Reducing Hyperexponential Functions over Monomial Extensions*

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Abstract We extend the shell and kernel reductions for hyperexponential functions over the field of rational functions to a monomial extension. Both of the reductions are incorporated into one algorithm. As an application, we present an additive decomposition in rationally hyperexponential towers. The decomposition yields an alternative algorithm for computing elementary integrals over such towers. The alternative can find some elementary integrals that are unevaluated by the integrators in the latest versions of MAPLE and MATHEMATICA.

Keywords Additive decomposition, Hyperexponential function, Reduction, Symbolic integration

1 Introduction

Symbolic integration aims at developing algorithms to compute integrals in closed form. One of its classical topics is to determine whether an integrand has an elementary integral, and compute such an integral if there exists one. Fundamental results on this topic are collected and reviewed in [9]. The monograph [2] presents algorithms for integrating transcendental functions.

In symbolic integration, an integrand $f(x)$ is decomposed in one way or another as

$$f = \frac{dg}{dx} + r, \tag{1}$$

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where g and r are functions of the same "type" as f's, and r is minimal in some technical sense. Such a process is referred to as reduction for f in this paper. A reduction (1) yields an additive decomposition $g' + r$ of f, provided that $r = 0$ if and only if the integral of f is of the same "type" as f's.

Let C be a field of characteristic zero throughout the paper, and $C(x)$ be the field of rational functions in x. The Hermite-Ostrogradsky reduction in [2, §2.2] computes $g, r \in C(x)$ such that (1) holds. Moreover, r is proper with a squarefree denominator, and the integral of f belongs to $C(x)$ if and only if r is equal to zero. In other words, the Hermite-Ostrogradsky reduction computes an additive decomposition of every element in $C(x)$. A nonzero function is hyperexponential if its logarithmic derivative belongs to $C(x)$. Nonzero rational functions are a special instance for hyperexponential functions. The Hermite reduction in [1, §4.2] computes an additive decomposition of a hyperexponential function. Reductions do not always yield additive decompositions. For instance, the algorithm **HermiteReduce** in [2, §5.3] decomposes an integrand as the sum of a derivative, a simple function and a reduced function.

The goal of this paper is to generalize several reductions for hyperexponential functions over $C(x)$ to monomial extensions. We extend and unify the shell and kernel reductions in [1, 7] (see Theorem 3.12), and generalize the Hermite reduction in [1] to an additive decomposition algorithm in rationally hyperexponential towers (see Theorem 4.4). A method is presented in Theorem 4.10 for determining elementary integrals over such towers.

Example 1.1 Let

$$y = \exp\left(\int \frac{1}{x^3 - x - 2}\right) \quad \text{and} \quad f = \frac{(x^3 - x - 3)\exp(x)}{(x^3 - x - 2)(\exp(x) + y)}.$$

Then

$$\int f = \log(\exp(x) + y) - \sum_{\alpha^3 - \alpha - 2 = 0} \frac{1}{3\alpha^2 - 1} \log(x - \alpha),$$

which is elementary over $C(x, \exp(x), y)$. However, neither "int( )" command in Maple nor "Integrate[ ]" command in Mathematica finds this closed form. We shall show how to find it by Theorems 4.4 and 4.10.

The rest of this paper is organized as follows. We describe basic notions in symbolic integration, and review shell, kernel and polynomial reductions for hyperexponential functions over the field of rational functions in Section 2. The shell and kernel reductions are extended and unified in Section 3. As a generalization of the Hermite reduction for hyperexponential functions in [1, §4.2], we present an algorithm for computing additive decompositions of all elements in a rationally hyperexponential tower in Section 4.

2 Preliminaries

This section consists of two parts. We present basic terminologies for symbolic integration in Section 2.1, and then recall reductions that will be generalized in Section 2.2.

2.1 Basic definitions and facts

In the sequel, F stands for a field of characteristic zero. Let f belong to $F(t)$, where t is an indeterminate over F. The numerator and denominator of f are denoted by $\mathrm{num}(f)$ and $\mathrm{den}(f)$, respectively. They are coprime polynomials in $F[t]$. In particular, $\mathrm{num}(0) = 0$ and $\mathrm{den}(0) = 1$. We say that f is *t-proper* if $\deg_t(\mathrm{num}(f)) < \deg_t(\mathrm{den}(f))$. Let p be a polynomial in $F[t]$ with positive degree. The *order* of f at p is defined to be $-m$ if $p^m \mid \mathrm{den}(f)$ but $p^{m+1} \nmid \mathrm{den}(f)$ for some $m \in \mathbb{Z}^+$; while it is defined to be m if $p^m \mid \mathrm{num}(f)$ but $p^{m+1} \nmid \mathrm{num}(f)$ for some $m \in \mathbb{N}$. The order is denoted by $\nu_p(f)$.

A *derivation* δ on a field F is an additive map from F to itself satisfying the usual product rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in F$. The pair (F, δ) is called a *differential field*. An element of F is called a *constant* if its derivative is zero. The set of constants in F is a subfield of F, which is denoted by C_F. An element of F is called a *logarithmic derivative* if it is equal to $\delta(a)/a$ for some $a \in F \setminus \{0\}$. Let (E, Δ) and (F, δ) be two differential fields. We call E a *differential field extension of* F if $F \subset E$ and $\Delta|_F = \delta$. When there is no confusion, we still denote the derivation Δ on E by δ.

Convention. *From now on, we let* $(F, {}')$ *be a differential field*, and $F' := \{f' \mid f \in F\}$.

Given an element f of F, a reduction decomposes f as $g' + r$, where $g, r \in F$ and r is minimal in some technical sense. We call $g' + r$ an *additive decomposition* of f, and r a *remainder* of f, provided that $r = 0$ if and only if $f \in F'$. Remainders are not necessarily unique. Algorithms for computing additive decompositions of all elements in F are particularly useful to determine elementary integrals over F, as indicated in [3, Theorem 6.1] and [6, Theorem 4.10]. The Hermite-Ostrogradsky reduction in [2, §2.2] yields an additive decomposition in $(C(x), d/dx)$. Additive decompositions are computed in a finite extension of $C(x)$ in [4] and in primitive towers of some kinds over $C(x)$ in [3, 6].

Let E be a differential field extension of F. An element t of E is called a *monomial* over F if it is transcendental over F and its derivative belongs to $F[t]$. According to [10, page 7], a monomial t over F is said to be *regular* if $C_F = C_{F(t)}$. A nonzero element of E is said to be *hyperexponential* over F if its logarithmic derivative belongs to F.

Let t be a monomial over F. For $p \in F[t]$ with $p \neq 0$, p is *normal* if $\gcd(p, p') = 1$. It is *special* if $p \mid p'$. Nonzero elements of F are both normal and special, and vice versa. They are said to be trivially normal and special. Basic properties of normal and special polynomials are presented in [2, §3.4].

An element f of $F(t)$ is said to be *normally t-proper* if it is t-proper and every irreducible factor of $\mathrm{den}(f)$ is normal. It is said to be *t-simple* if it is t-proper with a normal denominator, and it is *t-reduced* if $\mathrm{den}(f)$ is special. Clearly, t-simple elements are normally t-proper and nonzero polynomials in $F[t]$ are t-reduced.

Example 2.1 Let t be a regular and hyperexponential monomial over F. Then $p \in F[t]$ is special if and only if $p = at^m$ for some nonzero element $a \in F$ and $m \in \mathbb{N}$ by [2, Theorem 5.1.2]. The t-reduced elements in $F(t)$ are exactly Laurent polynomials in t over F.

Let $f \in F(t)$ and p be a nontrivial normal factor of $\operatorname{den}(f)$. By [2, Theorem 4.4.2 (i)],

$$\nu_p(f') = \nu_p(f) - 1 < \nu_p(f) \le -1.$$

For brevity, the use of this fact will be referred to as *an argument on orders* in the sequel. For instance, $f \notin F(t)'$ for every nonzero and t-simple element $f \in F(t)$ by such an argument.

Let y be hyperexponential over F. Then y can be expressed as $\exp\left(\int f\right)$, where f stands for the logarithmic derivative y'/y. Issues related to integrating y include: (i) determining whether $y \in F(y)'$, (ii) reducing y modulo $F(y)'$, (iii) developing an additive decomposition in $F(y)$, (iv) determining whether y has an elementary integral over $F(y)$.

Under some additional assumptions, we can replace the additive group $F(y)'$ in reduction algorithms with its subgroup $\{(ay)' \mid a \in F\}$, as described in the following proposition.

Proposition 2.2 *Let y be a regular and hyperexponential monomial over F. Then*

$$y \in F(y)' \iff y = (ay)' \quad \textit{for some } a \in F.$$

Proof By definition, $y \in F(y)'$ if and only if there exists $z \in F(y)$ such that $y = z'$. Then z is y-reduced by an argument on orders. So $z \in F[y^{-1}, y]$ by Example 2.1. It follows from the transcendence of y that $z = ay$ for some $a \in F$. □

2.2 Shell and kernel reductions in $C(x)$

The first three issues listed above are well-handled by the algorithms in [1] when $F = C(x)$.

A rational function $\xi \in C(x)$ is *differential reduced* if $\gcd(\operatorname{num}(\xi) - i\operatorname{den}(\xi)', \operatorname{den}(\xi)) = 1$ for all $i \in \mathbb{Z}$. For $f \in C(x)$, there exist $\xi, \eta \in C(x)$ with $\eta \neq 0$ such that (i) $f = \eta'/\eta + \xi$, (ii) ξ is differential reduced, (iii) $\operatorname{num}(\eta)$, $\operatorname{den}(\eta)$ and $\operatorname{den}(\xi)$ are coprime pairwise.

It is shown in [7, §3] that ξ is unique and that η is unique up to a multiplicative constant. They are called the *kernel* and *shell* of f, respectively.

Let y be hyperexponential over $C(x)$, and ξ, η be the kernel and shell of y'/y, respectively. Algorithm **ReduceCert** in [7, §4] decomposes

$$\eta = u' + u\xi + \underbrace{v + \frac{p}{\operatorname{den}(\xi)}}_{h}, \tag{2}$$

where $u, v \in C(x)$, v is x-simple, $\operatorname{den}(v) \mid \operatorname{den}(\eta)$, and $p \in C[x]$, and h is minimal in the sense that $\operatorname{den}(v)$ divides $\operatorname{den}(\widetilde{v})$ if there is another reduction $\eta = \widetilde{u}' + \widetilde{u}\xi + \widetilde{v} + \widetilde{p}/\operatorname{den}(\xi)$ for some $\widetilde{u}, \widetilde{v} \in C(x)$ and $\widetilde{p} \in C[x]$. In terms of hyperexponential functions, (2) can be expressed as

$$y = \eta \exp\left(\int \xi\right) = \left(u \exp\left(\int \xi\right)\right)' + h \exp\left(\int \xi\right),$$

which is a reduction for y. By [7, Theorem 4], y is the derivative of another hyperexponential element if and only if $v = 0$ and $z' + z\xi = p/\operatorname{den}(\xi)$ for some $z \in C[x]$. When this is the case, $y = \left(u\eta^{-1}y + z\eta^{-1}y\right)'$. The equality (2) is called a shell reduction in [1].

To avoid solving the above differential equation for z, we recall the C-linear map

$$\begin{array}{rccc} \phi_\xi: & C[x] & \to & C[x] \\ & a & \mapsto & \operatorname{den}(\xi)a' + \operatorname{num}(\xi)a \end{array}$$

defined in [1]. The map is injective, because ξ is differential reduced. The injectivity allows us to construct a C-basis of $\operatorname{im}(\phi_\xi)$ in a straightforward manner. This construction also yields a finite-dimensional C-linear subspace $\mathcal{N}$ with $C[x] = \operatorname{im}(\phi_\xi) \oplus \mathcal{N}$. Let q be the projection of p in (2) to $\mathcal{N}$ with respect to the above direct sum. Then (2) can be refined into

$$\eta = w' + w\xi + \underbrace{v + \frac{q}{\operatorname{den}(\xi)}}_{r}$$

for some $w \in C(x)$. We call r a *residual form* with respect to ξ. Translating the above equality in terms of hyperexponential functions, we have $y = (w\eta^{-1}y)' + r\eta^{-1}y$. By [1, Lemma 11], y is the derivative of another hyperexponential element over $C(x)$ if and only if $r = 0$. In other words, $(w\eta^{-1}y)' + r\eta^{-1}y$ is an additive decomposition of y. The algorithm for computing this additive decomposition in [1] is called the Hermite reduction for hyperexponential functions, which, together with the algorithm **HermiteReduce** in [2, §5.2], leads to an algorithm for computing additive decompositions in $C(x, y)$ whenever y is a regular monomial over $C(x)$.

Another useful reduction is introduced on the way of computing telescopers for bivariate hyperexponential functions in [1, §6]. Let $\xi \in C(x)$ be differential reduced. By [1, Lemma 16], for every $p \in C[x]$ and $m \geq 1$, we can compute $u \in C(x)$ and $q \in C[x]$ such that

$$\frac{p}{\operatorname{den}(\xi)^m} = u' + u\xi + \frac{q}{\operatorname{den}(\xi)}, \tag{3}$$

which is called a kernel reduction.

3 Generalizing shell and kernel reductions

This section consists of four parts. We generalize the shell reduction (2) and notion of kernels to monomial extensions in Sections 3.1 and 3.2, respectively. The kernel reduction (3) is generalized in Section 3.3. A generalized kernel-shell reduction in Section 3.4 unifies the algorithms in Sections 3.1 and 3.3.

Throughout this section, t stands for a monomial over F. For $f \in F(t)$, we denote by V_f the additive group $\{a' + af \,|\, a \in F(t)\}$. Let y be hyperexponential over $F(t)$ with logarithmic derivative f. Then an element $a' + af$ corresponds to $(ay)'$, because $(ay)' = (a' + af)y$.

3.1 A generalized shell reduction

First, we generalize the shell reduction (2) to $F(t)$ locally.

Lemma 3.1 *Let $f \in F(t)$, $p \in F[t]$ be normal and coprime with* $\operatorname{den}(f)$*, and $m \in \mathbb{Z}^+$. Then, for every $q \in F[t]$, there exist $v, w \in F[t]$ with $\deg_t(v) < \deg_t(p)$ such that*

$$\frac{q}{p^m} \equiv \frac{v}{p} + \frac{w}{\operatorname{den}(f)} \mod V_f.$$

Proof We set $v = 0$ and $w = q\,\mathrm{den}(f)/p^m$ if $p \in F$. Set v to be the remainder of q by p, and w to be the product of $\mathrm{den}(f)$ and the quotient of q by p if $m = 1$.

Assume that $\deg_t(p) > 0$ and $m > 1$. Since p is normal, there exist $u_1, u_2 \in F[t]$

$$u_1 p + u_2 p' = q. \tag{4}$$

With the aid of integration by parts, we deduce that

$$\frac{q}{p^m} = \frac{u_1}{p^{m-1}} + \frac{u_2 p'}{p^m} = \left(\frac{-(m-1)^{-1}u_2}{p^{m-1}}\right)' + \frac{(m-1)^{-1}u_2' + u_1}{p^{m-1}}.$$

Setting $u_3 = -(m-1)^{-1}u_2$ and $u_4 = (m-1)^{-1}u_2' + u_1$, we have

$$\begin{aligned}
\frac{q}{p^m} &= \left(\frac{u_3}{p^{m-1}}\right)' + \frac{u_4}{p^{m-1}} \\
&= \left(\frac{u_3}{p^{m-1}}\right)' + \frac{u_3}{p^{m-1}}f - \frac{u_3}{p^{m-1}}f + \frac{u_4}{p^{m-1}} \\
&\equiv -\frac{u_3}{p^{m-1}}f + \frac{u_4}{p^{m-1}} \mod V_f \\
&\equiv \frac{u_5}{p^{m-1}} + \frac{u_6}{\mathrm{den}(f)} \mod V_f \quad \text{for some } u_5, u_6 \in F[t].
\end{aligned}$$

The last congruence is derived by a partial fraction decomposition for $-u_3 f/p^{m-1}$ and the assumption $\gcd(p, \mathrm{den}(f)) = 1$. Applying the same argument to u_5/p^{m-1} inductively, we get

$$\frac{q}{p^m} \equiv \frac{\widetilde{v}}{p} + \frac{\widetilde{w}}{\mathrm{den}(f)} \mod V_f$$

for some $\widetilde{v}, \widetilde{w} \in F[t]$. Let v be the remainder of $\widetilde{v}$ and p. Then the lemma holds by a similar operation used in the case $m = 1$. □

A special case of the above lemma plays a key role in Section 4.

Corollary 3.2 *With the notation introduced in Lemma 3.1, assume further that*

$$\deg_t(t') \le 1,\ \deg_t(p) > \deg_t(q) \ge 0,\ \text{and } f \in F.$$

Then $q/p^m \equiv v/p \mod V_f$ *for some* $v \in F[t]$ *with* $\deg_t(v) < \deg_t(p)$.

Proof Let us go through the second paragraph of the above proof with the additional assumptions in mind. Since $\deg_t(t') \le 1$, we have $\deg_t(p') \le \deg_t(p)$, which, together with $\deg_t(q) < \deg_t(p)$, implies that u_1 and u_2 in (4) can be chosen to have degrees less than $\deg_t(p)$. Thus, u_3 and u_4 given in the proof of the above lemma satisfy the same degree constraints. Let $\widetilde{u}_5 = u_4 - u_3 f$. Then $\widetilde{u}_5$ is a polynomial of degree less than $\deg_t(p)$, because $f \in F$. From

$$\frac{q}{p^m} \equiv \frac{\widetilde{u}_5}{p^{m-1}} \mod V_f,$$

and a straightforward induction on m, the corollary follows. □

Example 3.3 Let $F = C(x, y)$ with $x' = 1$ and $y' = y$, and $t = \exp\left(x^2/2\right)$. Then t is a regular and hyperexponential monomial over F. Let $f = y/(t-x)$ and $g = x/(t+1)^2$. Note that $t \nmid \operatorname{den}(g)$ and $\gcd(\operatorname{den}(g), \operatorname{den}(f)) = 1$. So Lemma 3.1 is applicable. Then the algorithm implicitly described in the proof of Lemma 3.1 yields.

$$g = \left(\frac{1}{t+1}\right)' + \frac{1}{t+1} f + \frac{x^2+x+y}{(x+1)(t+1)} - \frac{y}{(x+1)(t-x)} \equiv \frac{v}{t+1} + \frac{w}{t-x} \mod V_f,$$

where $v = (x^2+x+y)/(x+1)$ and $w = -y/(x+1)$.

The next lemma helps us prove certain minimality and uniqueness.

Lemma 3.4 *Let $f, h \in F(t)$, h be t-simple, and $\gcd(\operatorname{den}(f), \operatorname{den}(h)) = 1$. If there exists a t-reduced element $r \in F(t)$ such that $h + r/\operatorname{den}(f) \in V_f$, then $h = 0$.*

Proof. Since $h + r/\operatorname{den}(f) \in V_f$, there exists $a \in F(t)$ such that

$$h + \frac{r}{\operatorname{den}(f)} = a' + af. \tag{5}$$

Suppose that h is nonzero. Then $\nu_p(h) = -1$ for some nontrivial normal polynomial p, because h is t-simple. It follows from $\gcd(\operatorname{den}(h), \operatorname{den}(f)) = 1$ and (5) that $\nu_p(a) < 0$. By an argument on orders, the left and right-hand sides of (5) have distinct orders at p, a contradiction. □

A generalized shell reduction in $F(t)$ is described in the following proposition.

Proposition 3.5 *Let $f, g \in F(t)$. If $\operatorname{den}(g)$ is free of any nontrivial special factor and coprime with $\operatorname{den}(f)$, then there exists a unique t-simple $h \in F(t)$ and $q \in F[t]$ such that*

$$g \equiv h + \frac{q}{\operatorname{den}(f)} \mod V_f \quad \textit{and} \quad \operatorname{den}(h) \mid \operatorname{den}(g).$$

Proof Applying Lemma 3.1 to each t-proper component in the squarefree partial fraction decomposition of g, we see that there exists a t-simple $h \in F(t)$ and a polynomial $q \in F[t]$ such that $g \equiv h + q/\operatorname{den}(f) \mod V_f$, and that $\operatorname{den}(h)$ divides $\operatorname{den}(g)$. The uniqueness of h is immediate from Lemma 3.4. □

Corollary 3.6 *With the notation introduced in Proposition 3.5, assume further that $\deg_t(t') \leq 1$, $f \in F$ and that g is normally t-proper. Then there exists a unique t-simple element $h \in F[t]$ such that $g \equiv h \mod V_f$.*

Proof Since $f \in F$, we have $\gcd(\operatorname{den}(g), \operatorname{den}(f)) = 1$. Since $\operatorname{den}(g)$ has no nontrivial special factor, the above proposition is applicable. Each component in the squarefree partial fraction decomposition of g is of the form a/p^m for some $a, p \in F[t]$ with $p \mid \operatorname{den}(g)$ and $\deg(a) < \deg(p)$, because g is t-proper. The corollary follows from Corollary 3.2 and the above proposition. □

For a convenience of later references, we specify the input and output of the generalized shell reduction. Its pseudo-code can be easily written down according to the proofs of Lemma 3.1 and Proposition 3.5.

Algorithm GSR (Generalized Shell Reduction)

Input: a monomial extension $F(t)$ and $f, g \in F(t)$, where $\operatorname{den}(g)$ has no nontrivial special factor and is coprime with $\operatorname{den}(f)$

Output: (a, h, q), where $a, h \in F(t)$ and $q \in F[t]$ such that $g = a' + af + h + q/\operatorname{den}(f)$, and h is t-simple with $\operatorname{den}(h) | \operatorname{den}(g)$

We can set q in the output of Algorithm **GSR** to be zero if $\deg_t(t') \leq 1$, $f \in F$ and g is normally t-proper by Corollary 3.6.

3.2 Weak normalization

In this subsection, we adapt some results scattered in [2, §6.1] to our later use. An element f of $F(t)$ is said to be *weakly normalized* (resp. normalized) if $\gcd(\operatorname{num}(f) - i\operatorname{den}(f)', \operatorname{den}(f)) = 1$ for all $i \in \mathbb{Z}^+$ (resp. $i \in \mathbb{Z}$). Note that f is normalized if and only if f is differential reduced when $F = C$ and $t = x$. We prefer the word "normalized" rather than the phrase "differential reduced", because the former is concise and compatible with the phrase "weakly normalized" coined in [2, Definition 6.1.1].

The next lemma enables us to generalize the notion of kernels and that of shells from rational functions to elements in a monomial extension.

Lemma 3.7 *Let $f \in F(t)$ and $\lambda \in F$. Then $\operatorname{num}(f) - \lambda\operatorname{den}(f)'$ and $\operatorname{den}(f)$ are not coprime if and only if there exists a nontrivially normal and irreducible polynomial p such that*

$$\nu_p(f) = -1 \quad \textit{and} \quad \nu_p(f - \lambda p'/p) \geq 0. \tag{6}$$

Proof Let p be a factor of $\operatorname{den}(f)$. Then $\operatorname{den}(f) = pq$ for some $q \in F[t]$, and

$$f - \lambda\frac{p'}{p} = \frac{\operatorname{num}(f) - \lambda p'q}{\operatorname{den}(f)}. \tag{7}$$

Assume that p is nontrivially normal and irreducible, and that it satisfies the constraints in (6). By (6) and (7), p divides $\operatorname{num}(f) - \lambda p'q$, which, together with $\nu_p(f) = -1$, implies that p is a common factor of $\operatorname{num}(f) - \lambda\operatorname{den}(f)'$ and $\operatorname{den}(f)$. Conversely, let p be a nontrivial irreducible factor of $\gcd(\operatorname{num}(f) - \lambda\operatorname{den}(f)', \operatorname{den}(f))$. Since $\operatorname{num}(f) - \lambda\operatorname{den}(f)' = \operatorname{num}(f) - \lambda(p'q + pq')$, we see that $\operatorname{num}(f) - \lambda p'q$ is divisible by p, which, together with $p \nmid \operatorname{num}(f)$, implies that $p \nmid p'q$. Thus $p \nmid p'$ and $p \nmid q$, which imply that p is normal and $\nu_p(f) = -1$, respectively. The inequality $\nu_p(f - \lambda p'/p) \geq 0$ in (6) holds owing to (7), $\nu_p(f) = -1$ and $p \mid (\operatorname{num}(f) - \lambda p'q)$. □

We present an algorithm to construct $\xi, \eta \in F(t)$ with $\eta \neq 0$ for a given $f \in F(t)$ such that (i) $f = \eta'/\eta + \xi$, (ii) ξ is weakly normalized (resp. normalized), (iii) $\operatorname{den}(\eta)$ is free of any nontrivial special factor, (iv) $\gcd(\operatorname{den}(\eta), \operatorname{den}(\xi)) = 1$ and $\gcd(\operatorname{num}(\eta), \operatorname{den}(\xi)) = 1$.

Algorithm GKS (Generalized Kernel and Shell)

Input: a monomial extension $F(t)$ and $f \in F(t)$

Output: $\xi, \eta \in F(t)$ satisfy the four requirements listed above

(1) **if** $f \in F$ **then return** $f, 1$

(2) $\xi \leftarrow f$ and $\eta \leftarrow 1$

(3) $g \leftarrow$ the product of the normal and irreducible factors of $\operatorname{den}(\xi)$ with multiplicity 1

(4) factor g over F to get its nontrivial irreducible factors: $g_1, \ldots, g_k$

(5) **for** i **from** 1 **to** k **do**

  (5.1) $p \leftarrow \text{num}\left(\xi - zg_i'/g_i\right)$, where z is a constant indeterminate

  (5.2) $r \leftarrow$ the remainder of p by g_i

  (5.3) set $r = 0$ to obtain a system of linear equations in z over F.

  (5.4) **if** the system has a solution $m \in \mathbb{Z}^+$ (resp. $m \in \mathbb{Z}$) **then**

    $\eta \leftarrow \eta g_i^m$ and $\xi \leftarrow \xi - mg_i'/g_i$

  end if

end do

(6) **return** ξ, η

Note that g in step (3) of Algorithm **GKS** can be found as follows. Compute

$$w = \frac{\text{den}(f)}{\gcd(\text{den}(f), \text{den}(f)')},$$

which is the product of all normal and irreducible factors of $\text{den}(f)$ by [2, Lemma 3.4.4]. Then g is equal to $w/\gcd(w, \text{den}(f)')$ by a straightforward calculation. The correctness of the algorithm then follows from Lemma 3.7. We call ξ and η computed by Algorithm **GKS**$(F(t), f)$ the *weakly normalized (resp. normalized) kernel and the corresponding shell in* $F(t)$, respectively. Searching for $m \in \mathbb{Z}^+$ in step (5.3) finds a weakly normalized kernel; while looking for $m \in \mathbb{Z}$, we get a normalized one.

Example 3.8 Let $F = C(x)$ and t be a regular and hyperexponential monomial over F with $t'/t = 1/(x^2+1)$. Applying Algorithm **GKS** to

$$f = \frac{x^3t + x^2t + 2xt + t + 1}{(xt+1)(x^2+1)}$$

yields the weakly normalized kernel $1/(x^2+1)$ and shell $xt+1$.

3.3 A generalized kernel reduction

We extend the kernel reduction (3) to $F(t)$.

Proposition 3.9 *Let $f \in F(t)$ be weakly normalized. Then, for every $p \in F[t]$ and a positive integer m, there exists $q \in F[t]$ such that $p/\text{den}(f)^m \equiv q/\text{den}(f) \mod V_f$.*

Proof If $m = 1$ or $p = 0$, then set $q = p$. Otherwise, there exist $u, v \in F[t]$ such that

$$u\left(\text{num}(f) - (m-1)\,\text{den}(f)'\right) + v\,\text{den}(f) = p$$

since f is weakly normalized. Using integration by parts, we have

$$\frac{p}{\text{den}(f)^m} = \left(\frac{u}{\text{den}(f)^{m-1}}\right)' + \left(\frac{u}{\text{den}(f)^{m-1}}\right) f + \frac{v-u'}{\text{den}(f)^{m-1}} \equiv \frac{v-u'}{\text{den}(f)^{m-1}} \mod V_f.$$

The proposition then follows from a straightforward induction on m. □

The algorithm described in the above proof is specified below.

Algorithm GKR (Generalized Kernel Reduction)

Input: a monomial extension $F(t)$, a weakly normalized element $f \in F(t)$, a polynomial $p \in F[t]$ and a positive integer m

Output: $a \in F(t)$ and $q \in F[t]$ such that $p/\text{den}(f)^m = a' + af + q/\text{den}(f)$

Example 3.10 Let $F(t)$ and f be given in Example 3.3 and $g = (y+1-xt)/(t-x)^2$. Since f is weakly normalized, Algorithm **GKR** yields $g = (1/(t-x))' + f/(t-x) \equiv 0 \mod V_f$.

Example 3.11 Let $F = C(x, y)$ with $x' = 1$ and $y' = xy$, and let $t = \exp(y)$. Then t is a regular and hyperexponential monomial over F. Let

$$f = \frac{1}{t+y} \quad \text{and} \quad g = \frac{(y+1-x^2y)t - x^2y + y^2 + x + y}{(t+y)^2}.$$

Note that f is weakly normalized and $\operatorname{den}(g) = \operatorname{den}(f)^2$. Algorithm **GKR** yields

$$g = \left(\frac{x}{t+y}\right)' + \left(\frac{x}{t+y}\right) f + \frac{y}{t+y} \equiv \frac{y}{t+y} \mod V_f.$$

3.4 A generalized kernel-shell reduction

We are ready to extend and unify shell and kernel reductions.

Theorem 3.12 *Let $f, g \in F(t)$ and f be weakly normalized. Then the following assertions hold.*

(i) There exists a unique t-simple element h with $\operatorname{den}(h) \mid \operatorname{den}(g)$ and $\gcd(\operatorname{den}(h), \operatorname{den}(f)) = 1$, and a t-reduced element r such that

$$g \equiv h + \frac{r}{\operatorname{den}(f)} \mod V_f. \tag{8}$$

(ii) If $g \equiv \widetilde{h} + \widetilde{r}/\operatorname{den}(f) \mod V_f$, where $\widetilde{h} \in F(t)$ and $\widetilde{r}$ is t-reduced, then $\operatorname{den}(h) \mid \operatorname{den}(\widetilde{h})$.

(iii) $g \in V_f$ if and only if $h = 0$ and there exists a t-reduced element $a \in F(t)$ such that

$$\frac{r}{\operatorname{den}(f)} = a' + fa.$$

(iv) Assume further that $\deg_t(t') \le 1$, $f \in F$, and that g is normally t-proper. Then (8) can be rewritten as $g \equiv h \mod V_f$. Moreover, $g \in V_f$ if and only if $h = 0$.

Proof. By a partial fraction decomposition for g, we have

$$g = g_1 + g_2 + g_3, \tag{9}$$

where $g_1, g_2, g_3 \in F(t)$, g_1 and g_2 are t-proper, all the irreducible factors of $\operatorname{den}(g_1)$ are normal and coprime with $\operatorname{den}(f)$, those of $\operatorname{den}(g_2)$ are factors of $\operatorname{den}(f)$, and those of $\operatorname{den}(g_3)$ are special and coprime with $\operatorname{den}(f)$.

(i) By Proposition 3.5, there exists a t-simple element h with $\operatorname{den}(h) | \operatorname{den}(g_1)$, and $q_1 \in F[t]$ such that $g_1 \equiv h + q_1/\operatorname{den}(f) \mod V_f$. Note that g_2 can be written as $p/\operatorname{den}(f)^m$ for some $p \in F[t]$ and $m \in \mathbb{Z}^+$. By Proposition 3.9, there exists $q_2 \in F[t]$ such that $g_2 \equiv q_2/\operatorname{den}(f) \mod V_f$. Since g_3 is t-reduced, the above two congruences and (9) lead to $g \equiv h + r/\operatorname{den}(f) \mod V_f$, where $r = q_1 + q_2 + g_3 \operatorname{den}(f)$. The uniqueness of h is evident by Lemma 3.4.

(ii) By (i), $\widetilde{h} \equiv h^* + r^*/\operatorname{den}(f) \mod V_f$ with $\operatorname{den}(h^*) | \operatorname{den}(\widetilde{h})$ and $\gcd(\operatorname{den}(h^*), \operatorname{den}(f)) = 1$ for some t-simple element h^* and t-reduced element r^*. Therefore, $g \equiv h^* + (\widetilde{r} + r^*)/\operatorname{den}(f) \mod V_f$. Since $\widetilde{r} + r^*$ is t-reduced, we have $h = h^*$ by (i). Consequently, $\operatorname{den}(h) \mid \operatorname{den}(\widetilde{h})$.

(iii) By (8), $g \in V_f$ if $h = 0$ and $r/\operatorname{den}(f) \in V_f$. Conversely, assume that $g \in V_f$. Then $h = 0$ by (ii). It follows from (8) that there exists $a \in F(t)$ such that

$$r = a' \operatorname{den}(f) + a \operatorname{num}(f). \tag{10}$$

It remains to show that a is t-reduced. Suppose that p is a nontrivial irreducible and normal polynomial with $m := \nu_p(a) < 0$. Then $\nu_p(a') = m - 1$. It follows from (10), $\nu_p(r) \geq 0$ and an argument on orders that neither $\nu_p(f) \geq 0$ nor $\nu_p(f) < -1$. So $\nu_p(f) = -1$. Consequently, $\nu_p(a\,\mathrm{den}(f)) \leq 0$ so that the order of $r/(a\,\mathrm{den}(f))$ at p is nonnegative. Hence, (10) implies $\nu_p(f + a'/a) \geq 0$. By the logarithmic derivative identity, there exist $n_i \in \mathbb{Z}$ and $q_i \in F[t]$ with $\gcd(p, q_i) = 1$ such that

$$f + \frac{a'}{a} = f + m\frac{p'}{p} + \sum_i n_i \frac{q_i'}{q_i},$$

which, together with $\nu_p\left(f + a'/a\right) \geq 0$ and $\nu_p\left(\sum_i n_i q_i'/q_i\right) \geq 0$, implies that $\nu_p\left(f + mp'/p\right) \geq 0$. Then $\mathrm{num}(f) + m\,\mathrm{den}(f)'$ and $\mathrm{den}(f)$ are not coprime by Lemma 3.7. Since m is a negative integer, f is not weakly normalized, a contradiction.

(iv) If $\deg_t(t') \leq 1$, $f \in F$ and g is normally t-proper, then both g_2 and g_3 in (9) are equal to zero. By Corollary 3.6, $g \equiv h \mod V_f$ for some t-simple $h \in F(t)$. The other conclusion holds by (ii). □

Based on Theorem 3.12 and its proof, we present a generalized kernel-shell reduction.

Algorithm GKSR (Generalized Kernel-Shell Reduction)

Input: a monomial extension $F(t)$, a weakly normalized element $f \in F(t)$ and $g \in F(t)$

Output: $a, h, r \in F(t)$ with h being t-simple and r being t-reduced such that

$$\mathrm{den}(h) \mid \mathrm{den}(g), \quad \gcd(\mathrm{den}(h), \mathrm{den}(f)) = 1 \quad \text{and} \quad g = a' + af + h + \frac{r}{\mathrm{den}(f)}$$

(1) use a partial fraction decomposition to compute $g_1, g_2, g_3 \in F(t)$ such that

$$g = g_1 + g_2 + g_3,$$

where g_1 is normally t-proper with $\gcd(\mathrm{den}(g_1), \mathrm{den}(f))=1$, g_2 is t-proper, every irreducible factor of $\mathrm{den}(g_2)$ divides $\mathrm{den}(f)$, and g_3 is t-reduced

(2) $(a_1, h, r_1) \leftarrow$ Algorithm $\mathbf{GSR}(F(t), f, g_1)$

(3) find $p \in F[t]$ and the minimal $m \in \mathbb{Z}^+$ such that $g_2 = p/\mathrm{den}(f)^m$,
and $(a_2, q) \leftarrow$ Algorithm $\mathbf{GKR}(F(t), f, p, m)$

(4) $(a, r) \leftarrow (a_1 + a_2,\ r_1 + q + g_3\,\mathrm{den}(f))$

(5) **return** a, h, r

The correctness of Algorithm **GKSR** is immediate from Propositions 3.5, 3.9 and Theorem 3.12 (i). The t-reduced element r is zero if $\deg_t(t') \leq 1$, $f \in F$ and g is normally t-proper by Theorem 3.12 (iv).

Let y be hyperexponential over $F(t)$. By Algorithm **GKS**, we compute the weakly normalized kernel ξ and the corresponding shell η of y'/y in $F(t)$. Set $z = y/\eta$. Then $z'/z = \xi$. Let us reduce gz for an element $g \in F(t)$. By Algorithm **GKSR**, we compute an element $u \in F(t)$, a t-simple element h and a t-reduced element r such that $g = u' + u\xi + h + r/\mathrm{den}(\xi)$. It follows that

$$gz = (uz)' + \left(h + \frac{r}{\mathrm{den}(\xi)}\right) z.$$

This is a reduction for gz for all $g \in F(t)$. In particular, setting $g = \eta$ yields a reduction for y.

Corollary 3.13 *Let $\deg_t(t') \leq 1$, y be a regular and hyperexponential monomial over $F(t)$ with $y'/y \in F$, and $g \in F(t)$ be nonzero and normally t-proper. Then there exists a unique t-simple element $h \in F(t)$ such that $gy = (uy)' + hy$, which is an additive decomposition of gy.*

Proof Since $y'/y \in F$, its kernel and shell in $F(t)$ are y'/y and 1, respectively. There exists a unique t-simple element $h \in F(t)$ such that $gy = (uy)' + hy$ by Theorem 3.12 (iv). Assume $gy \in F(t, y)'$. By Proposition 2.2, there exists $v \in F$ such that $gy = (vy)'$. Consequently, $g \in V_f$, which, together with Theorem 3.12 (iv), implies that $h = 0$. □

Example 3.14 Let $F(t)$ and f be given by Example 3.3. Consider

$$g = \frac{-xt^3 + (y - x + 1)t^2 + (2y - 2x^2 - x + 2)t + x^3 + y + 1}{(1 + t)^2(t - x)^2} \in F(t)$$

Since f is weakly normalized, its kernel is f and the shell is 1. First, we decompose g as

$$g = \underbrace{\frac{x}{(1+t)^2}}_{g_1} + \underbrace{\frac{y + 1 - xt}{(t-x)^2}}_{g_2},$$

where g_1 is normally t-proper with $\gcd(\mathrm{den}(g_1), \mathrm{den}(f)) = 1$, the irreducible factor $t - x$ of $\mathrm{den}(g_2)$ divides $\mathrm{den}(f)$. By Algorithm **GKSR**, Examples 3.3 and 3.10,

$$\begin{aligned} g &= \left(\frac{1}{t-x} + \frac{1}{1+t}\right)' + \left(\frac{1}{t-x} + \frac{1}{1+t}\right) f + \frac{x^2 + x + y}{(x+1)(1+t)} + \frac{-y}{(x+1)(t-x)} \\ &\equiv \underbrace{\frac{x^2 + x + y}{(x+1)(1+t)}}_{h} + \frac{-y}{(x+1)(t-x)} \quad \mathrm{mod}\ V_f. \end{aligned}$$

In other words,

$$\underbrace{g \exp\left(\int f\right)}_{z} = \left(\left(\frac{1}{t-x} + \frac{1}{1+t}\right) z\right)' + \left(h + \frac{-y}{(x+1)(t-x)}\right) z.$$

Since $h \neq 0$, we have that $gz \notin F(t, z)'$ by Theorem 3.12 (ii).

4 An additive decomposition in rationally hyperexponential towers

This section has four parts. In Section 4.1, we present a variant of the Matryoshka decomposition in [6]. An algorithm is developed for computing additive decompositions in rationally hyperexponential towers in Section 4.2. We describe the projections of logarithmic derivatives in terms of residues, and present a criterion on elementary integrability over such towers in Sections 4.3 and 4.4, respectively.

4.1 Laurent-Matryoshka decompositions

For $n \in \mathbb{Z}^+$, we denote $\{1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, n\}$ by $[n]$ and $[n]_0$, respectively. Let F_0 be a field. For every $i \in [n]$, we further let $F_i = F_{i-1}(t_i)$, where t_i is transcendental over F_{i-1}. Then there is a chain of field extensions:

$$\begin{array}{ccccccc} F_0 & \subset & F_1 & \subset & \cdots & \subset & F_n \\ & & \shortparallel & & & & \shortparallel \\ & & F_0(t_1) & \subset & \cdots & \subset & F_{n-1}(t_n). \end{array} \tag{11}$$

For each $i \in [n]$, $f \in F_n$ is said to be t_i*-proper* if $f \in F_i$ and $\deg_{t_i}(\mathrm{num}(f)) < \deg_{t_i}(\mathrm{den}(f))$. By a power product of $t_1, \ldots, t_n$, we mean the product $t_1^{\ell_1} \cdots t_n^{\ell_n}$, where the ℓ_i's are integers. For all $i \in [n-1]_0$, we denote by $\mathbb{T}_i$ the set of power products of $t_{i+1}, \ldots, t_n$, and set $\mathbb{T}_n = \{1\}$.

In the rest of this paper, we let F_0 be a differential field. Assume that each generator t_i is regular and hyperexponential over F_{i-1}. We call (11) a *hyperexponential tower*. For $i \in [n]$, an element f of F_n is t_i*-simple* if it is t_i-proper and $\operatorname{den}(f)$ is normal as an element of $F_{i-1}[t_i]$, and it is t_i*-reduced* if it belongs to $F_{i-1}[t_i^{-1}, t_i]$ (see Example 2.1). Similarly, f is normally t_i-proper if f is t_i-proper and t_i does not divide $\operatorname{den}(f)$ in $F_{i-1}[t_i]$. Zero is normally t_i-proper for all $i \in [n]$. An element of F_i can be written uniquely as the sum of a normally t_i-proper element and an element of $F_{i-1}[t_i^{-1}, t_i]$.

Let L_n be the additive subgroup consisting of all normally t_n-proper elements in F_n. For $i \in [n-1]$, let L_i be the additive group generated by elements of the form aT, where $a \in F_i$ is normally t_i-proper and $T \in \mathbb{T}_i$. Moreover, let L_0 be the ring of Laurent polynomials in $t_1, \ldots, t_n$ over F_0. Then $F_n = L_0 \oplus L_1 \oplus \cdots \oplus L_n$ by a straightforward verification. Let π_i be the projection from F_n onto L_i with respect to the above direct sum for all $i \in [n]_0$. For $f \in F_n$, $f = \pi_0(f) + \pi_1(f) + \cdots + \pi_n(f)$ is called the *Laurent-Matryoshka decomposition* of f.

Example 4.1 Let $F_0 = \mathbb{Q}(x)$. A Laurent-Matryoshka decomposition in F_3 is

$$\underbrace{\frac{t_2 t_3 (x - t_3)}{t_1(t_2+1)(t_3-1)}}_{f} = \underbrace{-t_3 t_1^{-1} + (x-1) t_1^{-1}}_{\pi_0(f)} + \underbrace{0}_{\pi_1(f)} + \underbrace{\frac{t_3}{t_1(t_2+1)} - \frac{x-1}{t_1(t_2+1)}}_{\pi_2(f)} + \underbrace{\frac{(x-1)t_2}{t_1(t_2+1)(t_3-1)}}_{\pi_3(f)}.$$

4.2 Rationally hyperexponential towers

Rationally hyperexponential towers are hyperexponential towers of a special type. They allow us to apply the Hermite reduction in [1] and additive decomposition in Corollary 3.13 directly.

Definition 4.2 The tower F_n in (11) is said to be *rationally hyperexponential* if $t_i'/t_i \in F_0$ for every $i \in [n]$, $(F_0, {}') = (C(x), d/dx)$ and $C_{F_n} = C$.

Lemma 4.3 *Let the tower F_n in (11) be rationally hyperexponential and g belong to F_n. If*

$$g = \sum_{i \in [n]_0} \underbrace{\sum_{T \in \mathbb{T}_i} g_T T}_{\pi_i(g)}, \tag{12}$$

where the coefficient g_T in $\pi_0(g)$ belongs to F_0 and g_T in $\pi_i(g)$ with $i \in [n]$ is normally t_i-proper, then

$$g' = \sum_{i \in [n]_0} \underbrace{\sum_{T \in \mathbb{T}_i} \left(g_T' + \frac{T'}{T} g_T \right) T}_{\pi_i(g')}. \tag{13}$$

Consequently, $\pi_i(g)' = \pi_i(g')$ for all $i \in [n]_0$.

Proof Since $T'/T \in F_0$, we have that $g_T' + (T'/T)\, g_T$ belongs to F_0 if g_T is a coefficient in $\pi_0(g)$, and it is normally t_i-proper if g_T is a coefficient in $\pi_i(g)$ for $i \in [n]$. Therefore, the lemma follows from the identity that $(g_T T)' = (g_T' + (T'/T)\, g_T)\, T$. □

We need some notation to describe remainders in a rationally hyperexponential tower F_n in (11). For $i \in [n]$, set R_i to be the additive group generated by $\{hT \mid h \in F_i \text{ is } t_i\text{-simple and } T \in \mathbb{T}_i\}$. For $T \in \mathbb{T}_0$, we let ξ_T be the normalized kernel and η_T the corresponding shell of T'/T in F_0. Set R_0 to be the additive group generated by

$$\left\{ r\left(\eta_T^{-1} T\right) \mid T \in \mathbb{T}_0 \setminus \{1\} \text{ and } r \text{ is a residual form w.r.t. } \xi_T \right\} \cup \{ s \mid s \in F_0 \text{ is } x\text{-simple} \}.$$

Finally, we let $R = \sum_{i \in [n]_0} R_i$, which is a direct sum by the observation that $R_i \subset L_i$ for all $i \in [n]_0$.

Theorem 4.4 *With the notation just introduced, we have* $F_n = F_n' \oplus R$.

Proof First, we show that $F_n = F_n' + R$. It suffices to show that $fT \in F_n' + R_i$ for every normally t_i-proper element f, $T \in \mathbb{T}_i$ and $i \in [n]$, and that $fT \in F_n' + R_0$ for all $f \in F_0$ and $T \in \mathbb{T}_0$.

Let $i \in [n]$. By Corollary 3.13, there exists a t_i-simple h such that $fT \equiv hT \mod F_n'$, where T is regarded as a hyperexponential element over F_i. Thus, $fT \in F_n' + R_i$ by the definition of R_i.

We regard each $T \in \mathbb{T}_0 \setminus \{1\}$ as a hyperexponential element over F_0. Let ξ_T and η_T be the normalized kernel and shell of T'/T in F_0, respectively. For $f \in F_0$, a partial fraction decomposition for $f\eta_T$ yields $f\eta_T = a + b$ with $\gcd(\mathrm{den}(a), \mathrm{den}(\xi_T)) = 1$ and $\mathrm{den}(b) \mid \mathrm{den}(\xi_T)^m$ for some $m \in \mathbb{N}$. Then there exists an x-simple element $h \in C(x)$ and $u, v \in C[x]$ such that

$$a\left(\eta_T^{-1}T\right) \equiv \left(h + \frac{u}{\mathrm{den}(\xi_T)}\right)\left(\eta_T^{-1}T\right) \mod F_n' \quad \text{and} \quad b\left(\eta_T^{-1}T\right) \equiv \frac{v}{\mathrm{den}(\xi_T)}\left(\eta_T^{-1}T\right) \mod F_n'$$

by the shell and kernel reductions in [1], respectively. It follows that

$$fT \equiv \left(h + \frac{u+v}{\mathrm{den}(\xi_T)}\right)\left(\eta_T^{-1}T\right) \mod F_n'.$$

The polynomial reduction in [1] finds a polynomial $p \in C[x]$ such that

$$fT \equiv \underbrace{\left(h + \frac{p}{\mathrm{den}(\xi_T)}\right)\left(\eta_T^{-1}T\right)}_{r} \mod F_n',$$

where r is a residual form with respect to ξ_T. So $fT \in F_n' + R_0$ for all $f \in F_0$ and $T \in \mathbb{T}_0 \setminus \{1\}$. In addition, there exists an x-simple element $s \in F_0$ such that $f \equiv s \mod F_n'$ by the Hermite-Ostrogradsky reduction. Therefore, $fT \in F_n' + R_0$ for all $f \in F_0$ and $T \in \mathbb{T}_0$. Consequently, $F_n = F_n' + R$.

It remains to show that $F_n' \cap R = \{0\}$. For $f \in F_n' \cap R$, there exists $g \in F_n$ such that $f = g'$. Let the Laurent-Matryoshka decomposition of g be given in (12). Then the Laurent-Matryoshka decomposition of f is given in (13) by Lemma 4.3. On the other hand, $f \in R$ implies that for all $i \in [n]$,

$$\pi_i(f) = \sum_{T \in \mathbb{T}_i} r_T T,$$

where r_T is t_i-simple. It follows that $r_T = g_T' + (T'/T)g_T$ for all $T \in \mathbb{T}_i$ and $i \in [n]$. Consequently, $g_T = r_T = 0$ by an argument on orders, that is, $g \in L_0$ and $f \in R_0$ with $f = g'$. For $T \in \mathbb{T}_0 \setminus \{1\}$,

$$\sum_{T \in \mathbb{T}_0} r_T\left(\eta_T^{-1}T\right) = \sum_{T \in \mathbb{T}_0}\left(g_T' + \frac{T'}{T}g_T\right)T,$$

where ξ_T and η_T are the same as above, r_T is a residual form with respect to ξ_T, and $g_T \in F_0$. So $r_T\left(\eta_T^{-1}T\right) = (g_T T)'$. By [1, Lemma 11], $r_T = g_T = 0$. Accordingly, $f, g \in F_0$ and $f = g'$. We have that f is x-simple by $f \in R_0$. Hence, $f = 0$, that is, $F_n' + R$ is a direct sum. □

By the above theorem, for every element $f \in F_n$, there exists $g \in F_n$ and a unique $r \in R$ such that $f = g' + r$. In other words, $g' + r$ is an additive decomposition of f. Moreover, the remainder r is unique due to the direct sum $F_n' \oplus R$.

Algorithm AD_RHT (Additive Decomposition in a Rationally Hyperexponential Tower)

Input: a rationally hyperexponential tower $F_n = C(x)(t_1, t_2, \ldots, t_n)$ and $f \in F_n$ with $f \neq 0$

Output: $g, r \in F_n$ such that $g' + r$ is an additive decomposition of f

(1) $(g, r, 0) \leftarrow$ Algorithm **GKSR**$(F_n, 0, \pi_n(f))$

(2) **for** i **from** 1 **to** $n-1$ **do**

(2.1) write $\pi_i(f) = \sum_{j \in J} a_j T_j$, where $a_j \in F_i \setminus \{0\}$ and $T_j \in \mathbb{T}_i$

(2.2) **for** each $j \in J$ **do**

$(u_j, v_j, 0) \leftarrow$ Algorithm **GKSR**$(F_i, T_j'/T_j, a_j)$ and $(g, r) \leftarrow (g + u_j T_j, r + v_j T_j)$

end do

end do

(3) write $\pi_0(f) = s + \sum_{j \in J} a_j T_j$, where $s, a_j \in C(x)$ with $a_j \neq 0$, and $T_j \in \mathbb{T}_0 \setminus \{1\}$

(4) find $u, v \in C(x)$ such that $s = u' + v$ with v being x-simple by the Hermite-Ostrogradsky reduction, and $(g, r) \leftarrow (g + u, r + v)$

(5) **for** each $j \in J$ **do**

compute $g_j, r_j \in F_n$ such that $a_j T = g_j' + r_j$ by the Hermite reduction in [1]
$(g, r) \leftarrow (g + g_j, r + r_j)$

end do

(6) **return** g, r

The correctness of Algorithm **AD_RHT** is immediate from Corollary 3.13 and the paragraphes for establishing $F_n = F_n' + R$ in the proof of Theorem 4.4.

Example 4.5 Find an additive decomposition of

$$f = -\frac{\exp(x)(x-1)}{\exp(x^2/2)} + \frac{\exp(-1/x)}{(1+\exp(x^2/2))^2} + \frac{x}{(\exp(-1/x)+x)^2}$$

in the rationally hyperexponential tower

$$F_3 = C(x)\big(\underbrace{\exp(x)}_{t_1}, \underbrace{\exp\left(x^2/2\right)}_{t_2}, \underbrace{\exp(-1/x)}_{t_3}\big).$$

In the tower F_3, $f = -(x-1)t_1 t_2^{-1} + t_3/(1+t_2)^2 + x/(t_3+x)^2$. Algorithm **AD_RHT** yields

$$f = \left(-\frac{x^2}{(x-1)(t_3+x)} + \frac{t_3}{x(1+t_2)} + t_1 t_2^{-1}\right)' + \underbrace{\frac{(x^3+x-1)t_3}{x^3(1+t_2)} + \frac{x^2-3x+1}{(x-1)^2(t_3+x)}}_{r}.$$

We conclude $f \notin F_3'$ by $r \neq 0$.

Example 4.6 Let $t_1 = \exp(x)$, $t_2 = y$, where y is given in Example 1.1, and let f be the same as that in Example 1.1. By Algorithm **AD_RHT**,

$$f = \left(\frac{1}{t_2+1}\right)' + \underbrace{\frac{(x^3-x-3)t_1}{(x^3-x-2)(t_1+t_2)}}_{r}. \tag{14}$$

So $f \notin F_2'$, because $r \neq 0$.

4.3 Logarithmic derivatives in rationally hyperexponential towers

The notion and basic properties of residues are described in [2, §4.4]. Let f be a nonzero element of $F(t)$, and $\operatorname{den}(f)$ be nontrivially normal. Then the nonzero residues of f are exactly the roots of its Rothstein-Trager resultant by [2, Theorem 4.4.3]. Residues are closely related to elementary integrals according to [8, Theorem 3.1].

Example 4.7 Let $p \in F[t]$ be a normal polynomial of positive degree d. Then the Rothstein-Trager resultant of p'/p is equal to $(-1)^d \operatorname{resultant}_t(p', p)(z-1)^d$. Thus, all nonzero residues of p'/p are equal to 1. It follows from the logarithmic derivative identity in [2, Theorem 3.1.1] that the residues of a logarithmic derivative in $F(t)$ are integers.

We are going to describe the residues of the projections of logarithmic derivatives in a rationally hyperexponential tower.

Lemma 4.8 *Let t be a hyperexponential monomial over F, and $p \in F[t]$ be normal. Then*

$$\frac{p'}{p} = \deg_t(p)\frac{t'}{t} + \frac{a'}{a} + h$$

for some $a \in F$ and some t-simple $h \in F(t)$. Moreover, all the nonzero residues of h are equal to 1.

Proof Let $p = at^m + q$, where $m > 0$, $a \in F \setminus \{0\}$ and $q \in F[t]$ with degree lower than m. Then $p' = (a' + mat'/t)\, t^m + q'$ and $\deg(q') < m$. So $p'/p = mt'/t + a'/a + r/p$, where r is the remainder of p' and p with respect to t. Setting $h = r/p$ proves the first conclusion. By [2, Theorem 4.4.1], taking residues is F-linear. Therefore, the residues of p'/p are equal to those of h, because both t'/t and a'/a are free of t. The second conclusion then follows from Example 4.7. □

Proposition 4.9 *Let the tower F_n in (11) be rationally hyperexponential, and let $f \in F_n$ be nonzero. Then $\pi_0(f'/f) \in F_0$ and $\pi_i(f'/f)$ is a t_i-simple element with integral residues for all $i \in [n]$.*

Proof Let $i \in [n]$, and let $p \in F_{i-1}[t_i]$ be nontrivially normal with respect to t_i.

Claim. The projection $\pi_i\,(p'/p)$ is t_i-simple, and $p'/p = s + a'/a + \pi_i\,(p'/p)$ for some $s \in F_0$ and $a \in F_{i-1}$. Moreover, all nonzero residues of $\pi_i\,(p'/p)$ are equal to 1.

Proof of the claim. By Lemma 4.8, there exists an element $a \in F_{i-1}$ and a t_i-simple element $q \in F_i$ such that $p'/p = \deg_{t_i}(p)t_i'/t_i + a'/a + q$, and that all nonzero residues of q are equal to 1. Since $t_i'/t_i \in F_0$, we can set $s = \deg_{t_i}(p)t_i'/t_i$. By the definition of Laurent-Matryoshka decompositions, we have that $q = \pi_i(p'/p)$. The claim is proved.

Based on the claim, we proceed by induction on n. For $n = 1$, the logarithmic derivative identity implies $f'/f = f_0 + \sum_{j\in[k]} m_j p_j'/p_j$, where $f_0 \in F_0$, p_j is a nontrivially normal polynomial in $F_0[t_1]$, and m_j is a nonzero integer. It follows that $\pi_1\,(f'/f) = \sum_{j\in[k]} m_j \pi_1\left(p_j'/p_j\right)$. The claim implies that $\pi_1\,(f'/f)$ is t_1-simple and its residues belong to $\mathbb{Z}$. The claim also implies that $\pi_0(f'/f)$ is equal to $f_0 + u + v'/v$ for some $u, v \in F_0$. So $\pi_0(f'/f)$ belongs to F_0.

Assume that $n > 1$ and the lemma holds for $n-1$. Again, the logarithmic derivative identity implies $f'/f = u + g'/g + \sum_{j\in[k]} m_j p_j'/p_j$, where $u \in F_0$, $g \in F_{n-1}$, $p_j \in F_{n-1}[t_n]$ is nontrivially normal, and $m_j \in \mathbb{Z}$. Then $\pi_n\,(f'/f) = \sum_{j\in[k]} m_j \pi_n\left(p_j'/p_j\right)$, which is t_n-simple and has only integral residues by the claim. Moreover, there exists $v \in F_0$ and $h \in F_{n-1}$ such that $f'/f = v + h'/h + \pi_n(f'/f)$. By the induction hypothesis, $\pi_\ell(h'/h)$ is t_ℓ-simple and has only integral residues, and $\pi_0\,(h'/h) \in F_0$. The induction is completed by the observation that $\pi_\ell(f'/f) = \pi_\ell(h'/h)$ for all $\ell \in [n-1]$. □

4.4 Computing elementary integrals

An element f of F has an elementary integral over F if there exists an elementary extension E of F such that $f \in E'$. With the aid of remainders and residues, we determine whether f has an elementary integral over a rationally hyperexponential tower.

Theorem 4.10 *Let F_n in (11) be a rationally hyperexponential tower, and $f \in F_n$ with the remainder r. Then f has an elementary integral over F_n if and only if $\pi_0(r) \in F_0$ is x-simple, and, for all $i \in [n]$, $\pi_i(r) \in F_i$ is a t_i-simple element whose residues are in the algebraic closure $\overline{C}$ of C.*

Proof Assume that $\pi_0(r) \in F_0$ is x-simple, $\pi_i(r) \in F_i$ is t_i-simple, and all the residues of $\pi_i(r)$ belong to $\overline{C}$ for all $i \in [n]$. Then $\pi_i(r)$ is the sum of a $\overline{C}$-linear combination of logarithmic derivatives and a polynomial u in $F_{i-1}[t_i]$ by [5, Lemma 3.1(i)]. Since $t_i'/t_i \in F_0$, the polynomial u belongs to F_0 by the expression for u in the proof of [5, Lemma 3.1(i)]. It follows that $\pi_i(r)$ has an elementary integral over F_i for all $i \in [n]$, which, together with $\pi_0(r) \in F_0$, implies that r has an elementary integral over F_n, and so does f.

To show the converse, we assume that f has an elementary integral over F_n. Then r also has an elementary integral over the same tower. By [2, Theorem 5.5.3], there exist $g, u_1, \ldots, u_k \in F_n$ and $c_1, \ldots, c_k \in \overline{C}$ such that

$$r = g' + \sum_{j \in [k]} c_j \frac{u_j'}{u_j}.$$

Accordingly, for each $i \in [n]_0$,

$$\pi_i(r) = \pi_i(g') + \sum_{j \in [k]} c_j \pi_i \left(\frac{u_j'}{u_j} \right). \tag{15}$$

First, we show that $\pi_0(r)$ is x-simple. By Proposition 4.9, $\pi_0\left(u_j'/u_j\right)$ in (15) belongs to F_0 for all $j \in [k]$. So $\pi_0(r) \equiv w \mod F_n'$ for some $w \in F_0$ by $\pi_0(g') = \pi_0(g)'$ in Lemma 4.3. We may further assume that w is x-simple by the Hermite-Ostrogradsky reduction. Therefore, $\pi_0(r) - w \in R_0 \cap F_n'$. By Theorem 4.4, we have that $\pi_0(r) = w$. Consequently, $\pi_0(r)$ is x-simple.

It remains to show that $\pi_i(r)$ is t_i-simple and has only constant residues for all $i \in [n]$. By Proposition 4.9, $\pi_i\left(u_j'/u_j\right)$ in (15) is t_i-simple for all $j \in [k]$. Since the coefficient of every power of $t_{i+1}, \ldots, t_n$ in $\pi_i(r)$ is t_i-simple, we have that $\pi_i(g') = 0$ by $\pi_i(g') = \pi_i(g)'$ in Lemma 4.3 and an argument on orders. Then $\pi_i(r) = \sum_{j \in [k]} c_j \pi_i\left(u_j'/u_j\right)$ by (15). Consequently, $\pi_i(r)$ is a t_i-simple element whose residues belong to $\overline{C}$ by Proposition 4.9. □

Algorithms for determining constant residues are given in [2, §5.6] and [5, 8].

Example 4.11 Let us reconsider the tower F_3 and the function f in Example 4.5. Note that

$$\pi_2(r) = \frac{x^3 + x - 1}{x^3(t_2 + 1)} t_3,$$

which is not t_2-simple. So f has no elementary integral over F_3 by Theorem 4.10.

Example 4.12 Let us reconsider the tower F_2 and the function f, which are the same as those in Examples 1.1 and 4.6. By (14), the remainder r of f has only one nonzero projection, which is

$$\pi_2(r) = \frac{(x^3 - x - 3)t_1}{(x^3 - x - 2)(t_1 + t_2)}.$$

It is t_2-simple. An algorithm for determining constant residues yields

$$\pi_2(r) = \frac{(t_1 + t_2)'}{t_1 + t_2} - \frac{1}{x^3 - x - 2}.$$

The nonzero residues of $\pi_2(r)$ are all equal to 1. So f has an elementary integral over F_2, which is

$$\int f = \frac{1}{1+t_2} + \log(t_1+t_2) - \sum_{\alpha^3-\alpha-2=0} \frac{1}{3\alpha^2-1}\log(x-\alpha).$$

References

[1] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, Ziming Li, and Guoce Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *Proceedings of ISSAC'13*, pages 77–84. ACM, New York, 2013.

[2] Manuel Bronstein. *Symbolic Integration I: Transcendental Functions*, volume 1 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2005.

[3] Shaoshi Chen, Hao Du, and Ziming Li. Additive decompositions in primitive extensions. In *Proceedings of ISSAC'18*, pages 135–142. ACM, New York, 2018.

[4] Shaoshi Chen, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for algebraic functions. In *Proceedings of ISSAC'16*, pages 175–182. ACM, New York, 2016.

[5] Hao Du, Yiman Gao, Jing Guo, and Ziming Li. Computing logarithmic parts by evaluation homomorphisms. In *Proceedings of ISSAC'23*, pages 242-250. ACM, New York, 2023.

[6] Hao Du, Jing Guo, Ziming Li, and Elaine Wong. An additive decomposition in logarithmic towers and beyond. In *Proceedings of ISSAC'20*, pages 146–153. ACM, New York, 2020.

[7] Keith O. Geddes, Ha Q. Le, and Ziming Li. Differential rational normal forms and a reduction algorithm for hyperexponential functions. In *Proceedings of ISSAC'04*, pages 183–190. ACM, New York, 2004.

[8] Clemens G. Raab. Using Gröbner bases for finding the logarithmic part of the integral of transcendental functions. *Journal of Symbolic Computation*, 47(10):1290–1296, 2012.

[9] Clemens G. Raab and Michael F. Singer. *Integration in Finite Terms: Fundamental Sources*. Text & Monographs in Symbolic Computation, Springer, 2022.

[10] Michael Rothstein. *Aspects of Symbolic Integration and Simplification of Exponential and Primitive Functions*. PhD thesis, University of Wisconsin, Madison, 1976.