Complete Reduction for Derivatives in a Transcendental Liouvillian Extension*

Shaoshi Chen[a,b], Hao Du[c], Yiman Gao[d,†], Hui Huang[e],
Wenqiao Li[a,b], Ziming Li[a,b]

[a] MMRC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing (100190), China
[b] School of Mathematical Sciences, University of Chinese Academy of Sciences, Chinese Academy of Sciences, Beijing (100049), China
[c] School of Mathematical Sciences, Beijing University of Posts and Telecommunications, Beijing (102206), China
[d] Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC), Altenberger Straße 69, 4040, Linz, Austria
[e] School of Mathematics and Statistics, Fuzhou University, Fuzhou (350108), China

Email: schen@amss.ac.cn, haodu@bupt.edu.cn, ymgao@risc.jku.at, huanghui@fzu.edu.cn, liwenqiao@amss.ac.cn, zmli@mmrc.iss.ac.cn

Abstract

Transcendental Liouvillian extensions are differential fields, in which one can model poly-logarithmic, hyperexponential, and trigonometric functions, logarithmic integrals, and their (nested) rational expressions. For such an extension $(F, ')$ with the subfield C of constants, we construct a complementary subspace W for the C-subspace of derivatives in F, and develop an algorithm that, for every $f \in F$, computes a pair $(g, r) \in F \times W$ such that $f = g' + r$. Moreover, f is a derivative in F if and only if $r = 0$. The algorithm enables us to determine elementary integrability over F by computing parametric logarithmic parts, and leads to a reduction-based approach to constructing telescopers for functions that can be represented by elements in F.

Keywords: Additive decomposition, Complete reduction, Creative telescoping, Elementary integration, In-field integration, Liouvillian extension, Risch equation, Symbolic integration

1 Introduction

A classical topic of computer algebra is to express indefinite integrals in closed forms. Historical developments and fundamental results on this topic are reviewed in [31]. On the computational

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†Corresponding author; Present address: Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

side, the most systematic developments are due to Risch [32, 33] and researchers who have clarified, improved, extended and implemented Risch's algorithm [34, 17, 36, 18, 8, 9, 10, 30].

For an elementary function, Risch's algorithm determines whether it has an elementary integral, and computes such an integral if there exists one.

Example 1.1. *Let* $f(x) = \dfrac{x\log(x)^3+1}{x\log(x)}$ *and* $g(x) = \dfrac{x\log(x)^3+1}{(x+3)\log(x)}$. *By Risch's algorithm,*

$$\int f = x\log(x)^2 - 2x\log(x) + 2x + \log(\log(x)) \quad \text{and} \quad \int g = \int \frac{x\log(x)^3+1}{(x+3)\log(x)}.$$

The algorithm yields an elementary integral of f*, and reveals that* g *has no elementary integral.*

The most basic routine in Risch's algorithm is the Hermite-Ostrogradsky reduction for rational functions [26, 29]. We refer to [10, Chapter 2] and [23, Chapter 11] for its modern presentations.

Example 1.2. *Let* C *be a field of characteristic zero. For every element* $f \in C(x)$*, the Hermite-Ostrogradsky reduction computes* $g, s \in C(x)$ *such that* $f = g' + s$*, where* $'$ *stands for the usual derivation* d/dx*, and* s *is a proper fraction with a squarefree denominator. Moreover,* $f \in C(x)'$ *if and only if* $s = 0$*, where* $C(x)'$ *denotes the set of derivatives in* $C(x)$*.*

Let S_x be the set $\{s \in C(x) \mid s \text{ is proper and has a squarefree denominator}\}$. Then S_x is a C-linear subspace of $C(x)$ and $C(x) = C(x)' \oplus S_x$. The projection from $C(x)$ to S_x with respect to the direct sum is a complete reduction for $C(x)'$ in $C(x)$ according to [27, Definition 5.67]. In this paper, complete reductions are for images of linear operators. The following definition is equivalent to [27, Definition 5.67], but more convenient for the present paper.

Definition 1.3. *Let* U *be a* C*-linear space and* ρ *be a linear operator on* U*. Another linear operator* ϕ *on* U *is called a* complete reduction for the pair (U, ρ) *if* $U = \operatorname{im}(\rho) \oplus \operatorname{im}(\phi)$ *and* ϕ *is the projection from* U *to* $\operatorname{im}(\phi)$ *with respect to the direct sum.*

Another equivalent definition is that a linear operator ϕ on U is a complete reduction for (U, ρ) if $\phi^2 = \phi$ and $\ker(\phi) = \operatorname{im}(\rho)$.

The next definition describes how a complete reduction decomposes an element additively.

Definition 1.4. *Let* ϕ *be a complete reduction for* (U, ρ) *and* $u \in U$*. We call* $\phi(u)$ *the* remainder of u with respect to ϕ*. Let* $v \in U$ *be such that* $u = \rho(v) + \phi(u)$*. Then* $(v, \phi(u))$ *is called a* reduction pair of u with respect to ϕ *or an* R-pair with respect to ϕ *for brevity.*

The Hermite-Ostrogradsky reduction not only induces a complete reduction for $(C(x), ')$, but also yields an algorithm for computing the corresponding R-pairs. An algorithm based on the reduction is developed in [3] to construct telescopers for bivariate rational functions. It avoids the costly computation of certificates, and has motivated a rapid development of complete reductions and additive decompositions in various settings.

Example 1.5. *Let* h *be a nonzero rational function in* $C(x)$*, and* U *be the space spanned by the hyperexponential function* $\exp(\int h)$ *over* $C(x)$*. For every element* $f \in U$*, the Hermite reduction for hyperexponential functions in [4] is a complete reduction for* $(U, d/dx)$*.*

Recall that a *derivation* $' : R \to R$ on a commutative ring R is an additive map satisfying the usual Leibniz's rule: $(ab)' = ba' + ab'$ for all $a, b \in R$. The pair $(R, ')$ is called a *differential ring* and a *differential field* if R is a field. The set $\{c \in R \mid c' = 0\}$ forms a subring whose elements are called *constants*. In particular, the subring becomes a subfield if R is a field.

Let $(R, ')$ be a differential ring whose subring C of constants is a field. Then R is a C-linear space. Complete reductions have been established for such differential rings with instances of R including the field of differential fractions [6], the field of algebraic functions [15], (sub)rings of D-finite functions [16, 37, 12], exponential towers [22] and primitive towers [20]. A complete reduction for the pair $(C(x), L)$ is developed in [5], where L denotes a linear differential operator. The reader is referred to [7, 14] for complete reductions related to symbolic summation.

A complete reduction for $(R, ')$ decomposes an element f as $g' + r$ for some $g, r \in R$ such that r is minimal with respect to a partial ordering defined in [22, Section 1.1]. In particular, f is a derivative in R if and only if $r = 0$. So a complete reduction for $(R, ')$ is an additive decomposition in the sense given by [13, 21]. It provides us with an alternative and possibly more informative way to express integrals.

Example 1.6. *Let f and g be the same as in Example 1.1. They can be regarded as elements of $\mathbb{Q}(x, \log(x))$, which is a differential field with derivation $' = d/dx$.*

The complete reduction in [20] yields

$$f(x) = h(x)' + \frac{1}{x\log(x)} \quad \text{and} \quad g(x) = h(x)' + \frac{1}{(x+3)\log(x)} - \frac{3\log(x)^2}{x+3},$$

where $h(x) = x\log(x)^2 - 2x\log(x) + 2x$.

Note that $f(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder $1/(x\log(x))$. The remainder has an elementary integral $\log(\log(x))$. In general, the elementary integrability of remainders in logarithmic extensions can be determined by computing parametric logarithmic parts by [20, Theorem 5.3]. On the other hand, $g(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder, which is not elementarily integrable. Unlike in Example 1.1, the complete reduction renders us a maximal integrable part $h(x)$ of $g(x)$ in $\mathbb{Q}(x, \log(x))$. The above two decompositions of f and g can also be obtained from the algorithms in [13, 21].

Now, let us define transcendental Liouvillian extensions. Such extensions are more general than transcendental elementary extensions (see [10, Definition 5.1.4]), and do not contain any complicated reduced elements (see [10, Definition 3.5.2]).

Definition 1.7. *Let C be a field of characteristic zero. A differential field $C(t_1, \ldots, t_n)$ with derivation $'$ is called a* transcendental Liouvillian extension *of C if*

(i) $t_1, \ldots, t_n$ are algebraically independent over C,

(ii) either t_i' or $t_i'/t_i \in C(t_1, \ldots, t_{i-1})$ for all $i \in \{1, 2, \ldots, n\}$,

(iii) C is the subfield of constants in $C(t_1, \ldots, t_n)$.

Our initial attempts to inductively construct a complete reduction for derivatives in a transcendental Liouvillian extension revealed the necessity of a more general inductive framework (see Example 3.1). A similar phenomenon also occurs in the inductive proof of the main theorem in [32]. In fact, the main theorem has two assertions. The first is of interest in elementary integration, and the second serves to facilitate the proof. In the second assertion, Risch writes down a differential equation

$$y' + fy = \sum_{i=1}^{m} c_i g_i, \tag{1}$$

where f and the g_i's belong to a differential field F, and the c_i's are constants of F. The equation has been called the (parametric) Risch differential equation in the literature.

To develop complete reductions for symbolic integration, we define Risch operators using the homogeneous part of (1).

Definition 1.8. *Let $(F, ')$ be a differential field and $h \in F$. The map*

$$\begin{array}{rccc} \mathcal{R}_h: & F & \to & F \\ & y & \mapsto & y' + hy. \end{array}$$

is called the Risch operator associated to *h.*

The letter "f" is replaced with "h" in the above definition because f usually stands for an arbitrary element of a differential field in this paper. Risch operators are linear over the subfield of constants in F. In particular, $\mathcal{R}_0$ is the derivation operator on F.

Assume that there is a complete reduction ϕ_f for $(F, \mathcal{R}_f)$, where f is given in the left-hand side of (1). Then each g_i in the right-hand side of (1) has an R-pair (p_i, q_i) with respect to ϕ_f. Therefore, (1) can be rewritten as $\mathcal{R}_f(y) = \sum_{i=1}^m c_i \left(\mathcal{R}_f(p_i) + q_i\right)$, which is equivalent to

$$\mathcal{R}_f\left(y - \sum_{i=1}^m c_i p_i\right) = \sum_{i=1}^m c_i q_i.$$

It follows from Definition 1.3 that there exist $y \in F$ and $c_1, \ldots, c_m \in C$ such that (1) holds if and only if $\sum_{i=1}^m c_i q_i = 0$ and $\mathcal{R}_f\left(y - \sum_{i=1}^m c_i p_i\right) = 0$. The first constraint leads to a linear system over the subfield C of constants in F, and the second amounts to determining the kernel of $\mathcal{R}_f$. In other words, the complete reduction finds all solutions of (1) in F by solving a linear algebraic system over C and $y' + fy = 0$ in F.

Let $F_n = C(t_1, \ldots, t_n)$ be a transcendental Liouvillian extension of C with $n > 0$, and set $F_0 = C$. In this paper, we construct a complete reduction $\phi_{n,h}$ for $(F_n, \mathcal{R}_h)$, where $h \in F_n$, in the sense that we fix a complement of $\operatorname{im}(\mathcal{R}_h)$ in F_n with the corresponding projection $\phi_{n,h}$ from F_n to the complement, and develop an algorithm for computing R-pairs with respect to $\phi_{n,h}$. Setting $h = 0$ yields a complete reduction $\phi_{n,0}$ for $(F_n, ')$. In doing so, we generalize the complete reductions in Examples 1.2, 1.5, and those in [22, 20] in one fell swoop.

Our construction is based on the assumption that there is a complete reduction for $(F_{n-1}, \mathcal{R}_z)$ for all $z \in F_{n-1}$. It proceeds in three steps:

Outline 1.9. *1. Reduce the problem of constructing a complete reduction for $(F_n, \mathcal{R}_h)$ to that for $(F_n, \mathcal{R}_\xi)$, where $\xi \in F_n$ is t_n-normalized (see Proposition 3.3).*

2. Reduce the problem of constructing a complete reduction for $(F_n, \mathcal{R}_\xi)$ to that for $(R_n, \mathcal{P}_\xi)$, where R_n is either $F_{n-1}[t_n]$ or $F_{n-1}[t_n, t_n^{-1}]$, and $\mathcal{P}_\xi$ is the companion operator of $\mathcal{R}_\xi$ (see Definition 3.4 and Proposition 3.5).

3. Construct a complete reduction for $(R_n, \mathcal{P}_\xi)$ (see Theorem 7.1).

The idea for the first two steps can be traced back to [18, Condition b) on page 905] and [10, Section 6.1], in which the problem of solving Risch differential equations in a monomial extension is transformed to that in its subring of reduced elements. Similar transformations are also given in [24, 11], which can, respectively, be viewed as a differential analogue and a generalization of the minimal decomposition for hypergeometric terms in [2, 1].

The idea for step 3 is to construct an auxiliary subspace A_ξ such that $R_n = \operatorname{im}(\mathcal{P}_\xi) + A_\xi$, and compute an echelon basis of $\operatorname{im}(\mathcal{P}_\xi) \cap A_\xi$. The basis enables us to construct a complement of $\operatorname{im}(\mathcal{P}_\xi)$ in a dual manner (see Section 2.3). Although this idea has appeared in [4, Section 4.1], [22, Chaper 4] and [20, Section 3] in one form or another, a comprehensive generalization is required, because F_n is more involved than the differential fields appearing therein. Step 3 is different from the way to handle elements of R_n in Risch's algorithm. It is also the most technical part in this paper.

Applications of the complete reduction in Theorem 7.1 include two algorithms: one is for in-field integration, and the other for elementary integration. In addition, the complete reduction leads to a reduction-based approach to constructing telescopers (up to a given order) for functions that can be interpreted as elements in a transcendental Liouvillian extension.

The next example illustrates an application of the complete reduction to in-field integration.

Example 1.10. *Let*

$$f = \frac{x}{1+\exp(x)} \cdot \exp\left(\frac{x}{1+\exp(x)}\right),$$

which can be regarded as an element in the transcendental elementary extension $\mathbb{C}(x,t,y)$ *of* $\mathbb{C}$, *where* $t=\exp(x)$ *and* $y = \exp\left(\frac{x}{1+\exp(x)}\right)$. *In fact,* $f = \frac{xy}{1+\exp(x)}$, *which belongs to* $\mathbb{C}(x,t)$. *We determine whether* f *has an integral in the extension* $\mathbb{C}(x,t,y)$.

The complete reduction in this paper confirms that f *is in-field integrable, and finds that*

$$\int f = -\big(1+\exp(-x)\big) \cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

The `int()` *command (with option* `method=_RETURNVERBOSE`*) in* MAPLE *2023 leaves the integral unevaluated, and so does* `Integrate[]` *command in* MATHEMATICA *14.3.*

Indeed, computing this integral amounts to handling a special case in Risch's algorithm carefully. It corresponds to Proposition 6.13 (ii). Details are given in Example 7.3.

In [30, Chapters 3 and 4], Raab provides several building blocks that were either missing or incomplete for the description of Risch's algorithm in [10], and presents a method for computing elementary integrals over an admissible differential field with some mild restrictions. Such fields are more general than transcendental Liouvillian extensions. Raab's method needs all the ingredients from Risch's algorithm in the transcendental case. Our algorithm for elementary integration is based on the complete reduction and some special structure of remainders described in Proposition 8.4. It merely needs to solve the problem of recognizing logarithmic derivatives [10, Section 5.12] and parametric logarithmic derivative problem [10, Section 7.3].

The last example of this section is for definite integration.

Example 1.11. *By the complete reduction, we find that* $(k+1)\sigma_k - k$ *is a minimal telescoper for both* $R(k)$ *and* $I(k)$, *where* σ_k *stands for the shift operator* $k \mapsto k+1$,

$$R(k) = \int_0^{\frac{\pi}{2}} \cos(2kx)\log(\sin(x))\ dx \quad \text{and} \quad I(k) = \int_0^{\frac{\pi}{2}} \sin(2kx)\log(\sin(x))\ dx.$$

The telescoper, together with the corresponding certificates, yields that

$$R(k+1) - \frac{k}{k+1}R(k) = 0 \quad \text{and} \quad I(k+1) - \frac{k}{k+1}I(k) = \frac{(-1)^k - 2k - 1}{4k(k+1)^2},$$

which help us express $R(k)$ *and* $I(k)$ *in closed forms. Details are given in Example 8.8.*

Preliminary experiments in MAPLE 2021 illustrated that our complete reduction were more efficient than the `int()` command in MAPLE, when both were applied to the derivatives of elements in some transcendental elementary extension of $C(x)$. In particular, a significant speed-up was observed when integrands are derivatives of polynomials in the generators of such an extension (see Tables 4 and 6 in Section 9).

The rest of this paper is organized as follows. In Section 2, we present basic terminologies in symbolic integration and a dual description of subspaces. Steps 1 and 2 in Outline 1.9 are taken in Section 3. The induction base and hypothesis are given in Section 4. Step 3 is carried out in Section 5 for primitive extensions and Section 6 for hyperexponential extensions, respectively. In Section 7, we complete the induction and present examples for in-field integration. The complete reduction is applied to elementary integration and creative telescoping in Section 8. Experimental results on the efficiency of the complete reduction are presented in Section 9. Concluding remarks are given in Section 10. In the appendix, we present the pseudo-code descriptions of algorithms based on some constructive proofs in Sections 5 and 6.

2 Preliminaries

This section has three parts. First, we present notation to be used in the sequel, and basic terminologies in symbolic integration. Next, some useful properties of residues are recalled for elementary integration. At last, we describe a dual presentation for complementary subspaces, which enables us to compute in subspaces of infinite dimension in Sections 5 and 6.

2.1 Basic terminologies

Besides the usual notation in textbooks, we let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{N}^-$ be the sets of positive, nonnegative and negative integers, respectively. Set $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$, and $[n]_0 := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$. For an abelian group $(G, +, 0)$, $G \setminus \{0\}$ is denoted by $G^\times$.

All the fields are of characteristic zero in the sequel. Let F be a field and $p \in F[t]^\times$. The degree and leading coefficient of p are denoted by $\deg_t(p)$ and $\operatorname{lc}_t(p)$, respectively. In addition, $\deg_t(0) := -\infty$ and $\operatorname{lc}_t(0) := 0$. We set $F[t]_{<d} := \{p \in F[t] \mid \deg_t(p) < d\}$ for all $d \in \mathbb{N}_0$.

The ring of Laurent polynomials in t over F is denoted by $F[t, t^{-1}]$. Let $q \in F[t, t^{-1}]$ be of the form $q_k t^k + q_{k-1} t^{k-1} + \cdots + q_{l+1} t^{l+1} + q_l t^l$, where $k, l \in \mathbb{Z}$, $k \geq l$, $q_k, q_{k-1}, \ldots, q_{l+1}, q_l \in F$, and $q_k q_l \neq 0$. The *head degree, head coefficient, tail degree and tail coefficient* of q are defined to be k, q_k, l and q_l, denoted by $\operatorname{hdeg}_t(q)$, $\operatorname{hc}_t(q)$, $\operatorname{tdeg}_t(q)$ and $\operatorname{tc}_t(q)$, respectively. In addition, $\operatorname{hdeg}_t(0) := -\infty$, $\operatorname{hc}_t(0) = 0$, $\operatorname{tdeg}_t(0) := \infty$, and $\operatorname{tc}_t(0) := 0$. Since $F[t, t^{-1}] = F[t] \oplus t^{-1} F[t^{-1}]$, we set q^+ and q^- to be the respective projections of q to $F[t]$ and $t^{-1} F[t^{-1}]$. For $Q \subset F[t, t^{-1}]$ and $l \in \mathbb{Z}$, the set $Q_{>l}$ denotes the set of elements in Q whose tail degrees are greater than l.

For an element f of $F(t)$, its numerator and denominator are denoted by $\operatorname{num}_t(f)$ and $\operatorname{den}_t(f)$, respectively. Moreover, $\operatorname{den}_t(f)$ is set to be monic. We say that f is *t-proper* if the degree of $\operatorname{num}_t(f)$ is less than that of $\operatorname{den}_t(f)$.

The subscript t will be omitted for brevity when the indeterminate t is clear from context.

Let $f \in F(t)^\times$ with $a = \operatorname{num}(f)$ and $b = \operatorname{den}(f)$. For $p \in F[t]$ with $\deg(p) > 0$, the *order* of f at p, denoted by $\nu_p(f)$, is defined to be $-m$ if p is of multiplicity $m > 0$ in b. Otherwise, the order is the multiplicity of p in a. In addition, $\nu_p(0)$ is set to be ∞. The *order of f at infinity* is $\deg(b) - \deg(a)$ and denoted by $\nu_\infty(f)$.

Let $(F, ')$ be a differential field. We denote $\{f' \mid f \in F\}$ by F', which is a linear space over the subfield of constants in F. An element of F is called a *logarithmic derivative* if it is equal to g'/g for some $g \in F^\times$. A differential field (E, δ) is called a *differential field extension* of $(F, ')$ if F is a subfield of E and $\delta|_F = '$. The derivation δ will still be denoted by $'$ when there is no confusion arising.

Let E be a differential field extension of F and $f \in E$. Then f is said to be *primitive (resp. hyperexponential) over* F if $f' \in F$ (resp. $f \neq 0$ and $f'/f \in F$). An element t in a differential field extension of F is called a *monomial over* F if t is transcendental over F and $t' \in F[t]$. A monomial t over F is said to be *regular* if $F(t)$ and F have the same subfield of constants. By Definition

1.7, a transcendental Liouvillian extension $C(t_1, \ldots, t_n)$ is a differential field extension of C, in which t_i is a regular monomial over $C(t_1, \ldots, t_{i-1})$, and is either primitive or hyperexponential over the same field for all $i \in [n]$.

In the rest of this section, we let $(F, ')$ be a differential field, C be its subfield of constants, and t be a monomial over F. Then $F[t]$ is a differential subring of $F(t)$. A nonzero polynomial $p \in F[t]$ is said to be *normal (resp. special)* if $\gcd(p, p') = 1$ (resp. $\gcd(p, p') = p$). Set N_t to be the set of all monic, irreducible and normal polynomials of positive degrees.

An element of $F(t)$ is t-*simple* if it is t-proper and has a normal denominator according to [13, Section 1]. All t-simple elements in $F(t)$ form an F-subspace, which is denoted by S_t. Let

$$F\langle t\rangle := \{f \in F(t) \mid \mathrm{den}(f) \text{ is special}\}.$$

Then $F[t] \subset F\langle t\rangle$, and $F\langle t\rangle$ is a differential subring (see [10, Corollary 4.4.1]). We remark that t-simple elements are not required to be t-proper in [10, Definition 3.5.2]. Our further requirement for t-properness results in $S_t \cap F\langle t\rangle = \{0\}$, which is necessary for the direct sum (4) in Section 3.

Some basic facts about generators in a transcendental Liouvillian extension are collected in the following lemma for a convenience of later references.

Lemma 2.1. *Let t be regular over F, $d \in \mathbb{Z}$, $u \in F$ and $p \in F[t]$.*

(i) If $d > 0$ and t is primitive over F, then $\left(ut^d\right)' = u't^d + udt't^{d-1}$ whose degree is equal to d if u is not a constant, and $d - 1$, otherwise.

(ii) If t is hyperexponential over F, then $\left(ut^d\right)' = (u' + ud(t'/t))\, t^d$ whose head and tail degrees are both equal to d.

(iii) If t is primitive over F, then p is normal if and only if p is squarefree.

(iv) If t is hyperexponential over F, then p is normal if and only if p is squarefree and $t \nmid p$.

Proof. (i) By Leibniz's rule, $\left(ut^d\right)' = u't^d + udt't^{d-1}$. Since t is regular over F, we have that $t' \in F^\times$, which implies that the degree of $\left(ut^d\right)'$ is at least $d - 1$.

(ii) Similarly, Leibniz's rule implies that $\left(ut^d\right)' = (u' + ud(t'/t))\, t^d$, which is nonzero because t is regular. Since $t'/t \in F$, both head and tail degrees of $\left(ut^d\right)'$ are equal to d.

(iii) and (iv) hold by [10, Lemma 3.4.4, Theorems 5.1.1 and 5.1.2]. □

We use sequences in algorithmic descriptions in order to avoid clumsy names of list operations. Let $L : l_1, \ldots, l_k$ be a sequence. Then l_i is denoted by $L[i]$ for all $i \in [k]$. The empty sequence is written as NIL. The *length* of L is defined to be k and denoted by $\mathrm{len}(L)$. In the description of an algorithm, comments are placed between (*···*).

2.2 Residues and elementary integration

The reader is referred to [10, Section 4.4] for the definition of residues. Let us mention two well-known facts on residues: First, all residues of an element in $F[t]$ are zero by the definition. Second, all residues of a logarithmic derivative in $F(t)$ are integers by [10, Corollary 4.4.2 (iii)].

The next two lemmas are useful to prove Theorem 8.3, which enables us to determine elementary integrability of remainders.

Lemma 2.2. *Let t be regular over F, $p \in F[t]^\times$ with $d = \deg(p)$ and $p_d = \mathrm{lc}(p)$.*

(i) If t is primitive over F, then $p'/p = s + p_d'/p_d$ for some $s \in S_t$.

(ii) If t is hyperexponential over F, then $p'/p = s + p_d'/p_d + dt'/t$ for some $s \in S_t$.

(iii) All residues of s appearing in (i) or (ii) are integers.

Proof. (i) Let $q = p_d^{-1}p$. Then $p'/p = p_d'/p_d + q'/q$ by the logarithmic derivative identity in [10, Exercis 3.1], So $q'/q \in S_t$ by Lemma 2.1 (i) and (iii). The conclusion holds by setting $s := q'/q$.

(ii) There exists an integer $m \in \mathbb{N}_0$ and a monic polynomial $r \in F[t]$ with $t \nmid r$ such that $p = p_d t^m r$. Again, the logarithmic derivative identity implies that $p'/p = p_d'/p_d + mt'/t + r'/r$. It follows from $t'/t \in F$, Lemma 2.1 (ii) and (iv) that $s := r'/r - (d-m)t'/t$ is an element of S_t. Furthermore, $p'/p = p_d'/p_d + dt'/t + s$.

(iii) The t-simple element s in (i) or (ii) is the difference of p'/p and an element in F. Since all residues of a logarithmic derivative are integers, so are the residues of s. $\square$

Lemma 2.3. *Let t be regular over F and $f \in S_t^{\times}$. Assume that C is algebraically closed, and that t is either primitive or hyperexponential over F. If all residues of f belong to C, then f has an elementary integral over F.*

Proof. Let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of $\mathrm{den}(f)$ in the algebraic closure of F. Since all residues of f in $\overline{F}$ are constants, there exist $c_1, \ldots, c_k \in C$ and $p \in F[t]$ such that

$$f = c_1 \frac{(t-\alpha_1)'}{t-\alpha_1} + \cdots + c_k \frac{(t-\alpha_k)'}{t-\alpha_k} + p$$

by [19, Lemma 3.1 (i)]. If t is primitive over F, then $p = 0$ by $f \in S_t$ and Lemma 2.1 (i). If t is hyperexponential over F, then $p = ct'/t$ for some $c \in C$ by $f \in S_t$ and Lemma 2.1 (ii). Thus, f has an elementary integral over F. $\square$

2.3 A dual presentation for complementary subspaces

Let U be a C-linear space. Every subspace of U is the intersection of kernels of some linear functions on U. Such an intersection is understood as a dual presentation of the subspace.

For a nonempty subset S of U, $\mathrm{span}_C(S)$ stands for the C-subspace spanned by S. In addition, the empty set spans $\{0\}$. Let Θ be a basis of U. For $\theta \in \Theta$, θ^* stands for the linear function on U that maps θ to 1 and other elements of Θ to 0. The basis Θ is said to be *effective* if there are two algorithms available: one finds an element $\theta \in \Theta$ such that $\theta^*(\mathbf{u}) \neq 0$ for all $\mathbf{u} \in U^{\times}$, and the other computes $\theta^*(\mathbf{v})$ for all $\mathbf{v} \in U$ and $\theta \in \Theta$ (see [20, Section 2.2]).

Definition 2.4. *Let Θ be given above, V be a subspace of U and $\mathbb{I} \subset \mathbb{N}_0$. A basis $\{\mathbf{v}_i \mid i \in \mathbb{I}\}$ of V is called an* echelon basis of V with respect to Θ *if, for each $i \in \mathbb{I}$, there exists an element $\theta_i \in \Theta$ such that $\mathbf{v}_i \notin \ker(\theta_i^*)$ for all $i \in \mathbb{I}$ and $\mathbf{v}_j \in \ker(\theta_i^*)$ for all $j \in \mathbb{I}$ with $j < i$. In this case, we also call $\{\mathbf{v}_i\}_{i\in\mathbb{I}}$ an* echelon basis of V with pivots $\{\theta_i\}_{i\in\mathbb{I}}$.

Intuitively, an echelon basis $\{\mathbf{v}_i \mid i \in \mathbb{I}\}$ of V with pivots $\{\theta_i\}_{i\in\mathbb{I}}$ is a basis of V, in which θ_i appears in the C-linear combination of $\mathbf{v}_i$ in terms of elements in Θ, but it does not appear in the C-linear combination of $\mathbf{v}_j$ in terms of the same elements for all $j < i$.

Lemma 2.5. *Let U be a C-linear space and Θ be a basis of U. Let V be a subspace of U and $\{\mathbf{v}_i\}_{i\in\mathbb{I}}$ be an echelon basis of V with pivots $\{\theta_i\}_{i\in\mathbb{I}}$. Then $U = V \oplus \left(\cap_{i\in\mathbb{I}} \ker(\theta_i^*)\right)$.*

Proof. Let $W = \cap_{i\in\mathbb{I}} \ker(\theta_i^*)$. For every $\mathbf{u} \in U$, $\mathbf{u}$ is a linear combination of elements in Θ, in which a finite number of elements in $\{\theta_i\}_{i\in\mathbb{I}}$ appear effectively. Eliminating these pivots by the echelon basis successively, we find $\mathbf{v} \in V$ such that $\mathbf{u} - \mathbf{v} \in W$. It follows that $U = V + W$.

Suppose that $\mathbf{w}$ is a nonzero element of $V \cap W$. Then it is a C-linear combination of elements in the echelon basis. So $\mathbf{w}$ does not belong to $\ker(\theta_j^*)$, where j is the maximal index such that the jth element in the basis appears in the linear combination. Accordingly, $\mathbf{w} \notin W$. It follows that $V \cap W = \{0\}$, a contradiction. Hence, $V \cap W = \{0\}$ and W is a complement of V. □

Example 2.6. *Let $U = C[x]$ with a C-basis $\Theta = \{x^i \mid i \in \mathbb{N}_0\}$. Let V be the ideal generated by x^2+1 in $C[x]$. Then V has an echelon basis $\{x^{i+2}+x^i \mid i \in \mathbb{N}_0\}$ with pivots $\{x^{i+2} \mid i \in \mathbb{N}_0\}$. By Lemma 2.5, a complement of V in U is $\cap_{i\in\mathbb{N}_0} \ker(\theta_i^*)$ with $\theta_i = x^{i+2}$, which is equal to $C[x]_{<2}$.*

When developing complete reductions for a transcendental Liouvillian extension with a Risch operator, we face situations, in which a C-space U, an effective basis Θ, and a subspace V_1 are given, but no echelon basis of V_1 is known. To construct a complement of V_1 in U, we seek another subspace V_2 such that $U = V_1 + V_2$ and that $V_1 \cap V_2$ has an echelon C-basis. Then a complement of V_1 in U can be obtained from the next lemma.

Lemma 2.7. *Let U and Θ be given as in Lemma 2.5, and let V_1 and V_2 be two subspaces of U such that $U = V_1 + V_2$. If $V_1 \cap V_2$ has an echelon basis with pivots $\{\theta_i\}_{i\in\mathbb{I}}$, then $V_2 \cap \left(\bigcap_{i\in\mathbb{I}} \ker(\theta_i^*)\right)$ is a complement of V_1 in U.*

Proof. Let $W = \cap_{i\in\mathbb{I}} \ker(\theta_i^*)$. Then $U = V_1 \cap V_2 + W$ by Lemma 2.5. Since $V_1 \cap V_2 \subset V_2$, we have $V_2 = V_1 \cap V_2 + V_2 \cap W$ by the modular law of subspaces. Then

$$U = V_1 + V_2 = V_1 + V_1 \cap V_2 + V_2 \cap W = V_1 + V_2 \cap W.$$

The conclusion follows immediately from $V_1 \cap V_2 \cap W = \{0\}$. □

Example 2.8. *Let $U = C[x]$ with a C-basis $\Theta = \{x^i \mid i \in \mathbb{N}_0\}$. Let V_1 and V_2 be the ideals generated by x^2+1 and x^2-1 in $C[x]$, respectively. Then $U = V_1 + V_2$ and $V_1 \cap V_2$ is the ideal generated by x^4-1. The intersection has an echelon basis $\{x^{i+4}-x^i \mid i \in \mathbb{N}_0\}$ with pivots $\{\theta_i\}_{i\in\mathbb{N}_0}$, where $\theta_i = x^{i+4}$. It follows from Lemma 2.7 that a complement of V_1 is $V_2 \cap (\cap_{i\in\mathbb{N}_0} \ker(\theta_i^*))$, which is equal to $\operatorname{span}_C\{x^2-1, x^3-x\}$.*

As mentioned in Section 1, we consider linear spaces equipped with an operator ρ. Reductions usually take place in or modulo a subspace of $\operatorname{im}(\rho)$. This motivates us to define the notion of echelon sequences so as to find pre-images under ρ without any additional effort.

Definition 2.9. *Let U be a C-linear space, ρ be a linear operator on U, Θ be a basis of U, and V be a subspace contained in $\operatorname{im}(\rho)$. Assume that $\{\mathbf{v}_i\}_{i\in\mathbb{I}}$ is an echelon basis of V with pivots $\{\theta_i\}_{i\in\mathbb{I}}$, where $\theta_i \in \Theta$. If $\mathbf{v}_i = \rho(\mathbf{u}_i)$ for some $\mathbf{u}_i \in U$ and for every $i \in \mathbb{I}$, then we call $\{(\mathbf{u}_i, \mathbf{v}_i, \theta_i)\}_{i\in\mathbb{I}}$ an* echelon sequence of V with respect to (U, ρ) and Θ.

Example 2.10. *Let U, Θ and V be the same as those in Example 2.6, and $\rho : U \to U$ map p to $(x^2+1)p$ for all $p \in U$. Then $V = \operatorname{im}(\rho)$ and has an echelon sequence $\left\{\left(x^i, x^{i+2}+x^i, x^{i+2}\right)\right\}_{i\in\mathbb{N}_0}$.*

The last lemma is a preparation for computing R-pairs (see Definition 1.4).

Lemma 2.11. *With the notation introduced in Definition 2.9, we further assume that Θ is effective. If $\{(\mathbf{u}_i, \mathbf{v}_i, \theta_i)\}_{i\in\mathbb{I}}$ is an echelon sequence of V with respect to (U, ρ) and Θ, then, for every $\mathbf{x} \in U$, one can compute $\mathbf{y} \in U$ and $\mathbf{z} \in V$ such that $\mathbf{z} = \rho(\mathbf{y})$ and $\mathbf{x} - \mathbf{z} \in \cap_{i\in\mathbb{I}} \ker(\theta_i^*)$.*

Proof. Since Θ is effective, one can express $\mathbf{x}$ as a linear combination of elements in Θ.

Let $\Theta_{\mathbf{x}}$ be the finite set consisting of all the pivots appearing in the linear combination of $\mathbf{x}$ in terms of elements in Θ. It suffices to set both $\mathbf{y}$ and $\mathbf{z}$ to be zero if $\Theta_{\mathbf{x}} = \emptyset$. Otherwise, let j be the maximal index such that $\theta_j \in \Theta_{\mathbf{x}}$. Set $c_{\mathbf{x}} := \theta_j^*(\mathbf{x})$ and $c_j := \theta_j^*(\mathbf{v}_j)$. Then $c_j \neq 0$. Set $\mathbf{y}_j := c_{\mathbf{x}} c_j^{-1} \mathbf{u}_j$ and $\mathbf{z}_j = c_{\mathbf{x}} c_j^{-1} \mathbf{v}_j$. Then $\mathbf{z}_j = \rho(\mathbf{y}_j)$, and $\mathbf{x} - \mathbf{z}_j \in \ker(\theta_i^*)$ for all $i \geq j$ by Definition 2.4. The lemma follows from a straightforward induction on j. □

3 Normalization and companion operators

Similar to the previous section, we let $(F, ')$ be a differential field, C be the subfield of constants in F, and t be a monomial over F.

Our first attempt was to construct a complete reduction for $(F(t), ')$ under the assumption that there is a complete reduction for $(F, ')$. But such an assumption is insufficient as long as t is a hyperexponential monomial.

Example 3.1. *Let t be both regular and hyperexponential. Set $h = t'/t$. For an element $u \in F$, we determine whether $ut \in F(t)'$. By an order argument, $ut \in F(t)'$ if and only if $ut \in F[t, t^{-1}]'$. Then $ut \in F(t)'$ if and only if $ut = (vt)'$ for some $v \in F$ by Lemma 2.1 (ii). In other words, v is a solution of the Risch differential equation $y' + hy = u$. So complete reductions for $(F(t), ')$ are linked with Risch differential equations inevitably.*

We are going to construct a complete reduction ϕ_h for $(F(t), \mathcal{R}_h)$ for all $h \in F(t)$ under the assumption that a complete reduction for $(F, \mathcal{R}_z)$ is available for every element $z \in F$, where $\mathcal{R}_h : F(t) \to F(t)$ and $\mathcal{R}_z : F \to F$ are Risch operators given in Definition 1.8.

Example 3.2. *Let $F = C$ and $c \in C$. The Risch operator $\mathcal{R}_c : C \to C$ is given by $y \mapsto cy$ for all $y \in C$. Then $\operatorname{im}(\mathcal{R}_c) = C$ if $c \neq 0$, and $\operatorname{im}(\mathcal{R}_c) = \{0\}$ if $c = 0$. Hence, the complete reduction for $(C, \mathcal{R}_c)$ is the zero map if $c \neq 0$. Otherwise, it is the identity map.*

An element h of $F(t)$ is *t-normalized* if $\gcd(\operatorname{num}(h) - i \operatorname{den}(h)', \operatorname{den}(h)) = 1$ for all $i \in \mathbb{Z}$ (see [11, Section 3.2]). The gcd-condition is equivalent to that any nonzero residue of h is not an integer by [10, Theorem 4.4.3]. All elements of F are t-normalized.

This normalization simplifies our induction significantly in three aspects:

(i) It suffices to construct complete reductions for the images of Risch operators associated to t-normalized elements (see Proposition 3.3).

(ii) t-Normalized elements enable us to transform the problem of constructing complete reductions in $F(t)$ to that in $F\langle t \rangle$ (see Proposition 3.5).

(iii) The Risch operator associated to a t-normalized element h is injective if $h \in F(t) \setminus F$, and t is regular and is either primitive or hyperexponential over F (see Lemma 4.4).

An element of $F(t)$ is said to be *weakly t-normalized* if it does not have any positive integer residue (see [10, Section 6.1]). Note that (i) and (ii) given above hold for weakly t-normalized elements, but (iii) hinges on the assumption that h does not have any residue in $\mathbb{Z}^\times$.

For every element $h \in F(t)$, there exists a pair $(\xi, \eta) \in F(t) \times F(t)^\times$ such that ξ is t-normalized, and $h = \xi + \eta'/\eta$. We call (ξ, η) a *normal form* of h. Normal forms are not unique. Among them, the canonical one is computed by Algorithm GKS in [11, Section 3.2].

Proposition 3.3. *Let $h \in F(t)$ with a normal form (ξ, η). Then the following assertions hold.*

(i) $\mathcal{R}_h = \eta^{-1} \circ \mathcal{R}_\xi \circ \eta$, where η is understood as the map with $f \mapsto \eta f$ for all $f \in F(t)$.

(ii) Assume further that ϕ_ξ is a complete reduction for $(F(t), \mathcal{R}_\xi)$. Then $\phi_h := \eta^{-1} \circ \phi_\xi \circ \eta$ is a complete reduction for $(F(t), \mathcal{R}_h)$.

(iii) For all $f \in F(t)$, $\left(\eta^{-1} g, \eta^{-1} r\right)$ is an R-pair of f with respect to ϕ_h if (g, r) is an R-pair of ηf with respect to ϕ_ξ.

Proof. (i) For all $f \in F(t)$, $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \eta^{-1}\left((\eta f)' + \xi(\eta f)\right)$. Then $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \mathcal{R}_h(f)$ by $h = \xi + \eta'/\eta$ and a straightforward calculation.

(ii) Since ϕ_ξ is an idempotent, so is ϕ_h. Hence, $\operatorname{im}(\phi_h)$ is a complement of its kernel in $F(t)$. It suffices to verify that $\ker(\phi_h) = \operatorname{im}(\mathcal{R}_h)$. By (i), $\operatorname{im}(\mathcal{R}_h) = \eta^{-1} \cdot \operatorname{im}(\mathcal{R}_\xi)$. By $\phi_h = \eta^{-1} \circ \phi_\xi \circ \eta$, we have $\ker(\phi_h) = \eta^{-1} \cdot \ker(\phi_\xi)$. Thus, $\ker(\phi_h) = \operatorname{im}(\mathcal{R}_h)$ by $\ker(\phi_\xi) = \operatorname{im}(\mathcal{R}_\xi)$.

(iii) Let (g, r) be an R-pair of ηf with respect to ϕ_ξ. It follows from $\eta f = \mathcal{R}_\xi(g) + r$ and $r \in \operatorname{im}(\phi_\xi)$ that $f = \mathcal{R}_h(\eta^{-1} g) + \eta^{-1} r$, which, together with $\eta^{-1} r \in \operatorname{im}(\phi_h)$, implies that $\left(\eta^{-1} g, \eta^{-1} r\right)$ is an R-pair of f with respect to ϕ_h. □

In the rest of this section, we let $h \in F(t)$ be t-normalized, $a = \operatorname{num}(h)$ and $b = \operatorname{den}(h)$. Set

$$S_{t,h} := \{s \in S_t \mid \gcd(\operatorname{den}(s), b) = 1\}, \tag{2}$$

which is also a C-subspace. For all $f \in F(t)$, Algorithm GKSR in [11, Section 3.4] computes $(g, r, s) \in F(t) \times F\langle t \rangle \times S_{t,h}$ such that

$$f = \mathcal{R}_h(g) + \frac{r}{b} + s. \tag{3}$$

Moreover, s in the above equality is unique. Therefore,

$$F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot F\langle t \rangle\right) \oplus S_{t,h}. \tag{4}$$

This direct sum generalizes $F(t) = (F(t)' + F\langle t \rangle) \oplus S_t$ induced by Algorithm HermiteReduce in [10, Section 5.6]. It remains to find a C-subspace $W \subset F\langle t \rangle$ with

$$\operatorname{im}(\mathcal{R}_h) + \left(b^{-1} \cdot F\langle t \rangle\right) = \operatorname{im}(\mathcal{R}_h) \oplus \left(b^{-1} \cdot W\right).$$

Such a subspace is found via a polynomial reduction map in [4, Section 4.1] when $F(t) = C(x)$. This motivates us to define the notion of companion operators.

Definition 3.4. *The map*

$$\begin{array}{rccc} \mathcal{P}_h : & F\langle t \rangle & \to & F\langle t \rangle \\ & r & \mapsto & br' + ar. \end{array}$$

is called the companion operator *of* $\mathcal{R}_h$.

The following proposition reduces the problem of constructing a complete reduction for $(F(t), \mathcal{R}_h)$ to that for $(F\langle t \rangle, \mathcal{P}_h)$.

Proposition 3.5. *If* W *is a complement of* $\operatorname{im}(\mathcal{P}_h)$ *in* $F\langle t \rangle$*, then*

$$F(t) = \operatorname{im}(\mathcal{R}_h) \oplus \left(b^{-1} \cdot W \oplus S_{t,h}\right).$$

Proof. By (4), $F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot \operatorname{im}(\mathcal{P}_h) + b^{-1} \cdot W\right) \oplus S_{t,h}$. Then

$$F(t) = \left(\operatorname{im}(\mathcal{R}_h) + b^{-1} \cdot W\right) \oplus S_{t,h}$$

by $b^{-1} \cdot \operatorname{im}(\mathcal{P}_h) \subset \operatorname{im}(\mathcal{R}_h)$. It remains to show that $\operatorname{im}(\mathcal{R}_h) \cap \left(b^{-1} \cdot W\right) = \{0\}$. Assume that $v \in \operatorname{im}(\mathcal{R}_h) \cap \left(b^{-1} \cdot W\right)$. Then $v = w/b$ for some $w \in W$, and there exists $g \in F(t)$ such that $bg' + ag = w$. Since h is t-normalized, we have $g \in F\langle t \rangle$ by either [10, Theorem 6.1.2] or [11, Theorem 3.12 (iii)]. Hence, $w \in \operatorname{im}(\mathcal{P}_h)$. By $\operatorname{im}(\mathcal{P}_h) \cap W = \{0\}$, w is equal to 0, and so is v. □

Remark 3.6. *With the notation and direct sum given in the above proposition, we see that the projection $\Phi_{h,W}$ from $F(t)$ to $(b^{-1} \cdot W \oplus S_{t,h})$ is a complete reduction for $(F(t), \mathcal{R}_h)$. For every element $f \in F(t)$, an R-pair of f with respect to $\Phi_{h,W}$ can be computed as follows.*

Let $(g, r, s) \in F(t) \times F\langle t \rangle \times S_{t,h}$ satisfy (3), and $\pi_{h,W}$ be the projection from $F\langle t \rangle$ to W with respect to $F\langle t \rangle = \operatorname{im}(\mathcal{P}_h) \oplus W$. Then $\pi_{h,W}$ is a complete reduction for $(F\langle t \rangle, \mathcal{P}_h)$.

Let (u, v) be an R-pair of r with respect to $\pi_{h,W}$. It follows from (3), $r = \mathcal{P}_h(u) + v$ and $\mathcal{R}_h(u) = b^{-1}\mathcal{P}_h(u)$ that $f = \mathcal{R}_h(g) + b^{-1}(\mathcal{P}_h(u) + v) + s = \mathcal{R}_h(g + u) + b^{-1}v + s$. Since $b^{-1}v + s$ belongs to $(b^{-1} \cdot W \oplus S_{t,h})$, an R-pair of f with respect to $\Phi_{h,W}$ is $(g + u,\ b^{-1}v + s)$.

Remark 3.7. *If $h = 0$, then the direct sum in Proposition 3.5 becomes $F(t) = F(t)' \oplus (W \oplus S_t)$, which leads to a complete reduction for derivatives in $F(t)$.*

4 Induction hypothesis

By Proposition 3.5 and Remark 3.6, our task boils down to constructing a complete reduction for $(F\langle t \rangle, \mathcal{P}_h)$, where t is a regular monomial, and is either primitive or hyperexponential over F. It is realistic due to equation (5.1) in [10, §5.1], which states that $F\langle t \rangle = F[t]$ (resp. $F\langle t \rangle = F[t, t^{-1}]$) if t is primitive (resp. hyperexponential).

Convention 4.1. *In the next two sections, the following are assumed.*

(i) Let $n \in \mathbb{N}$ and $C(t_1, \ldots, t_{n-1}, t_n)$ be a transcendental Liouvillian extension of C. For $n \geq 1$, we set $F := C(t_1, \ldots, t_{n-1})$ and $t := t_n$.

(ii) Let h be a t-normalized element of $F(t)$.

(iii) Set $a := \operatorname{num}(h)$, $b := \operatorname{den}(h)$, $m := \max(\deg(a), \deg(b))$, $a := a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0$ and $b := b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0$, where $a_i, b_i \in F$ and $b_m = 1$ if it is nonzero.

(iv) Let Θ be an effective basis of F over C.

Unless mentioned otherwise, all linear spaces in the rest of this paper are regarded as linear spaces over C, and a basis means a C-basis. Note that $\mathcal{R}_h$ may stand for the Risch operator associated to h on either F or $F(t)$ when h belongs to F. We do not distinguish them, because the former is the restriction of the latter to F.

Hypothesis 4.2. *In the next two sections, we assume that there is a map that assigns to each element z of F a complete reduction ϕ_z for $(F, \mathcal{R}_z)$.*

By the hypothesis, $\operatorname{im}(\mathcal{R}_z) \cap \operatorname{im}(\phi_z) = \{0\}$ for all $z \in F$, and every element $f \in F$ has an R-pair (g, r) with respect to ϕ_z, that is, $f = \mathcal{R}_z(g) + r$ and $r = \phi_z(f)$ (see Definition 1.4).

The base case of our induction is given by the following example.

Example 4.3. *Let $F = C$. By Example 3.2, the map in Hypothesis 4.2 assigns to zero the identity operator and to every element of $C^\times$ the zero operator*

We show that one can construct a complete reduction for $(F\langle t \rangle, \mathcal{P}_h)$ in three steps:

1. Construct an auxiliary subspace $A_h \subset F\langle t \rangle$ such that $F\langle t \rangle = \operatorname{im}(\mathcal{P}_h) + A_h$.

2. Compute an echelon sequence of $\operatorname{im}(\mathcal{P}_h) \cap A_h$.

3. Determine a complement W_h of $\operatorname{im}(\mathcal{P}_h) \cap A_h$ in A_h.

Auxiliary subspaces come from our attempt to simplify elements of $F\langle t\rangle$ modulo $\operatorname{im}(\mathcal{P}_h)$. Such a simplification is an elaborate application of integration by parts when $h=0$ (see [13, 21]).

A basis of $\operatorname{im}(\mathcal{P}_h)\cap A_h$ is constructed based on two algorithms in [30, §4.3.1]. One is to recognize logarithmic derivatives in F (see [10, §5.12]). The other is to find parametric logarithmic derivatives in F (see [10, §7.3]). Although such a basis may not be finite, its regular shape described in Proposition 5.15 allows us to find an echelon sequence of the intersection, which yields a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F\langle t\rangle$ by Lemma 2.7, and leads to an algorithm for computing R-pairs by Lemma 2.11.

As the final preparation for the next two sections, we present a lemma allowing us to perform a basis transformation whenever h does not belong to F.

Lemma 4.4. *If $h\in F(t)\setminus F$, then both $\mathcal{R}_h$ and $\mathcal{P}_h$ are injective.*

Proof. Suppose that $u\in\ker(\mathcal{R}_h)^\times$. Then $h=-u'/u$ by Definition 1.8. The logarithmic derivative identity implies that h is a $\mathbb{Z}$-linear combination of logarithmic derivatives of polynomials in $F[t]$. Since h is t-normalized, no logarithmic derivative of any element in N_t appears in the $\mathbb{Z}$-linear combination effectively. It follows that $h\in F$, a contradiction. Thus, $\mathcal{R}_h$ is injective. Consequently, $\mathcal{P}_h$ is injective, because it is $b\mathcal{R}_h$ restricted to $F\langle t\rangle$. □

5 The primitive case

In this section, t is assumed to be primitive. Then $F\langle t\rangle=F[t]$. We define a C-linear operator

$$\begin{array}{rccc}\mathcal{L}: & F & \to & F\\ & z & \mapsto & b_m z' + a_m z\end{array} \tag{5}$$

to describe the coefficient of the possibly highest power in an element of $\operatorname{im}(\mathcal{P}_h)$. More precisely, let $f\in F[t]$ with $d=\deg(f)$ and $f_d=\operatorname{lc}(f)$. By Lemma 2.1 (i) and Definition 3.4, we have

$$\mathcal{P}_h(f)\equiv(b_m t^m)f_d' t^d+(a_m t^m)f_d t^d \mod F[t]_{<m+d}.$$

It follows that

$$\mathcal{P}_h(f)\equiv\mathcal{L}(f_d)t^{m+d} \mod F[t]_{<m+d}. \tag{6}$$

Remark 5.1. *By Convention 4.1 and* (5)*, the following two assertions hold.*

(i) If $\nu_\infty(h)<0$, then $a_m\neq 0$ and $b_m=0$. So $\mathcal{L}$ is injective.

(ii) If $\nu_\infty(h)\geq 0$, then $b_m=1$. By Definition 1.8, $\mathcal{L}=\mathcal{R}_{a_m}$.

The next lemma is will be frequently used in constructing echelon sequences.

Lemma 5.2. *If $u\in\ker(\mathcal{L})^\times$, then $\phi_{a_m}(ut')\neq 0$.*

Proof. Since $\ker(\mathcal{L})\neq\{0\}$, we have that $\nu_\infty(h)\geq 0$ by Remark 5.1 (i), and $u\in\ker(\mathcal{R}_{a_m})^\times$ by Remark 5.1 (ii). Suppose that $\phi_{a_m}(ut')=0$. Then $ut'\in\operatorname{im}(\mathcal{R}_{a_m})$. Hence, there exists $v\in F$ such that $ut'=v'+a_m v$. On the other hand, $u'+a_m u=0$ by $u\in\ker(\mathcal{R}_{a_m})^\times$. The two equalities involving a_m imply that $t'=(v/u)'$, a contradiction to [10, Theorem 5.1.1]. □

The rest of this section consists of three parts. We define the auxiliary subspace U_h of $F[t]$ and develop a corresponding reduction from $F[t]$ to U_h modulo $\operatorname{im}(\mathcal{P}_h)$ in Section 5.1. A criterion is presented for deciding if $\operatorname{im}(\mathcal{P}_h)\cap U_h=\{0\}$ in Section 5.2. An echelon sequence of the intersection and a complete reduction for $(F(t),\mathcal{R}_h)$ are presented in Section 5.3.

The results of this section have been presented in [20] when $h=0$.

5.1 Auxiliary subspaces

Let $f \in F[t]$, $d = \deg(f)$ and $f_d = \mathrm{lc}(f)$. Assume that $d \geq m$. We reduce the coefficients of f modulo $\mathrm{im}(\mathcal{P}_h)$ with the aid of the complete reduction ϕ_{a_m} for $(F, \mathcal{R}_{a_m})$ in Hypothesis 4.2.

Case 1. If $\nu_\infty(h) < 0$, then (5) and Remark 5.1 (i) imply that $\mathcal{L}(a_m^{-1} f_d) = f_d$. By (6), we have

$$f \equiv \mathcal{P}_h \left(a_m^{-1} f_d t^{d-m}\right) \mod F[t]_{<d}. \tag{7}$$

Case 2. Let $\nu_\infty(h) \geq 0$ and (g_d, r_d) be an R-pair of f_d with respect to ϕ_{a_m}. By Definitions 1.4 and 1.8, $f_d = \mathcal{R}_{a_m}(g_d) + r_d$, which, together with $\mathcal{L} = \mathcal{R}_{a_m}$ in Remark 5.1 (ii), implies that $f_d = \mathcal{L}(g_d) + r_d$. Then $f_d t^d = \mathcal{L}(g_d)t^d + r_d t^d$. So $f \equiv \mathcal{L}(g_d)t^d + r_d t^d \mod F[t]_{<d}$. It follows from (6) that

$$f \equiv \mathcal{P}_h \left(g_d t^{d-m}\right) + r_d t^d \mod F[t]_{<d}. \tag{8}$$

Applying the above two congruences to $F[t]_{<d}$ repeatedly, we arrive at the following definition.

Definition 5.3. *Set $U_h := F[t]_{<m}$ if $\nu_\infty(h) < 0$, and $U_h := F[t]_{<m} + \mathrm{im}(\phi_{a_m}) \otimes (t^m C[t])$ if $\nu_\infty(h) \geq 0$, where $\otimes$ is taken over C. We call U_h the* auxiliary subspace associated to $(F[t], \mathcal{P}_h)$.

Example 5.4. *Let $F = C$, $t = x$ and $' = d/dx$. If $h = 0$, then $m = 0$ and $a_0 = 0$. Thus, $U_h = C[x]$ by Example 4.3. If $h = x$, then $m = 1$ and $a_1 = 1$. So $U_h = C$ by the same example.*

Proposition 5.5. *Let $f \in F[t]^\times$ with $d = \deg(f)$. Then there exist $g \in F[t]$ and $r \in U_h$ with $\deg(g) \leq d - m$ and $\deg(r) \leq d$ such that $f = \mathcal{P}_h(g) + r$. Consequently, $F[t] = \mathrm{im}(\mathcal{P}_h) + U_h$.*

Proof. If $d < m$, then $f \in U_h$ by Definition 5.3. So it suffices to set $g := 0$ and $r := f$.

Assume that $d \geq m$, and that the conclusion holds for lower values of d. Let $f_d = \mathrm{lc}(f)$. If $\nu_\infty(h) < 0$, then (7) holds. Otherwise, $\nu_\infty(h) \geq 0$. Then $f \equiv \mathcal{P}_h(g_d t^{d-m}) \mod U_h + F[t]_{<d}$ by (8) and Definition 5.3. Hence, there exists a polynomial $p \in F[t]$ with $\deg(p) \leq d - m$ such that $f \equiv \mathcal{P}_h(p) \mod U_h + F[t]_{<d}$ for all values of $\nu_\infty(h)$. It follows from the induction hypothesis and $F[t]_{<m} \subset U_h$ that $f \equiv \mathcal{P}_h(g) \mod U_h$ for some $g \in F[t]$ with $\deg(g) \leq d - m$.

Let $r = f - \mathcal{P}_h(g)$. Then $r \in U_h$ and $\deg(r) \leq d$. □

Definition 5.6. *Let f and (g, r) be given in the above proposition. We call (g, r) an* auxiliary pair *of f with respect to $(F[t], \mathcal{P}_h)$ or an* auxiliary pair *of f if $(F[t], \mathcal{P}_h)$ is clear from context.*

The proof of Proposition 5.5 leads to Algorithm A.1 for computing auxiliary pairs, where the pseudo-code is presented in Section A.1.

Of particular interests in elementary integration is the case in which h belongs to F.

Corollary 5.7. *Let $f, h \in F$, and (g, r) be an auxiliary pair of f with respect to $(F[t], \mathcal{P}_h)$. Then (g, r) is an R-pair of f with respect to ϕ_h, where ϕ_h is the complete reduction for $(F, \mathcal{R}_h)$ given by Hypothesis 4.2.*

Proof. Note that $m = 0$, $b = b_0 = 1$ and $a = a_0 = h$ by Convention 4.1 and $h \in F$. It follows from Definitions 1.8 and 3.4 that $\mathcal{P}_h(z) = \mathcal{R}_h(z)$ for all $f \in F$.

By Definition 5.6, we have $g, r \in F$ with $f = \mathcal{P}_h(g) + r$, which, together with $\mathcal{P}_h(g) = \mathcal{R}_h(g)$, implies that $f = \mathcal{R}_h(g) + r$. Furthermore, $U_h = \mathrm{im}(\phi_h) \otimes C[t]$ by Definition 5.3, $m = 0$ and $h = a_0$. Then $r \in \mathrm{im}(\phi_h)$ by $r \in U_h$. Hence, $r = \phi_h(f)$ because both r and $\phi_h(f)$ are the projection of f from F to $\mathrm{im}(\phi_h)$ with respect to $F = \mathrm{im}(\mathcal{R}_h) \oplus \mathrm{im}(\phi_h)$. Consequently, (g, r) is an R-pair with respect to ϕ_h. □

5.2 Determining whether $\operatorname{im}(\mathcal{P}_h) \cap U_h$ is trivial

Set $I_h := \operatorname{im}(\mathcal{P}_h) \cap U_h$ in the rest of this section. We decide whether $I_h = \{0\}$ or not.

The next lemma connects I_h with $\ker(\mathcal{L})$.

Lemma 5.8. *If $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]$, then $\operatorname{lc}(f) \in \ker(\mathcal{L})$.*

Proof. Let $d = \deg(f)$ and $f_d = \operatorname{lc}(f)$. If $\nu_\infty(h) < 0$, then $U_h = F[t]_{<m}$ by Definition 5.3. So the degree of $\mathcal{P}_h(f)$ is less than m. It follows from (6) that $\mathcal{L}(f_d) = 0$. Otherwise, $\nu_\infty(h) \geq 0$. By Definition 5.3, $\mathcal{L}(f_d) \in \operatorname{im}(\phi_{a_m})$. By Remark 5.1 (ii), $\mathcal{L} = \mathcal{R}_{a_m}$. Since $\operatorname{im}(\mathcal{R}_{a_m}) \cap \operatorname{im}(\phi_{a_m})$ is equal to $\{0\}$, so is $\operatorname{im}(\mathcal{L}) \cap \operatorname{im}(\phi_{a_m})$. Accordingly, $\mathcal{L}(f_d) = 0$. $\square$

Proposition 5.9. *$I_h = \{0\}$ if and only if $\ker(\mathcal{L}) = \{0\}$. In particular, $I_h = \{0\}$ if $\nu_\infty(h) < 0$.*

Proof. Assume that $\ker(\mathcal{L}) = \{0\}$. Let $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]$. By Lemma 5.8, $\operatorname{lc}(f) = 0$ and so is f. Thus, $I_h = \{0\}$.

Conversely, assume that $I_h = \{0\}$. Let $u \in \ker(\mathcal{L})$. Then $\mathcal{P}_h(u) \in F[t]_{<m}$ by (6), and, hence, $\mathcal{P}_h(u) \in U_h$ by Definition 5.3. Consequently, $\mathcal{P}_h(u) = 0$ by $I_h = \{0\}$. It remains to show $u = 0$.

If $h \in F(t) \setminus F$, then $u = 0$ by Lemma 4.4. Otherwise, $h \in F$. Then $m = 0$, $b = b_0 = 1$ and $a = a_0 = h$. It follows from $u \in \ker(\mathcal{L})$ and $h \in F$ that $\mathcal{P}_h(ut) = ut'$. Since $ut' \in F$, it has an R-pair $(g, \phi_h(ut'))$ for some $g \in F$. Then $\mathcal{P}_h(ut) = \mathcal{R}_h(g) + \phi_h(ut')$. Again, $b = b_0 = 1$ and $a = a_0 = h$ imply that $\mathcal{P}_h(g) = \mathcal{R}_h(g)$. So $\mathcal{P}_h(ut) = \mathcal{P}_h(g) + \phi_h(ut')$. Consequently, $\mathcal{P}_h(ut - g) = \phi_h(ut')$, which, together with $\phi_h(ut') \in U_h$, implies that $\phi_h(ut') \in I_h$. We conclude that $\phi_h(ut') = 0$ by $I_h = \{0\}$. Then $u = 0$ by Lemma 5.2.

If $\nu_\infty(h) < 0$, then $\ker(\mathcal{L}) = \{0\}$ by Remark 5.1 (i). So $I_h = \{0\}$. $\square$

We distinguish whether I_h is trivial or not by types defined below.

Definition 5.10. *The* type *of I_h is set to be* 0 *if $I_h = \{0\}$, that is,* $\ker(\mathcal{L}) = \{0\}$*. Otherwise, the* type *is set to be an element $u \in \ker(\mathcal{L})^\times$.*

The type of I_h is unique up to a multiplicative constant in $C^\times$, because $\dim_C \ker(\mathcal{L}) \leq 1$. It can be found by recognizing logarithmic derivatives in F (see [10, §5.12] and [30, §4.3.1]).

Example 5.11. *Let $h = 0$. Then $m = 0$, $a_0 = 0$ and $b_0 = 1$. Thus, $\mathcal{L} = {}'$. It follows that the type of I_0 can be taken as* 1.

5.3 An echelon sequence of I_h

In this subsection, we assume that $I_h \neq \{0\}$, and construct an echelon sequence of I_h in the following three steps:

1. A basis of $\mathcal{P}_h^{-1}(I_h)$ is constructed in Lemma 5.14.
2. A basis of I_h is presented in Proposition 5.15 by mapping the basis of $\mathcal{P}_h^{-1}(I_h)$ to I_h via $\mathcal{P}_h$ with a minor modification.
3. The above basis of I_h is converted into an echelon basis in Corollaries 5.17 and 5.18.

Unless mentioned otherwise, we let the type of I_h be a fixed element $u \in \ker(\mathcal{L})^\times$ in the rest of this subsection. Then $\nu_\infty(h) \geq 0$ by Proposition 5.9. In particular, $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1 (ii).

For all $i \in \mathbb{N}_0$, Definition 3.4 and $\mathcal{L}(u) = 0$ imply that

$$\mathcal{P}_h(ut^i) \equiv (iut' + b_{m-1}u' + a_{m-1}u)t^{m+i-1} \mod F[t]_{<m+i-1}, \tag{9}$$

where a_{m-1} and b_{m-1} are set to be zero if $m = 0$. This congruence leads to the next definition.

Definition 5.12. *The* first R-pair associated to $(F[t], \mathcal{P}_h)$ *is an R-pair of* ut' *with respect to* ϕ_{a_m}*, and the* second *is an R-pair of* $b_{m-1}u' + a_{m-1}u$ *with respect to* ϕ_{a_m}.

Since R-pairs of an element with respect to ϕ_{a_m} may have distinct first components, we fix the first and second associated R-pairs, denoted by $(\tilde{v}, v)$ and $(\tilde{w}, w)$, once and for all in the sequel. Note that both a_{m-1} and b_{m-1} are equal to zero if $m = 0$. So $(\tilde{w}, w) := (0, 0)$ if $m = 0$.

Remark 5.13. *The element* v *in the first associated R-pair* $(\tilde{v}, v)$ *is nonzero by Lemma 5.2.*

First, we define a sequence $\{p_i\}_{i\in\mathbb{N}_0}$ in $F[t]$ such that $i = \deg(p_i)$, $u = \mathrm{lc}(p_i)$ and $p_i \in \mathcal{P}_h^{-1}(I_h)$.

For $i = 0$, we let $p_0 = u$. Then $\mathcal{P}_h(p_0) \in F[t]_{<m}$ by (6). So $\mathcal{P}_h(p_0) \in I_h$ by $F[t]_{<m} \subset U_h$, and, consequently, $p_0 \in \mathcal{P}_h^{-1}(I_h)$.

By Definitions 1.4 and 5.12, $ut' = \mathcal{R}_{a_m}(\tilde{v}) + v$ and $b_{m-1}u' + a_{m-1}u = \mathcal{R}_{a_m}(\tilde{w}) + w$. It follows from (9) and $\mathcal{L} = \mathcal{R}_{a_m}$ that, for all $i \in \mathbb{N}$,

$$\mathcal{P}_h(ut^i) \equiv \mathcal{L}(i\tilde{v} + \tilde{w})t^{m+i-1} + (iv + w)t^{m+i-1} \mod F[t]_{<m+i-1},$$

which, together with (6), implies that there exists a polynomial $g_i \in F[t]_{<m+i-1}$ such that

$$\mathcal{P}_h(ut^i) = \mathcal{P}_h\left((i\tilde{v} + \tilde{w})t^{i-1}\right) + (iv + w)t^{m+i-1} + g_i.$$

Let (q_i, r_i) be the auxiliary pair of g_i obtained from Algorithm A.1. Then $\deg(q_i) < i - 1$, $\deg(r_i) < m + i - 1$ by Proposition 5.5. Moreover,

$$\mathcal{P}_h(ut^i) = \mathcal{P}_h\left((i\tilde{v} + \tilde{w})t^{i-1}\right) + (iv + w)t^{m+i-1} + \mathcal{P}_h(q_i) + r_i.$$

Moving the images under $\mathcal{P}_h$ in the above equality to the left-hand side, we arrive at

$$\mathcal{P}_h\left(ut^i - (i\tilde{v} + \tilde{w})t^{i-1} - q_i\right) = (iv + w)t^{m+i-1} + r_i.$$

Set

$$p_i := ut^i - (i\tilde{v} + \tilde{w})t^{i-1} - q_i. \tag{10}$$

Then $\deg(p_i) = i$ and

$$\mathcal{P}_h(p_i) = (iv + w)t^{m+i-1} + r_i. \tag{11}$$

Since $v, w \in \mathrm{im}(\phi_{a_m})$ and $r_i \in U_h$, we see that $\mathcal{P}_h(p_i) \in U_h$, that is, $p_i \in \mathcal{P}_h^{-1}(I_h)$ for all $i \in \mathbb{N}$.

Lemma 5.14. *Let* $p_0 = u$ *and* p_i *be the same as those in* (10) *for all* $i \in \mathbb{N}$*. Then* $\{p_i\}_{i\in\mathbb{N}_0}$ *is a basis of* $\mathcal{P}_h^{-1}(I_h)$.

Proof. We have seen that $\{p_i\}_{i\in\mathbb{N}_0} \subset \mathcal{P}_h^{-1}(I_h)$. Since $\deg(p_i) = i$ for all $i \in \mathbb{N}_0$, the set $\{p_i\}_{i\in\mathbb{N}_0}$ is linearly independent. For $f \in F[t]$ with $\mathcal{P}_h(f) \in U_h$, we have $\mathrm{lc}(f) \in \ker(\mathcal{L})$ by Lemma 5.8. Then $\mathrm{lc}(f) = cu$ for some $c \in C$ by $\dim_C(\ker(\mathcal{L})) = 1$. Let $d = \deg(f)$. Then $f - cp_d \in \mathcal{P}_h^{-1}(I_h)$ with $\deg(f - cp_d) < d$. Thus, f is a linear combination of $p_d, \ldots, p_0$ by an induction on d. □

The sequence $\{p_i\}_{i\in\mathbb{N}_0}$ is called the *standard basis* of $\mathcal{P}_h^{-1}(I_h)$. It is unique, because u, $(\tilde{v}, v)$ and $(\tilde{w}, w)$ are all fixed, and the auxiliary pair (q_i, r_i) is computed by Algorithm A.1.

Proposition 5.15. *Let* r_i *be given in* (11) *for all* $i \in \mathbb{N}$*. Assume that* $\{p_i\}_{i\in\mathbb{N}_0}$ *is the standard basis of* $\mathcal{P}_h^{-1}(I_h)$.

(i) If $h \in F$*, then* $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ *is a basis of* I_h*, and* $\mathcal{P}_h(p_i) = ivt^{i-1} + r_i$ *with degree* $i - 1$.

(ii) If $h \in F(t) \setminus F$*, then* $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}_0}$ *is a basis of* I_h*,* $\mathcal{P}_h(p_0) \in F[t]^{\times}_{<m}$*, and, for all* $i \in \mathbb{N}$*,*

$$\mathcal{P}_h(p_i) = (iv + w)t^{m+i-1} + r_i.$$

Proof. (i) Since $h \in F$, we have that the restriction of $\mathcal{P}_h$ to F is equal to $\mathcal{L}$. Then $\mathcal{P}_h(p_0) = 0$ by $p_0 = u$. Hence, $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ spans I_h over C by Lemma 5.14. It follows from (11), $m = 0$ and $w = 0$ that $\mathcal{P}_h(p_i) = ivt^{i-1} + r_i$, which is of degree $i - 1$ by Remark 5.13 and $\deg(r_i) < i - 1$. Thus, $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ is linearly independent. It is a basis of I_h accordingly.

(ii) Since $h \in F(t) \setminus F$, the companion operator $\mathcal{P}_h$ is injective by Lemma 4.4, which, together with Lemma 5.14, implies that $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}_0}$ is a basis of I_h. Note that $\mathcal{P}_h(p_0) = \mathcal{P}_h(u)$, which is a nonzero polynomial of degree less than m by $u \in \ker(\mathcal{L})$ and (6). The rest is due to (11). □

The following example rediscovers the complete reduction in Example 1.2.

Example 5.16. *Let $F = C$, $t = x$ with $x' = 1$ and $h = 0$. Then $u = 1$ by Example 5.11. So $ux' = 1$ and $(\tilde{v}, v) = (0, 1)$. Moreover, $(\tilde{w}, w) = (0, 0)$ by $m = 0$. It follows from Proposition 5.15 (i) that I_0 has a basis $\{\mathcal{P}_0(p_i)\}_{i\in\mathbb{N}}$, in which the degree of $\mathcal{P}_0(p_i)$ is equal to $i - 1$. Then $I_0 = C[x]$, and, consequently, $C[x] = \operatorname{im}(\mathcal{P}_0)$. Hence, $\{0\}$ is the only complement of $\operatorname{im}(\mathcal{P}_0)$ in $C[x]$. By Proposition 3.5, $C(x) = C(x)' \oplus S_x$.*

At last, we construct an echelon sequence of I_h by Proposition 5.15. Recall that Θ is an effective basis of F by Convention 4.1 (iv). Then $\Gamma = \{\theta t^i \mid \theta \in \Theta, i \in \mathbb{N}_0\}$ is also an effective basis of $F[t]$ by [20, Remark 2.7]. Let us fix an element $\theta_v \in \Theta$ such that $\theta_v^*(v) \neq 0$ in the rest of this section. Such an element can be found by Remark 5.13 and the effectiveness of Θ.

Corollary 5.17. *Let $h \in F$ and $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}}$ be the basis of I_h given by Proposition 5.15 (i). Then $\left\{\left(p_i, \mathcal{P}_h(p_i), \theta_v t^{i-1}\right)\right\}_{i\in\mathbb{N}}$ is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Γ.*

Proof. It is immediate from Proposition 5.15 (i) and the definition of θ_v. □

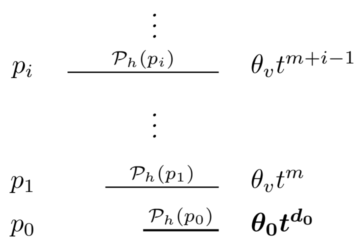
When $h \in F(t) \setminus F$, constructing an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Γ amounts to a careful and tedious case study, because the elements of $\{\mathcal{P}_h(p_i)\}_{i\in\mathbb{N}_0}$ in Proposition 5.15 (ii) do not necessarily have distinct degrees.

Corollary 5.18. *Let $h \in F(t) \setminus F$ and $\{p_i\}_{i\in\mathbb{N}_0}$ be the standard basis of $\mathcal{P}_h^{-1}(I_h)$. Furthermore, assume that d_0 and l_0 are the degree and leading coefficient of $\mathcal{P}_h(p_0)$, respectively, and θ_0 is an element of Θ such that $\theta_0^*(l_0) \neq 0$.*

(i) If $\theta_v^*(iv + w) \neq 0$ for all $i \in \mathbb{N}$, then

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \theta_v t^m\right), \left(p_2, \mathcal{P}_h(p_2), \theta_v t^{m+1}\right), \ldots, \left(p_i, \mathcal{P}_h(p_i), \theta_v t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h, which is described pictorially as



(ii) If $\theta_v^*(jv + w) = 0$ for some $j \in \mathbb{N}$ but $jv + w \neq 0$, then there exists an element $\theta \in \Theta$ with $\operatorname{lc}(\mathcal{P}_h(p_j)) \notin \ker(\theta^*)$ such that

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \theta_v t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_v t^{m+j-2}\right),$$

$$\left(p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1}\right), \left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \theta_v t^{m+j}\right), \ldots \left(p_i, \mathcal{P}_h(p_i), \theta_v t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h, which is described pictorially as

	$\vdots$	
p_i	$\mathcal{P}_h(p_i)$	$\theta_v t^{m+i-1}$
	$\vdots$	
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$\theta_v t^{m+j}$
p_j	$\mathcal{P}_h(p_j)$	$\boldsymbol{\theta t^{m+j-1}}$
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$\theta_v t^{m+j-2}$
	$\vdots$	
p_1	$\mathcal{P}_h(p_1)$	$\theta_v t^m$
p_0	$\mathcal{P}_h(p_0)$	$\boldsymbol{\theta_0 t^{d_0}}$

(iii) If $jv + w = 0$ for some $j \in \mathbb{N}$, then there exist $q \in F[t]^\times$, $d \in \mathbb{N}_0$ and $\theta \in \Theta$ with $\operatorname{lc}(\mathcal{P}_h(q)) \notin \ker(\theta^*)$ such that

$$\left(q, \mathcal{P}_h(q), \theta t^d\right), \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \theta_v t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_v t^{m+j-2}\right),$$
$$\left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \theta_v t^{m+j}\right), \ldots \left(p_i, \mathcal{P}_h(p_i), \theta_v t^{m+i-1}\right), \ldots$$

is an echelon sequence of I_h, which is described pictorially as

	$\vdots$	
p_i	$\mathcal{P}_h(p_i)$	$\theta_v t^{m+i-1}$
	$\vdots$	
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$\theta_v t^{m+j}$
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$\theta_v t^{m+j-2}$
	$\vdots$	
p_1	$\mathcal{P}_h(p_1)$	$\theta_v t^m$
p_0	$\mathcal{P}_h(p_0)$	$\boldsymbol{\theta_0 t^{d_0}}$
q	$\mathcal{P}_h(q)$	$\boldsymbol{\theta t^d}$

Proof. (i) The conclusion is immediate from Proposition 5.15 (ii).

(ii) Since $jv + w \neq 0$, we see that $\mathcal{P}_h(p_j)$ is of degree $m + j - 1$ by Proposition 5.15 (ii). Furthermore, $\theta_v^*(jv + w) = 0$ implies that $\theta_v^*(iv + w) \neq 0$ for all $i \in \mathbb{N}$ with $i \neq j$. Hence, the degree of $\mathcal{P}_h(p_i)$ is $m + i - 1$ for all $i \in \mathbb{N}$ with $i \neq j$. Let θ be an element of Θ be such that $\operatorname{lc}(\mathcal{P}_h(p_j)) \notin \ker(\theta^*)$. We see that (ii) holds.

(iii) Since $jv + w = 0$, the degree of $\mathcal{P}_h(p_j)$ is less than $m + j - 1$ by Proposition 5.15 (ii). Let V be the subspace spanned by $\mathcal{P}_h(p_0)$, $\mathcal{P}_h(p_1)$, …, $\mathcal{P}_h(p_{j-1})$ over C. Then

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left(p_1, \mathcal{P}_h(p_1), \theta_v t^m\right), \ldots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_v t^{m+j-2}\right)$$

is an echelon sequence of V. Let $\gamma_0 := \theta_0 t^{d_0}$, $\gamma_1 := \theta_v t^m$, …, $\gamma_{j-1} := \theta_v t^{m+j-2}$. It follows from Lemma 2.11 that there exists a polynomial $r \in F[t]$ such that $\mathcal{P}_h(p_j) - \mathcal{P}_h(r) \in \cap_{k=0}^{j-1} \ker(\gamma_k^*)$

and $\mathcal{P}_h(r) \in V$. Set $q := p_j - r$. Then $\mathcal{P}_h(q)$ lies in both I_h and $\cap_{k=0}^{j-1} \ker(\gamma_k^*)$. Hence, $\mathcal{P}_h(q), \mathcal{P}_h(p_0), \mathcal{P}_h(p_1), \ldots, \mathcal{P}_h(p_{j-1}), \mathcal{P}_h(p_{j+1}), \mathcal{P}_h(p_{j+2}), \ldots$ form a basis of I_h.

Set d to be the degree of $\mathcal{P}_h(q)$ and choose an element $\theta \in \Theta$ such that $\operatorname{lc}(\mathcal{P}_h(q)) \notin \ker(\theta^*)$. We conclude that (iii) holds. □

Let $E := \{(f_i, \mathcal{P}_h(f_i), \gamma_i)\}_{i\in\mathbb{N}}$ be an echelon sequence of I_h. By Lemma 2.7, the subspace

$$U_h \cap \big(\bigcap_{i\in\mathbb{N}} \ker(\gamma_i^*) \big) \tag{12}$$

is a complement of $\operatorname{im}(\mathcal{P}_h)$ in $F[t]$, which is called the *complement of* $\operatorname{im}(\mathcal{P}_h)$ *induced by* E.

The next example rediscovers the results of the case distinction made in [4, Section 4.1], which is the key to Hermite reduction for hyperexponential functions in Example 1.5.

Example 5.19. *Let $F = C$, $t = x$ and $' = d/dx$. We take $\Theta = \{1\}$ as a basis of C. Let h in Convention 4.1 (ii) be a nonzero element of $C(x)$. Then $U_h = C[x]_{<m}$ if $a_m \neq 0$, and $U_h = C[x]$ if $a_m = 0$ by Example 4.3.*

By Proposition 5.9, the type of I_h is zero if and only if $a_m \neq 0$. When this is the case, $U_h = C[x]_{<m}$, because ϕ_{a_m} is the zero operator on C. Thus, $C[x] = \operatorname{im}(\mathcal{P}_h) \oplus C[x]_{<m}$. This corresponds to Case 1 in [4, §4.1].

Assume that $a_m = 0$ in the rest of this example. Then $I_h \neq \{0\}$. Moreover, $U_h = C[x]$, because ϕ_0 is the identity operator on C. In addition, $m > 0$ by $h \neq 0$ and $a_m = 0$.

Since $a_m = 0$, we have $\ker(\mathcal{L}) = C$ by (5). Then the type of I_h is set to be 1. So $p_0 = 1$, which implies $\mathcal{P}_h(p_0) = a$. In particular, d_0 in Corollary 5.18 is equal to $\deg(a)$.

Set the first associated R-pair $(\tilde{v}, v) := (0, 1)$, the second pair $(\tilde{w}, w) := (0, a_{m-1})$, and $\theta_v := 1$. It suffices to examine whether $iv + w$ is zero for $i \in \mathbb{N}$, since θ_v^* is the identity operator on C.

Note that $iv + w = i + a_{m-1}$. An echelon sequence of I_h is given by Corollary 5.18 (i) if a_{m-1} is not a negative integer. Otherwise, such a sequence is given by Corollary 5.18 (iii). Pivots in the echelon sequences and bases of the induced complements are listed below.

$a_{m-1} = -j$	*Pivots in the echelon sequence*	*Basis of the induced complement*
$j \notin \mathbb{N}_0$	$x^{m-1}, \quad x^m, x^{m+1}, \ldots$	$1, x, \ldots, x^{m-2}$
$j = 0$	$x^{d_0}, \quad x^m, x^{m+1}, \ldots$	$1, x, \ldots, x^{d_0-1}, \quad x^{d_0+1}, \ldots, x^{m-1}$
$j \in \mathbb{N}$	$x^d, \quad x^{m-1}, x^m, \ldots, x^{m+j-2}, x^{m+j}, x^{m+j+1}, \ldots$	$1, x, \ldots, x^{d-1}, \quad x^{d+1}, \ldots, x^{m-2}, \quad x^{m+j-1}$

The results in the second, third and last row correspond to Cases 2, 3 and 4 in [4, Section 4.1], respectively.

A complementary subspace for $\operatorname{im}(\mathcal{P}_h)$ in $F[t]$ is of finite dimension over C if $F = C$ by Examples 5.16 and 5.19. But such a subspace is of infinite dimension if $\dim_C(F) = \infty$.

Example 5.20. *Let $F = C(x)$, $t = \log(x)$ and d/dx be the derivation on $F(t)$. Let*

$$h = \frac{2x^2 - 2t}{xt^2 + x}.$$

We determine an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Γ, where

$$\Gamma = \{\theta t^i \mid \theta \in \Theta, i \in \mathbb{N}_0\}$$

and Θ is given by equation (2) in [20]. By Convention 4.1,

a	b	m	a_m	b_m	a_{m-1}	b_{m-1}
$-2x^{-1}t+2x$	t^2+1	2	0	1	$-2x^{-1}$	0

Since $a_m = 0$, we may further assume that the complete reduction ϕ_0 in Hypothesis 4.2 is induced by the Hermite-Ostrogradsky reduction. Then the auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ is equal to $F[t]_{<2} + S_x \otimes \left(t^2 C[t]\right)$ by Definition 5.3 and $\operatorname{im}(\phi_0) = S_x$.

Note that $\mathcal{L} = d/dx$ by $a_2 = 0$ and $b_2 = 1$. Then we can set the type u of I_h to be 1. Then $(\tilde{v}, v) = \left(0, x^{-1}\right)$ and $(\tilde{w}, w) = \left(0, -2x^{-1}\right)$ by Algorithm A.1. Since $v = x^{-1}$, we choose θ_v given before Corollary 5.17 to be x^{-1}.

We need Corollary 5.18 (iii) to construct an echelon sequence E, because $2v + w = 0$. At first, we compute p_0 and p_1 in the standard basis $\{p_i\}_{i\in\mathbb{N}_0}$ of $\mathcal{P}_h^{-1}(I_h)$.

By $u = 1$, we have that $p_0 = 1$. Then $\mathcal{P}_h(p_0) = -2x^{-1}t + 2x$. Hence, $d_0 = 1$. Choose θ_0 to be x^{-1}. Then the second member of E is

$$\left(p_0,\, \mathcal{P}_h(p_0),\, \theta_0 t^{d_0}\right) = \left(1,\, -2x^{-1}t + 2x,\, x^{-1}t\right). \tag{13}$$

Note that $\theta_0 = \theta_v$ is just a coincidence. By (10) and $\tilde{v} = \tilde{w} = 0$, we see that $p_1 = t$. Thus, the third member of E is

$$\left(p_1,\, \mathcal{P}_h(p_1),\, \theta_v t^2\right) = \left(t,\, -x^{-1}t^2 + 2xt + x^{-1},\, x^{-1}t^2\right). \tag{14}$$

Since $2v + w = 0$, we compute the first member of E by p_2. By (10) and Algorithm A.1, we find that $p_2 := t^2 - x^2$. Then $\mathcal{P}_h(p_2) = (2x + 2x^{-1})t - 2x^3 - 2x$. Eliminating $x^{-1}t$ from $\mathcal{P}_h(p_2)$ by $\mathcal{P}_h(p_0)$ yields $q = t^2 - x^2 + 1$ and $\mathcal{P}_h(q) = 2xt - 2x^3$. Choose θ to be x. Then the first member of the echelon sequence is

$$\left(q,\, \mathcal{P}_h(q),\, \theta t\right) = \left(t^2 - x^2 + 1,\, 2xt - 2x^3,\, xt\right). \tag{15}$$

For $k > 3$, the kth member of E can be found by (10), (11) and Algorithm A.1. Its pivot is fixed to be $x^{-1}t^{k-1}$ by Corollary 5.18 (iii). We summarize the results in the following table.

u	$(\tilde{v}, v)$	$(\tilde{w}, w)$	θ_v	j	$(q, \mathcal{P}_h(q), \theta t)$	$(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0})$	$(p_1, \mathcal{P}_h(p_1), \theta_v t^2)$
1	$\left(0, x^{-1}\right)$	$\left(0, -2x^{-1}\right)$	x^{-1}	*2*	*r.h.s. of* (15)	*r.h.s. of* (13)	*r.h.s. of* (14)

where j stands for the positive integer with $jv + w = 0$.

We have made three choices for pivots in the above example. Together with (10) and Algorithm A.1, these choices determine an echelon sequence of I_h uniquely. In the appendix, we introduce the notion of initial sequences in Definition A.2. Such a sequence is finite and contains all the information to determine a unique echelon sequence. In fact, the last table in Example 5.20 records an initial sequence of I_h. Moreover, Algorithm A.3 computes initial sequences, and Algorithm A.4 generates members of the echelon sequence determined by an initial sequence, as many as required. We are ready to present the main result of this section.

Theorem 5.21. *With Hypothesis 4.2, one can construct a complete reduction for $(F(t), \mathcal{R}_h)$.*

Proof. If the type u of I_h is zero, then we set W to be the auxiliary subspace U_h. Otherwise, let $E = \{(q_i, Q_i, \gamma_i)\}_{i\in\mathbb{N}}$ be an echelon sequence of I_h, where γ_i belongs to $\{\theta t^i \mid \theta \in \Theta, i \in \mathbb{N}\}$. Moreover, let W be the complement of $\operatorname{im}(\mathcal{P}_h)$ induced by E. Since $F[t] = \operatorname{im}(\mathcal{P}_h) \oplus W$, a complete reduction Φ_h for $(F(t), \mathcal{R}_h)$ is given by Proposition 3.5.

It remains to describe an algorithm for computing R-pairs with respect to Φ_h. Let Ψ_h be the projection from $F[t]$ to W with respect to $F[t] = \operatorname{im}(\mathcal{P}_h) \oplus W$. By Remark 3.6, it suffices to develop an algorithm for computing R-pairs with respect to Ψ_h.

For every element $p \in F[t]$, an auxiliary pair (q, r) of p can be found by Algorithm A.1. If $u = 0$, then (q, r) is an R-pair with respect to Ψ_h. Otherwise, one can compute $\tilde{q} \in F[t]$ and $\tilde{r} \in I_h$ such that $\tilde{r} = \mathcal{P}_h(\tilde{q})$ and $r - \tilde{r} \in \cap_{i \in \mathbb{N}} \ker(\gamma_i^*)$ by Lemma 2.11, in which $U = F[t]$, $\rho = \mathcal{P}_h$ and $V = I_h$. On the other hand, $r - \tilde{r} \in U_h$ by $r \in U_h$ and $\tilde{r} \in U_h$. Thus, $r - \tilde{r} \in W$ because W is defined by (12). It follows that $(q + \tilde{q},\, r - \tilde{r})$ is an R-pair of p with respect to Ψ_h. □

At the end of this section, we connect the complete reduction Φ_h for $(F(t), \mathcal{R}_h)$ in the above theorem with ϕ_h for $(F, \mathcal{R}_h)$ in Hypothesis 4.2 when $h \in F$. Such a connection is important for elementary integration in Section 8.

Corollary 5.22. *Let Φ_h be the complete reduction for $(F(t), \mathcal{R}_h)$ constructed in the proof of Theorem 5.21. Assume that $h \in F$, and that ϕ_h is the complete reduction for $(F, \mathcal{R}_h)$ given in Hypothesis 4.2. Then, for all $f \in F$, $\Phi_h(f) = \phi_h(f)$ if $I_h = \{0\}$, and, otherwise, there exists $c \in C$ such that $\Phi_h(f) = \phi_h(f) + c\phi_h(ut')$, where u is the type of I_h.*

Proof. For every $f \in F$, its projection to $S_{t,h}$ is zero, where $S_{t,h}$ is given in (2) and the projection is with respect to the direct sum in Proposition 3.5.

Let (g, r) be an auxiliary pair of f. Then $r = \phi_h(f)$ by Corollary 5.7. If $I_h = \{0\}$, then $r = \Phi_h(f)$. Otherwise, let $(\tilde{v}, v)$ be the first R-pair associated to $(F(t), \mathcal{R}_h)$, where $v = \phi_h(ut')$ by Definition 5.12. Then an echelon sequence E of I_h is given by Corollary 5.17, in which the first member is $(ut - \tilde{v}, v, \theta_v)$. Since $\deg(r) \le 0$ and $\deg(v) = 0$, the projection of r to the complement induced by E is equal to $r + cv$, where $c = -\theta_v^*(v)^{-1} \cdot \theta_v^*(r)$. The corollary holds by noting that $\Phi_h(f) = r + cv$ and $v = \phi_h(ut')$. □

6 The hyperexponential case

In this section, t is assumed to be hyperexponential. Then $F\langle t \rangle = F[t, t^{-1}]$. For every integer d, we define two C-linear operators:

$$\begin{array}{rccc} \mathcal{H}_d: & F & \to & F \\ & z & \mapsto & b_m z' + \left(a_m + d b_m \dfrac{t'}{t}\right) z \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathcal{T}_d: & F & \to & F \\ & z & \mapsto & b_0 z' + \left(a_0 + d b_0 \dfrac{t'}{t}\right) z \end{array} \tag{16}$$

to describe the coefficients of the possibly highest and lowest powers in an element of $\operatorname{im}(\mathcal{P}_h)$. Namely, for an element $f \in F[t, t^{-1}]^\times$ with $k = \operatorname{hdeg}(f)$, $f_k = \operatorname{hc}(f)$, $l = \operatorname{tdeg}(f)$ and $f_l = \operatorname{tc}(f)$,

$$\mathcal{P}_h(f) = \mathcal{H}_k(f_k) t^{m+k} + \cdots + \mathcal{T}_l(f_l) t^l \tag{17}$$

by Lemma 2.1 (ii). It follows that $\mathcal{P}_h(f^+) \in F[t]$ and $\mathcal{P}_h(f^-) \in F[t]_{<m} + t^{-1}F[t^{-1}]$.

Remark 6.1. *By Convention 4.1 and (16), the following assertions hold for all $k, l \in \mathbb{Z}$.*

(i) If $\nu_\infty(h) < 0$, then $a_m \neq 0$, $b_m = 0$ and $\ker(\mathcal{H}_k) = \{0\}$.

(ii) If $\nu_\infty(h) \ge 0$, then $b_m = 1$ and $\mathcal{H}_k = \mathcal{R}_{\lambda_k}$, where $\lambda_k = a_m + k\dfrac{t'}{t}$.

(iii) If $\nu_t(h) < 0$, then $a_0 \neq 0$, $b_0 = 0$, and $\ker(\mathcal{T}_l) = \{0\}$.

(iv) If $\nu_t(h) \ge 0$, then $b_0 \neq 0$ and $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$, where $\mu_l = \dfrac{a_0}{b_0} + l\dfrac{t'}{t}$.

The λ_k's and μ_l's defined above will be used throughout the rest of this section.

In this section, we will mainly prove conclusions concerning tail degrees. In the head-degree case, arguments are similar to those in the previous section. Actually, the head-degree case is easier than the tail-degree case because $b_m = 1$ whenever it is nonzero.

The next lemma is a reformulation of [10, Lemma 6.2.2] in terms of $\mathcal{H}_k$ and $\mathcal{T}_l$.

Lemma 6.2. *There exists at most an integer k (resp. an integer l) such that* $\ker(\mathcal{H}_k) \neq \{0\}$ *(resp.* $\ker(\mathcal{T}_l) \neq \{0\}$*).*

Proof. Suppose that $i, j \in \mathbb{Z}$ such that $u_i \in \ker(\mathcal{T}_i)^\times$ and $u_j \in \ker(\mathcal{T}_j)^\times$. Then $\nu_t(h) \geq 0$ by Remark 6.1 (iii). Thus, $b_0 \neq 0$ and $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$ by Remark 6.1 (iv). It follows from $\mathcal{T}_i(u_i) = 0$ and $\mathcal{T}_j(u_j) = 0$ that $u_i'/u_i + \mu_i = 0$ and $u_j'/u_j + \mu_j = 0$. Hence, $(i-j)t'/t$ is a logarithmic derivative in F. Since t is both regular and hyperexponential, we see that $i = j$ by [10, Theorem 5.1.2]. The conclusion on $\ker(\mathcal{H}_k)$ is proved similarly by Remark 6.1 (i) and (ii). □

The rest of this section is organized in the same way as in Section 5, In terms of content, there are two differences. First, the definition of auxiliary subspaces is more complicated due to the presence of negative powers. Second, the construction of echelon sequences becomes much simpler, since the intersection of $\mathrm{im}(\mathcal{P}_h)$ with an auxiliary subspace is of dimension at most two. The results in this section generalize those in [22, Chapter 4], in which t'/t is a derivative in F.

6.1 Auxiliary subspaces

Let $f \in F[t, t^{-1}]$ with $k = \mathrm{hdeg}(f)$, $f_k = \mathrm{hc}(f)$, $l = \mathrm{tdeg}(f)$ and $f_l = \mathrm{tc}(f)$. Assume further that $k \geq m$ and $l < 0$. Similar to the case distinction at the beginning of Section 5.1, we have the following four congruences.

Case 1. If $\nu_\infty(h) < 0$, then (17) and Remark 6.1 (i) imply that

$$f^+ \equiv \mathcal{P}_h\left(a_m^{-1} f_k t^{k-m}\right) \mod F[t]_{<k}. \tag{18}$$

Case 2. Let $\nu_\infty(h) \geq 0$ and (g_k, r_k) be an R-pair of f_k with respect to $\phi_{\lambda_{k-m}}$. By (17), Remark 6.1 (ii), and the same argument used in Case 2 in Section 5.1, we have

$$f^+ \equiv \mathcal{P}_h(g_k t^{k-m}) + r_k t^k \mod F[t]_{<k}. \tag{19}$$

Case 3. If $\nu_t(h) < 0$, then (17) and Remark 6.1 (iii) imply that

$$f^- \equiv \mathcal{P}_h\left(a_0^{-1} f_l t^l\right) \mod F[t]_{<m} + t^{-1}F[t^{-1}]_{>l}. \tag{20}$$

Case 4. Let $\nu_t(h) \geq 0$ and (g_l, r_l) be an R-pair of $b_0^{-1} f_l$ with respect to ϕ_{μ_l}. By Definition 1.4, we have that $b_0^{-1} f_l = \mathcal{R}_{\mu_l}(g_l) + r_l$, which, together with $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$ in Remark 6.1 (iv), implies that $f_l = \mathcal{T}_l(g_l) + b_0 r_l$. Then $f_l t^l = \mathcal{T}_l(g_l) t^l + (b_0 r_l) t^l$. So

$$f^- \equiv \mathcal{T}_l(g_l) t^l + (b_0 r_l) t^l \mod F[t]_{<m} + t^{-1}F[t^{-1}]_{>l}.$$

It follows from (17) that

$$f^- \equiv \mathcal{P}_h(g_l t^l) + (b_0 r_l) t^l \mod F[t]_{<m} + t^{-1}F[t^{-1}]_{>l}. \tag{21}$$

This case distinction leads to the notion of auxiliary subspaces in the hyperexponential case.

Definition 6.3. *The auxiliary subspace associated to* $(F[t,t^{-1}],\mathcal{P}_h)$ *is*

$$V_h = V_h^+ + F[t]_{<m} + V_h^-,$$

where

$$V_h^+ = \begin{cases} \{0\} & \text{if } \nu_\infty(h) < 0, \\ \sum_{i\geq m} \operatorname{im}\left(\phi_{\lambda_{i-m}}\right)\cdot t^i & \text{if } \nu_\infty(h)\geq 0, \end{cases} \quad \text{and } V_h^- = \begin{cases} \{0\} & \text{if } \nu_t(h) < 0, \\ \sum_{j<0} b_0 \cdot \operatorname{im}\left(\phi_{\mu_j}\right)\cdot t^j & \text{if } \nu_t(h)\geq 0 \end{cases}$$

with $\operatorname{im}\left(\phi_{\lambda_{i-m}}\right)\cdot t^i = \left\{zt^i \mid z\in \operatorname{im}\left(\phi_{\lambda_{i-m}}\right)\right\}$ *and* $b_0\cdot \operatorname{im}\left(\phi_{\mu_j}\right)\cdot t^j = \left\{b_0 z t^j \mid z \in \operatorname{im}\left(\phi_{\mu_j}\right)\right\}$.

Example 6.4. *Let* $F = C$ *and* $t'/t = 1$*, that is,* t *models* $\exp(x)$*. Let* $h = 0$*. Then* $\lambda_k = k$ *by Remark 6.1 (ii) and* $\mu_l = l$ *by Remark 6.1 (iv). So* $V_h^+ = C$ *and* $V_h^- = \{0\}$ *by Example 4.3. Consequently,* $V_h = C$*.*

Similar to Proposition 5.5 and Definition 5.6, we have

Proposition 6.5. *Let* $f\in F[t,t^{-1}]$*,* $k = \operatorname{hdeg}(f)$ *and* $l = \operatorname{tdeg}(f)$*. Then*

(i) $f^+ = \mathcal{P}_h(p) + r$ *for some* $p \in F[t]$ *with* $\operatorname{hdeg}(p)\leq k-m$ *and* $r\in V_h$*,*

(ii) $f^- = \mathcal{P}_h(q) + s$ *for some* $q\in t^{-1}F[t^{-1}]$ *with* $\operatorname{tdeg}(q)\geq l$ *and* $s\in V_h$*.*

Consequently, $F[t,t^{-1}] = \operatorname{im}(\mathcal{P}_h) + V_h$*.*

Proof. (i) Using (18) and (19), we prove (i) in the same way as in the proof of Proposition 5.5.

(ii) If $f^- = 0$, then we set $q := 0$ and $s := 0$. Assume that $f^-\neq 0$. Then let $f_l = \operatorname{tc}(f^-)$.

If $\nu_t(h) < 0$, then (20) holds. Otherwise, $\nu_t(h)\geq 0$. Then $b_0\neq 0$ by Remark 6.1 (iv). Let (g_l, r_l) be an R-pair of $b_0^{-1}f_l$ with respect to ϕ_{μ_l}. It follows from (21) and $b_0 r_l t^l\in V_h^-$ that

$$f^- \equiv \mathcal{P}_h(g_l t^l) \mod V_h + t^{-1}F[t^{-1}]_{>l}.$$

Thus, there exists $g\in t^{-1}F[t^{-1}]$ with $\operatorname{tdeg}(g)\geq l$ such that $f^-\equiv \mathcal{P}_h(g) \mod V_h + t^{-1}F[t^{-1}]_{>l}$ for all values of $\nu_t(h)$. The conclusion holds by an induction on l.

By (i) and (ii), $f = \mathcal{P}_h(p+q) + r + s$ and $r + s \in V_h$. Hence, $F[t,t^{-1}] = \operatorname{im}(\mathcal{P}_h) + V_h$. □

Definition 6.6. *Let* f, p, q, r *and* s *be given in the above proposition. We call* $(p+q, r+s)$ *an* auxiliary pair *of* f *with respect to* $(F[t,t^{-1}],\mathcal{P}_h)$ *or an* auxiliary pair *of* f *if* $(F[t,t^{-1}],\mathcal{P}_h)$ *is clear from context.*

Auxiliary pairs always exist by Proposition 6.5. Based on (18), (19), (20) (21) and the proof of Proposition 6.5, an algorithm is developed for computing such pairs (see Algorithm A.6).

Corollary 6.7. *Let* $f, h\in F$ *and* (g, r) *be an auxiliary pair of* f *with respect to* $(F[t,t^{-1}],\mathcal{P}_h)$*. Then* (g,r) *is an R-pair of* f *with respect to* ϕ_h*, where* ϕ_h *is given by Hypothesis 4.2.*

Proof. Note that λ_0 defined in Remark 6.1 (ii) is equal to h by $h\in F$. The rest is similar to the proof of Corollary 5.7. □

6.2 Determining whether $\mathrm{im}(\mathcal{P}_h) \cap V_h$ is trivial

In the rest of this section, we let $J_h = \mathrm{im}(\mathcal{P}_h) \cap V_h$. The next two lemmas connect J_h with the kernels of $\mathcal{H}_d$ and $\mathcal{T}_d$ in (16). We will prove the second assertions in these two lemmas. Their first assertions can be shown likewise.

Lemma 6.8. *Let $f \in F[t, t^{-1}]^\times$, $k = \mathrm{hdeg}(f)$ and $l = \mathrm{tdeg}(f)$. Assume that $\mathcal{P}_h(f) \in V_h$. Then*

(i) $\mathrm{hc}(f) \in \ker(\mathcal{H}_k)$ *if $k \geq 0$, and*

(ii) $\mathrm{tc}(f) \in \ker(\mathcal{T}_l)$ *if $l < 0$.*

Proof. (ii) Let $f_l = \mathrm{tc}(f)$. Suppose on the contrary that $\mathcal{T}_l(f_l) \neq 0$. Then $\mathcal{T}_l(f_l)t^l \in V_h^-$ by $\mathcal{P}_h(f) \in V_h$. In particular, $V_h^- \neq \{0\}$. Hence, $\nu_t(h) \geq 0$ and $\mathcal{T}_l(f_l) \in b_0 \cdot \mathrm{im}(\phi_{\mu_l})$ by Definition 6.3. By Remark 6.1 (iv), $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$. Thus, $\mathcal{R}_{\mu_l}(f_l) \in \mathrm{im}(\phi_{\mu_l})$. Since $\mathrm{im}(\mathcal{R}_{\mu_l}) \cap \mathrm{im}(\phi_{\mu_l}) = \{0\}$, we have that $\mathcal{T}_l(f_l) = 0$, a contradiction. □

Lemma 6.9. *(i) If $\ker(\mathcal{H}_k) \neq \{0\}$ for some $k \in \mathbb{N}_0$, then, for all $u \in \ker(\mathcal{H}_k)^\times$, there exists $p \in F[t]$ with $\mathrm{hdeg}(p) = k$ and $\mathrm{hc}(p) = u$ such that $\mathcal{P}_h(p) \in V_h$. Moreover, for every element $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists an element $c \in C$ such that $f - cp \in t^{-1}F[t^{-1}]$.*

(ii) If $\ker(\mathcal{T}_l) \neq \{0\}$ for some $l \in \mathbb{N}^-$, then, for all $v \in \ker(\mathcal{T}_l)^\times$, there exists $q \in t^{-1}F[t^{-1}]$ with $\mathrm{tdeg}(q) = l$ and $\mathrm{tc}(q) = v$ such that $\mathcal{P}_h(q) \in V_h$. Moreover, for every element $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists an element $c \in C$ such that $f - cq \in F[t]$.

Proof. (ii) By Remark 6.1 (iii), $\nu_t(h) \geq 0$. By (17) and $v \in \ker(\mathcal{T}_l)$, $\mathcal{P}_h(vt^l) \in t^{-1}F[t^{-1}] + F[t]_{<m}$ has tail degree greater than l. Let (g, r) be an auxiliary pair of $\mathcal{P}_h\left(vt^l\right)$. Then

$$\mathcal{P}_h\left(vt^l\right) = \mathcal{P}_h(g) + r,$$

$g \in t^{-1}F[t^{-1}]$ with $\mathrm{tdeg}(g) > l$ and $r \in V_h$ by Proposition 6.5 (ii). Set $q = vt^l - g$. Then $\mathrm{tdeg}(q) = l$, $v = \mathrm{tc}(q)$ and $\mathcal{P}_h(q) \in V_h$.

Assume that $f \in F[t, t^{-1}]$ with $d = \mathrm{tdeg}(f)$ and $\mathcal{P}_h(f) \in V_h$. If $d \geq 0$, then we set $c := 0$. Otherwise, $\mathrm{tc}(f) \in \ker(\mathcal{T}_d)$ by Lemma 6.8 (ii). It follows from Lemma 6.2 that $d = l$. Since $\ker(\mathcal{T}_l)$ is one-dimensional over C, there exists $c \in C$ such that $\mathrm{tc}(f) = cv$. Then $\mathcal{P}_h(f - cq) \in V_h$ and $\mathrm{tdeg}(f - cq) > l$. So $\mathrm{tdeg}(f - cq) \geq 0$ by Lemma 6.8 (ii) and Lemma 6.2. □

A criterion for $J_h = \{0\}$ is given below.

Proposition 6.10. *$J_h = \{0\}$ if and only if one of the following two conditions holds.*

(i) $h \in F$,

(ii) $\ker(\mathcal{H}_k) = \{0\}$ *for all $k \in \mathbb{N}_0$ and* $\ker(\mathcal{T}_l) = \{0\}$ *for all $l \in \mathbb{N}^-$.*

In particular, $J_h = \{0\}$ if both $\nu_\infty(h)$ and $\nu_t(h)$ are negative.

Proof. Assume that $J_h = \{0\}$ and $h \notin F$. Suppose on the contrary that $\ker(\mathcal{H}_k)$ is nontrivial for some $k \in \mathbb{N}_0$. Then there exists $p \in F[t]^\times$ such that $\mathcal{P}_h(p) \in V_h$ by Lemma 6.9 (i). Moreover, $\mathcal{P}_h(p) \neq 0$ by Lemma 4.4. So J_h is nontrivial, a contradiction. The same contradiction is reached by Lemma 6.9 (ii) and Lemma 4.4 if $\ker(\mathcal{T}_l)$ is nontrivial for some $l \in \mathbb{N}^-$.

To show the converse, we let $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$. It suffices to show that $\mathcal{P}_h(f) = 0$.

First, we consider the case in which $h \in F$. Then $m = 0$, $b_0 = 1$, $\nu_\infty(h) \geq 0$ and $\nu_t(h) \geq 0$. It follows from Definition 6.3 that $V_h = \sum_{i \in \mathbb{N}_0} \mathrm{im}(\phi_{\lambda_i}) \cdot t^i + \sum_{j \in \mathbb{N}^-} \mathrm{im}(\phi_{\mu_j}) \cdot t^j$, where $\lambda_i = it'/t + a_0$ and $\mu_j = jt'/t + a_0$ by Remark 6.1 (ii) and (iv).

Set $f = \sum_{i=l}^{k} f_i t^i$ with $f_i \in F$. We have $\mathcal{P}_h(f) = \sum_{i=0}^{k} \mathcal{R}_{\lambda_i}(f_i)t^i + \sum_{j=l}^{-1} \mathcal{R}_{\mu_j}(f_j)t^j$ by Lemma 2.1 (ii) and $\mathcal{P}_h(f) = f' + a_0 f$. Hence, $\mathcal{R}_{\lambda_i}(f_i) \in \operatorname{im}(\phi_{\lambda_i})$ for all $i \in \mathbb{N}_0$ and $\mathcal{R}_{\mu_j}(f_j) \in \operatorname{im}(\phi_{\mu_j})$ for all $j \in \mathbb{N}^-$. Thus, $\mathcal{P}_h(f) = 0$ by $\operatorname{im}(\mathcal{R}_{\lambda_i}) \cap \operatorname{im}(\phi_{\lambda_i}) = \{0\}$ and $\operatorname{im}(\mathcal{R}_{\mu_j}) \cap \operatorname{im}(\phi_{\mu_j}) = \{0\}$.

Next, we assume that $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and that $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$. Suppose on the contrary that $\mathcal{P}_h(f) \neq 0$. If $k = \operatorname{hdeg}(f) \geq 0$, then $f_k \in \ker(\mathcal{H}_k)^\times$ by Lemma 6.8 (i), a contradiction. So $f^+ = 0$. Thus, we may further assume that $\operatorname{tdeg}(f) = l < 0$. It follows from Lemma 6.8 (ii) that $f_l \in \ker(\mathcal{T}_l)$, a contradiction.

If both $\nu_\infty(h)$ and $\nu_t(h)$ are negative, then $\mathcal{H}_k$ and $\mathcal{T}_l$ are injective by Remark 6.1 (i) and (iii). So $J_h = \{0\}$ by the second condition. ☐

Similar to Definition 5.10, we define the notion of types as follows.

Definition 6.11. *The* type *of J_h is defined to be*

(i) 0 *if either $h \in F$, or, $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$,*

(ii) (k, u) *if $h \notin F$, $u \in \ker(\mathcal{H}_k)^\times$ for some $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$,*

(iii) (l, v) *if $h \notin F$, $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $v \in \ker(\mathcal{T}_l)^\times$ for some $l \in \mathbb{N}^-$,*

(iv) $(k, u), (l, v)$ *if $h \notin F$, $u \in \ker(\mathcal{H}_k)^\times$ for some $k \in \mathbb{N}_0$ and $v \in \ker(\mathcal{T}_l)^\times$ for some $l \in \mathbb{N}^-$.*

The type of J_h can be computed by an algorithm for solving the parametric logarithmic derivative problem in F (see [30, Section 4.3.1]). The type is zero if and only if $J_n = \{0\}$ by Proposition 6.10. If it is nonzero, then k, l are unique by Lemma 6.2, and u, v are unique up to a multiplicative constant in $C^\times$. Similar to the primitive case, the type is fixed once and for all.

Example 6.12. *Let $F = C$, $t = \exp(x)$ and $h = -t^2/(t^2 + t + 1)$. Then $m = 2$, $a_m = -1$, $b_m = 1$, $a_0 = 0$ and $b_0 = 1$. It follows from Remark 6.1 (ii) and (iv) that $\mathcal{H}_k(z) = (k-1)z$ and $\mathcal{T}_l(z) = lz$ for all $k \in \mathbb{N}_0$, $l \in \mathbb{N}^-$ and $z \in F$. Hence, the type of J_h is equal to $(1, 1)$.*

6.3 An echelon sequence of J_h

Unlike the intersection I_h in the primitive case, we have

Proposition 6.13. *(i)* $\dim_C(J_h) = 1$ *if the type of J_h is equal to (i, w) for some $i \in \mathbb{Z}$.*

(ii) $\dim_C(J_h) = 2$ *if the type of J_h is equal to $(k, u), (l, v)$ for some $k \in \mathbb{N}_0$ and $l \in \mathbb{N}^-$.*

Proof. Since the type of J_h is nonzero, we have that $h \in F(t) \setminus F$ by Proposition 6.10. Consequently, $\mathcal{P}_h$ is injective by Lemma 4.4.

(i) Let us consider the case in which $i < 0$. By Lemma 6.9 (ii), there exists $q \in t^{-1}F[t^{-1}]$ with $\operatorname{tdeg}(q) = i$ such that $\mathcal{P}_h(q) \in V_h$ and that, for every $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, $g := f - cq \in F[t]$ for some $c \in C$. Then $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f) \in V_h$ and $\mathcal{P}_h(q) \in V_h$. Suppose that $g \neq 0$. Let $k = \operatorname{hdeg}(g)$. Then $k \geq 0$. By Lemma 6.8 (i), $\ker(\mathcal{H}_k) \neq \{0\}$, a contradiction to the type of J_h. Hence, $g = 0$, and, consequently, $\mathcal{P}_h(f) = c\mathcal{P}_h(q)$. Note that $\mathcal{P}_h(q) \neq 0$ by the injectivity of $\mathcal{P}_h$. We have that $\dim_C(J_h) = 1$. Likewise, one shows that $\dim_C(J_h) = 1$ if $i \geq 0$.

(ii) By Lemma 6.9, there exist $p \in F[t]$ with $\operatorname{hdeg}(p) = k$ and $q \in t^{-1}F[t^{-1}]$ with $\operatorname{tdeg}(q) = l$ such that $\mathcal{P}_h(p), \mathcal{P}_h(q) \in V_h$. Let $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$. Then $g := f - c_p p \in t^{-1}F[t^{-1}]$ for some $c_p \in C$ by Lemma 6.9 (i). Moreover, $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f) \in V_h$ and $\mathcal{P}_h(p) \in V_h$. It follows from Lemma 6.9 (ii) that $g - c_q q \in F[t]$ for some $c_q \in C$. Then $g - c_q q = 0$ because both g and q belong to $t^{-1}F[t^{-1}]$. Thus, $f = c_p p + c_q q$. Consequently, $\mathcal{P}_h(f) = c_p\mathcal{P}_h(p) + c_q\mathcal{P}_h(q)$. Since $p \in F[t]$ and $q \in t^{-1}F[t^{-1}]$, they are linearly independent. So are $\mathcal{P}_h(p)$ and $\mathcal{P}_h(q)$ by the injectivity of $\mathcal{P}_h$. Thus, $\dim_C(J_h) = 2$. ☐

Similar to Section 5, we let $\Gamma = \{\theta t^i \mid i \in \mathbb{Z}\}$. It is an effective basis of $F[t, t^{-1}]$ by Convention 4.1 (iv) and [20, Remark 2.7].

Proposition 6.14. *If J_h is nontrivial, then one can construct an echelon sequence of J_h with respect to $\big(F[t, t^{-1}], \mathcal{P}_h\big)$ and Γ.*

Proof. Assume that $\dim_C(J_h) = 1$. Then $J_h = \mathrm{span}_C\{\mathcal{P}_h(p)\}$ for some $p \in F[t, t^{-1}]$. Let d and w be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively. We choose an element $\theta \in \Theta$ such that $\theta^*(w) \neq 0$. Then $\big(p, \mathcal{P}_h(p), \theta t^d\big)$ is an echelon sequence of J_h

Otherwise, $\dim_C(J_h) = 2$ by Proposition 6.13. Then $J_h = \mathrm{span}_C\{\mathcal{P}_h(p), \mathcal{P}_h(q)\}$ for some $p, q \in F[t, t^{-1}]$. Let d and w be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively, and let $\theta_p \in \Theta$ with $\theta_p^*(w) \neq 0$. Set $\gamma_p := \theta_p t^d$, $c_p := \gamma_p^*\left(\mathcal{P}_h(p)\right)$, $c_q := \gamma_p^*\left(\mathcal{P}_h(q)\right)$, and $r := q - c_p^{-1} c_q p$. Then $\mathcal{P}_h(r)$ belongs to $\ker(\gamma_p^*)$ and $\{\mathcal{P}_h(p), \mathcal{P}_h(r)\}$ is also a basis of J_h. Let e and z be the head degree and head coefficient of $\mathcal{P}_h(r)$, respectively, and choose $\theta_r \in \Theta$ with $\theta_r^*(z) \neq 0$. Set $\gamma_r = \theta_r t^e$. Then $(r, \mathcal{P}_h(r), \gamma_r)$, $(p, \mathcal{P}_h(p), \gamma_p)$ is an echelon sequence of J_h. □

An algorithm for constructing an echelon sequence of J_h can be easily developed according to the proofs of Propositions 6.13 and 6.14 (see Algorithm A.7 in the appendix). For the sake of completeness, the echelon sequence is set to be NIL if the type of J_h is equal to 0.

Corollary 6.15. *Let E be an echelon sequence of J_h.*

(i) If E is NIL*, then V_h is a complement of $\mathrm{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.*

(ii) If E is $(p, \mathcal{P}_h(p), \gamma)$, then $V_h \cap \ker(\gamma^*)$ is a complement of $\mathrm{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.

(iii) If E is $(p_1, \mathcal{P}_h(p_1), \gamma_1)$, $(p_2, \mathcal{P}_h(p_2), \gamma_2)$, then $V_h \cap \ker(\gamma_1^*) \cap \ker(\gamma_2^*)$ is a complement of $\mathrm{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.

Proof. It is immediate from Proposition 6.5 and Lemma 2.7. □

The complement in the above proposition is also called the *complement of* $\mathrm{im}(\mathcal{P}_h)$ *in* $F[t, t^{-1}]$ *induced by E*. The main result of this section is

Theorem 6.16. *With Hypothesis 4.2, one can construct a complete reduction for $(F(t), \mathcal{R}_h)$.*

Proof. It is similar to the proof of Theorem 5.21. □

Corollary 6.17. *Let $h \in F$, Φ_h be the complete reduction for $(F(t), \mathcal{R}_h)$ in Theorem 6.16, and ϕ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.2. Then $\Phi_h(f) = \phi_h(f)$ for all $f \in F$.*

Proof. By Proposition 6.10 and $h \in F$, we have $J_h = \{0\}$. The corollary follows from Corollary 6.7 and the same argument used for proving that $\Phi_h(f) = \phi_h(f)$ for all $f \in F$ when $I_h = \{0\}$ in Corollary 5.22. □

7 Completing the induction

In this section, we let $F_n = C(t_1, \ldots t_n)$ be a transcendental Liouvillian extension given in Definition 1.7. In addition, $F_0 = C$ during an induction or a recursion.

Theorem 7.1. *For every element $h \in F_n$, one can construct a complete reduction for $(F_n, \mathcal{R}_h)$.*

Proof. We proceed by induction on n. If $n = 0$, then the conclusion holds by Example 3.2.

Let $n > 0$. Assume that, for every $z \in F_{n-1}$, one can construct a complete reduction $\phi_{n-1,z}$ for $(F_{n-1}, \mathcal{R}_z)$, and assume that the map in Hypothesis 4.2 is defined by $z \mapsto \phi_{n-1,z}$.

For $h \in F_n$, let (ξ, η) be the canonical form of h. Then one can construct a complete reduction for $(F_n, \mathcal{R}_\xi)$ by either Theorem 5.21 or Theorem 6.16. It follows from Proposition 3.3 (ii) that a complete reduction $\phi_{n,h}$ for $(F_n, \mathcal{R}_h)$ is obtained.

The map in Hypothesis 4.2 is then redefined by $h \mapsto \phi_{n,h}$ for all $h \in F_n$. □

In practice, we need to maintain some initial data in order to avoid choosing elements from a set in different ways (see [20, Remark 2.5]). Let $\Theta_0 = \{1\}$. For each $i \in [n]$, we have an effective basis Θ_i of F_i described in [20, Section 2.2]. Furthermore, we set up a table $\mathbb{T}$ indexed by (i, z), where $i \in [n]$ and $z \in F_i$ is t_i-normalized. The value of (i, z) is an initial (resp. echelon) sequence of (i, z), where $i \geq 1$ and t_i is primitive (resp. hyperexponential). See Definition A.2 for the notion of initial sequences. The table can be understood as the map in Hypothesis 4.2 for each F_i with $i \in [n]$. It is global and will be updated as long as a complete reduction for $(F_i, \mathcal{R}_z)$ is constructed. These considerations lead to

Outline 7.2. *Given $f, h \in F_n$, we compute an R-pair of f with respect to $\phi_{n,h}$.*

1. (*Base case*) *If $n = 0$, then an R-pair of f w.r.t. $\phi_{n,h}$ is $(f/h, 0)$ if $h \neq 0$, and $(0, f)$ if $h = 0$. The algorithm terminates.* (*See Example 3.2 and 4.3*)

2. (*Preprocessing*)

   (2.1) *Find the canonical form (ξ, η) of h w.r.t. t_n by Algorithm* GKS *in [11, Section 3.2].*

   (2.2) *Compute $(g, r, s) \in F_n \times F_{n-1}\langle t_n \rangle \times S_{t_n,\xi}$ s.t. $\eta f = \mathcal{R}_\xi(g) + r/b + s$ by Algorithm* GKSR *in [11, Section 3.4], where $b = \operatorname{den}(\xi)$ and $S_{t_n,\xi}$ is given in (2).* (*See (3)*)

   (2.3) *If $r = 0$, then an R-pair of f w.r.t. $\phi_{n,h}$ is $(\eta^{-1} g, \eta^{-1} s)$. The algorithm terminates.* (*See Propositions 3.5 and 3.3*)

3. (*Recursion*) *Assume that t_n is primitive (resp. hyperexponential) over F_{n-1}.*

   (3.1) *Compute an auxiliary pair (p, q) of r w.r.t. $(F_{n-1}\langle t_n \rangle, \mathcal{P}_\xi)$ by Algorithm A.1 (resp. Algorithm A.6);*

   (3.2) *If $q = 0$, an R-pair of f w.r.t. $\phi_{n,h}$ is $(\eta^{-1}(g + p), \eta^{-1} s)$. The algorithm terminates.* (*See Propositions 5.5, 6.5 and Remark 3.6*).

   (3.3) (*Updating $\mathbb{T}$*) *If (n, ξ) is an index of $\mathbb{T}$, then set L to be the value of (n, ξ). Otherwise, compute an initial sequence L of I_h by Algorithm A.3 (resp. an echelon sequence E. of J_h by Algorithm A.7), and add $(n, \xi) = L$ (resp. $(n, \xi) = E$) to $\mathbb{T}$.*

   (3.4) (*trivial intersection*) *If $L =$ NIL or $L[1] = 0$, then an R-pair of f w.r.t. $\phi_{n,h}$ is $\left(\eta^{-1}(g + p), \eta^{-1}(q/b + s)\right)$. The algorithm terminates.* (*See Propositions 5.9, 6.10 and Remark 3.6*)

   (3.5) (*Nontrivial intersection*) *By Algorithm A.5 (resp. Algorithm A.8), compute $\tilde{q} \in F_{n-1}\langle t_n \rangle$ and $\tilde{r} \in W$ such that $q = \mathcal{P}_\xi(\tilde{q}) + \tilde{r}$, where W is the complement of $\operatorname{im}(\mathcal{P}_\xi)$ in $F_{n-1}\langle t_n \rangle$ induced by L (resp. E). An R-pair of f w.r.t. $\phi_{n,h}$ is*

   $$\left(\eta^{-1}(g + p + \tilde{q}), \eta^{-1}(\tilde{r}/b + s)\right).$$

   (*See Theorems 5.21, 6.16 and Remark 3.6*)

In the following two examples, all steps are referred to as those in the above outline. The first example provides details of Example 1.10.

Example 7.3. *Determine whether* $f = \frac{x}{1+\exp(x)} \cdot \exp\left(\frac{x}{1+\exp(x)}\right)$ *is a derivative in the transcendental elementary extension* $(\mathbb{C}(x,t,y), d/dx)$*, where* $t = \exp(x)$ *and* $y = \exp\left(\frac{x}{1+\exp(x)}\right)$.

Note that $f = gy$*, where* $g = \frac{x}{1+\exp(x)}$*. To avoid describing too many recursions, we set* $h := y'/y$*, which is equal to* g'*. By the same argument as in Example 3.1, it suffices to compute an R-pair of* g *with respect to the complete reduction* $\phi_{2,h}$ *for* $(\mathbb{C}(x,t), \mathcal{R}_h)$.

Initially, we set $F_0 = \mathbb{C}$, $F_1 = F_0(x)$, $F_2 = F_1(t)$*, and the table* $\mathbb{T}$ *to be empty.*

Since h *is a derivative in* F_2*, it is* t*-normalized. So* $(\xi, \eta) = (h, 1)$ *in step 2.1. With Convention 4.1 and* $h = \xi$*, we have*

a	b	m	a_m	b_m	a_0	b_0
$1+(1-x)t$	$(1+t)^2$	2	0	1	1	1

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b) + 0$*, where* $r = x + xt$*. Step 2.3 is skipped.*

In step 3.1, Algorithm A.6 computes an auxiliary pair $(p, q) = (0, x + xt)$ *of* r*, where* r *belongs to the auxiliary subspace* V_h *associated to* $(F_1[t, t^{-1}], \mathcal{P}_h)$.

Step 3.2 is skipped since q *is nonzero. In step 3.3, we find that* $(2, h)$ *is not an index of* $\mathbb{T}$*. So an echelon sequence of* J_h *is constructed as follows.*

First, we obtain the type of J_h *is* $(0,1), (-1,1)$ *by solving a parametric logarithmic derivative problem, where* $J_h = \operatorname{im}(\mathcal{P}_h) \cap V_h$ *as in Section 6.2. Second, Algorithm A.6 computes the auxiliary pairs of* $\mathcal{P}_h(1)$ *and* $\mathcal{P}_h(t^{-1})$*, which are* $(0, -xt + t + 1)$ *and* $(0, -t - 1 - x)$*, respectively. Third, choose a pivot* xt *for* $\mathcal{P}_h(1)$ *and a pivot* t *for* $\mathcal{P}_h(t^{-1})$*, we have an echelon sequence*

$$E : (t^{-1}, -t - 1 - x, t),\ (1, -xt + t + 1, xt)$$

of J_h*. At last, the entry* $(2, h) = E$ *is added to* $\mathbb{T}$.

Step 3.4 is skipped since E *is not* NIL.

In step 3.5, Algorithm A.8 projects q *to* $\operatorname{im}(\mathcal{P}_h)$ *and the complement of* $\operatorname{im}(\mathcal{P}_h)$ *induced by* E*. The respective projections are* $\mathcal{P}_h(-1 - t^{-1})$ *and* 0*. Then an R-pair of* g *with respect to* $\phi_{2,h}$ *is* $(-1 - t^{-1}, 0)$*, that is,* $g = \mathcal{R}_h(-1 - t^{-1})$*. It follows from* $h = y'/y$ *and* $f = gy$ *that* $f = \left(\left(-1 - t^{-1}\right) y\right)'$*. In functional notation, we have*

$$\int f = -\left(1 + \exp(-x)\right) \cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

In the second example, F_n is neither a primitive tower nor an exponential one.

Example 7.4. *Let* $t = \log(x)$*. Determine whether*

$$f = \frac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2+1)} \cdot \exp\left(\int \frac{2x^2 - 2t}{xt^2 + x}\right)$$

is a derivative in $\mathbb{Q}(x, t, f)$*, where the derivation is* d/dx.

We rewrite f *in another form so as to make intermediate expressions more compact.*

Let $y = \exp\left(\int \frac{2x^2 - 2t}{xt^2 + x}\right)$*. Then* $f = gy$*, where* $g = \frac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2+1)}$. *Moreover,* $\mathbb{Q}(x, t, f) = \mathbb{Q}(x, t, y)$*. It follows from the same reasoning in Example 3.1 that* f

belongs to $\mathbb{Q}(x,t,y)'$ *if and only if there exists* $w \in \mathbb{Q}(x,t)$ *such that* $\mathcal{R}_h(w) = g$*, where*

$$h := \frac{y'}{y} = \frac{2x^2 - 2t}{xt^2 + x}.$$

Thus, it suffices to compute an R-pair of g *with respect to the complete reduction* $\phi_{2,h}$ *for* $(\mathbb{Q}(x,t), \mathcal{R}_h)$ *given in the proof of Theorem 7.1.*

Let $F_0 = \mathbb{Q}$*,* $F_1 = F_0(x)$ *and* $F_2 = F_1(t)$*. Then it is* t*-normalized by a straightforward verification. So* $(\xi, \eta) = (h, 1)$ *in step 2.1. With Convention 4.1 and* $\xi = h$*, we have*

a	b	m	a_m	b_m	a_0	b_0
$-2x^{-1}t + 2x$	t^2+1	2	0	1	$2x$	1

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b)$*, where* $r = -x^{-2}t^3 - x^{-2}t^2 + (2x+2-x^{-2})t - 2x^3 + x^{-2}$*.*

In step 3.1, we compute an auxiliary pair

$$(p,q) = \left(x^{-1}t, 2xt - 2x^3\right) \tag{22}$$

of r *with respect to* $(F_1[t], \mathcal{P}_h)$*. During the process of the auxiliary reduction, an R-pair of* $\mathrm{lc}_t(r)$ *with respect to* $\phi_{1,0}$ *is computed. So an entry* $(1,0) = 1, (0,1), (0,0), 1$ *is added to* $\mathbb{T}$*.*

Note that h *is the same as that in Example 5.20, in which the last table presents an initial sequence* L *with* $L[1] = 1, L[2] = (0, x^{-1}), L[3] = (0, 2x^{-1}), L[4] = x^{-1}, L[5] = 2,$

$$L[6] = \left(t^2 - x^2 + 1, 2xt - 2x^3, xt\right), \; L[7] = \left(1, -2x^{-1}t + 2x, x^{-1}t\right),$$

and $L[8] = \left(t, -x^{-1}t^2 + 2xt + x^{-1}, x^{-1}t^2\right)$*. Step 3.3 is then completed by adding* $(2,h) = L$ *to* $\mathbb{T}$*.*

Step 3.4 is skipped because $L[1] \neq 0$*.*

Let E *denote the echelon sequence induced by* L*. In step 3.5, we project* q *in* (22) *to* $\mathrm{im}(\mathcal{P}_h)$ *and* W*, respectively, where* W *is the complement of* $\mathrm{im}(\mathcal{P}_h)$ *induced by* E*. Since* $\deg_t(q) = 1$*, there is no need to compute more members in* E*. In fact,* q *is equal to the second component of* $L[6]$*. Then the respective projections of* q *are* $\mathcal{P}_h(t^2 - x^2 + 1)$ *and* 0*. It follows that an R-pair of* g *with respect to* $\phi_{2,h}$ *is* $(x^{-1}t + t^2 - x^2 + 1, 0)$*. In other words,* $g = \mathcal{R}_h(x^{-1}t + t^2 - x^2 + 1)$*, which implies that* $f = (gy)'$*. In functional notation, we have*

$$\int f = \left(\frac{\log(x)}{x} + \log(x)^2 - x^2 + 1\right) \cdot \exp\left(\int \frac{2x^2 - 2\log(x)}{x\log^2(x) + x}\right).$$

The same integral can be computed by the `int()` *command with option* `method=parallelrisch` *in* MAPLE *2023 and the* `Integrate[]` *command in* MATHEMATICA *14.3.*

8 Applications

Similar to the previous section, we let $n \in \mathbb{N}$ and $F_n = C(t_1, \ldots, t_n)$ be a transcendental Liouvillian extension. In addition, C is denoted by F_0 during an induction or a recursion. For all $i \in [n]_0$, we let ψ_i be the complete reduction $\phi_{i,0}$ for $(F_i, ')$ given in the proof of Theorem 7.1.

To determine elementary integrability, we need to distinguish primitive generators from hyperexponential ones. Set $\mathbb{P} := \{i \in [n] \mid t_i' \in F_{i-1}\}$ and $\mathbb{H} := [n] \setminus \mathbb{P}$. Moreover,

$$P := \mathrm{span}_C\{\psi_{i-1}(t_i') \mid i \in \mathbb{P}\} \quad \text{and} \quad H := \mathrm{span}_C\{\psi_{j-1}(t_j'/t_j) \mid j \in \mathbb{H}\}.$$

The following two technical lemmas present some useful properties of remainders.

Lemma 8.1. *For $i \in [n]_0$ and $f \in F_i$, $\psi_n(f) - \psi_i(f) \in P$.*

Proof. Let $k \geq i$. By Example 5.11 and Corollary 5.22, we have that $\psi_{k+1}(f) - \psi_k(f) \in P$ if $k+1 \in \mathbb{P}$. By Corollary 6.17, $\psi_{k+1}(f) - \psi_k(f) = 0$ if $k+1 \in \mathbb{H}$. It follows from

$$\psi_n(f) - \psi_i(f) = \sum_{k=i}^{n-1} (\psi_{k+1}(f) - \psi_k(f))$$

that $\psi_n(f) - \psi_i(f) \in P$. □

Let S_i be the set of t_i-simple elements in $F_{i-1}(t_i)$ for all $i \in [n]$, that is, $S_i := S_{t_i}$. Set $S := S_1 + \cdots + S_n$. The sum is direct. An element of F_n is said to be *simple* if it belongs to S.

Lemma 8.2. *For every element $s \in S$, $s - \psi_n(s) \in P$.*

Proof. Let $s = \sum_{i\in[n]} s_i$, where $s_i \in S_i$. By Proposition 3.5 and Remark 3.6, $\psi_i(s_i) = s_i$ for all $i \in [n]$. Then $s - \psi_n(s) = \sum_{i\in[n]} (\psi_i(s_i) - \psi_n(s_i))$ by $\psi_n(s) = \sum_{i\in[n]} \psi_n(s_i)$. The lemma follows from Lemma 8.1. □

Next, we define residues of elements in S without referring to a particular generator of F_n. For a simple element s, there exists a unique element $s_i \in S_i$ for each $i \in [n]$ such that $s = s_1 + \cdots + s_n$. Regarding s_i as an element of $F_{i-1}(t_i)$, we say that a residue of s_i is a *residue of s*.

We are ready to generalize [20, Theorem 5.3] from primitive towers to F_n.

Theorem 8.3. *Let C be algebraically closed. Then an element f of F_n has an elementary integral over F_n if and only if*

(i) there exists a simple element s such that $\psi_n(f) \equiv s \mod H + P$, and

(ii) all residues of s belong to C.

Proof. Assume that both (i) and (ii) hold. By (ii) and Lemma 2.3, s has an elementary integral over F_n. By (i), it suffices to show that every element of $H + P$ has an elementary integral over F_n. If $i \in \mathbb{H}$, then $t_i'/t_i = u_i' + \psi_{i-1}(t_i'/t_i)$ for some $u_i \in F_{i-1}$, which implies that $\psi_{i-1}(t_i'/t_i)$ is the derivative of $\log(t_i) - u_i$. If $i \in \mathbb{P}$, then $t_i' = v_i' + \psi_{i-1}(t_i')$ for some $v_i \in F_{i-1}$. Therefore, $\psi_{i-1}(t_i') \in F_n'$. Accordingly, $\psi_n(f)$ has an elementary integral over F_n, and so does f.

Conversely, assume that f has an elementary integral over F_n. Then there exists a C-linear combination g of logarithmic derivatives in F_n such that $f \equiv g \mod F_n'$ by [10, Theorem 5.5.2]. Since $\psi_n(F_n') = \{0\}$, we have $\psi_n(f) = \psi_n(g)$. It suffices to show that $\psi_n(g)$ satisfies (i) and (ii).

By Lemma 2.2 and the logarithmic derivative identity, there exists $s \in S$ whose residues are all constants, such that $g - s$ is a C-linear combination of t_i'/t_i for $i \in \mathbb{H}$. Then $\psi_n(g) - \psi_n(s) \in H + P$ by Lemma 8.1. So $\psi_n(g) - s \in H + P$ by Lemma 8.2. Both (i) and (ii) hold. □

To compute elementary integrals by the above theorem, we set $R_i := t_i F_{i-1}[t_i]$ if $i \in \mathbb{P}$, and $R_i := t_i F_{i-1}[t_i] + t_i^{-1} F_{i-1}[t_i^{-1}]$ if $i \in \mathbb{H}$. Moreover, let $R = C + R_1 + \cdots + R_n$. This sum is evidently direct, and so is $R + S$.

Proposition 8.4. *With the notation just introduced, we have that* $\operatorname{im}(\psi_n) \subset R \oplus S$.

Proof. We proceed by induction on n. For $n = 0$, $\operatorname{im}(\psi_0) = C$ by Example 3.2. So $\operatorname{im}(\psi_0) \subset R$. The conclusion holds. Assume that $n > 0$ and that the conclusion holds for $n - 1$. We need to consider the cases in which n belongs to either $\mathbb{P}$ or $\mathbb{H}$ separately.

Case 1. Assume $n \in \mathbb{P}$. The auxiliary subspace U_0 associated to $(F_{n-1}[t_n], \mathcal{P}_0)$ is $\operatorname{im}(\psi_{n-1}) \otimes C[t]$, which is contained in $\operatorname{im}(\psi_{n-1}) + R_n$. Since U_0 contains the complement W of $\operatorname{im}(\mathcal{P}_0)$ induced by an echelon sequence of $\operatorname{im}(\mathcal{P}_0) \cap U_0$, we have $W \subset \operatorname{im}(\psi_{n-1}) + R_n$. It follows from Remark 3.7 that $\operatorname{im}(\psi_n) = W \oplus S_n$. So $\operatorname{im}(\psi_n) \subset \operatorname{im}(\psi_{n-1}) + R_n + S_n$. Thus, $\operatorname{im}(\psi_n) \subset R + S$ by the induction hypothesis.

Case 2. Assume $n \in \mathbb{H}$. The auxiliary subspace V_0 associated to $(F[t_n, t_n^{-1}], \mathcal{P}_0)$ is equal to

$$\sum_{k \in \mathbb{N}} \operatorname{im}(\phi_{n-1,\lambda_k}) \cdot t_n^k + \operatorname{im}(\phi_{n-1,\lambda_0}) + \sum_{j \in \mathbb{N}^-} \operatorname{im}(\phi_{n-1,\mu_l}) \cdot t_n^l,$$

where $\phi_{n-1,\lambda_k}, \phi_{n-1,\lambda_0}$ and ϕ_{n-1,μ_l} are the complete reductions on F_{n-1} given by Theorem 7.1. It is contained in $\operatorname{im}(\phi_{n-1,\lambda_0}) + R_n$. Then V_0 is also a subset of $\operatorname{im}(\psi_{n-1}) + R_n$, because $\phi_{n-1,\lambda_0} = \psi_{n-1}$ by $h = 0$. The rest holds by the same argument as in Case 1. □

Next, we outline an algorithm for computing elementary integrals over F_n.

Outline 8.5. *For $f \in F_n$, we proceed as follows.*

1. *Compute an R-pair $(g, \psi_n(f))$. If $\psi_n(f) = 0$, then $\int f = g$ and the algorithm terminates.*
2. *By Proposition 8.4, we decompose $\psi_n(f) = r_f + s_f$, $\psi_{i-1}(t_i'/t_i) = r_i + s_i$ if $i \in \mathbb{H}$, and $\psi_{i-1}(t_i') = r_i + s_i$ if $i \in \mathbb{P}$, where $r_f, r_i \in R$ and $s_f, s_i \in S$.*
3. *Let $z_1, \ldots, z_n$ be constant indeterminates.*
   - 3.1. *Compute $M \in C^{k \times n}$ and $\mathbf{v} \in C^k$ such that $r_f - \sum_{i \in [n]} z_i r_i = 0$ if and only if the linear system in $z_1, \ldots, z_n$ given by the augmented matrix $(M \mid \mathbf{v})$ is consistent.*
   - 3.2. *Use [20, Algorithm 2.8] to compute a matrix $N \in C^{l \times n}$ and $\mathbf{w} \in C^l$ such that all residues of $s_f - \sum_{i \in [n]} z_i s_i$ are constants if and only if the linear system in $z_1, \ldots, z_n$ given by $(N \mid \mathbf{w})$ is consistent.*
   - 3.3. *Solve the linear system given by $\left(\begin{array}{c|c} M & \mathbf{v} \\ N & \mathbf{w} \end{array} \right)$. If the system has no solution, then f has no elementary integral over F_n by Theorem 8.3 and the fact that $R + S$ is direct. The algorithm terminates.*
4. *Let $c_1, \ldots, c_n$ be a solution of the above system. Set $s := s_f - \sum_{i \in [n]} c_i s_i$. Then*

$$\begin{aligned} \psi_n(f) &= r_f + s_f \\ &= \left(r_f - \sum_{i=1}^n c_i r_i \right) + \left(s_f - \sum_{i=1}^n c_i s_i \right) + \sum_{j \in \mathbb{H}} c_j \psi_{j-1} \left(\frac{t_j'}{t_j} \right) + \sum_{k \in \mathbb{P}} c_k \psi_{k-1} (t_k') \\ &= s + \sum_{j \in \mathbb{H}} c_j \psi_{j-1} \left(\frac{t_j'}{t_j} \right) + \sum_{k \in \mathbb{P}} c_k \psi_{k-1}(t_k'). \end{aligned}$$

   The integral of s is elementary over F_n by Lemma 2.3. Let $(q_j, \psi_{j-1}(t_j'/t_j))$ be an R-pair of t_j'/t_j for all $j \in \mathbb{H}$, and $(q_k, \psi_{k-1}(t_k'))$ be an R-pair of t_k' for all $k \in \mathbb{P}$. It follows from the proof of Theorem 8.3 that

$$\int f = g + \int s + \sum_{j \in \mathbb{H}} c_j (\log(t_j) - q_j) + \sum_{k \in \mathbb{P}} c_k (t_k - q_k),$$

   where s is integrated by determining its residues (see [10, Theorem 4.4.3] and [19]).

Example 8.6. *Let us integrate*

$$f := \frac{(\log x + 1)\ \mathrm{Li}(x^\alpha)}{x^{\alpha+1}},$$

where α is a constant indeterminate, $\mathrm{Li}(x)$ *is the logarithmic integral, i.e.,* $\mathrm{Li}(x)' = 1/\log x$*, where $'$ stands for the usual derivation d/dx.*

Let C be the algebraic closure of $\mathbb{Q}(\alpha)$, $t_1 = x$, $t_2 = x^\alpha$, $t_3 = \log(x)$, and $t_4 = \mathrm{Li}(x^\alpha)$. Then $t_1' = 1$, $t_2'/t_2 = \alpha/t_1$, $t_3' = 1/t_1$, $t_4' = t_2/(t_1t_3)$, and $F_4 := C(t_1, t_2, t_3, t_4)$ is a transcendental Liouvillian extension. Moreover, $f = (t_3t_4 + t_4)/(t_1t_2)$.

1. *By the algorithm in Outline 7.2, (p, q) is an R-pair of f with respect to ψ_4, where*

$$p = \frac{\alpha t_2t_3 - \alpha t_3t_4 - (\alpha+1)t_4}{\alpha^2 t_2} \quad \textit{and} \quad q = \frac{\alpha+1}{\alpha^2 t_1t_3}.$$

2. *Projecting remainders $\psi_4(f)$, $\psi_0(t_1')$, $\psi_1(t_2'/t_2)$, $\psi_2(t_3')$ and $\psi_3(t_4')$ with respect to $R \oplus S$, we obtain the following table*

	$\psi_0(t_1')$	$\psi_1(t_2'/t_2)$	$\psi_2(t_3')$	$\psi_3(t_4')$	$\psi_4(f)$
Proj. in R	$r_1 = 1$	$r_2 = 0$	$r_3 = 0$	$r_4 = 0$	$r_f = 0$
Proj. in S	$s_1 = 0$	$s_2 = \alpha/t_1$	$s_3 = 1/t_1$	$s_4 = t_2/(t_1t_3)$	$s_f = q$

3. *Let z_1, z_2, z_3 and z_4 be constant indeterminates. We make two ansatzes:*

   (i) $r_f = z_1r_1 + z_2r_2 + z_3r_3 + z_4r_4$, and

   (ii) $s_f - z_1s_1 - z_2s_2 - z_3s_3 - z_4s_4$ has merely constant residues.

   They lead to the linear system in z_1, z_2, z_3 and z_4 whose augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 0 \end{array}\right).$$

   It has a solution $(0, 0, 0, 0)$. Hence, f has an elementary integral over F_4.

4. *Computing the residues of s_f yields that*

$$\int f = \frac{1}{\alpha}\log(x) + \frac{\alpha+1}{\alpha^2}\log(\log(x)) - \frac{\alpha\log x + (\alpha+1)}{\alpha^2 x^\alpha}\mathrm{Li}(x^\alpha).$$

Example 8.7. *Let $F_0 = \mathbb{C}$ and $F_3 = F_0(t_1, t_2, t_3)$, where*

$$\frac{t_1'}{t_1} = \sqrt{-1}, \quad t_2' = \frac{\sqrt{-1}(t_1^2+1)}{t_1^2 - 1} \quad \textit{and} \quad \frac{t_3'}{t_3} = \frac{t_2^2 - 1}{2\sqrt{-1}t_2}.$$

Then F_3 is a transcendental Liouvillian extension, where t_1, t_2 and t_3 model

$$\exp(\sqrt{-1}x), \quad \log(\sin x) \quad \textit{and} \quad \exp\left(\int \frac{\log(\sin x)^2 - 1}{2\sqrt{-1}\log(\sin x)}\right),$$

respectively. Note that any nonzero constant is not a derivative in F_3.

Let us try to integrate

$$f := \frac{(\sqrt{-1}t_2^2 - \sqrt{-1}t_1^2t_2^2 - 2)t_3 + \sqrt{-1}t_1^2t_2 + 3\sqrt{-1}t_2 - 2t_2}{2(t_1^2-1)t_2(t_3+t_2)}$$

over F_3 by the above algorithm.

1. *An R-pair of f with respect to ψ_3 is $(0, f)$. So $\psi_3(f) = f$ and $f \notin F_3'$.*
2. *Projecting the corresponding remainders with respect to $R \oplus S$ yields that*

	$\psi_0(t_1'/t_1)$	$\psi_1(t_2')$	$\psi_2(t_3'/t_3)$	$\psi_3(f)$
Proj. in R	$r_1 = \sqrt{-1}$	$r_2 = \sqrt{-1}$	$r_3 = t_2/\left(2\sqrt{-1}\right)$	r_f
Proj. in S	$s_1 = 0$	$s_2 = 2\sqrt{-1}/(t_1^2 - 1)$	$s_3 = -1/\left(2\sqrt{-1}t_2\right)$	s_f

where

$$r_f = \frac{t_2}{2\sqrt{-1}} \quad \text{and} \quad s_f = \frac{1}{(1-t_1^2)t_2} + \frac{\sqrt{-1}(t_1^2 t_2^2 + t_1^2 - t_2^2 + 3)}{2(t_1^2-1)(t_2+t_3)}.$$

3. *Let z_1, z_2, z_3 be three constant indeterminates. Ansatzes similar to those in the third step of the above example lead to the linear system in z_1, z_2 and z_3 whose augmented matrix is*

$$\left(\begin{array}{ccc|c} \sqrt{-1} & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \sqrt{-1} & 1 \end{array}\right).$$

Since the system is inconsistent, we conclude that f has no elementary integral over F_3.

Let us consider the case in which an integrand contains a shift variable k. More precisely, let $C = \mathbb{C}(k)$, where k is a constant indeterminate. Assume further that the shift operator $\sigma : k \mapsto k+1$ can be extended to F_n, and that the extended operator commutes with the derivation on F_n. For every element $f \in F_n$, one can compute an R-pair (g_i, r_i) of $\sigma^i(f)$ for every $i \in \mathbb{N}_0$ with respect to ψ_n. Then there exists a recurrence operator $\mathcal{L} \in C[\sigma]^\times$ of order no more than i such that $\mathcal{L}(f) \in F_n'$ if and only if $r_0, r_1, \ldots, r_i$ are C-linearly dependent. So we can construct a telescoper for f up to a given order. Unfortunately, we have not found any criterion for the existence of telescopers in such extensions. Discussions related to the construction of telescopers in primitive towers are given in [20, Section 5.2].

At the end of this section, we illustrate how to construct telescopers via the complete reduction in Theorem 7.1 by an example.

Example 8.8. *For all $k \in \mathbb{N}$, compute*

$$\int_0^{\frac{\pi}{2}} \cos(2kx)\log(\sin(x))\, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin(2kx)\log(\sin(x))\, dx.$$

Let $f(k,x) = \exp(2k\sqrt{-1}x)\log(\sin(x))$ and $A(k) = \int_0^{\frac{\pi}{2}} f(k,x)\, dx$. The integrand $f(k,x)$ is not a D-finite function over $\mathbb{C}(k,x)$. The two integrals are the real and imaginary parts of $A(k)$, which are denoted by $R(k)$ and $I(k)$, respectively.

Let $F_0 = \mathbb{C}(k)$ and $F_4 = F_0(t_1, t_2, t_3, t_4)$ with

$$t_1' = 1, \quad \frac{t_2'}{t_2} = \sqrt{-1}, \quad t_3' = \frac{\sqrt{-1}(t_2^2+1)}{t_2^2 - 1}, \quad \text{and} \quad \frac{t_4'}{t_4} = 2k\sqrt{-1}.$$

The generators t_1 t_2, t_3 and t_4 model x, $\exp(\sqrt{-1}x)$, $\log(\sin(x))$ and $\exp(2k\sqrt{-1}x)$, respectively. Then $f(k,x) = t_3 t_4$ and $f(k+1,x) = t_2^2 t_3 t_4$. Using ψ_4, we find an R-pair (g_0, r_0) of $f(k,x)$ and an R-pair (g_1, r_1) of $f(k+1,x)$, where

$$(g_0, r_0) = \left(-\frac{\sqrt{-1}(2kt_3 - 1)t_4}{4k^2}, -\frac{t_4}{k(t_2^2-1)}\right)$$

and

$$(g_1, r_1) = \left(-\frac{\sqrt{-1}(2k^2t_2^2t_3 + 2kt_2^2t_3 - kt_2^2 - 2k - 2)t_4}{4(k+1)^2k}, -\frac{t_4}{(k+1)(t_2^2-1)}\right).$$

Since $(k+1)r_1 - kr_0 = 0$, *we have*

$$(k+1)f(k+1,x) - kf(k,x) = (k+1)g_1' - kg_0'. \tag{23}$$

Although neither $\lim_{x\to 0^+} g_0$ *nor* $\lim_{x\to 0^+} g_1$ *exists,*

$$\lim_{x\to 0^+}((k+1)g_1 - kg_0) = \frac{\sqrt{-1}(2k+1)}{4k(k+1)}.$$

Integrating both sides of (23) *from* 0 *to* $\pi/2$, *we have*

$$A(k+1) - \frac{k}{k+1}A(k) = \frac{\sqrt{-1}((-1)^k - 2k - 1)}{4k(k+1)^2}. \tag{24}$$

To compute $A(k)$ *for all* $k \in \mathbb{N}$, *we need* $A(1)$, *that is, the value of* $\int_0^{\frac{\pi}{2}} \exp(2\sqrt{-1}x)\log(\sin(x))\,dx$. *Its integrand is represented by* $t_2^2t_3$ *in* F_3. *Using* ψ_3, *the integrand has an R-pair*

$$\left(-\frac{\sqrt{-1}}{2}t_2^2t_3 + \frac{\sqrt{-1}}{4}t_2^2 + \frac{\sqrt{-1}}{2}t_3 - \frac{t_1}{2}, 0\right).$$

So the integrand belongs to F_3'. *It follows that* $A(1) = -\pi/4 - \sqrt{-1}/2$.

Taking respective real and imaginary parts of both sides in (24) *and* $A(1)$, *we have*

$$\begin{cases} R(k+1) - \dfrac{k}{k+1}R(k) = 0 \\ R(1) = -\dfrac{\pi}{4} \end{cases} \quad \text{and} \quad \begin{cases} I(k+1) - \dfrac{k}{k+1}I(k) = \dfrac{(-1)^k - 2k - 1}{4k(k+1)^2} \\ I(1) = -\dfrac{1}{2}. \end{cases}$$

The first recurrence equation and initial value imply that $R(k) = -\pi/(4k)$. *Using the* MATHEMATICA *package* `Sigma` *(see [35]), we get*

$$I(k) = \frac{1-(-1)^k}{4k^2} - \frac{1}{2k}\left(\sum_{i=1}^{k}\frac{1-(-1)^i}{i}\right) + \frac{c}{k}$$

for some $c \in \mathbb{C}$ *in the setting of* $R\Pi\Sigma$*-rings. With* $I(1) = -1/2$, *we find*

$$I(k) = \frac{1-(-1)^k}{4k^2} - \frac{1}{2k}\left(\sum_{i=1}^{k}\frac{1-(-1)^i}{i}\right).$$

Remark 8.9. *The result* $R(k) = -\pi/(4k)$ *is given by Formula 4.384.7 in [25]. Integrals similar to* $I(k)$ *are evaluated in [28] and [30, Chapter 5].*

9 Experiments

We present empirical results about in-field integration obtained from the complete reduction (`CR`) in Theorem 7.1 and the command `int()` in MAPLE without any option. Experiments were carried out with MAPLE 2021 on a computer with iMac CPU 3.6GHZ, Intel Core i9, 16G memory.

Derivations acting on the differential fields appearing in this section are all equal to d/dx. An integrand is the derivative of a rational fraction in a transcendental Liouvillian extension $\mathbb{Q}(t_1, \ldots, t_n)$. The numerator and denominator of such a fraction are given by the MAPLE command `randpoly([t1, t2, ..., tn], degree=d)` with option either `dense` or `sparse`. All integrands in our experiments are derivatives, because `int` would try to integrate a non-derivative in other closed forms. For each degree d, five derivatives were integrated. Timings were measured by MAPLE CPU time in seconds. Computation would be aborted if one example took more than 3600 seconds. An entry was marked with "$\int$" if we detected that `int` returned an unevaluated integrals.

In the first suite, we let $n = 3$, $t_1 = x$, $t_2 = \exp(x)$ and $t_3 = \log(\exp(x) + x)$. Rational fractions were dense. The average timings are summarized in Table 1.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.23	0.01	0.02	0.14	0.62	2.18	7.93	21.97	55.35	>3600
`int`	0.04	0.05	0.13	0.36	1.17	3.64	13.49	35.22	98.43	>3600

Table 1: Dense rational fractions in x, $\exp(x)$ and $\log(\exp(x) + x)$

In the second suite, we let $t_1 = x$, $t_2 = \log(x^2 + 1)$ and $t_3 = \exp(x^2/2)$. An integrand was generated in the same way as in the first suite. The average timings are given in Table 2.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.02	0.01	0.03	0.13	0.44	1.61	4.16	12.39	35.62	88.48
`int`	0.02	0.07	0.14	0.51	1.63	5.74	16.61	44.73	145.48	>3600

Table 2: Dense rational fractions in x, $\log(x^2 + 1)$ and $\exp(x^2/2)$

We let $n = 4$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, $t_3 = \exp(x^2/2)$ and $t_4 = \exp(\exp(x^2/2))$ in the third suite. Rational fractions were sparse. The timings are given in Table 3.

d	1	2	3	4	5	6	7	8	9	10
`CR`	0.02	0.01	0.02	0.04	0.05	0.07	0.15	28.71	4.15	56.03
`int`	0.04	0.13	0.19	0.28	0.41	0.49	0.61	38.93	16.40	$\int$

Table 3: Sparse rational fractions in x, $\log(x^2 + 1)$, $\exp(x^2/2)$ and $\exp(\exp(x^2/2))$

The timings for $d = 8$ in Table 3 are very different from others. To have more coherent empirical results, we had changed the option from `sparse` to `dense`. But both `CR` and `int` took more than an hour without any outcome when $d > 6$. So we integrated the derivative of a polynomial in t_1, t_2, t_3 and t_4 with the option `dense`. The timings are given in Table 4.

d	10	11	12	13	14	15	16	17	18	19	20
`CR`	0.19	0.21	0.26	0.35	0.51	0.46	0.58	0.62	0.82	1.05	1.19
`int`	3.30	4.70	6.65	9.28	12.55	26.36	32.52	39.76	62.94	90.34	86.59

Table 4: Dense polynomials in x, $\log(x^2 + 1)$, $\exp(x^2/2)$ and $\exp(\exp(x^2/2))$

We let $n = 4$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, $t_3 = \exp(x^2/2)$ and $t_4 = \exp\left(x \log(x^2 + 1)\right)$ in the last suite. The timings for sparse fractions and dense polynomials are summarized in Tables 5 and 6, respectively.

d	1	2	3	4	5	6	7	8	9	10
CR	0.01	0.01	0.03	0.04	0.08	0.31	1.08	1.64	54.38	74.30
int	0.06	0.19	0.33	0.49	0.51	1.13	1.80	$\int$	$\int$	$\int$

Table 5: Sparse rational fractions in x, $\log(x^2+1)$, $\exp(x^2/2)$ and $\exp\big(x\log(x^2+1)\big)$

d	10	11	12	13	14	15	16	17	18	19	20
CR	0.31	0.32	0.47	0.67	0.87	1.08	1.05	1.23	1.68	1.85	2.48
int	3.99	5.78	7.94	10.77	15.04	20.71	27.13	35.65	46.04	59.96	82.78

Table 6: Dense polynomials in x, $\log(x^2+1)$, $\exp(x^2/2)$ and $\exp\big(x\log(x^2+1)\big)$

We did not compare CR with int for computing elementary integrals, because such a comparison would need an algorithm for determining constant residues. Our MAPLE scripts use an evaluation-based algorithm in [19]. It outperforms resultant-based algorithms developed in 1970's (see [10, Section 4.4]).

10 Concluding remarks

In this paper, a complete reduction is constructed for $(F, \mathcal{R}_h)$, where F is a transcendental Liouvillian extension and $\mathcal{R}_h$ is the Risch operator associated to an element h of F. The complete reduction yields an algorithm for in-field integration directly, and leads to a new algorithm for computing elementary integrals over F.

There are at least two challenging problems concerning complete reductions for symbolic integration. One is to develop a reduction-based algorithm for computing telescopers for elements in F without a given priori order bound. This would require a criterion on the existence of telescopers for elements in F. The other is how to adapt remainders of the complete reduction for algebraic functions in [15] so as to compute elementary integrals over algebraic-function fields more efficiently.

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A Algorithm descriptions

This appendix is devoted to algorithmic descriptions of some constructive proofs in Sections 5 and 6. Convention 4.1 and Hypothesis 4.2 are kept throughout the appendix. For brevity, we use the data in Convention 4.1 instead of $F(t)$, h, m, a, b, a_m, b_m, a_0 and b_0 in the specification of an algorithm. Furthermore, we abbreviate "with respect to" and "such that" as "w.r.t." and "s.t." in pseudo-codes, respectively.

A.1 Algorithms in the primitive case

In this subsection, t is a regular monomial and is primitive over F, U_h is the auxiliary subspace given in Definition 5.3, and I_h denotes the intersection of $\operatorname{im}(\mathcal{P}_h)$ and U_h.

The first algorithm describes an auxiliary reduction in $F[t]$. It is based on (7), (8) and the proof of Proposition 5.5.

Algorithm A.1. PrimAuxRed

Input: $f \in F[t]$ *and the data in Convention 4.1*

Output: (g, r)*, an auxiliary pair of* f *w.r.t.* $(F[t], \mathcal{P}_h)$

1. $p \leftarrow f, \quad (g, r) \leftarrow 0, 0, \quad (d, p_d) \leftarrow \deg(p), \operatorname{lc}(p)$
2. while $d \geq m$ do

   if $\nu_\infty(h) < 0$ then

   $(u, v) \leftarrow a_m^{-1} p_d t^{d-m}, 0$

   else

   $(g_d, r_d) \leftarrow$ *an R-pair of* p_d *w.r.t.* $\phi_{a_m}, \quad (u, v) \leftarrow g_d t^{d-m}, r_d t^d$

   $(g, r) \leftarrow g + u, r + v, \quad p \leftarrow p - \mathcal{P}_h(u) - v, \quad (d, p_d) \leftarrow \deg(p), \operatorname{lc}(p)$ (* by (7) and (8)*)
3. return $(g, r + p)$

We are going to define a finite sequence L that determines an echelon sequence of I_h uniquely as long as the above algorithm is used for computing auxiliary pairs.

Definition A.2. *Let* u *be the type of* I_h*. If* $u = 0$*, then* L *is set to be* 0*.*

Assume that $u \neq 0$*. Let* $(\tilde{v}, v)$ *and* $(\tilde{w}, w)$ *be the first and second R-pairs associated to* $(F[t], \mathcal{P}_h)$*, respectively, and we fix an element* $\theta_v \in \Theta$ *such that* $\theta_v^*(v) \neq 0$*.*

If $h \in F$*, then* L *is the sequence consisting of four members:*

$$u, (\tilde{v}, v), (\tilde{w}, w), \theta_v. \tag{25}$$

Assume further that $h \in F(t) \setminus F$*. If* $\theta_v^*(iv + w) \neq 0$ *for all* $i \in \mathbb{N}$*, then* L *is the sequence consisting of five members:*

$$u,\ (\tilde{v}, v),\ (\tilde{w}, w),\ \theta_v,\ \left(p_0, \mathcal{P}_h(p_0),\ \theta_0 t^{d_0}\right), \tag{26}$$

where the last member is given by Corollary 5.18 (i).

If $\theta_v^*(jv + w) = 0$ *for some* $j \in \mathbb{N}$ *but* $jv + w \neq 0$*, then* L *consists of seven members:*

$$u,\ (\tilde{v}, v),\ (\tilde{w}, w),\ \theta_v,\ j,\ \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right),\ \left(p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1}\right), \tag{27}$$

where the last two members are given by Corollary 5.18 (ii).

If $jv + w = 0$ *for some* $j \in \mathbb{N}$*, then* L *consists of* $j + 6$ *members:* u, $(\tilde{v}, v)$, $(\tilde{w}, w)$, θ_v, j,

$$\left(q, \mathcal{P}_h(q), \theta t^d\right),\ \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right),\ \left(p_1, \mathcal{P}_h(p_1), \theta_v t^m\right),\ \ldots,\ \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_v t^{m+j-2}\right), \tag{28}$$

where the last $j + 1$ *members are given by Corollary 5.18 (iii).*

We call L *an* initial sequence *of* I_h*.*

An initial sequence L with $L \neq 0$ determines an echelon sequence E of I_h uniquely by (10) and Algorithm A.1. Moreover, we do not need to recompute $\mathcal{P}_h(p_i)$ according to (11). The complement of $\operatorname{im}(\mathcal{P}_h)$ induced by E is also called the *complement induced by* L. The next algorithm computes initial sequences.

Algorithm A.3. InitSeq

Input: *the data in Convention 4.1*

Output: *an initial sequence of* I_h

1. $u \leftarrow$ *the type of* I_h, IF $u = 0$ THEN RETURN 0 (* $I_h = \{0\}$ *)
2. $(\tilde{v}, v) \leftarrow$ *a first R-pair associated to* $(F[t], \mathcal{P}_h)$

   $(\tilde{w}, w) \leftarrow$ *a second R-pair associated to* $(F[t], \mathcal{P}_h)$

   $\theta_v \leftarrow$ *an element of* Θ *s.t.* $\theta_v^*(v) \neq 0$, $L_0 \leftarrow u, (\tilde{v}, v), (\tilde{w}, w), \theta_v$
3. IF $h \in F$ THEN RETURN L_0 (* *by Corollary 5.17* and (25) *)
4. $p_0 \leftarrow u$, $(d_0, l_0) \leftarrow \deg\left(\mathcal{P}_h(p_0)\right), \mathrm{lc}\left(\mathcal{P}_h(p_0)\right)$

   $\theta_0 \leftarrow$ *an element of* Θ *s.t.* $\theta_0^*(l_0) \neq 0$, $\gamma_0 \leftarrow \theta_0 t^{d_0}$, $j \leftarrow -\theta_v^*(w)/\theta_v^*(v)$, $k \leftarrow -w/v$
5. IF $j \notin \mathbb{N}$ THEN RETURN $L_0, (p_0, \mathcal{P}_h(p_0), \gamma_0)$ (* *by Corollary 5.18 (i) and* (26) *)
6. IF $k \notin \mathbb{N}$ THEN *compute* p_j *and* $\mathcal{P}_h(p_j)$ *by* (10), (11) *and Algorithm A.1*

   $l \leftarrow \mathrm{lc}\left(\mathcal{P}_h(p_j)\right)$, *choose an element* $\theta \in \Theta$ *s.t.* $\theta^*(l) \neq 0$

   RETURN $L_0, j, (p_0, \mathcal{P}_h(p_0), \gamma_0), \left(p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1}\right)$

   (* *by Corollary 5.18 (ii)* and (27) *)
7. 7.1. *compute* $p_1, \mathcal{P}_h(p_1), \ldots, p_{j-1}, \mathcal{P}_h(p_{j-1}), p_j, \mathcal{P}(p_j)$ *by* (10), (11) *and Algorithm A.1*

   7.2. $(q, Q) \leftarrow p_j, \mathcal{P}_h(p_j)$

   7.3. FOR i FROM $j-1$ TO 0 BY -1 DO

   IF $i > 0$ THEN $\gamma_i \leftarrow \theta_v t^{m+i-1}$

   $c \leftarrow \gamma_i^*\left(\mathcal{P}_h(p_i)\right)^{-1} \cdot \gamma_i^*(Q)$, $(q, Q) \leftarrow q - cp_i, Q - c\mathcal{P}_h(p_i)$

   7.4. $(d, l) \leftarrow \deg(Q), \mathrm{lc}(Q)$, *choose an element* $\theta \in \Theta$ *s.t.* $\theta^*(l) \neq 0$

   7.5. RETURN $L_0, j, \left(q, Q, \theta t^d\right), (p_0, \mathcal{P}_h(p_0), \gamma_0), (p_1, \mathcal{P}_h(p_1), \gamma_1), \ldots, (p_{j-1}, \mathcal{P}_h(p_{j-1}), \gamma_{j-1})$

   (* *by Corollary 5.18 (iii)* and (28) *)

The correctness of this algorithm is direct from the definition of initial sequences and Corollaries 5.17 and 5.18. Note that j and k in step 4 are equal when both of them are positive integers. So j is used in step 7.

Given an initial sequence L with $L[1] \neq 0$ and a positive integer l, we can compute the first l members of the echelon sequence E by the following algorithm.

Algorithm A.4. PrimEchSeq

Input: $l \in \mathbb{N}$*, and the data in Convention 4.1, and an initial sequence* L *of* I_h *with* $L[1] \neq 0$

output: *the first* l *members of the echelon sequence induced by* L

remark: *In the following pseudo-code,* $p_0 = L[1]$ *and* p_i *and* $\mathcal{P}_h(p_i)$ *are computed by* (10), (11) *and Algorithm A.1 for* $i > 0$.

1. $u \leftarrow L[1]$, $(\tilde{v}, v) \leftarrow L[2]$, $(\tilde{w}, w) \leftarrow L[3]$, $\theta_v \leftarrow L[4]$
2. IF $\mathrm{len}(L) = 4$ THEN RETURN $\left\{\left(p_i, \mathcal{P}_h(p_i), \theta_v t^{i-1}\right)\right\}_{i \in [l]}$ (* *by Corollary 5.17* and (25) *)
3. IF $\mathrm{len}(L) = 5$ THEN

   RETURN $L[5]$, $\left\{\left(p_i, \mathcal{P}_h(p_i), \theta_v t^{m+i-1}\right)\right\}_{i \in [l-1]}$ (* *by Corollary 5.18 (i)* and (26) *)
4. $j \leftarrow L[5]$

5. IF $\text{len}(L) = 7$ AND $jv + w \neq 0$ THEN $(q_1, Q_1, \gamma_1) \leftarrow L[6]$

   FOR i FROM 2 TO l DO

   IF $i \neq j+1$ THEN $(q_i, Q_i, \gamma_i) \leftarrow p_{i-1}, \mathcal{P}_h(p_{i-1}), \theta_v t^{m+i-2}$ ELSE $(q_i, Q_i, \gamma_i) \leftarrow L[7]$

   RETURN $(q_1, Q_1, \gamma_1), (q_2, Q_2, \gamma_2), \ldots, (q_l, Q_l, \gamma_l)$ (*by Corollary 5.18 (ii) and (27)*)

6. 6.1. FOR i FROM 1 TO $j+1$ DO $(q_i, Q_i, \gamma_i) \leftarrow L[5+i]$

   6.2. FOR i FROM $j+2$ TO l DO $(q_i, Q_i, \gamma_i) \leftarrow p_i, \mathcal{P}_h(p_{i-1}), \theta_v t^{m+i-2}$

   6.3. RETURN $(q_1, Q_1, \gamma_1), (q_2, Q_2, \gamma_2), \ldots, (q_l, Q_l, \gamma_l)$ (*by Corollary 5.18 (iii) and (28)*)

The following algorithm computes the respective projections of a given element in $F[t]$ to $\text{im}(\mathcal{P}_h)$ and its complement induced by an initial sequence.

Algorithm A.5. PRIMPROJ

INPUT: *$f \in F[t]$, the data in Convention 4.1, and the initial sequence L of I_h*

OUTPUT: *$(g, r) \in F[t] \times W$ s.t. $f = \mathcal{P}_h(g) + r$, where W is the complement induced by L*

1. $u \leftarrow L[1]$ (*find the type*)
2. $(g, r) \leftarrow$ *an auxiliary pair of f by Algorithm A.1* (*auxiliary reduction*)
3. IF $u = 0$ OR $r = 0$ THEN RETURN (g, r) (*$I_h = \{0\}$ or the trivial projection*)
4. *use Algorithm A.4 to compute the first k members*

$$(q_1, Q_1, \gamma_1), \ldots, (q_k, Q_k, \gamma_k)$$

   in the echelon sequence E induced by L s.t. other members in E are of degrees $> \deg(r)$

5. FOR i FROM k TO 1 BY -1 DO $c \leftarrow \gamma_i^*(Q_i)^{-1}\gamma_i^*(r)$, $(g, r) \leftarrow g + cq_i, r - cQ_i$ (*elimination*)
6. RETURN (g, r)

The first four steps of the above algorithm are evidently correct. Step 5 is correct by Lemma 2.11 and its proof. In step 6, r belongs to the complement of $\text{im}(\mathcal{P}_h)$ induced by L, because $\gamma_i^*(r) = 0$ for all $i \in [k]$ and the pivot of the lth member does not appear in r for all $l > k$ due to the choice of k in step 4.

A.2 Algorithms in the hyperexponential case

In this subsection, t is a regular monomial, and is hyperexponential over F, V_h is the auxiliary space given in Definition 6.3, and J_h denotes the intersection of $\text{im}(\mathcal{P}_h)$ and V_h. The elements λ_k and μ_l of F are defined in Remark 6.1 (ii) and (iv), respectively.

The first algorithm is for the auxiliary reduction in $F[t, t^{-1}]$. It is based on (18), (19), (20), (21) and the proof of Proposition 6.5.

Algorithm A.6. HYPEREXPAUXRED

INPUT: *$f \in F[t, t^{-1}]$ and the data in Convention 4.1*

OUTPUT: *(g, r), an auxiliary pair of f w.r.t. $(F[t, t^{-1}], \mathcal{P}_h)$*

1. $p \leftarrow f^+$, $(k, p_k) \leftarrow \text{hdeg}(p), \text{hc}(p)$, $q \leftarrow f^-$, $(l, q_l) \leftarrow \text{tdeg}(q), \text{tc}(q)$, $(g, r) \leftarrow 0, 0$
2. WHILE $k \geq m$ DO

   IF $\nu_\infty(h) < 0$ THEN

   $(u, v) \leftarrow a_m^{-1} p_k t^{k-m}, 0$, $g \leftarrow g + u$

ELSE

$(g_k, r_k) \leftarrow$ *an R-pair of* p_k *w.r.t.* $\phi_{\lambda_{k-m}}$, $(u,v) \leftarrow g_k t^{k-m}, r_k t^k$, $(g,r) \leftarrow g+u,\ r+v$

$p \leftarrow p - \mathcal{P}_h(u) - v$, $(k, p_k) \leftarrow \mathrm{hdeg}(p), \mathrm{hc}(p)$ (*by (18) and (19)*)

3. WHILE $l < 0$ DO

IF $\nu_t(h) < 0$ THEN

$(u,v) \leftarrow a_0^{-1} q_l t^l, 0$, $g \leftarrow g+u$

ELSE

$(g_l, r_l) \leftarrow$ *an R-pair of* $b_0^{-1} q_l$ *w.r.t.* ϕ_{μ_l} $(u,v) \leftarrow g_l t^l, b_0 r_l t^l$, $(g,r) \leftarrow g+u, r+v$

$q \leftarrow q - \mathcal{P}_h(u) - v$, $(l, q_l) \leftarrow \mathrm{tdeg}(q), \mathrm{tc}(q)$ (*by (20) and (21)*)

4. RETURN $(g, r+p+q)$

Next, we determine whether $J_h = \{0\}$, and find an echelon sequence of J_h with respect to $(F[t,t^{-1}], \mathcal{P}_h)$ if $J_h \neq \{0\}$. The next algorithm is based on Propositions 6.13 and 6.14.

Algorithm A.7. HYPEREXPECHSEQ

INPUT: *the data in Convention 4.1*

OUTPUT: NIL *if* $J_h = \{0\}$, *and an echelon sequence of* J_h, *otherwise.*

1. $M \leftarrow$ *the type of* J_h
2. IF $M = 0$ THEN RETURN NIL (*$J_n = \{0\}$*)
3. (*$\dim(J_h) = 1$*) IF $\mathrm{len}(M) = 1$ THEN

   $(k, u) \leftarrow M[1]$ (*find the type*)

   $(g, r) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(ut^k)$ *by Algorithm A.6*

   $(p, P) \leftarrow ut^k - g, r$, $\theta \leftarrow$ *an element of* Θ *s.t.* $\mathrm{hc}(P) \notin \ker(\theta^*)$

   RETURN $\left(p, P, \theta t^{\mathrm{hdeg}(P)}\right)$ (*by Proposition 6.14 (i)*)

4. (*$\dim(J_h) = 2$*) $(k,u) \leftarrow M[1]$, $(l, v) \leftarrow M[2]$ (*find the type*)

   4.1. $(g_k, r_k) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(ut^k)$ *by Algorithm A.6*

   $(g_l, r_l) \leftarrow$ *the auxiliary pair of* $\mathcal{P}_h(vt^l)$ *by Algorithm A.6*

   $(p_k, P_k) \leftarrow ut^k - g_k, r_k$, $(p_l, P_l) \leftarrow vt^l - g_l, r_l$ (*by Proposition 6.13 (ii)*)

   4.2. $d_k \leftarrow \mathrm{hdeg}(P_k)$, $\theta_k \leftarrow$ *an element of* Θ *s.t.* $\mathrm{hc}(P_k) \notin \ker(\theta_k^*)$, $\gamma_k \leftarrow \theta_k t^{d_k}$

   $c \leftarrow \gamma_k^*(P_k)^{-1} \gamma_k(P_l)$, $(q, Q) \leftarrow p_l - cp_k, P_l - cP_k$, $d \leftarrow \mathrm{hdeg}(Q)$

   $\theta \leftarrow$ *an element of* Θ *s.t.* $\mathrm{hc}(Q) \notin \ker(\theta)$, $\gamma \leftarrow \theta t^d$, RETURN $(q, Q, \gamma), (p_k, P_k, \gamma_k)$

   (*by Proposition 6.14 (ii)*)

The last algorithm computes the respective projections of an element in $F[t,t^{-1}]$ to $\mathrm{im}(\mathcal{P}_h)$ and the complement induced by an echelon sequence of J_h. It is simpler than Algorithm A.5, because an echelon sequence of J_h has at most two members. So we merely provide a specification.

Algorithm A.8. HYPEREXPPROJ

INPUT: $f \in F[t,t^{-1}]$, *the data in Convention 4.1, and an echelon sequence* E *of* J_h

OUTPUT: $(g,r) \in F[t,t^{-1}] \times W$ *s.t.* $f = \mathcal{P}_h(g) + r$, *where* W *is the complement induced by* E