Stability Problems on D-finite Functions

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ABSTRACT

This paper continues the dynamical studies of symbolic integration by focusing on the stability problems on D-finite functions. We introduce the notion of stability index in order to investigate the order growth of the differential operators satisfied by iterated integrals of D-finite functions and determine bounds and exact formulae for stability indices of several special classes of differential operators. With the basic properties of stability index, we completely solve the stability problem on general hyperexponential functions.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

D-finite functions; stable Sets; symbolic integration

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1 INTRODUCTION

Motivated by irrationality proofs in number theory, Chen initialized the dynamical aspect of symbolic integration by studying some stability problems in differential fields in [4]. For a given special function f in certain differential ring (R, D), the stability problem is deciding whether there exists a sequence $\{g_i\}_{i\geq 0}$ in R such that

$$f = D^i(g_i) \quad \text{for all } i \in \mathbb{N}.$$

Focusing on elementary functions, Chen in [4] had solved the stability problems on three families of special functions including rational functions, logarithmic functions, and exponential functions. The main goal of this paper is to continue the work [4] by focusing on D-finite functions using the language of differential operators.

The notion of D-finite power series was first introduced by Stanley [11] in 1980 in the univariate case and later studied by Lipshitz in the multivariate case [8]. D-finite functions are also called holonomic functions that had played a significant role in Zeilberger's method of creative telescoping [13, 14]. These series like algebraic numbers can be algorithmically manipulated via its defining linear differential equations [2, 10]. For a comprehensive introduction to theory and algorithms for D-finite functions, one can see the coming monograph by Kauers [6]. The indefinite integration problem on D-finite functions was studied by Abramov and van Hoeij in [3]. The main goal of this paper is to investigate Abramov-van Hoeij's algorithm through the dynamical lens.

The remainder of this paper is organized as follows. We recall some basic terminologies in differential algebra and then define the notion of principal integrals related to iterated integration in Section 2. We will focus on studying the order of the differential operators satisfied by principal integrals in Section 3, which inspires us to introduce the notion of stability index. In Section 4, we will determine the stability indices of two special classes of differential operators including Katz's operators and first-order operators. With the help of stability index, we will completely solve the stability problem on general hyperexponential functions in Section 5 which generalizes the results in [4]. As an application, we present an exact formula for the stability index of a special class of rational functions in Section 6.

2 PRINCIPAL INTEGRALS

Throughout this paper, let C be any algebraically closed field of characteristic zero and $k = C(x)$ be the differential field with the usual derivation $'$ such that $x' = 1$ and $c' = 0$ for all $c \in C$. Let $\mathcal{K}$ be the differential closure of $C(x)$ (see Kolchin's book [7] for the existence of such closures). Any system of algebraic differential equations with coefficients from k has solutions in $\mathcal{K}$ if it has any solution in some differential extension of k. Let $k\langle D\rangle$ be the ring of linear differential operators with coefficients in k in which we have

$$D \cdot f = f \cdot D + f', \quad \text{for any } f \in k.$$

The ring $k\langle D\rangle$ is a left Euclidean domain in which any left ideal is principal [9]. For an $m \in \mathbb{Z}$, $\mathbb{Z}_{\leq m}$ (resp., $\mathbb{Z}_{\geq m}$) stands for the set of all integers not greater than m (resp., not less than m).

Let L be a nonzero operator in $k\langle D\rangle$. Write

$$L = \sum_{i=0}^{n} a_i D^i, \quad \text{where } a_i \in k \text{ and } a_n \neq 0.$$

The integer n is called the order of L, denoted by $\operatorname{ord}(L)$, and a_n is called the leading coefficient of L, denoted by $\operatorname{lc}(L)$. If $a_n = 1$, then L is called a monic operator. The adjoint operator of L, denoted by L^*, is defined to be

$$L^* := \sum_{i=0}^{n} (-D)^i a_i.$$

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The field $\mathcal{K}$ can be equipped with a left $k\langle D\rangle$-module structure with the action $L(f) = \sum_{i=0}^{n} a_i f^{(i)}$ for any $L = \sum_{i=0}^{n} a_i D^i \in k\langle D\rangle$ and $f \in \mathcal{K}$. The annihilating ideal

$$\mathrm{Ann}_{k\langle D\rangle}(f) := \{P \in k\langle D\rangle \mid P(f) = 0\}$$

is a left ideal of $k\langle D\rangle$. Since $k\langle D\rangle$ is a left Euclidean domain, we have $\mathrm{Ann}_{k\langle D\rangle}(f) = \langle L\rangle$ for some monic operator $L \in k\langle D\rangle$.

Definition 2.1. *An element $f \in \mathcal{K}$ is said to be D-*finite *over k if $Ann_{k\langle D\rangle}(f) = \langle L\rangle$ for some nonzero and monic operator $L \in k\langle D\rangle$. We call L the defining operator for f, denoted by L_f, whose order is called the order of f, denoted by* $\mathrm{ord}(f)$*. For convenience, we set* $\mathrm{ord}(f) = \infty$ *if f is not D-finite.*

Lipshitz [8] showed that the set of all D-finite elements over k forms a subalgebra of $\mathcal{K}$, which is also closed under taking derivatives and integrals. We will investigate how the order changes when integrating a D-finite function.

Definition 2.2. *For $f \in \mathcal{K}$ and $i \in \mathbb{Z}_{>0}$, $g \in \mathcal{K}$ is called an i-th primitive or an i-th integral of f if $f = D^i(g)$. When $i = 1$, we simply call g an integral of f.*

Suppose f is a D-finite function and g_i is an i-th integral of f. Then g_i is also D-finite and $\mathrm{ord}(g_i) \le \mathrm{ord}(f) + i$. Set

$$\mathcal{N}_i(f) = \min\{\mathrm{ord}(g) \mid g \text{ is an } i\text{-th integral of } f\}.$$

We now introduce the notion of principal integrals of nonzero operators in $k\langle D\rangle$.

Definition 2.3. *The i-th integrals of order $\mathcal{N}_i(f)$ are called the principal i-th integrals of f.*

It was proved in [1, 3] that $\mathcal{N}_1(f) = \mathrm{ord}(f)$ if and only if $L_f^*(y) = 1$ has a solution in k. If $L_f^*(y) = 1$ has no solution in k then $\mathcal{N}_1(f) = \mathrm{ord}(f) + 1$ and moreover $L_g = L_f D$ for any integrals g of f. Suppose that $L_f^*(y) = 1$ has a solution l in k. Then $(1 - lL_f)^*(1) = 0$ and so there is a unique $H_l \in k\langle D\rangle$ of order $n - 1$ such that

$$1 - lL_f = DH_l$$

where $n = \mathrm{ord}(f)$. Note that $-l$ is the leading coefficient of H_l. Set $g = H_l(f)$. Then g is an integral of f and

$$L_g = \frac{1}{l}(1 - H_l D).$$

For a nonzero operator $P \in k\langle D\rangle$ and $q \in C[x]$, define

$$\delta(P, q) = \begin{cases} 1 & P^*(y) = q \text{ has a solution in } k \\ 0 & \text{otherwise} \end{cases}.$$

When $q = 1$, we abbreviate $\delta(P, q)$ to $\delta(P)$. The above discussions motivate the definition below.

Definition 2.4. *Suppose that L is a nonzero monic operator in $k\langle D\rangle$. A principal integral of L is defined to be*

$$\begin{cases} LD & \delta(L) = 0 \\ \frac{1}{l}(1 - H_l D) & \delta(L) = 1 \end{cases}$$

where $l \in k$ is a solution of $L^*(y) = 1$ and H_l is the unique operator in $k\langle D\rangle$ of order $\mathrm{ord}(L) - 1$ *such that $lL + DH_l = 1$. We also call $\frac{1}{l}(1 - H_l D)$ the principal integral of L with respect to l. An i-th principal integral of L is defined iteratively.*

Remark 2.5. *Suppose that $f \in \mathcal{K}$ and g is a principal integral of f. Then from Definition 2.4 one sees that L_g is a principal integral of L_f. On the other hand, suppose that $\tilde{L}$ is a principal integral of L and $f \in \mathcal{K}$ satisfies that $L = L_f$. Then there is a principal integral g of f such that $\tilde{L} = L_g$.*

Remark 2.6. *Suppose P is a principal integral of L. From the definition, one sees that* $\mathrm{ord}(P) = \mathrm{ord}(L) - \delta(L) + 1$.

Remark 2.6 indicates that all principal integrals of L have the same order. In next section, we shall show that all principal i-th integrals of L also have the same order.

3 ORDERS OF PRINCIPAL INTEGRALS

Throughout this section, L is always a monic operator in $k\langle D\rangle$. Suppose that L_i is a principal i-th integral of L for any positive integer i. We are going to prove that $\mathrm{ord}(L_{i+1}) = \mathrm{ord}(L_i)$ if and only if there is a polynomial $p \in C[x]$ of degree i such that $L_i^*(y) = p$ has a solution in k. This generalizes the results in [1, 3] for the case when $i = 1$. For a rational function $l \in k$, l is said to be rationally integrable if $l = D(h)$ for some $h \in k$.

Lemma 3.1. *Suppose that $P = 1 + MD$ and l is a rational solution of $P^*(y) = p$, where $M \in k\langle D\rangle$ and $p \in C[x]$. Then l is rationally integrable.*

Proof. Note that $P^* = 1 - DM^*$. Since l is a solution of $P^*(y) = p$ in k, we have $l - D(M^*(l)) = p$. Let $q \in C[x]$ be such that $p = q'$. Then $l = D(q + M^*(l))$. The lemma then follows from the fact that $q + M^*(l) \in k$. ∎

Proposition 3.2. *Suppose that $L_0 = L$ and L_{i+1} is a principal integral of L_i for any $i \ge 0$. Set $I_{-1} = 1$ and for each $i \ge 0$ set*

$$I_i = \begin{cases} 1 & \delta(L_i) = 0 \\ D + \frac{D(l_i)}{l_i} & \delta(L_i) = 1 \end{cases}$$

where $l_i \in k$ is a solution of $L_i^*(y) = 1$ such that L_{i+1} is the principal integral of L_i with respect to l_i. Then for each $n \ge 0$,

$$LD^n = \left(\prod_{i=-1}^{n-1} I_i\right) L_n. \tag{1}$$

Proof. We shall prove the proposition by induction on n. The case $n = 0$ is clear. Assume that $n > 0$ and the assertion holds for $n - 1$. By induction hypothesis,

$$LD^{n-1} = \left(\prod_{i=-1}^{n-2} I_i\right) L_{n-1}.$$

If $\delta(L_{n-1}) = 0$ then $L_n = L_{n-1}D$ and $I_{n-1} = 1$. Thus

$$LD^n = LD^{n-1}D = \left(\prod_{i=-1}^{n-2} I_i\right) L_{n-1}D = \left(\prod_{i=-1}^{n-1} I_i\right) L_n.$$

If $\delta(L_{n-1}) = 1$ then $l_{n-1}L_{n-1} + DH_{l_{n-1}} = 1$ and $L_n = \frac{1}{l_{n-1}}(1 - H_{l_{n-1}}D)$. These imply that

$$\begin{aligned} L_{n-1}D &= \frac{1}{l_{n-1}}(1 - DH_{l_{n-1}})D = \frac{1}{l_{n-1}}D(1 - H_{l_{n-1}}D) \\ &= \frac{1}{l_{n-1}}Dl_{n-1}L_n = \left(D + \frac{D(l_{n-1})}{l_{n-1}}\right) L_n = I_{n-1}L_n. \end{aligned}$$

Therefore

$$LD^n = LD^{n-1}D = \left(\prod_{i=-1}^{n-2} I_i\right) L_{n-1}D = \left(\prod_{i=-1}^{n-1} I_i\right) L_n.$$

The assertion holds for all nonnegative integers n. ∎

Lemma 3.3. *Suppose that L_n is a principal integral of L_{n-1}. Then for every $p \in C[x]$, $\delta(L_n, p) = \delta(L_{n-1}, q)$ for some $q \in C[x]$ with $p = q'$. In other words, $L_n^*(y) = p$ has a solution in k if and only if $L_{n-1}^*(y) = q$ has a solution in k.*

Proof. Assume that $\delta(L_{n-1}) = 0$. Then $L_n = L_{n-1}D$ and thus $L_n^* = -DL_{n-1}^*$. One can verify that for each $h \in k$, $L_n^*(h) = p$ if and only if $L_{n-1}^*(-h) = q$ for some $q \in C[x]$ with $p = q'$. Now suppose that $\delta(L_{n-1}) = 1$ and L_n is the principal integral of L_{n-1} with respect to l_{n-1}. Then $L_n = 1/l_{n-1}(1 - H_{l_{n-1}}D)$, where $H_{l_{n-1}} \in k\langle D\rangle$ satisfies that $l_{n-1}L_{n-1} + DH_{l_{n-1}} = 1$. Suppose that $h \in k$ is a solution of $L_n^*(y) = p$. Then h/l_{n-1} is a solution of $(1 - H_{l_{n-1}}D)^*(y) = p$ in k. Due to Lemma 3.1, there is a $g \in k$ such that $D(g) = h/l_{n-1}$. Since $l_{n-1}L_{n-1}D = D - DH_{l_{n-1}}D = D(1 - H_{l_{n-1}}D)$, one has that

$$\begin{aligned} DL_{n-1}^*(l_{n-1}g) &= DL_{n-1}^* l_{n-1}(g) = (1 - H_{l_{n-1}}D)^* D(g) \\ &= (1 - H_{l_{n-1}}D)^*(h/l_{n-1}) = p. \end{aligned}$$

Let $\tilde{q} \in C[x]$ be such that $D(\tilde{q}) = p$. Then $D(L_{n-1}^*(l_{n-1}g) - \tilde{q}) = 0$ and thus $L_{n-1}^*(l_{n-1}g) - \tilde{q} = c \in C$. Set $q = \tilde{q} + c$. Then $L_{n-1}^*(l_{n-1}g) = q$. Conversely, assume that h is a solution of $L_{n-1}^*(y) = q$ for some $q \in C[x]$ with $p = q'$. Then $DL_{n-1}^*(h) = p$. Set $\tilde{h} = l_{n-1}D(h/l_{n-1})$. One then has that

$$\begin{aligned} L_n^*(\tilde{h}) &= (1 - H_{l_{n-1}}D)^*(\tilde{h}/l_{n-1}) = (1 - H_{l_{n-1}}D)^* D(h/l_{n-1}) \\ &= (1 + DH_{l_{n-1}}^*) D \frac{1}{l_{n-1}}(h) = D(1 + H_{l_{n-1}}^* D)\frac{1}{l_{n-1}}(h) \\ &= D\left(\frac{1}{l_{n-1}}(1 - DH_{l_{n-1}})\right)^*(h) = DL_{n-1}^*(h) = p. \end{aligned}$$

In other words, $L_n^*(y) = p$ has a solution in k. ∎

Proposition 3.4. *Let L_i be as in Proposition 3.2. Then*

$$\delta(L_n, p) = \delta(LD^n, p)$$

for any $p \in C[x] \setminus \{0\}$ and nonnegative integer n.

Proof. We shall prove the assertion by induction on n. The case $n = 0$ is clear, because $L_0 = L = LD^0$ in this case. Suppose that $n > 0$ and the assertion holds for the case $n - 1$. Consider the case n. It suffices to show that $\delta(L_n, p) = 1$ if and only if $\delta(LD^n, p) = 1$. Assume that $\delta(L_n, p) = 1$. Lemma 3.3 implies that $\delta(L_{n-1}, q) = 1$ for some $q \in C[x]$ with $p = q'$. By induction hypothesis, $\delta(LD^{n-1}, q) = 1$, i.e. there is an $r \in k$ such that $(LD^{n-1})^*(r) = q$. Since $(LD^n)^* = -D(LD^{n-1})^*$, one sees that $(LD^n)^*(-r) = D((LD^{n-1})^*(r)) = q' = p$, i.e. $\delta(LD^n, p) = 1$. Conversely, assume that $\delta(LD^n, p) = 1$. Then $\delta(LD^{n-1}, \tilde{q}) = 1$ for some $\tilde{q} \in C[x]$ with $D(\tilde{q}) = p$. By induction hypothesis again, $\delta(L_{n-1}, \tilde{q}) = 1$. By Lemma 3.3 again, $\delta(L_n, p) = 1$. Hence the assertion holds for the case n. ∎

Corollary 3.5. *All i-th principal integrals of L have the same order.*

Proof. We shall prove the corollary by induction on i. The case $i = 0$ is clear. Suppose $i > 0$ and the assertion holds for the case $i - 1$. Consider the case i. Suppose that P_1 and P_2 are two i-th principal integrals of L. Suppose further that P_1, P_2 are principal integrals of Q_1, Q_2 respectively. Then Q_1 and Q_2 are $i-1$-th principal integrals of L. By induction hypothesis, $\mathrm{ord}(Q_1) = \mathrm{ord}(Q_2)$. Due to Proposition 3.2 and Remark 2.6,

$$\begin{aligned} \mathrm{ord}(P_1) &= \mathrm{ord}(Q_1) - \delta(Q_1) + 1 = \mathrm{ord}(Q_2) - \delta(LD^{i-1}) + 1 \\ &= \mathrm{ord}(Q_2) - \delta(Q_2) + 1 = \mathrm{ord}(P_2). \end{aligned}$$

The second and third equalities hold because $\delta(Q_1) = \delta(LD^{i-1}) = \delta(Q_2)$ by Proposition 3.4 with $p = 1$. ∎

Corollary 3.6. *Let L_i be as in Proposition 3.2. Then $\mathrm{ord}(L_{i+1}) = \mathrm{ord}(L_i)$ if and only if $\delta(L, q) = 1$ for some $q \in C[x]$ of degree i.*

Proof. From Definition 2.4, $\mathrm{ord}(L_{i+1}) = \mathrm{ord}(L_i)$ if and only if $\delta(L_i) = \delta(L_i, 1) = 1$. Note that $\delta(L_i) = \delta(L_i, c)$ for any nonzero $c \in C$. Using Lemma 3.3 repeatedly, there is a $q \in C[x]$ of degree i such that

$$\delta(L, q) = \delta(L_1, q') = \cdots = \delta(L_i, i!\,\mathrm{lc}(q)) = \delta(L_i).$$

In other words, for such q, $\delta(L_i) = 1$ if and only if $\delta(L, q) = 1$. Hence $\mathrm{ord}(L_{i+1}) = \mathrm{ord}(L_i)$ if and only if $\delta(L, q) = 1$ for some $q \in C[x]$ of degree i. ∎

Given a nonzero $P \in k\langle D\rangle$, there exist a nonzero polynomial $\mathrm{ind}^P(s) \in C[s]$ and an integer σ^P such that for any $s \in \mathbb{Z}$,

$$P(x^s) = \mathrm{ind}^P(s) x^{s+\sigma^P}(1 + c_1 x^{-1} + c_2 x^{-2} + \dots)$$

where $c_i \in C$ (see [12, p. 102]). The polynomial ind^P is usually called the indicial polynomial of P at ∞. Remark that Pd and P have the same indicial polynomial and $\sigma^{Pd} = \sigma^P + \deg(d)$, where d is a nonzero monic polynomial. In the following, $V_{\mathbb{Z}_{\geq 0}}(\mathrm{ind}^P)$ stands for the set of nonnegative integer solutions of $\mathrm{ind}^P(s) = 0$ and we agree with $\max \emptyset = -1$.

Theorem 3.7. *Let d be a nonzero monic polynomial of minimal degree such that dL is an operator of polynomial coefficients. Set*

$$\mathcal{B}(L) = \max\{0, \max V_{\mathbb{Z}_{\geq 0}}(\mathrm{ind}^{L^*}) + 1 + \sigma^{L^*} + \deg(d)\}. \qquad (2)$$

Then for all $i \geq \mathcal{B}(L)$, there exists a polynomial p of degree i such that $\delta(L, p) = 1$.

Proof. It is clear that $\delta(dL, p) = \delta(L, p)$ for any nonzero polynomial p. So it suffices to consider dL. Note that $\mathrm{ind}^{(dL)^*} = \mathrm{ind}^{L^*}$ and $\sigma^{(dL)^*} = \sigma^{L^*} + \deg(d)$. Suppose that $i \geq \mathcal{B}(L)$. Set $\ell = i - \sigma^{(dL)^*}$. Then $\ell \geq 0$ and $(dL)^*(x^\ell)$ is a polynomial. Furthermore,

$$\begin{aligned} (dL)^*(x^\ell) &= \mathrm{ind}^P(\ell) x^{\ell + \sigma^{(dL)^*}} + \text{lower terms} \\ &= \mathrm{ind}^P(\ell) x^i + \text{lower terms}. \end{aligned}$$

Since $\ell \geq \mathcal{B}(L) - \sigma^{(dL)^*} \geq \max V_{\mathbb{Z}_{\geq 0}}(\mathrm{ind}^P) + 1$, $\mathrm{ind}^{(dL)^*}(\ell) \neq 0$ and thus $\deg((dL)^*(x^\ell)) = i$. Set $p = (dL)^*(x^\ell)$. Then p is a polynomial as required. ∎

In the case of D-finite power series, it has been proved in [4, Theorem 4.5] that the order of i-th principal integrals of a given D-finite function is uniformly bounded. We now provide a more explicit order bound in terms of the information of indicial polynomials.

Definition 3.8. *The stability index of L, denoted by* $\mathrm{Sind}(L)$, *is defined to be the minimal nonnegative integer m satisfying that for each $i \geq m$ there exists a polynomial p_i of degree i such that $\delta(L, p_i) = 1$. For an $f \in \mathcal{K}$, the stability index of f is defined to be the stability index of L_f, also denoted by* $\mathrm{Sind}(f)$.

Remark 3.9. *It is clear that* $\mathrm{Sind}(L) \leq \mathcal{B}(L)$.

Definition 3.10. *A nonzero operator $L \in k\langle D\rangle$ is said to be stable under integration, or simply stable, if* $\mathrm{Sind}(L) = 0$. *An $f \in \mathcal{K}$ is said to be stable if* $\mathrm{Sind}(f) = 0$.

Remark 3.11. *Suppose that L is stable and L_i is an i-th principal integral of L for all $i \geq 0$. Then Corollary 3.6 implies that* $\mathrm{ord}(L_i) = \mathrm{ord}(L)$ *for all $i \geq 0$.*

Below are several examples of stable operators.

Example 3.12. *Suppose that $\beta \in C$ is not a positive integer and $L = D + \beta/(x - c)$ for some $c \in C$. Then for each $s \geq 1$,*

$$L^*((x-c)^s) = -s(x-c)^{s-1} + \beta(x-c)^{s-1} = (\beta - s)x^{s-1}.$$

Since β is not a positive integer, $\beta - s \neq 0$ for any $s \geq 1$. Hence $\deg(L^*((x-c)^s)) = s - 1$ *for all $s \geq 1$ and then* $\delta(L, (\beta - s)(x-c)^{s-1}) = 1$ *for all $s \geq 1$. So* $\mathrm{Sind}(L) = 0$, *i.e. L is stable.*

Example 3.13. *Suppose that $L = p(D)$, where $p \in C[z]$ is a polynomial of positive degree. Let s be the maximal integer such that z^s divides p. Then for each $i \geq s$,*

$$L^*(x^s) = (-1)^s c_s \binom{i}{s} x^{i-s} + \textit{lower terms}$$

where c_s is the trailing coefficient of p. Thus $\mathrm{Sind}(L) = 0$, *i.e. L is stable.*

Example 3.14. *Suppose $\alpha, \beta, \gamma \in C$. Consider hypergeometric differential operator*

$$L = D^2 + \frac{(\alpha + \beta + 1)x - \gamma}{x(x-1)} D + \frac{\alpha\beta}{x(x-1)}.$$

For each $s \geq 0$, an easy calculation yields that

$$L^*(x^{s+1}(x-1)) = (s+1-\alpha)(s+1-\beta)x^s - s(s+1-\gamma)x^{s-1}.$$

Suppose that $\alpha - 1 \notin \mathbb{Z}_{\geq 0}$ and $\beta - 1 \notin \mathbb{Z}_{\geq 0}$. Then $\deg((L^*(x^{s+1}(x-1))) = s$. *Hence L is stable if neither $\alpha - 1$ nor $\beta - 1$ is a nonnegative integer.*

4 STABILITY INDICES OF SPECIAL OPERATORS

For a general operator even for a first order operator, it is difficult to compute its stability index. In this section, we shall study the stability indices of some special operators.

Proposition 4.1. *Suppose that L is stable. Then $\delta(L, p) = 1$ for any nonzero polynomial $p \in C[x]$.*

Proof. Since L is stable, for each $i = 0, 1, \ldots$, there is some nonzero $h_i \in C[x]$ of degree i such that $\delta(L, h_i) = 1$. Let $r_i \in k$ be a solution of $L^*(y) = h_i$. Note that $\{h_i \mid i \geq 0\}$ is a basis of $C[x]$ as a vector space over C. Assume that p is a nonzero polynomial in $C[x]$. Write $p = \sum_{i=0}^{\ell} c_i h_i$ where $\ell = \deg(p)$. Then $\sum_{i=0}^{\ell} c_i r_i$ is a solution of $L^*(y) = p$. ∎

4.1 Stability indices of Katz's operators

In [5], Katz introduced a class of operators of the following form:

$$p(D) + q \tag{3}$$

where $p, q \in C[z] \setminus C$, $p(0) = 0$ and $\gcd(\deg(p), \deg(q)) = 1$. He proved that differential Galois groups of the operators of the above form are large. In this subsection, let us compute the stability indices of Katz's operators.

Lemma 4.2. *Suppose that L is of the form*

$$D^n + \frac{a_{n-1}}{a_n} D^{n-1} + \cdots + \frac{a_0}{a_n}, \ a_i \in C[x], \ a_n \neq 0.$$

Suppose further that $\gcd(a_0, a_1, \ldots, a_n) = 1$ *and* $\deg(a_0) > \deg(a_i) - i$ *for all $i = 1, 2, \ldots, n$. Then* $\mathrm{Sind}(L) \leq \deg(a_0)$.

Proof. For each $s \geq 0$, a simple calculation yields that $L^*(a_n x^s)$ is a polynomial of degree $\deg(a_0) + s$. Thus $\mathrm{Sind}(L) \leq \deg(a_0)$. ∎

Proposition 4.3. *Suppose that $L = p(D) + q(x)$, where $p, q \in C[z]$ and $p(0) = 0$. Then* $\mathrm{Sind}(L) = \max\{\deg(q), 0\}$.

Proof. If $q(x) = 0$ then from Example 3.13 one sees that

$$\mathrm{Sind}(L) = 0 = \max\{\deg(q), 0\}.$$

Assume that $q(x) \neq 0$. By Lemma 4.2, $\mathrm{Sind}(L) \leq \deg(q)$. It remains to show that $\delta(L, d) = 0$ for any polynomial d of degree $\deg(q) - 1$. Suppose on the contrary that there is an $l \in k$ such that $L^*(l)$ is a polynomial of degree $\deg(q) - 1$. Suppose that l is not a polynomial and $c \in C$ is a pole of l. Then c must be a pole of $L^*(l)$ and so $L^*(l)$ can not be a polynomial. However, this implies that $\deg(L^*(l)) = \deg(q) + \deg(l) > \deg(q) - 1$, a contradiction. Hence $\delta(L, d) = 0$ for any polynomial d of degree $\deg(q) - 1$. ∎

4.2 Stability indices of first-order operators

In this subsection, we shall focus on studying the stability indices of first-order operators. Suppose that $f \in k$ and $c \in C$. Let

$$f = \sum_{i \geq \ell} a_i (x-c)^i \quad \text{with } a_i \in C \text{ and } a_\ell \neq 0$$

be the power series expansion of f at c. Then ℓ is called the order of f at c, denoted by $\mathrm{ord}_c(f)$, and a_{-1} is called the residue of f at c, denoted by $\mathrm{res}_c(f)$. Similarly, we can define the order and residue of f at ∞. Let

$$f = \sum_{i \geq \ell} a_i \left(\frac{1}{x}\right)^i, a_i \in C, a_\ell \neq 0$$

be the power series expansion of f at ∞. Then ℓ is called the order of f at ∞, denoted by $\mathrm{ord}_\infty(f)$, and a_1 (not a_{-1} in this case) is called the residue of f at ∞, denoted by $\mathrm{res}_\infty(f)$. Later one will see that the residues of f play an important role in estimating the stability index of $D + f$. We shall use $\mathcal{S}(f)$ to denote the set of $c \in C$ such that c is a simple pole of f and $\mathrm{res}_c(f)$ is a negative integer. As usual, we use $\mathrm{den}(f)$ to denote the denominator of f.

Notation 4.4. *For a rational function $f \in k$, set*

$$\Delta(f) = \begin{cases} 1 & \mathcal{S}(f) = \emptyset \\ \prod_{c \in \mathcal{S}} (x-c)^{-\mathrm{res}_c(f)} & \mathcal{S}(f) \neq \emptyset \end{cases}.$$

Lemma 4.5. *Suppose that $L = D+f$ and $r \in k$. If $L^*(r)$ is a nonzero polynomial then the denominator of r divides $\Delta(f)$.*

Proof. Set $r_2 = \mathrm{den}(r)$. If $r_2 \in C$ then there is nothing to prove. Suppose that $r_2 \notin C$. Let $c \in C$ be a zero of r_2 with multiplicity m. It suffices to show that $(x-c)^m$ divides $\Delta(f)$. One has that $\mathrm{ord}_c(r) = -m$ and $\mathrm{ord}_c(r') = -m-1$. Write $r = r_0(x-c)^{-m} + \cdots$ and $f = f_0(x-c)^{\mu} + \cdots$, where $r_0, f_0 \in C$ and $\mu = \mathrm{ord}_c(f)$. From $-r' + fr = d$, one sees that $-m-1 = -m+\mu$ and $mr_0 + f_0 r_0 = 0$. These imply that $\mu = -1$ and $f_0 = -m$. Consequently, c is a simple pole of f and $\mathrm{res}_c(f) = f_0 = -m$. So $(x-c)^m$ divides $\Delta(f)$ as desired. ∎

The following result reduces the stability problem on first order operators into that on operators of special form.

Proposition 4.6. *Suppose that $L = D+f$ and $h \in C[x]$ is a non zero polynomial. If L is stable then $L - \frac{h'}{h}$ is stable.*

Proof. For each $i \geq 0$, let r_i be a solution of $L^*(y) = x^i h$ in k. Such r_i exists because of Proposition 4.1. A simple calculation yields that

$$\left(L - \frac{h'}{h}\right)^* \left(\frac{r_i}{h}\right) = -\left(\frac{r_i}{h}\right)' + \frac{fr_i}{h} - \frac{r_i h'}{h} = \frac{-r_i' + fr_i}{h} = x^i$$

i.e. $\delta(L - h'/h, x^i) = 1$ for all $i \geq 0$. So $L - h'/h$ is stable. ∎

The converse of Proposition 4.6 is not true any more. For example, D is stable but $D+1/x$ is not (see Corollary 6.4 for a proof). However, if h is a special divisor of the denominator of f then the converse is still true.

Lemma 4.7. *Suppose that $L = D+f$ and $c \in C$ is a pole of f.*

(1) For each $i \geq \mathrm{Sind}(L)+1$, there is a polynomial $p_i \in C[x]$ of degree i such that $L^*(y) = p_i/(x-c)$ has a solution in k.
(2) If L is stable then for each $i \geq 0$, $L^*(y) = x^i/(x-c)$ has a solution in k.

Proof. (1). Write $\mathrm{den}(f) = (x-c)M$. One has that

$$L^*(M) = -M' + fM = -M' + f_1/(x-c) = \tilde{M} + \alpha/(x-c)$$

where f_1 is the numerator of f, $\tilde{M} \in C[x]$ and $\alpha \in C$. For each $j \geq \mathrm{Sind}(L)$, let $d_j \in C[x]$ be a polynomial of degree j such that $L^*(y) = d_j$ has a solution $r_j \in k$. Fix an $i \geq \mathrm{Sind}(L)+1$. It is easy to see that there are $\ell > 0, \beta_1, \ldots, \beta_\ell \in C$ such that $h = \tilde{M} + \sum_{j=\mathrm{Sind}(L)}^{\ell} c_j d_j$ is of degree $i-1$. Then

$$L^*\left(M + \sum_{j=\mathrm{Sind}(L)}^{\ell} \beta_j r_j\right) = h + \frac{\alpha}{x-c} = \frac{(x-c)h+\alpha}{x-c}.$$

Since $\deg((x-c)h+\alpha) = i$, $(x-c)h+\alpha$ is a polynomial as required.

(2). As in (1), one has that $L^*(M) = \tilde{M} + \alpha/(x-c)$. As $\gcd(f_1, x-c) = 1$, $\alpha \neq 0$. If $\tilde{M} = 0$ then M/α is a solution of $L^*(y) = 1/(x-c)$. Otherwise, let r be a solution of $L^*(y) = \tilde{M}$. Such r exists due to Proposition 4.1. Then $(M-r)/\alpha$ is a solution of $L^*(y) = 1/(x-c)$. This proves the case $i = 0$. For the case $i > 0$, write $x^i/(x-c) = p + \beta/(x-c)$ where $p \in C[x]$ and $\beta \in C$. Let r_1 be a solution of $L^*(y) = p$ in k and r_2 a solution of $L^*(y) = \beta/(x-c)$. Then $r_1 + r_2$ is a solution of $L^*(y) = x^i/(x-c)$ in k. ∎

Proposition 4.8. *Suppose that $L = D+f$ and $c \in C$ is a pole of f. Then*

(1) $\mathrm{Sind}(L + 1/(x-c)) \leq \mathrm{Sind}(L) + 1$;
(2) if L is stable then $L + 1/(x-c)$ is also stable.

Proof. (1). For each $i \geq \mathrm{Sind}(L)+1$, by Lemma 4.7, there exists a polynomial p_i of degree i such that $L^*(y) = p_i$ has a solution r_i in k. Then

$$\left(L + \frac{1}{x-c}\right)^* ((x-c)p_i) = (x-c)(-r_i' + fr_i) = p_i.$$

Hence $\mathrm{Sind}(L + 1/(x-c)) \leq \mathrm{Sind}(L) + 1$.

(2). Use a similar argument as in (1). ∎

We now apply Theorem 3.7 to the first order operators to obtain an upper bound on the stability indices of such operators. We use $\mathrm{num}(f)$ to denote the numerator of a rational function f.

Proposition 4.9. *Suppose that $L = D+f$ and $v = \mathrm{res}_\infty(f)$. Then*

$$\mathrm{Sind}(L) \leq \begin{cases} \max\{\deg(f_1), \deg(f_2)\} & \mathrm{ord}_\infty(f) \neq 1 \\ \deg(f_2) - 1 & \mathrm{ord}_\infty(f) = 1 \text{ and } v \notin \mathbb{Z}_{\leq 0} \\ -v + \deg(f_2) & \text{otherwise} \end{cases}$$

where $f_1 = \mathrm{num}(f)$ and $f_2 = \mathrm{den}(f)$.

Proof. It suffices to show that $\mathcal{B}(L)$ equals the corresponding bounds. Let $f = \alpha\left(\frac{1}{x}\right)^t + \ldots$ be the power series expansion of f at ∞, where $t = \mathrm{ord}_\infty(f)$. Then for each $s \geq 0$,

$$L^*(x^s) = -sx^{s-1} + \alpha x^{s-t} + \ldots.$$

Suppose that $t > 1$. Then $\mathrm{ind}^{L^*}(s) = -s$ and $\sigma^{L^*} = -1$. Thus $\mathcal{B}(L) = \deg(f_2) = \max\{\deg(f_1), \deg(f_2)\}$ because $\deg(f_1) < \deg(f_2)$. Suppose that $t < 1$. Then $\mathrm{ind}^{L^*}(s) = \alpha$ and $\sigma^{L^*} = -t$. Hence $\mathcal{B}(L) = -t + \deg(f_2) = \deg(f_1) = \max\{\deg(f_1), \deg(f_2)\}$. Now suppose that $t = 1$. Then $\alpha = -\mathrm{res}_\infty(f) = -v$. Furthermore, $\mathrm{ind}^{L^*}(s) = -(s-\alpha)$ and $\sigma^{L^*} = -1$. If v is not a nonnegative integer then $\max V_{\mathbb{Z}_{\geq 0}}(\mathrm{ind}^{L^*}) = -1$ and so $\mathcal{B}(L) = \deg(f_2) - 1$. Otherwise if v is a nonnegative integer then $\max V_{\mathbb{Z}_{\geq 0}}(\mathrm{ind}^{L^*}) = -v$ and thus $\mathcal{B}(L) = -v + \deg(f_2)$. ∎

In what below, we shall present a lower bound.

Lemma 4.10. *Suppose that $L = D+f$ with $\mathcal{S}(f) = \emptyset$. Set $N = \deg(\mathrm{den}(f))$ and $v = \mathrm{res}_\infty(f)$. Then*

$$\mathrm{Sind}(L) = \begin{cases} N - \min\{1, \mathrm{ord}_\infty(f)\} & \mathrm{ord}_\infty(f) \neq 1 \text{ or } v \notin \mathbb{Z}_{\leq -N} \\ -v & \text{otherwise} \end{cases}.$$

Proof. Suppose that $r \in k$ satisfies that $L^*(r) = d$ for some nonzero polynomial d. By Corollary 3.6, r must be a polynomial. Furthermore, from $-r' + rf = d$, one sees that $\mathrm{den}(f)$ divides r. In particular, $\deg(r) \geq \deg(\mathrm{den}(f)) = N$. Let

$$r = cx^s + \ldots, \quad f = \alpha x^{-t} + \ldots,$$

be the power series expansions of r and f respectively, where $s = \deg(r)$ and $t = \mathrm{ord}_\infty(f)$. From $-r' + rf = d$ again, one has that

$$-scx^{s-1} + \cdots + c\alpha x^{s-t} + \cdots = d. \tag{4}$$

Suppose that $t = \mathrm{ord}_\infty(f) < 1$. Then $\deg(d)$ must be equal to $s-t$ that is not less than $N-t$. In other words, if $\delta(L,d) = 1$ for some nonzero polynomial d then $\deg(d) \ge N - t$. So $\mathrm{Sind}(L) \ge N - t$. On the other hand, for each $i \ge 0$,

$$L^*(\mathrm{den}(f)x^i) = -\alpha x^{N+i-t} + \text{lower terms}.$$

This implies that for each $s \ge N - t$, $\mathrm{den}(f)x^{s-N+t}$ is a solution of $L^*(y) = -\alpha x^s$ + lower terms, i.e. $\mathrm{Sind}(L) \le N - t$. Consequently, $\mathrm{Sind}(L) = N - t$. Similarly, one can prove the case $t > 1$.

Now assume that $t = 1$. In this case, $\alpha = -\mathrm{res}_\infty(f) = -\nu$ and $s = \deg(r) \ge N > 0$. Moreover, the equality (4) becomes

$$-c(s+\nu)x^{s-1} + \text{lower terms} = d. \tag{5}$$

Suppose that $\nu \notin \mathbb{Z}_{\le -N}$. Then $s + \nu \ne 0$ for all $s \ge N$. Hence $\deg(d)$ must be equal to $s - 1$. Using a similar argument as in the case $t < 1$, one has that $\mathrm{Sind}(L) = N - 1$. It remains to show the case $\nu \in \mathbb{Z}_{\le -N}$. We first show that $\delta(L, d) = 0$ for any polynomial $d \in C[x]$ of degree $-\nu - 1$. Suppose on the contrary that there is a polynomial $d \in C[x]$ of degree $-\nu - 1$ such that $L^*(y) = d$ has a solution $r \in C[x]$. If $s = \deg(r) > -\nu$ then $s + \nu = 0$ by (5). This implies that $s = -\nu$, a contradiction. So $s \le -\nu$. From (5), it is obvious that s can not be less than $-\nu$. Thus $s = -\nu$. While, this means that the degree of the left-hand side of (5) is less than $-\nu - 1$. This contradicts with the assumption that $\deg(d) = -\nu - 1$. Consequently, $\delta(L, d) = 0$ for any polynomial d of degree $-\nu - 1$ and so $\mathrm{Sind}(L) \ge -\nu$. Finally, for each $i \ge 0$,

$$L^*\left(\mathrm{den}(f)x^{i+1-\nu-N}\right) = -(i+1)x^{i-\nu} + \text{lower terms}.$$

This implies that $\mathrm{Sind}(L) \le -\nu$. Therefore $\mathrm{Sind}(L) = -\nu$. ∎

Proposition 4.11. *Suppose that $L = D + f$. Set*

$$N = \deg(\mathrm{den}(f)) - |\mathcal{S}(f)| - \deg(\Delta(f))$$

and $\nu = \mathrm{res}_\infty(f)$. Then

$$\mathrm{Sind}(L) \ge \begin{cases} N - \min\{1, \mathrm{ord}_\infty(f)\} & \mathrm{ord}_\infty(f) \ne 1 \text{ or } \nu \notin \mathbb{Z}_{\le -N} \\ -\nu & \text{otherwise} \end{cases}.$$

Proof. Set $M_1 = \prod_{c \in \mathcal{S}(f)}(x - c)$ and write $\mathrm{den}(f) = M_1 M_2$. Then $\deg(M_2) = \deg(\mathrm{den}(f)) - |\mathcal{S}(f)|$. Since $\gcd(M_1, M_2) = 1$, there exist $q_1, q_2 \in C[x]$ such that $f = q_1/M_1 + q_2/M_2$. Since M_1 is square-free and $\mathrm{res}_c(q_1/M_1) = \mathrm{res}_c(f)$ for each $c \in \mathcal{S}(f)$, one has that

$$\frac{q_1}{M_1} = \sum_{c \in \mathcal{S}(f)} \frac{\mathrm{res}_c(f)}{x - c} = -\frac{\Delta(f)'}{\Delta(f)}.$$

So $f = -\Delta(f)'/\Delta(f) + q_2/M_2$. Set $\tilde{L} = L + \Delta(f)'/\Delta(f) = D + q_2/M_2$. Using Proposition 4.8 repeatedly yields that $\mathrm{Sind}(\tilde{L}) \le \mathrm{Sind}(L) + \deg(\Delta(f))$. Since $\mathcal{S}(q_2/M_2) = \emptyset$, by Lemma 4.10, one has that if $\mathrm{ord}_\infty(q_2/M_2) \ne 1$ or $\mathrm{res}_\infty(q_2/M_2) \notin \mathbb{Z}_{\le -\deg(M_2)}$ then $\mathrm{Sind}(\tilde{L}) = \deg(M_2) - \min\{1, \mathrm{ord}_\infty(q_2/M_2)\}$, otherwise $\mathrm{Sind}(\tilde{L}) = -\mathrm{res}_\infty(q_2/M_2)$.

Note that $\mathrm{ord}_\infty(-\Delta(f)'/\Delta(f)) = 1$ and $\mathrm{res}_\infty(-\Delta(f)'/\Delta(f)) = \deg(\Delta(f))$. Furthermore, one has that

$$\begin{aligned} \mathrm{res}_\infty(f) &\ge \min\left\{\mathrm{ord}_\infty\left(-\frac{\Delta(f)'}{\Delta(f)}\right), \mathrm{ord}_\infty\left(\frac{q_2}{M_2}\right)\right\} \\ &= \min\left\{1, \mathrm{ord}_\infty\left(\frac{q_2}{M_2}\right)\right\} \end{aligned}$$

and the equality holds if $\mathrm{ord}_\infty(q_2/M_2) \ne 1$. If $\mathrm{ord}_\infty(f) < 1$ then $\mathrm{ord}_\infty(q_2/M_2) = \mathrm{ord}_\infty(f) < 1$ and so

$$\begin{aligned} \mathrm{Sind}(L) &\ge \mathrm{Sind}(\tilde{L}) - \deg(\Delta(f)) \\ &= \deg(M_2) - \min\{1, \mathrm{ord}_\infty(q_2/M_2)\} - \deg(\Delta(f)) \\ &= N - \min\{1, \mathrm{ord}_\infty(f)\}. \end{aligned}$$

Suppose that $\mathrm{ord}_\infty(f) > 1$. Then $\mathrm{ord}_\infty(q_2/M_2) = 1$ and moreover $\mathrm{res}_\infty(q_2/M_2) = -\mathrm{res}_\infty(-\Delta(f)'/\Delta(f)) = -\deg(\Delta(f))$. Assume that $\deg(\Delta(f)) \ge \deg(M_2)$. Then $\mathrm{res}_\infty(q_2/M_2) \in \mathbb{Z}_{\le -\deg(M_2)}$ and so $\mathrm{Sind}(\tilde{L}) = -\mathrm{res}_\infty(q_2/M_2) = \deg(\Delta(f))$. Thus $\mathrm{Sind}(L) \ge 0$. On the other hand, in this case,

$$N - \min\{1, \mathrm{ord}_\infty(f)\} = \deg(M_2) - \deg(\Delta(f)) - 1 < 0 \le \mathrm{Sind}(L).$$

Assume that $\deg(\Delta(f)) < \deg(M_2)$. Then we have $\mathrm{res}_\infty(q_2/M_2) \notin \mathbb{Z}_{\le -\deg(M_2)}$. So $\mathrm{Sind}(\tilde{L}) = \deg(M_2) - 1$ and $\mathrm{Sind}(L) \ge \mathrm{Sind}(\tilde{L}) - \deg(\Delta(f)) = N - 1$. Since $\mathrm{ord}_\infty(f) > 1$, we have $N - 1 = N - \min\{1, \mathrm{ord}_\infty(f)\}$ and then the assertion holds.

Note that $\nu = \mathrm{res}_\infty(f) = \mathrm{res}_\infty(-\Delta(f)'/\Delta(f)) + \mathrm{res}_\infty(q_2/M_2)$ and $\mathrm{res}_\infty(-\Delta(f)'/\Delta(f)) = \deg(\Delta(f))$. This implies that $\nu \notin \mathbb{Z}_{\le -N}$ if and only if $\mathrm{res}_\infty(q_2/M_2) \notin \mathbb{Z}_{\le -\deg(M_2)}$. Suppose that $\mathrm{ord}_\infty(f) = 1$ and $\nu \notin \mathbb{Z}_{\le -N}$. Then $\mathrm{ord}_\infty(q_2/M_2) \ge 1$ and $\mathrm{Sind}(\tilde{L}) = \deg(M_2) - 1$. Thus $\mathrm{Sind}(L) \ge \mathrm{Sind}(\tilde{L}) - \deg(\Delta(f)) \ge N - 1$. The assertion then follows from the fact that $\min\{1, \mathrm{ord}_\infty(f)\} = 1$. Finally, assume that $\mathrm{ord}_\infty(f) = 1$ and $\nu \in \mathbb{Z}_{\le -N}$. Then $\mathrm{res}_\infty(q_2/M_2) \in \mathbb{Z}_{\le -\deg(M_2)}$. Hence $\mathrm{Sind}(\tilde{L}) = -\mathrm{res}_\infty(q_2/M_2)$ and then $\mathrm{Sind}(L) \ge -\mathrm{res}_\infty(q_2/M_2) - \deg(\Delta(f)) = -\nu$. ∎

5 STABLE FIRST ORDER OPERATORS

Any solution of a first order operator over k is called a hyperexponential function over k. A first order operator is stable if and only if its corresponding hyperexponential solution is stable. The stability problem on hyperexponential functions of the form $f\exp(g)$ with $f, g \in k$ has been solved in [4]. In this section, we shall solve the stability problem on general hyperexponential functions by presenting a necessary and sufficient condition on the stability of first order operators.

Theorem 5.1. *Suppose that $L = D + f$ where $f \in C(x)$. Then L is stable if and only if f admits one of the following forms:*

$$-\frac{h'}{h} + \alpha \text{ or } -\frac{h'}{h} + \frac{\beta}{x - c}, \ h \in C[x] \setminus \{0\}, \alpha, \beta, c \in C$$

and β is not a positive integer.

Proof. Suppose that f admits one of the above forms. By Proposition 4.6, it suffices to show that $D + \alpha$ and $D + \beta/(x - c)$ are stable. The case $D + \alpha$ follows from Proposition 4.3 and the case $D + \beta/(x - c)$ follows from Example 3.12.

Now suppose that L is stable. Set $h = \Delta(f)$ and $\tilde{f} = f + h'/h$. If $\tilde{f} = 0$ then there is nothing to prove. Suppose that $\tilde{f} \ne 0$. Then there is no simple pole of $\tilde{f}$ at which the residue of $\tilde{f}$ is a negative integer, i.e. $\mathcal{S}(\tilde{f}) = \emptyset$ and $\Delta(\tilde{f}) = 1$. Using Proposition 4.8 repeatedly, one has that $\tilde{L} = L + h'/h$ is stable. Note that $\tilde{L} = D + \tilde{f}$.

Set $\tilde{f}_1 = \mathrm{num}(\tilde{f}_1)$ and $\tilde{f}_2 = \mathrm{den}(\tilde{f})$. Since $\mathcal{S}(\tilde{f}) = \emptyset$, $\mathrm{Sind}(\tilde{L}) = \deg(\tilde{f}_2) - 1$ if $\mathrm{ord}_\infty(\tilde{f}) \ge 1$ and $\mathrm{Sind}(\tilde{L}) = \deg(\tilde{f}_2) - \mathrm{ord}_\infty(\tilde{f})$ if $\mathrm{ord}_\infty(\tilde{f}) < 1$ due to Lemma 4.10. As $\mathrm{Sind}(\tilde{L}) = 0$, one has that $\deg(\tilde{f}_2) = 1$ if $\mathrm{ord}_\infty(\tilde{f}) \ge 1$ and $\deg(\tilde{f}_2) = \mathrm{ord}_\infty(\tilde{f})$ if $\mathrm{ord}_\infty(\tilde{f}) < 1$.

Suppose that $\mathrm{ord}_\infty(\tilde{f}) = 0$. Then $\deg(\tilde{f}_2) = \mathrm{ord}_\infty(\tilde{f}_2) = 0$. Hence one has that $\alpha = \tilde{f} \in C \setminus \{0\}$ and then $f = -h'/h + \alpha$. Now we consider the case $\mathrm{ord}_\infty(\tilde{f}) \geq 1$. Then $\deg(\tilde{f}_2) = 1$. Moreover, as $\mathrm{ord}_\infty(\tilde{f}) = \deg(\tilde{f}_2) - \deg(\tilde{f}_1) \geq 1$, $\deg(\tilde{f}_1) = 0$. Set $\beta = \tilde{f}_1$ and write $\tilde{f}_2 = x - c$ for some $c \in C$. Then $f = -h'/h + \beta/(x-c)$.

It remains to show that β is not a positive integer. Assume on the contrary that β is a positive integer. Since $\tilde{L}$ is stable, by Proposition 4.1, there is an $r \in k$ such that $\tilde{L}^*(r) = (x-c)^{\beta-1}$. As $\Delta(\tilde{f}) = 1$, Lemma 4.5 implies that r is a polynomial. From $\tilde{L}^*(r) = (x-c)^{\beta-1}$, one has that

$$-r'(x-c) + r\beta = (x-c)^\beta.$$

Write $r = \alpha x^s + \dots$, where $s = \deg(r)$. Then

$$-\alpha(s-\beta)x^s + \text{lower terms} = (x-c)^\beta.$$

Comparing the degrees of both sides of the above equality, one sees that there does not exist any integer s such that the above equality holds, a contradiction. Therefore β is not a positive integer. ■

6 ITERATED INTEGRATION OF SPECIAL RATIONAL FUNCTIONS

In the last section, we shall consider the iterated integrals of the rational function $1/q$, where q is a nonzero polynomial in $C[x]$. Write $q = \mathrm{lc}(q)\prod_{i=1}^n (x-c_i)^{m_i}$. Then the first order operator with $1/q$ as a fundamental solution is of the form

$$L = D + \sum_{i=1}^n \frac{m_i}{x - c_i}. \tag{6}$$

Lemma 6.1. *Suppose that $L = D - f'/f$ where $f \in k \setminus \{0\}$. Then for a nonzero $d \in C[x]$, $\delta(L, d) = 1$ if and only if df is rationally integrable.*

Proof. For $r \in k \setminus \{0\}$, one has that

$$L^*(r) = -r' - \frac{f'}{f}r = -\frac{f'r + fr'}{f} = -\frac{1}{f}(rf)'.$$

So $L^*(r) = d$ if and only if $(-fr)' = df$. In other words, $\delta(L, d) = 1$ if and only if df is rationally integrable. ■

Lemma 6.2. *Suppose that $n > 1$ and $p \in C[x]$ is a nonzero polynomial. If $\deg(p) < n - 1$ or $\deg(p) = \deg(q) - 1$ then p/q is not rationally integrable.*

Proof. Assume that $\deg(p) < n-1$ and assume on the contrary that p/q is rationally integrable, i.e. there is an $r \in C(x)$ such that $r' = p/q$. Set $r_1 = \mathrm{num}(r)$ and $r_2 = \mathrm{den}(r)$. We claim that $q = \tilde{q}r_2\prod_{i=1}^n(x-c_i)$ for some nonzero $\tilde{q} \in C[x]$. Note that $c \in C$ is a pole of r_1/r_2 if and only if c is a pole of r. Hence we may write $r_2 = \prod_{i=1}^n (x-c_i)^{l_i}$ where $l_i > 0$. For each $1 \leq i \leq n$, $\mathrm{ord}_{c_i}(r') = \mathrm{ord}_{c_i}(r) - 1 = -l_i - 1$ and $\mathrm{ord}_{c_i}(p/q) \geq -m_i$. These imply that $m_i \geq l_i + 1$ for all $1 \leq i \leq n$. So $q = \tilde{q}r_2\prod_{i=1}^n(x-c_i)$ and our claim holds. Now from $r' = p/q$, one has that $q(r_1'r_2 - r_1r_2') = pr_2^2$. Cancelling r_2 from both sides yields that

$$\tilde{q}(r_1'r_2 - r_1r_2')\prod_{i=1}^n(x-c_i) = pr_2. \tag{7}$$

If $\deg(r_1) \neq \deg(r_2)$ then $\deg(r_1'r_2 - r_1r_2') = \deg(r_1) + \deg(r_2) - 1$. Suppose that $\deg(r_1) = \deg(r_2)$. Write $r_1 = cr_2 + h$ where $c \in C$ and $h \in C[x]$ with $\deg(h) < \deg(r_2)$. Since $\gcd(r_1, r_2) = 1$, $h \neq 0$. One sees that $r_1'r_2 - r_1r_2' = h'r_2 - hr_2'$ and then $\deg(r_1'r_2 - r_1r_2') = \deg(h'r_2 - hr_2') = \deg(h) + \deg(r_2) - 1$. In either case, one sees that $\deg(r_1'r_2 - r_1r_2') \geq \deg(r_2) - 1$. Then the degree of the left-hand side of (7) is not less than $\deg(r_2) - 1 + n$. However the degree of the right-hand side of (7) equals $\deg(p) + \deg(r_2)$ that is less than $\deg(r_2) + n - 1$. We obtain a contradiction. Thus p/q is not rationally integrable.

Finally suppose that $\deg(p) = \deg(q) - 1$. Then the residue of p/q at infinity is not zero because $\mathrm{ord}(x^2p/q) = -1$. It is well-known that a rational function is rationally integrable if and only if all of its residues vanish. So p/q is not rationally integrable. ■

Proposition 6.3. *Let L be of the form (6) and let L_j be a j-th principal integral of L. Then*

$$\mathrm{ord}(L_j) = \begin{cases} \mathrm{ord}(L) + j, & 0 \leq j \leq n-1 \\ \mathrm{ord}(L) + n - 1, & n - 1 < j \leq \sum_{i=1}^n m_i - 1 \\ \mathrm{ord}(L) + n, & j \geq \sum_{i=1}^n m_i \end{cases}.$$

Proof. Note that $\frac{q'}{q} = \sum_{i=1}^n \frac{m_i}{x-c_i}$. Due to Lemmas 6.1 and 6.2, $\delta(L, d) = 0$ for any nonzero polynomial d of degree less than $n - 1$. By Corollary 3.6, $\mathrm{ord}(L_{j+1}) = \mathrm{ord}(L_j) + 1$ for all $0 \leq j < n-1$. Thus $\mathrm{ord}(L_j) = \mathrm{ord}(L) + j$ for all $0 \leq j \leq n-1$.

Set $N = \sum_{i=1}^n m_i$. Assume that $N > n$. Then there is at least one m_i that is greater than 1. Without loss of generality, we assume that all of $m_1, \dots, m_s$ are greater than 1 and $m_{s+1} = \cdots = m_n = 1$. Suppose that j is an integer satisfying that $n - 1 \leq j < N - 1$. Then $\sum_{i=1}^s (m_i - 1) = \sum_{i=1}^n (m_i - 1) \geq N - 1 - j$. Hence there exist integers $t, l_1, \dots, l_t$ satisfying that $1 \leq t \leq s, 0 < l_i < m_i$ and $\sum_{i=1}^t l_i = N - 1 - j$. Set $r = 1/\prod_{i=1}^t (x-c_i)^{l_i}$. Then

$$\left(\frac{1}{r}\right)' = \frac{h}{\prod_{i=1}^t (x-c_i)^{l_i+1}}$$

where h is a nonzero polynomial of degree $t - 1$. Let $\tilde{d} = q(1/r)'$. Then $\tilde{d}$ is a nonzero polynomial because $l_i + 1 \leq m_i$, and

$$\deg(\tilde{d}) = \deg(h) + \deg(q) - \sum_{i=1}^t (l_i + 1) = j.$$

As $(1/r)' = \tilde{d}/q$, Lemma 6.1 implies that $\delta(L, \tilde{d}) = 1$. Due to Corollary 3.6 again, $\mathrm{ord}(L_{j+1}) = \mathrm{ord}(L_j)$ for all $n - 1 \leq j < N - 1$. Thus $\mathrm{ord}(L_j) = \mathrm{ord}(L_{n-1}) = \mathrm{ord}(L) + n - 1$ for all $n - 1 < j \leq N - 1$.

By Lemma 6.2, we have that p/q is not rationally integrable if $\deg(p) = \deg(q) - 1$. Therefore $\delta(L, d) = 0$ for any $d \in C[x]$ of degree $N-1$. Corollary 3.6 then implies that $\mathrm{ord}(L_N) = \mathrm{ord}(L_{N-1}) + 1 = \mathrm{ord}(L) + n$. For each $j \geq N$, one has that

$$L^*(qx^{j-N+1}) = (j - N + 1)qx^{j-N}.$$

In other words, $L^*(qx^{j-N+1})$ is a polynomial of degree j. By Corollary 3.6, $\mathrm{ord}(L_{j+1}) = \mathrm{ord}(L_j)$ for all $j \geq N$. Consequently, $\mathrm{ord}(L_j) = \mathrm{ord}(L) + n$ for all $j \geq N$. ■

Corollary 6.4. *Let L be of the form (6). Then* $\mathrm{Sind}(L) = \deg(q)$.

REFERENCES

[1] Sergei A. Abramov and Mark Van Hoeij. Integration of solutions of linear functional equations. *Integral Transforms and Special Functions*, 8(1-2):3–12, 1999.

[2] Sergei A. Abramov, Ha Quang Le, and Ziming Li. Univariate Ore polynomial rings in computer algebra. *J. of Mathematical Sci.*, 131(5):5885–5903, 2005.

[3] Sergei A. Abramov and Mark van Hoeij. A method for the integration of solutions of ore equations. In *ISSAC '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, pages 172–175, New York, NY, USA, 1997. ACM.

[4] Shaoshi Chen. Stability problems in symbolic integration. In *ISSAC'22: Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, ISSAC '22, pages 517–524, New York, NY, USA, 2022. ACM.

[5] Nicholas M. Katz. On the calculation of some differential galois groups. *Inventiones mathematicae*, 87:13–61, 1987.

[6] Manuel Kauers. *D-Finite Fuctions.* Springer Nature, August 26, 2022.

[7] E. R. Kolchin. *Differential algebra and algebraic groups.* Academic Press, New York-London, 1973. Pure and Applied Mathematics, Vol. 54.

[8] Leonard M. Lipshitz. D-finite power series. *J. Algebra*, 122(2):353–373, 1989.

[9] Oystein Ore. Theory of non-commutative polynomials. *Ann. of Math. (2)*, 34(3):480–508, 1933.

[10] Bruno Salvy. Linear differential equations as a data structure. *Found. Comput. Math.*, 19(5):1071–1112, 2019.

[11] Richard P. Stanley. Differentiably finite power series. *European J. Combin.*, 1(2):175–188, 1980.

[12] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2003.

[13] Herbert S. Wilf and Doron Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Invent. Math.*, 108(3):575–633, 1992.

[14] Doron Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32:321–368, 1990.