ring $\mathbb{K}(x)[S_x]$.

Lemma 4.1. *Suppose*

$$[T_0, \ldots, T_{m-1}] \cdot \mathcal{L}_m = 0 \tag{4.3}$$

with each $T_k \in \mathbb{K}(x)[S_x]$. *Then* $T_0 + \cdots + T_{m-1} = 0$.

Proof. Note that $\mathcal{L}_m \cdot [1, \ldots, 1]^T = [L, \ldots, L]^T$. Hence any solution of (4.3) implies that

$$(T_0 + \cdots + T_{m-1}) \cdot L = 0.$$

Since L is nonzero and $\mathbb{K}(x)[S_x]$ is a left Euclidean domain we have $T_0 + \cdots + T_{m-1} = 0$. ∎

In fact our goal is to show that each component T_k of (4.3) is zero, that is, the left kernel of $\mathcal{L}_m$ is trivial. In order to do this we do an m-exponent separation of each T_k and look at the resulting decomposition. Suppose first that

$$[T_0, \ldots, T_{m-1}] \cdot \mathcal{L}_m = [R_0, \ldots, R_{m-1}]$$

and that for each k

$$\begin{aligned} T_k &= T_{k,0} + T_{k,1} + \cdots + T_{k,m-1} \\ R_k &= R_{k,0} + R_{k,1} + \cdots + R_{k,m-1} \end{aligned}$$

are the m-exponent separations for T_k and R_k, respectively. Let $\mathcal{T}$ and $\mathcal{R}$ be the $m \times m$ matrices defined as

$$\mathcal{T} = \begin{bmatrix} T_{0,0} & T_{1,m-1} & T_{2,m-2} & \ldots & T_{m-1,1} \\ T_{0,1} & T_{1,0} & T_{2,m-1} & \ldots & T_{m-1,2} \\ T_{0,2} & T_{1,1} & T_{2,0} & \ldots & T_{m-1,3} \\ \vdots & \vdots & \vdots & & \vdots \\ T_{0,m-1} & T_{1,m-2} & T_{2,m-3} & \ldots & T_{m-1,0} \end{bmatrix} \tag{4.4}$$

and

$$\mathcal{R} = \begin{bmatrix} R_{0,0} & R_{1,m-1} & R_{2,m-2} & \dots & R_{m-1,1} \\ R_{0,1} & R_{1,0} & R_{2,m-1} & \dots & R_{m-1,2} \\ R_{0,2} & R_{1,1} & R_{2,0} & \dots & R_{m-1,3} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{0,m-1} & R_{1,m-2} & R_{2,m-3} & \dots & R_{m-1,0} \end{bmatrix}.$$

Then it is straightforward to show that

$$\mathcal{T} \cdot \mathcal{L}_m = \mathcal{R}. \tag{4.5}$$

Proposition 4.2. *Suppose*

$$[T_0, \dots, T_{m-1}] \cdot \mathcal{L}_m = 0 \tag{4.6}$$

with each $T_k \in \mathbb{K}(x)[S_x]$*. Then* $T_k = 0$ *for each* k*.*

Proof. From (4.5) and (4.6) we have that each $R_k = 0$ and hence also that each $R_{k,j} = 0$. Thus $\mathcal{T} \cdot \mathcal{L}_m = 0$ and so for each $j = 1, 2, \dots, m$ we have

$$[T_{0,j-1}, \dots, T_{j-1,0}, T_{j,m-1}, \dots, T_{m-1,j}] \cdot \mathcal{L}_m = 0.$$

From Lemma 4.1 we get for each j

$$T_{0,j} + T_{1,j-1} + \dots + T_{m-1,j-m+1} = 0.$$

This implies $T_k = 0$ for all k. ∎

We will also later need to use the following:

Proposition 4.3. *There is a matrix* $\mathcal{M} \in \mathbb{K}(x)[S_x]^{m\times m}$ *such that*

$$\mathcal{M} \cdot \mathcal{L}_m = \textit{diagonal}(T_0, T_1, \cdots, T_{m-1}) \tag{4.7}$$

with nonzero $T_i \in \mathbb{K}(x)[S_x]$*.*

Proof. From the definition of LCLM,we know for any nonzero $A, B \in \mathbb{K}(x)[S_x]$, there always exist nonzero $A', B' \in \mathbb{K}(x)[S_x]$ such that $A' \cdot A + B' \cdot B = 0$. Similar to the use of division-free the Gaussian elimination over a Euclidean domain, we can find $\mathcal{M} \in \mathbb{K}(x)[S_x]^{m\times m}$ satisfying (4.7) (c.f. [10]. That each diagonal element is nonzero follows directly from Proposition 4.2 since otherwise there would be a nonzero element of the right kernel of $\mathcal{L}_m$. ∎

5 Reduction to simple fractions

In this section, we will reduce the existence problem of telescopers for rational functions in $\mathbb{E}$ into the same problem but for simpler rational functions.

Let $f \in \mathbb{E}$ be nonzero with $f = \Delta_y(g) + \Delta_z(h) + r$ and (g, h, r) be an additive decomposition of f with respect to y and z. Then f has a telescoper in $\mathbb{K}(x)[S_x]$

if and only if r has a telescoper in $\mathbb{K}(x)[S_x]$. As such, we need only study the existence problem for rational functions of the form (3.1).

For any $\sigma \in \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $a, b \in \mathbb{E}$, we have

$$\frac{a}{\sigma^n(b)} = \sigma(g) - g + \frac{\sigma^{-n}(a)}{b}, \tag{5.1}$$

where

$$g = \begin{cases} \sum_{i=0}^{n-1} \frac{\sigma^{i-n}(a)}{\sigma^i(b)}, & \text{if } n \geq 0; \\ -\sum_{i=0}^{-n-1} \frac{\sigma^i(a)}{\sigma^{n+i}(b)}, & \text{if } n < 0. \end{cases}$$

Suppose now that $d_{i'} = \sigma_x^m \sigma_y^n \sigma_z^k d_i$ for some index $i \neq i'$ and $m, n, k \in \mathbb{Z}$ with $m \geq 0$. Applying the formula (5.1) repeatedly yields

$$\frac{b_{i',j}}{d_{i'}^j} = \Delta_y(u) + \Delta_z(v) + \frac{\sigma_y^{-n}\sigma_z^{-k}(b_{i',j})}{\sigma_x^m d_i^j}$$

for some $u, v \in \mathbb{E}$. With this reduction, we can always decompose r of the form (3.1) into the form

$$r = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,\ell}}{\sigma_x^\ell d_i^j} \tag{5.2}$$

with $b_{i,j,\ell} \in \mathbb{K}(x,y)[z], d_i \in \mathbb{K}[x,y,z]$, $\deg_z(b_{i,j,\ell}) < \deg_z(d_i)$, and d_i are irreducible polynomials with d_i and $d_{i'}$ being in distinct $\langle \sigma_x, \sigma_y, \sigma_z \rangle$-orbits for any $1 \leq i \neq i' \leq m$.

Let $\mathcal{O} = \{p/q \in \mathbb{E} \mid \deg_z(p) < \deg_z(q)\}$ and V_m be the set of all rational functions of the form $\sum_{i=1}^{I} a_i/b_i^m$, where $a_i, b_i, \in \mathbb{K}(x,y)[z]$, $\deg_z(a_i) < \deg_z(b_i)$ and b_i's are distinct irreducible polynomials in the ring $\mathbb{K}(x,y)[z]$. By definition, the set V_m forms a subspace of $\mathcal{O}$ as vector spaces over $\mathbb{K}(x,y)$. By the irreducible partial fraction decomposition, any $f \in \mathcal{O}$ can be uniquely decomposed into $f = f_1 + \cdots + f_n$ with $f_i \in V_i$ and so $\mathcal{O} = \bigoplus_{i=1}^{\infty} V_i$. The following lemma shows that the space V_m is invariant under certain linear recurrence operators.

Lemma 5.1. *Let $f \in V_m$ and $P \in \mathbb{K}(x,y)[S_x, S_y, S_z]$. Then $P(f) \in V_m$.*

Proof. Let $f = \sum_{t=1}^{n} a_t/b_t^m$ and $P = \sum_{i,j,k} p_{i,j,k} S_x^i S_y^j S_z^k$. For any $\sigma = \sigma_x^i \sigma_y^j \sigma_y^k$ with $i, j, k \in \mathbb{Z}$, $\sigma(b)$ is irreducible and $\deg_z(\sigma(a)) < \deg_z(\sigma(b))$. Then all of the simple fractions $\frac{p_{i,j,k} S_x^i S_y^j S_z^k(a)}{S_x^i S_y^j S_z^k(b)}$ appearing in $P(f)$ are proper in z and have irreducible denominators. If some of denominators are the same, we can simplify them by adding the numerators to get a simple fraction. After this simplification, we see that $P(f)$ can be written in the same form as f, so it is in V_m. ∎

Lemma 5.2. *Let $r \in \mathbb{E}$ be of the form (5.2). Then r has a telescoper if and only if the summand $\sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,\ell}}{\sigma_x^\ell d_i^j}$ has a telescoper for all i, j with $1 \leq i \leq I$ and $1 \leq j \leq J_i$.*

Proof. From Lemma 5.1 we see that any r as in (5.2) has a telescoper if and only if $\sum_{i=1}^{I}\sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,\ell}}{\sigma_x^\ell d_i^j}$ has a telescoper for all different multiplicities j. Also, from Lemma 3.2 in [21] we have that $\sum_{i=1}^{I}\sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,\ell}}{\sigma_x^\ell d_i^j}$ has a telescoper if and only if $\sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,l}}{\sigma_x^\ell d_i^j}$ has a telescoper for all i with $1 \leq i \leq I$. ∎

At this stage we have reduced the existence of telescopers problem for general rational functions to those having the simple form $r = \sum_{\ell=0}^{\ell_{i,j}} \frac{b_{i,j,\ell}}{\sigma_x^\ell d_i^j}$. If $\sigma_x^{\ell'} d_i = \sigma_x^\ell \sigma_y^n \sigma_z^k d_i$ for some $\ell \neq \ell'$ and $n, k \in \mathbb{Z}$, then applying the formula (5.1), we get

$$\frac{b_{i,j,\ell'}}{\sigma_x^{\ell'} d_i^j} = \frac{b_{i,j,\ell'}}{\sigma_x^\ell \sigma_y^n \sigma_z^k d_i^j} = \Delta_y(u_{i,j}) + \Delta_z(v_{i,j}) + \frac{\sigma_y^{-n}\sigma_z^{-k} b_{i,j,\ell'}}{\sigma_x^\ell d_i^j}$$

for some $u_{i,j}, v_{i,j} \in \mathbb{K}(x, y, z)$. Repeating the above transformation gives a decomposition

$$r = \Delta_y(u) + \Delta_z(v) + \sum_{i=0}^{I'} \frac{b_i'}{\sigma_x^i d^j},$$

where $u, v \in \mathbb{K}(x, y, z)$ and $\sigma_x^i(d)$ and $\sigma_x^{i'}(d)$ are not $\langle\sigma_y, \sigma_z\rangle$-equivalent for $0 \leq i \neq i' \leq I'$.

The following lemma reduces the existence problem for rational functions into one whose denominators have distinct orbits.

Lemma 5.3. *Let*

$$r = \sum_{i=0}^{I} \frac{b_i}{\sigma_x^i d^j} \quad \text{with} \quad b_i \in \mathbb{K}(x, y)[z], \quad d \in \mathbb{K}[x, y, z].$$

Suppose b_i, d are irreducible polynomials, $\deg_z(b_i) < \deg_z(d)$ with $\sigma_x^i d$ and $\sigma_x^{i'} d$ in distinct $\langle\sigma_y, \sigma_z\rangle$-orbits, for $0 \leq i \neq i' \leq I$. Then r has a telescoper if and only if each simple fraction $\frac{b_i}{\sigma_x^i d^j}$ has a telescoper for $0 \leq i \leq I$.

Proof. Sufficiency follows from Remark 2.3. For the other direction assume that $L = \sum_{i=0}^{\rho} \ell_i S_x^i$ (with $\ell_0 \neq 0$) is a telescoper for r. There are two cases to be considered according to whether there exists a positive integer m such that $\sigma_x^m d = \sigma_y^n \sigma_z^k d$.

Case 1. There is no positive integer m such that

$$\sigma_x^m d = \sigma_y^n \sigma_z^k d \quad \text{for some} \quad n, k \in \mathbb{Z}.$$

In this case, $\sigma_x^i d$ and $\sigma_x^{i'} d$ are in distinct $\langle\sigma_y, \sigma_z\rangle$-orbits for any $i \neq i'$. We claim that $\frac{b_i}{\sigma_x^i d^j}$ is (σ_y, σ_z)-summable for $0 \leq i \leq I$. Since

$$L(r) = \sum_{i=0}^{\rho}\sum_{t=0}^{I} \ell_i \sigma_x^i \left(\frac{b_t}{\sigma_x^t d^j}\right) = \sum_{p=0}^{\rho+I}\sum_{i=0}^{p} \ell_i \sigma_x^i \left(\frac{b_{p-i}}{\sigma_x^{p-i} d^j}\right)$$

is (σ_y, σ_z)-summable, according to Lemma 3.2 in [21], we get

$$\sum_{i=0}^{p} \ell_i \sigma_x^i \left(\frac{b_{p-i}}{\sigma_x^{p-i} d^j} \right) = \Delta_y(u_p) + \Delta_z(v_p) \tag{5.3}$$

for any $0 \le p \le \rho + I$.

We prove the claim by induction. The result is true for $p = 0$ in (5.3) since then $\frac{b_0}{d^j} = \Delta_y(\frac{u_0}{\ell_0}) + \Delta_z(\frac{v_0}{\ell_0})$. Suppose we have shown that $\frac{b_i}{\sigma_x^i d^j}$ is (σ_y, σ_z)-summable for $i = 0, 1, \ldots, k-1$ with $k \le I$. Letting $p = k$ in (5.3), we get

$$\sum_{i=0}^{k} \ell_i \sigma_x^i \left(\frac{b_{k-i}}{\sigma_x^{k-i} d^j} \right) = \Delta_y(u_k) + \Delta_z(v_k).$$

As $\frac{b_{k-i}}{\sigma_x^{k-i} d^j}$ is (σ_y, σ_z)-summable for all $1 \le i \le k$, it is easy to check that $\sum_{i=1}^{k} \ell_i \sigma_x^i (\frac{b_{k-i}}{\sigma_x^{k-i} d^j})$ is also (σ_y, σ_z)-summable. Thus $\frac{b_k}{\sigma_x^k d^j}$ is (σ_y, σ_z)-summable.

Case 2. Suppose $\sigma_x^m d = \sigma_y^n \sigma_z^k d$ for m a positive integer and n, k some integers. Let m_0 be the smallest such integer and $\sigma_x^{m_0} d = \sigma_y^{n_0} \sigma_z^{k_0} d$. Since $\sigma_x^i d$ and $\sigma_x^{i'} d$ are in distinct (σ_y, σ_z)-orbits, we can assume $r = \sum_{i=0}^{m_0-1} \frac{b_i}{\sigma_x^i d^j}$. Suppose the m_0-exponent separation of L is

$$L = L_0 + L_1 + \cdots + L_{m_0-1}.$$

According to Lemma 3.1 and Lemma 3.2 in [21], we have

$$\begin{cases} L_0 \dfrac{b_0}{d^j} + L_{m_0-1} \dfrac{b_1}{\sigma_x d^j} + \cdots + L_1 \dfrac{b_{m_0-1}}{\sigma_x^{m_0-1} d^j} \equiv_{y,z} 0 \\ L_1 \dfrac{b_0}{d^j} + L_0 \dfrac{b_1}{\sigma_x d^j} + \cdots + L_2 \dfrac{b_{m_0-1}}{\sigma_x^{m_0-1} d^j} \equiv_{y,z} 0 \\ \cdots \\ L_{m_0-1} \dfrac{b_0}{d^j} + L_{m_0-2} \dfrac{b_1}{\sigma_x d^j} + \cdots + L_0 \dfrac{b_{m_0-1}}{\sigma_x^{m_0-1} d^j} \equiv_{y,z} 0. \end{cases}$$

If we let

$$\mathcal{V} = \left[\frac{b_0}{d^j}, \ \frac{b_1}{\sigma_x d^j}, \ \ldots, \ \frac{b_{m_0-1}}{\sigma_x^{m_0-1} d^j} \right]$$

then we can write this as

$$\mathcal{L}_{m_0} \cdot \mathcal{V}^T \equiv_{y,z} 0,$$

with $\mathcal{L}_{m_0}$ from (4.2). From Proposition 4.3 there exists $T_0, \ldots, T_{m-1}$ and a matrix $\mathcal{M}$ having entries from $\mathbb{K}(x)[S_x]$ such that

$$\mathcal{M} \cdot \mathcal{L}_{m_0} = \text{diagonal}(T_0, \ldots, T_{m-1}).$$

By the commutativity between operators in $\mathbb{K}(x)[\sigma_x]$ and the difference operators Δ_y, Δ_z, we know T_i is a telescoper for $\frac{b_i}{\sigma_x^i d^j}$ for $0 \le i \le m_0 - 1$. ∎

6 Existence criteria

Lemma 5.3 from the previous section implies that the telescoper existence problem for rational functions is reduced to the case of a rational function of the form

$$f = \frac{b(x,y,z)}{c(x,y)d(x,y,z)^{\lambda}} \tag{6.1}$$

where $\lambda \in \mathbb{N}$, $b, d \in \mathbb{K}[x,y,z]$ with $\deg_z(b) < \deg_z(d)$. In this section, we will give a criterion for deciding the existence of telescopers for rational functions of the above form. If b and c are not primitive, that is, their contents are not 1, then we can write

$$b = b_0(x)b_1(x,y,z) \quad \text{and} \quad c = c_0(x)c_1(x,y),$$

where b_1, c_1 are primitive in y, z. Similar to the proof of Lemma 7.4 in [12], $\frac{b}{cd^{\jmath}}$ has a telescoper if and only if $\frac{b_1}{c_1 d^{\jmath}}$ has a telescoper. As such we can assume in form (6.1) that b, c, d are all primitive in y, z.

As we did in the proof of Lemma 5.3 we will proceed by case distinction according to whether or not d satisfies the condition that there exists a positive integer m such that

$$\sigma_x^m d = \sigma_y^n \sigma_z^k d \quad \text{for some } n, k \in \mathbb{Z}. \tag{6.2}$$

We may always assume m is the smallest integer satisfying the above condition. Let us first consider the case that the condition is not satisfied. In this case, the existence problem will be reduced to the summability problem. As the summability problem for bivariate rational functions has been solved in [13, 21], the existence problem becomes:

Theorem 6.1. *Let $f = b/(cd^{\lambda}) \in \mathbb{E}$ satisfy the same conditions as in (6.1) but that d does not satisfy condition (6.2). Then f has a telescoper if and only if f is (σ_y, σ_z)-summable.*

Proof. The sufficiency is obvious. For the necessity, we assume that $L = \sum_{i=0}^{I} \ell_i S_x^i \in \mathbb{K}(x)[S_x]$ with $\ell_0, \ell_I \neq 0$ is a telescoper for f. Then

$$L(f) = \sum_{i=0}^{I} \frac{\ell_i \sigma_x^i(b)}{\sigma_x^i(c)\sigma_x^i(d^{\lambda})} = \Delta_y(g) + \Delta_z(h)$$

for some $g, h \in \mathbb{E}$. Since $\sigma_x^m(d) \neq \sigma_y^n \sigma_z^k(d)$ for any positive integer m and $n, k \in \mathbb{Z}$, we have $\sigma_x^i(d)$ and $\sigma_x^{i'}(d)$ are in distinct (σ_y, σ_z)-orbits for any $i \neq i'$. By Lemma 3.2 in [21], the summands $\frac{\ell_i \sigma_x^i(b)}{\sigma_x^i(c)\sigma_x^i(d^{\lambda})}$ of $L(f)$ are all (σ_y, σ_z)-summable. In particular, $\ell_0 f$ is (σ_y, σ_z)-summable. As $\ell_0 \in \mathbb{K}(x) \setminus \{0\}$, f is (σ_y, σ_z)-summable. ∎

The second case where (6.2) is satisfied is considerably more involved. Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. An irreducible polynomial $q \in \overline{\mathbb{K}}$ is said to be *integer-linear* in x, y and z over $\overline{\mathbb{K}}$ if it is of the form $\alpha_i x + \beta_j y + \gamma_i z + \delta_i$, where $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$ and $\delta_i \in \overline{\mathbb{K}}$. A rational function $f \in \mathbb{E}$ is said to be *proper* if it can be written in the form $f = \frac{p}{\prod_{i=1}^{I} q_i}$, where $p, q_i \in \mathbb{K}[x, y, z]$ and all q_i are integer-linear in x, y and z over $\overline{\mathbb{K}}$. By the fundamental theorem in [27, p. 590], any proper rational function has a telescoper.

The following lemma describes some necessary conditions for the existence of telescopers.

Lemma 6.2. *Let $f = b/(cd^\lambda) \in \mathbb{E}$ satisfy the same conditions as in (6.1) and that d satisfies the condition (6.2). If one of the following conditions is also satisfied:*

- (i) *there exist $n_1, n_2, k_1, k_2 \in \mathbb{Z}$ with $n_1, n_2 > 0$ such that $\sigma_y^{n_1}(d) = \sigma_z^{k_1}(d)$ and $\sigma_x^{n_2}(c) = \sigma_y^{k_2}(c)$;*
- (ii) *there exists a positive integer t such that $\sigma_x^{tm}(c) = \sigma_y^{tn}(c)$,*

then f has a telescoper.

Proof. Suppose that the polynomials c and d satisfy the conditions (6.2) and (i). By Lemma 3 in [7], the equalities $\sigma_x^{n_2}(c) = \sigma_y^{k_2}(c)$ and $\sigma_x^m(d) = \sigma_y^n \sigma_z^k(d)$ imply that there exist $p \in \mathbb{K}[z]$ and $q \in \mathbb{K}[z_1, z_2]$ such that

$$c = p(y + \frac{k_2}{n_2}x) \quad \text{and} \quad d = q(y + \frac{n}{m}x, z + \frac{k}{m}x).$$

Furthermore, the equality $\sigma_y^{n_1}(d) = \sigma_z^{k_1}(d)$ implies that there exists $h \in \mathbb{K}[z]$ such that

$$d = h(z + \frac{k}{m}x + \frac{k_1}{n_1}(y + \frac{n}{m}x)).$$

Thus both c and d factor into products of integer-linear polynomials in x, y, and z over $\overline{\mathbb{K}}$. Therefore f is a proper rational function, and hence it has a telescoper.

Suppose that c satisfies the condition (ii). Set

$$L = \sum_{i=0}^{\rho} \ell_i S_x^{itm},$$

where $\rho \in \mathbb{N}$ and $\ell_i \in \mathbb{K}(x)$ are to be determined. Applying the reduction formula (5.1) yields

$$\begin{aligned} L(f) &= \sum_{i=0}^{\rho} \frac{\ell_i \sigma_x^{itm}(b)}{\sigma_x^{itm}(cd^\lambda)} = \sum_{i=0}^{\rho} \frac{\ell_i \sigma_x^{itm}(b)}{\sigma_y^{itn}(c)\sigma_y^{itn}\sigma_z^{itk}(d^\lambda)} \\ &= \Delta_y(u) + \Delta_z(v) + \frac{1}{cd^\lambda} \sum_{i=0}^{\rho} \ell_i \sigma_x^{itm} \sigma_y^{-itn} \sigma_z^{-itk}(b). \end{aligned}$$

Note that the degrees of $\sigma_x^{itm}\sigma_y^{-itn}\sigma_z^{-itk}(b)$ in y or z are the same as that of b. Thus all shifts of b lie in a finite dimensional linear space over $\mathbb{K}(x)$. If ρ is large enough, then there always exists $\ell_i \in \mathbb{K}(x)$, not all zero, such that

$$\sum_{i=0}^{\rho} \ell_i \sigma_x^{itm}\sigma_y^{-itn}\sigma_z^{-itk}(b) = 0.$$

As a result $L = \sum_{i=0}^{\rho} \ell_i S_x^{itm}$ is a telescoper for f. ∎

Example 6.3. *Let $f = 1/d$ with $d = x+y+z^2$. Since $\sigma_x(d) = \sigma_y(d)$ and $c = 1$, f has a telescoper by Lemma 6.2.*

Decompose the rational function $f = \frac{b}{cd^\lambda}$ into the form

$$f = \frac{1}{d^\lambda}\left(p + \frac{B}{C} + \sum_{i=1}^{I}\sum_{\ell=1}^{m_i} \frac{b_{i,\ell}}{c_i^\ell}\right),$$

where $p \in \mathbb{K}(x)[y,z]$, $B, b_{i,\ell} \in \mathbb{K}[x,y,z], C, c_i \in \mathbb{K}[x,y]$ with $\deg_y(B) < \deg_y(C)$, $\deg_y(b_{i,\ell}) < \deg_y(c_i)$, and all of the irreducible factors of C satisfy the condition (ii) as in Lemma 6.2, but all c_i do not satisfy this condition. By Lemma 6.2, $(p + B/C)/d^\lambda$ has a telescoper and so for the existence problem of telescopers we need only consider

$$r = \frac{1}{d^\lambda}\sum_{i=1}^{I}\sum_{\ell=1}^{m_i} \frac{b_{i,\ell}}{c_i^\ell}. \tag{6.3}$$

From now on, we always assume that d satisfies the condition 6.2. As before we consider two distinct cases, in this case according to whether or not d satisfies the condition:

$$\sigma_y^{n_1}(d) = \sigma_z^{k_1}(d) \text{ for some } n_1, k_1 \in \mathbb{Z} \text{ with } n_1 > 0. \tag{6.4}$$

Theorem 6.4. *Let $r \in \mathbb{E}$ be as in (6.3). Suppose that d satisfies the condition (6.2) and there are no integers n_1, k_1 with $n_1 > 0$ such that $\sigma_y^{n_1}(d) = \sigma_z^{k_1}(d)$. Then r has a telescoper if and only if $r = 0$.*

Proof. The sufficiency is clear. For the necessity, we assume $L = \sum_{i=0}^{\rho} \ell_i S_x^i \in \mathbb{K}(x)[S_x]$ with $\ell_0, \ell_\rho \neq 0$ is a telescoper for r. Let m be the smallest positive integer such that $\sigma_x^m(d) = \sigma_y^n\sigma_z^k(d)$ for some $n, k \in \mathbb{Z}$. Then $\sigma_x^i(d)$ and $\sigma_x^j(d)$ are in distinct $\langle\sigma_y, \sigma_z\rangle$-orbits if $m \nmid (i-j)$. Let $L = L_0 + \ldots + L_{m-1}$ be the m-exponent separation of L. Since the denominators of $L_i(r)$ are in distinct $\langle\sigma_y, \sigma_z\rangle$-orbits, Lemma 3.2 in [21] implies that $L_i(r)$ is (σ_y, σ_z)-summable for all i with $0 \leq i \leq m-1$. Then $L_0 \neq 0$ is a telescoper for r. Write $L_0 = \sum_{t=0}^{T} a_t S_x^{tm}$.

Then

$$\begin{aligned}L_0(r) &= \sum_{t=0}^{T}\sum_{i=1}^{I}\sum_{\ell=1}^{m_i}\frac{a_t\sigma_x^{tm}(b_{i,\ell})}{\sigma_x^{tm}(c_i^\ell)\sigma_x^{tm}(d^\lambda)}\\ &= \sum_{t=0}^{T}\sum_{i=1}^{I}\sum_{\ell=1}^{m_i}\frac{a_t\sigma_x^{tm}(b_{i,\ell})}{\sigma_x^{tm}(c_i^\ell)\sigma_y^{tn}\sigma_z^{tk}(d^\lambda)}\\ &= \Delta_y(u)+\Delta_z(v)+\frac{h}{d^\lambda}\end{aligned}$$

where

$$h=\sum_{t=0}^{T}\sum_{i=1}^{I}\sum_{\ell=1}^{m_i}\frac{a_t\sigma_x^{tm}\sigma_y^{-tn}\sigma_z^{-tk}(b_{i,\ell})}{\sigma_x^{tm}\sigma_y^{-tn}(c_i^\ell)}.$$

Since $L_0(r)$ is (σ_y,σ_z)-summable but d does not satisfy condition (6.4), Theorem 3.1 implies that $h=0$. By Lemma 5.1, for each multiplicity ℓ, we have

$$h_\ell=\sum_{t=0}^{T}\sum_{i=1}^{I}\frac{a_t\sigma_x^{tm}\sigma_y^{-tn}\sigma_z^{-tk}(b_{i,\ell})}{\sigma_x^{tm}\sigma_y^{-tn}(c_i^\ell)}=0.$$

We first claim that there exists a polynomial $p\in\Omega:=\{c_i\mid 1\leq i\leq I\}$ such that $p\neq\sigma_x^{\nu m}\sigma_y^{-\nu n}(q)$ for any $q\in\Omega$ and $\nu\in\mathbb{N}$. We prove this claim by contradiction. Suppose that for any $p_1\in\Omega$, there always exists $p_2\in\Omega$ such that $p_1=\sigma_x^{\nu_1 m}\sigma_y^{-\nu_1 n}(p_2)$ for some positive integer ν_1. If $p_1=p_2$, then we get a contraction with the assumption on c_i's in (6.3). If $p_1\neq p_2$, then there exists $p_3\in\Omega$ such that $p_2=\sigma_x^{\nu_2 m}\sigma_y^{-\nu_2 n}(p_3)$ for some positive integer ν_2. Continuing this process, we get a sequence of polynomials $p_1,p_2,\ldots\in\Omega$. Since Ω is a finite set, $p_i=p_j$ for some $i<j$ in this sequence. Then $p_i=\sigma_x^{\nu m}\sigma_y^{-\nu n}(p_i)$ with $\nu=\nu_i+\cdots\nu_{j-1}>0$, a contradiction. This completes the proof of the claim.

Suppose now that c_1 is such an element in Ω satisfying $c_1\neq\sigma_x^{\nu m}\sigma_y^{-\nu n}(q)$ for any $q\in\Omega$ and $\nu\in\mathbb{N}$. Then the fraction $\frac{a_0 b_{1,\ell}}{c_1^\ell}$ has a different irreducible denominator from the other fractions in h_ℓ which implies that $a_0 b_{1,\ell}=0$. Since $a_0\neq 0$ we have that $b_{1,\ell}=0$ for all ℓ. We can now repeat the argument for the set $\Omega\setminus\{c_1\}$ to get $b_{i,\ell}=0$ for all $i=2,\ldots,n$ and all ℓ. Thus, $r=0$. ∎

Example 6.5. *Let*

$$f=\frac{xy+xz+y^2+yz+1}{(x+y)\left((x+y)^2+z^2\right)}.$$

In order to decide whether there exists a telescoper for f, we first rewrite f into

$$f=\left(y+z+\frac{1}{x+y}\right)\cdot\frac{1}{(x+y)^2+z^2}.$$

Letting $d=(x+y)^2+z^2$ one has $\sigma_x d=\sigma_y d$ and hence from Remark 2.3 and Lemma 6.2 we see that f has a telescoper. In fact, following the proof of

Lemma 6.2, we can determine that

$$L_1 = S_x^2 - 2S_x + 1 = (S_x - 1)^2 \quad \text{and} \quad L_2 = S_x - 1$$

are telescopers for $\frac{y+z}{d}$ *and for* $\frac{1}{(x+y)d}$*, respectively. Thus* $L = (S_x - 1)^2$ *is a telescoper for* f.

We now study the case when d satisfies the condition (6.4). Assume that n_1 is the smallest positive integer such that $\sigma_y^{n_1}(d) = \sigma_z^{k_1}(d)$ for some $k_1 \in \mathbb{Z}$. By Lemma 6.2, all the fractions $\frac{b_{i,\ell}}{c_i^\ell d^\lambda}$ in (6.3) with c_i satisfying the condition: for all i,

$$\sigma_x^{n_i}(c_i) = \sigma_y^{k_i}(c_i) \text{ for some } n_i, k_i \in \mathbb{Z} \text{ with } n_i > 0 \tag{6.5}$$

have telescopers. It remains to study the existence problem of telescopers for rational functions of the form

$$r = \sum_{i=1}^{I} \sum_{\ell=1}^{m_i} \frac{b_{i,\ell}}{c_i^\ell d^\lambda}, \tag{6.6}$$

where $b_{i,\ell} \in \mathbb{K}[x, y, z], c_i \in \mathbb{K}[x, y], \deg_y(b_{i,\ell}) < \deg_y(c_i)$, where the c_i are irreducible polynomials such that condition (6.5) is not satisfied.

Theorem 6.6. *Let* r *be of the form (6.6) with* d *satisfying conditions (6.2) and (6.4) and where* c_i*'s do not satisfy the condition (6.5). Then* r *has a telescoper if and only if*

$$r_\ell \; := \; \sum_{i=1}^{I} \frac{b_{i,\ell}}{c_i^\ell d^\lambda}$$

is (σ_y, σ_z)*-summable for all* ℓ.

Proof. The sufficiency follows from Remark 2.3. For the necessity, we assume that L is a telescoper for r. By the same argument as in the proof of Theorem 6.4, we may always assume that $L = \sum_{t=0}^{T} a_t S_x^{tm}$ with $a_0 \neq 0$. The same calculation as in the proof of Theorem 6.4 then yields

$$L(r) = \Delta_y(u) + \Delta_z(v) + \frac{1}{d^\lambda} h,$$

where $u, v \in \mathbb{K}(x, y, z)$ and $h = Q(\sum_{i=1}^{I} \sum_{\ell=1}^{m_i} b_{i,\ell}/c_i^\ell)$ with

$$Q = \sum_{t=0}^{T} a_t S_x^{tm} S_y^{-tn} S_z^{-tk} \in \mathbb{K}(x)[S_x, S_y, S_z].$$

Since $L(r)$ is (σ_y, σ_z)-summable but d satisfies the condition (6.4), Theorem 3.1 implies that $h = \sigma_y^{n_1} \sigma_z^{-k_1}(p) - p$, where $p \in \mathbb{K}(x, y)[z]$ with $\deg_z(p) < \deg_z(d)$. By Lemma 5.1, for each multiplicity ℓ, we have

$$h_\ell = Q\left(\sum_{i=1}^{I} \frac{b_{i,\ell}}{c_i^\ell}\right) = \sigma_y^{n_1} \sigma_z^{-k_1}(p_\ell) - p_\ell.$$

Let $\triangle := \{c_i \mid 1 \leq i \leq I\}$. As in the argument for the proof of Theorem 6.4, we may assume $c_1 \in \triangle$ satisfying $c_1 \neq \sigma_x^m \sigma_y^n c_i$ for any $c_i \in \triangle$, when $m, n \in \mathbb{Z}$ with $m > 0$. Note that there may exist some $c_i \in \triangle \setminus \{c_1\}$ such that $c_1 = \sigma_y^n c_i$ for some $n \in \mathbb{Z}$, and we will let

$$\triangle_1 = \{i \mid 1 \leq i \leq I, c_i = \sigma_y^n c_1 \text{ for some } n \in \mathbb{Z}\}.$$

Continuing now with $\triangle \setminus \triangle_1$, we will find $c_1, c_2, \ldots, c_M \in \triangle$ and $\triangle_1, \triangle_2, \ldots, \triangle_M$ such that for $1 \leq i < i' \leq M$, we have $c_i \neq \sigma_x^m \sigma_y^n c_{i'}$, when $m, n \in \mathbb{Z},\ m > 0$ and $\{1, 2, \ldots, I\} = \bigcup_{i=1}^M \triangle_i$. We can therefore rewrite h_ℓ as

$$Q\left(\sum_{j=1}^{M} \sum_{i \in \triangle_j} \frac{b_{i,\ell}}{c_i^\ell}\right) = \sigma_y^{n_1} \sigma_z^{-k_1}(p_\ell) - p_\ell. \tag{6.7}$$

Since $p_\ell \in \mathbb{Q}(x, y)[z]$, we can decompose it into

$$p_\ell = \sum_{j=1}^{M} \sum_{t=\alpha_j}^{\beta_j} \frac{u_{j,t}}{\sigma_y^t(c_j^\ell)} + q_\ell,$$

where $\alpha_i, \beta_i \in \mathbb{Z}$ and q_ℓ contains no term of the form $\frac{u_{j,t}}{\sigma_y^t(c_j^\ell)}$ in its irreducible partial fraction decomposition with respect to y. According to Equation (6.7) and the uniqueness of irreducible partial fraction decomposition along with the fact that $a_0 \in \mathbb{K}(x) \setminus \{0\}$, we derive that

$$\sum_{i \in \triangle_1} \frac{b_{i,\ell}}{c_i^\ell} = \sigma_y^{n_1} \sigma_z^{-k_1}(h_{1,\ell}) - h_{1,\ell},$$

where $h_{1,\ell} = \frac{1}{a_0} \sum_{t=\alpha_1}^{\beta_1} \frac{u_{1,t}}{\sigma_y^t(c_1^\ell)}$. Collecting all the terms with the denominator $\langle \sigma_x, \sigma_y \rangle$-equivalent to c_1 in Equation (6.7), we obtain

$$Q\left(\sum_{i \in \triangle_1} \frac{b_{i,\ell}}{c_i^\ell}\right) = Q\left(\sigma_y^{n_1} \sigma_k^{-k_1}(h_{1,\ell}) - h_{1,\ell}\right) \tag{6.8}$$

$$= \sigma_y^{n_1} \sigma_k^{-k_1}(p_{1,\ell}) - p_{1,\ell} \tag{6.9}$$

with $p_{1,\ell} = Q(h_{1,\ell})$. Subtracting Equation (6.9) from Equation (6.7), we obtain

$$Q\left(\sum_{j=2}^{M} \sum_{i \in \triangle_j} \frac{b_{i,\ell}}{c_i^\ell}\right) = \sigma_y^{n_1} \sigma_z^{-k_1}(p_\ell^\star) - p_\ell^\star \tag{6.10}$$

with $p_\ell^\star = p_\ell - p_{1,\ell}$. Now we can repeat the arguments for the set $\Delta \setminus \{\Delta_1\}$ and Equation (6.10) to get

$$\sum_{i \in \triangle_j} \frac{b_{i,\ell}}{c_i^\ell} = \sigma_y^{n_1} \sigma_z^{-k_1}(h_{j,\ell}) - h_{j,\ell}$$

for all $j = 1, \ldots, M$ and all ℓ. Then $\sum_{i\in\triangle_j} \frac{b_{i,\ell}}{c_i^\ell d^\lambda}$ is (σ_y, σ_z)-summable by Theorem 3.1 and thus $\sum_{i=1}^{I} \frac{b_{i,\ell}}{c_i^\ell d^\lambda}$ is (σ_y, σ_z)-summable for all ℓ. This completes the proof. ∎

Example 6.7. *Let*

$$f = \frac{x^4 + 2x^2y^2 + y^4 + x^3 + 3yx^2 + y^3 - xy^2 + x^2 - xy}{(x+y)(x^2+y^2+2y+1)(x^2+y^2)(x+y+z)^2}.$$

To solve the existence problem of telescopers for f, we firstly need to decompose

$$f = \left(\frac{1}{x+y} + \frac{y+1}{x^2+y^2+2y+1} - \frac{y}{x^2+y^2}\right) \cdot \frac{1}{(x+y+z)^2}.$$

Letting $d = x + y + z$, we have $\sigma_x d = \sigma_y d$ and $\sigma_y d = \sigma_z d$. As in the proof of Lemma 6.2, we get that $L = S_x - 1$ is a telescoper for $\frac{1}{(x+y)(x+y+z)^2}$. Theorem 3.1 then guarantees

$$\left(\frac{y+1}{x^2+y^2+2y+1} - \frac{y}{x^2+y^2}\right) \cdot \frac{1}{(x+y+z)^2}.$$

is (σ_y, σ_z)-summable, so $L = S_x - 1$ is a telescoper for f.

Remark 6.8. *To test the existence of telescopers for a simple fraction, one first needs to test the conditions (6.2), (6.4) and (6.5) satisfied by the polynomials d and c_i's. This amounts to solving the following problem:*

Problem 6.9 (Integer Shift Equivalence Testing Problem)**.** *Let $\mathbb{K}$ be any computable field of characteristic zero and σ_i be the shift operator w.r.t. x_i on $\mathbb{K}[x_1, ..., x_n]$. Given $p \in \mathbb{K}[x_1, ..., x_n]$, to decide whether there exist integers $m_1, ..., m_n$ with $m_1 > 0$ such that $\sigma_1^{m_1} \cdots \sigma_n^{m_n}(p) = p$.*

This problem is a special case of the problem proposed and solved by Grigoriev in [19, 20] and more recently by Dvir et al. in [17]. Theorems 6.4 and 6.6 reduce the problem to that of testing the summability of bivariate rational functions. For this, we can apply the algorithm in [21]. As such the existence problem in this case is solved.

7 Conclusion

In this paper, we solve the existence problem of telescopers for rational functions in three discrete variables. We give a procedure which reduces the problem to a special shift equivalence testing problem and the summability problem of bivariate rational functions. Those problems have recently been solved.

In terms of future research, the first direction is to solve the existence problem of telescopers for multivariate rational functions or a more general class of functions, for example, hypergeometric terms. This would include both efficient algorithms and implementations. A crucial first step is solving the summability problem for these functions. This is also a challenging problem in symbolic summation as noted in [9].

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