Power series with coefficients from a finite set

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Abstract

We prove in this paper that a multivariate D-finite power series with coefficients from a finite set is rational. This generalizes a rationality theorem of van der Poorten and Shparlinski in 1996.

Keywords: Power series, Szegő's theorem, D-finiteness, integer points

1. Introduction

In his thesis [16], Hadamard began the study of the relationship between the coefficients of a power series and the properties of the function it represents, especially its singularities and natural boundaries. Two special cases of the problem have been extensively studied: one is on power series with integer coefficients and the other is on power series with finitely many distinct coefficients.

In the first case, Fatou [13] in 1906 proved a lemma on rational power series with integer coefficients, which is now known as Fatou's lemma [33, p. 275]. The next celebrated result is the Pólya-Carlson theorem, which asserts that a power series with integer coefficients and of radius of convergence 1 is either rational or has the unit circle as its natural boundary. This theorem was first conjectured in 1915 by Pólya [25] and later proved in 1921 by Carlson [7]. Several extensions of the Pólya-Carlson theorem have been presented in [26, 24, 15, 31, 22, 35, 2].

In the second case, Fatou [13] was also the first to investigate power series with coefficients from a finite set by showing that such power series are either

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rational or transcendental. The study was continued by Pólya [25] in 1916, Jentzsch [17] in 1917, Carlson [6] in 1918 and finally Szegő [36, 37] in 1922 settled the question by proving the following beautiful theorem (see [27, Chap. 11] and [10, Chap. 10] for its proof and related results).

Theorem 1 (Szegő, 1922)**.** *Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{C}$. If F is continuable beyond the unit circle then it is a rational function of the form $F = P(x)/(1-x^m)$, where P is a polynomial and m a positive integer.*

Szegő's theorem was generalized in 1945 by Duffin and Schaeffer [11] by assuming a weaker condition that f is bounded in a sector of the unit circle. In 2008, P. Borwein et al. in [5] gave a shorter proof of Duffin and Schaeffer's theorem. By using Szegő's theorem, van der Poorten and Shparlinski proved the following result [38].

Theorem 2 (van der Poorten and Shparlinski, 1996)**.** *Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{Q}$. If $f(n)$ satisfies a linear recurrence equation with polynomial coefficients, then F is rational.*

A univariate sequence $f : \mathbb{N} \to K$ is *P-recursive* if it satisfies a linear recurrence equation with polynomial coefficients in $K[n]$. A power series $F = \sum f(n)x^n$ is *D-finite* if it satisfies a linear differential equation with polynomial coefficients in $K[x]$. By [32, Theorem 1.5], a sequence $f(n)$ is P-recursive if and only if the power series $F := \sum f(n)x^n$ is D-finite. The notion of D-finite power series can be generalized to the multivariate case (see Definition 4). Our main result is the following multivariate generalization of Theorem 2.

Theorem 3. *Let K be a field of characteristic zero, and let Δ be a finite subset of K. Suppose that $f : \mathbb{N}^d \to \Delta$ with $d \geq 1$ is such that*

$$F(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d} \in K[[x_1, \ldots, x_d]]$$

is D-finite. Then F is rational.

We note that a multivariate rational power series

$$F(x_1, \cdots, x_d) = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d}$$

with all coefficients in $\{0, 1\}$ has a very restricted form. In particular, the set E of $(n_1, \ldots, n_d) \in \mathbb{N}^d$ for which $f(n_1, \ldots, n_d) \neq 0$ is *semilinear*; that is there exist $n \in \mathbb{N}$ and finite subsets $V_0, \ldots V_n$ of $\mathbb{N}^d$, and $b_1, \ldots, b_n \in \mathbb{N}^d$ such that

$$E = V_0 \bigcup \left\{ \bigcup_{i=1}^{n} \left(b_i + \sum_{v\in V_i} v \cdot \mathbb{N} \right) \right\}. \tag{1}$$

Although this result is known, we are unaware of a reference and give a proof of this fact in Proposition 11.

The remainder of this paper is organized as follows. The basic properties of D-finite power series are recalled in Section 2. The proof of Theorem 3 is given in Section 3. In Section 4, we present several applications of our main theorem on generating functions over nonnegative points on algebraic varieties.

2. D-finite power series

Throughout this paper, we let $\mathbb{N}$ denote the set of all nonnegative integers. Let K be a field of characteristic zero and let $K(\mathbf{x})$ be the field of rational functions in several variables $\mathbf{x} = x_1, \ldots, x_d$ over K. By $K[[\mathbf{x}]]$ we denote the ring of formal power series in $\mathbf{x}$ over K and by $K((\mathbf{x}))$ we denote the field of fractions of $K[[\mathbf{x}]]$. For two power series $F = \sum f(i_1, \ldots, i_d) x_1^{i_1} \cdots x_d^{i_d}$ and $G = \sum g(i_1, \ldots, i_d) x_1^{i_1} \cdots x_d^{i_d}$, the Hadamard product of F and G is defined by

$$F \odot G = \sum f(i_1, \ldots, i_d) g(i_1, \ldots, i_d) x_1^{i_1} \cdots x_d^{i_d}.$$

Let $D_{x_1}, \ldots, D_{x_d}$ denote the derivations on $K((\mathbf{x}))$ with respect to $x_1, \ldots, x_d$, respectively.

Definition 4 ([19]). *A formal power series $F(x_1, \ldots, x_d) \in K[[\mathbf{x}]]$ is said to be* D-finite *over $K(\mathbf{x})$ if the set of all derivatives $D_{x_1}^{i_1} \cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ span a finite-dimensional $K(\mathbf{x})$-vector subspace of $K((\mathbf{x}))$. Equivalently, for each $i \in \{1, \ldots, d\}$, F satisfies a nontrivial linear partial differential equation of the form*

$$\left\{p_{i,m_i} D_{x_i}^{m_i} + p_{i,m_1-1} D_{x_i}^{m_i-1} + \cdots + p_{i,0}\right\} F = 0 \quad \textit{with } p_{i,j} \in K[\mathbf{x}].$$

The notion of D-finite power series was first introduced in 1980 by Stanley [32], and has since become ubiquitous in algebraic combinatorics as an important part of the study of generating functions (see [34, Chap. 6]). We recall some closure properties of this class of power series.

Proposition 5 ([20]). *Let $\mathcal{D}$ denote the set of all D-finite power series in $K[[\mathbf{x}]]$. Then*

(i) *$\mathcal{D}$ forms a subalgebra of $K[[\mathbf{x}]]$, i.e., if $F, G \in \mathcal{D}$ and $\alpha, \beta \in K$, then $\alpha F + \beta G \in \mathcal{D}$ and $FG \in \mathcal{D}$.*

(ii) *$\mathcal{D}$ is closed under the Hadamard product, i.e., if $F, G \in \mathcal{D}$, then $F \odot G \in \mathcal{D}$.*

(iii) *If $F(x_1, \ldots, x_d)$ is D-finite, and*

$$\alpha_1(y_1, \ldots, y_d), \ldots, \alpha_d(y_1, \ldots, y_d) \in K[[y_1, \ldots, y_d]]$$

are algebraic over $K(y_1, \ldots, y_d)$ and the substitution makes sense, then $F(\alpha_1, \ldots, \alpha_d)$ is also D-finite over $K(y_1, \ldots, y_d)$.
In particular, if $F(x_1, \ldots, x_d)$ is D-finite and the evaluation of F at $x_d = 1$ makes sense, then $F(x_1, \ldots, x_{d-1}, 1)$ is D-finite.

The coefficients of a D-finite power series are highly structured. In the univariate case, a power series $f = \sum a(n)x^n$ is D-finite if and only if the sequence $a(n)$ is P-recursive, i.e., it satisfies a linear recurrence equation with polynomial coefficients in n [32]. The structure in the multivariate case is much more profound, which was explored by Lipshitz in [20]. We continue this exploration to study the position of nonzero coefficients. To this end, we recall a notion of size in the semigroup $(\mathbb{N}, +)$. A subset $S \subseteq \mathbb{N}$ is *syndetic* if there is some positive integer C such that if $n \in S$ then $n + i \in S$ for some $i \in \{1, \ldots, C\}$. Note that a syndetic subset of $\mathbb{N}$ has nonzero density. The term "syndetic" comes from the study of topological dynamics [14, Chapter 2] and further used by Bergelson et al. [3] for studying general semigroups. Syndetic sets are also closely related to the Cobham's theorem on automatic sequences [1, Chapter 11].

Example 6. *The subset of all even numbers in $\mathbb{N}$ is syndetic, but the subset $S := \{p_1^{m_1} \cdots p_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{N}\}$ with $p_1, \ldots, p_n$ being prime numbers is not syndetic since the difference between two successive integers $a_i, a_{i+1} \in S$ tends to infinity as i tends to infinity.*

Lemma 7. *Let K be a field of characteristic zero and let*

$$G(x_1, \ldots, x_d) = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} g(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in K[[\mathbf{x}]]$$

be a D-finite power series over $K(\mathbf{x})$. Then the set

$$\{n \in \mathbb{N} \mid \exists (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1} \text{ such that } g(n_1, \ldots, n_{d-1}, n) \neq 0\}$$

is either finite or syndetic.

Proof. We let L denote the field of fractions of $K[[x_1, \ldots, x_{d-1}]]$. Then we may regard G as a power series in $L[[x_d]]$ and it is D-finite in x_d over $L(x_d)$ and it is straightforward to see that the lemma reduces to the univariate case. Thus we now assume that $G(x) = \sum g(n)x^n \in L[[x]]$ is D-finite. Then there exist $m \geq 1$, distinct nonnegative integers $a_1 = 0, \ldots, a_m$, and nonzero polynomials $P_1, \ldots, P_m \in L[z]$ such that

$$\sum_{j=1}^{m} P_j(n) g(n + a_j) = 0$$

for all sufficiently large n. Then there is some M such that $P_1(n) \cdots P_m(n) \neq 0$ for $n > M$. If $m = 1$ then we then see that $g(n) = 0$ for $n > M$. Thus we assume that $m > 1$. Then if $n > M$ and $g(n)$ is nonzero then $g(n + a_j)$ is nonzero for some $1 < j \leq m$ and so we see that the set of n for which $g(n)$ is nonzero is syndetic. $\square$

3. Proof of the main theorem

The proof of Theorem 2 by van der Poorten and Shparlinski is based on the fact that any univariate D-finite power series represents an analytic function with only finitely many poles [32], so it is impossible to have the unit circle as its natural boundary. Then their result follows from Szegő's theorem. The singularities of analytic functions represented by multivariate D-finite power series are much more involved. It is not known how to extend Szegő's theorem to the multivariate case. Thus new ideas are needed in order to generalize Theorem 2 to the multivariate case.

Before the proof of our main theorem, we first prove a lemma about finitely generated $\mathbb{Z}$-algebras.

Lemma 8. *Let R be a finitely generated $\mathbb{Z}$-algebra that is an integral domain of characteristic zero. Then there is only a finite set of prime numbers that divide a given nonzero element of R; i.e., for any $x \in R \setminus \{0\}$, there exists finitely many prime numbers $p_1, \ldots, p_m$ such that $n \in \{p_1^{i_1} \cdots p_m^{i_m} \mid i_1, \ldots, i_m \in \mathbb{N}\}$ if $x \in nR$.*

Proof. Let U denote the group of units of R. By a result of Roquette [28] (or see [18, page 39, Corollary]) we have that U is a finitely generated abelian group and so U_0, the subgroup of U generated by the rational numbers in U is a finitely generated subgroup of $\mathbb{Q}^*$. In particular, there exist prime numbers $q_1, \ldots, q_t$ such that every positive rational number in U is in the multiplicative subgroup of $\mathbb{Q}^*$ generated by $\pm 1, q_1, \ldots, q_t$. Thus if x is a unit and $x \in nR$ then n is an integer unit of R and hence in the semigroup generated by $\pm 1, q_1, \ldots, q_t$.

For the general case, we let $S = R[1/x]$, which is still a finitely generated $\mathbb{Z}$-algebra that is an integral domain of characteristic zero. We observe that if $x \in nS$ then n is necessarily a unit in S and by the above remarks we have that n lies in a semigroup generated by ± 1 along with a finite set of prime numbers. We note that if $x \in nR$ then $x \in nS$ and so we obtain the desired result.

□

Proof of Theorem 3. We prove this by induction on d. When $d = 0$, F is constant and there is nothing to prove. We now suppose that the result holds whenever $d < k$ and we consider the case when $d = k$. Since F is D-finite, we have that $F(x_1, \ldots, x_k)$ satisfies a nontrivial linear differential equation of the form

$$\sum_{j=0}^{\ell} P_j(x_1, \ldots, x_k) \partial_{x_k}^j F = 0,$$

where $P_0, \ldots, P_\ell$ are polynomials in $K[x_1, \ldots, x_k]$. Translating this into a relation for the coefficients of F, we see that there exists some positive integer N and polynomials $Q_{a_1,\ldots,a_k}(t) \in K[t]$ for $(a_1, \ldots, a_k) \in \{-N, \ldots, N\}^k$, not all zero, such that

$$\sum_{-N \le a_1,\ldots,a_k \le N} Q_{a_1,\ldots,a_k}(n_k) f(n_1 - a_1, \ldots, n_k - a_k) = 0 \qquad (2)$$

for all $(n_1,\dots,n_k)\in\mathbb{N}^k$, where we take $f(i_1,\dots,i_k)=0$ if some i_j is negative. By dividing our polynomials $Q_{a_1,\dots,a_k}(t)$ by t^a for some nonnegative integer a if necessary, we may assume that $q(a_1,\dots,a_k):=Q_{a_1,\dots,a_k}(0)$ is nonzero for some $(a_1,\dots,a_k)\in\{-N,\dots,N\}^k$. We now let R denote the $\mathbb{Z}$-subalgebra of K generated by Δ and by the coefficients of $Q_{a_1,\dots,a_k}(t)\in K[t]$ with $(a_1,\dots,a_k)\in\{-N,\dots,N\}^k$. Then R is finitely generated. By construction, we have

$$\sum_{-N\le a_1,\dots,a_k\le N} q(a_1,\dots,a_k)f(n_1-a_1,\dots,n_k-a_k)\in n_kR$$

for all $(n_1,\dots,n_k)\in\mathbb{N}^k$. Now let Γ denote the set of all numbers of the form

$$\sum_{-N\le a_1,\dots,a_k\le N} q(a_1,\dots,a_k)s(a_1,\dots,a_k)$$

with $s(a_1,\dots,a_k)\in\Delta\cup\{0\}$. Then Γ is a finite set. By Lemma 8, there is a finite set of prime numbers $p_1,\dots,p_m$ such that for each nonzero $x\in\Gamma$ we have that if n is a positive integer with $x\in nR$ then n is in the semigroup generated by $p_1,\dots,p_m$. In particular,

$$\sum_{-N\le a_1,\dots,a_k\le N} q(a_1,\dots,a_k)f(n_1-a_1,\dots,n_k-a_k)=0$$

whenever n_k is not in the multiplicative semigroup generated by $p_1,\dots,p_m$. Equivalently,

$$G(x_1,\dots,x_k):=F(x_1,\dots,x_k)\left(\sum_{0\le a_1,\dots,a_k\le N} q(a_1,\dots,a_k)x_1^{a_1}\cdots x_k^{a_k}\right)x_1^N\cdots x_k^N$$

has the property that $g(n_1,\dots,n_k)=0$ whenever $n_k\ge N$ and n_k-N is not in the semigroup generated by $p_1,\dots,p_m$, where $g(n_1,\dots,n_k)$ denotes the coefficient of $x_1^{n_1}\cdots x_k^{n_k}$ in $G(x_1,\dots,x_k)$. Since G is just F multiplied by a polynomial, $G(x_1,\dots,x_k)$ is D-finite by Proposition 5 (i); moreover, all coefficients of G lie in the finite set Γ. Note that any translate of the multiplicative semigroup generated by $p_1,\dots,p_m$ cannot be syndetic by the same argument as in Example 6. Therefore, Lemma 7 implies that there is some positive integer M such that $g(n_1,\dots,n_k)=0$ whenever $n_k>M$. Thus we have

$$G=\sum_{i=0}^{M} G_i(x_1,\dots,x_{k-1})x_k^i$$

for some power series $G_0,\dots,G_M\in K[[x_1,\dots,x_{k-1}]]$. Then for $i\in\{0,\dots,M\}$, we have that $G_ix_k^i$ is the Hadamard product of G with $x_k^i\prod_{j=1}^{k-1}(1-x_j)^{-1}$ and so each $G_ix_k^i$ is D-finite by Proposition 5 (ii). Then specializing $x_k=1$ gives each G_i is D-finite by Proposition 5 (iii). Since each G_i has coefficients in a finite set, we see by the induction hypothesis that each G_i is rational and so G is rational. But this now gives that F is rational by our definition of G, completing the proof. □

4. Generating functions over nonnegative integer points on algebraic varieties

Let $V \subseteq \mathbb{A}_K^d$ be an affine algebraic variety over an algebraically closed field K of characteristic zero. We define the *generating function over nonnegative integer points* on V by

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}.$$

Then one can ask the following questions about the properties of F_V that often reflect the global geometric structure of V:

1. When F_V is zero? This is Hilbert Tenth Problem when K is the field of rational numbers. In 1970, Matiyasevich [23, 9] proved that this problem is undecidable.
2. When F_V is a polynomial? If so, V has only finitely many nonnegative integer points. Siegel's theorem on integral points answers this question for a smooth algebraic curve C of genus $g \geq 1$ defined over a number field K [4, Chap. 7].
3. When F_V is a rational function? This is always true when the variety V is defined by linear polynomials with integer coefficients [33, Chap. 4].
4. When F_V is D-finite? By our main theorem, we see that this question is the same as question (3), by taking $f(n_1, \ldots, n_d) = 1$ if $(n_1, \ldots, n_d) \in V \cap \mathbb{N}^d$ and $f(n_1, \ldots, n_d) = 0$ otherwise (see Corollary 9).
5. When F_V satisfies an algebraic differential equation?More precisely, we say that a power series $F(x_1, \ldots, x_d) \in K[[x_1, \ldots, x_d]]$ is *differentially algebraic* if the transcendence degree of the field generated by all of the derivatives $D_{x_1}^{i_1}\cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1, \ldots, x_d)$ is finite. If a power series is not differentially algebraic, then it is called *transcendentally transcendental.* For a nice survey on transcendentally transcendental functions, see Rubel [29].

Corollary 9. *Let $V \subseteq \mathbb{A}_K^d$ be an affine variety over an algebraically closed field K of characteristic zero. Then the power series*

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is D-finite if and only if it is rational.

To show an application of this corollary, let us consider the linear system $A\mathbf{x} = 0$, where A is a $d \times m$ matrix with integer entries. Let E be the set of all vectors $(n_1, \ldots, n_d) \in \mathbb{N}^d$ such that $A\mathbf{x} = 0$. We now give a proof of the following classical theorem in enumerative combinatorics.

Theorem 10 (Theorem 4.6.11 in [33]). *The generating function*

$$F_E(x_1,\dots,x_d) := \sum_{(n_1,\dots,n_d)\in E} x_1^{n_1}\cdots x_d^{n_d}$$

represents a rational function of $x_1,\dots,x_d$.

Proof. By Corollary 9, it suffices to show that F_E is D-finite. We first recall a fact proved by Lipshitz in [19, p. 377] that if the power series $G(\mathbf{x}) = \sum g(n_1,\dots,n_d)x_1^{n_1}\cdots x_d^{n_d}$ is D-finite and $C \subseteq \mathbb{N}^d$ is the set of elements of $\mathbb{N}^d$ satisfying a finite set of inequalities of the form $\sum a_i n_i + b \geq 0$, where the $a_i, b \in \mathbb{Z}$, then the power series

$$H(\mathbf{x}) := \sum_{(n_1,\dots n_k)\in C} g(n_1,\dots,n_d)x_1^{n_1}\cdots x_d^{n_d}$$

is D-finite. Note that $R(x_1,\dots,x_d) := \sum x_1^{n_1}\cdots x_d^{n_d} = 1/\prod_{i=1}^{d}(1-x_i)$ is D-finite and any equality $\sum a_i n_i = 0$ is equivalent to two inequalities $\sum a_i n_i \geq 0$ and $\sum(-a_i)n_i \geq 0$. Then the D-finiteness of F_E follows from the fact. □

We now derive some properties of an algebraic variety E from the generating function F_E when $d = 2$. We first prove a basic result that is probably well-known, but for which we are unaware of a reference.

Proposition 11. *Let*

$$F(x_1,\cdots,x_d) = \sum_{(n_1,\dots,n_d)\in\mathbb{N}^d} f(n_1,\dots,n_d)x_1^{n_1}\cdots x_d^{n_d} \in \mathbb{Q}[[x_1,\dots,x_d]]$$

with $f(n_1,\dots,n_d) \in \{0,1\}$ *for all* $(n_1,\dots,n_d) \in \mathbb{N}^d$. *Then* F *is rational if and only if the support set* $E := \{(n_1,\dots,n_d) \in \mathbb{N}^d \mid f(n_1,\dots,n_d) \neq 0\}$ *of* F *is* semilinear *(see Equation* (1) *for the definition of semilinearity).*

Proof. The sufficiency follows from Theorem 10. For the other direction assume that $F(x_1,\dots,x_d)$ is rational. Since $f(n_1,\dots,n_d) \in \{0,1\}$ for all $(n_1,\dots,n_d) \in \mathbb{N}^d$, we have that F can be written in the form $F = P/Q$ with $P, Q \in \mathbb{Z}[x_1,\dots,x_d]$ and the gcd of the collection of coefficients of P and Q equal to 1. For any prime number $p \in \mathbb{N}$, the modulo p mapping $\phi_p : \mathbb{Z}[[x_1,\dots,x_d]] \to \mathbb{F}_p[[x_1,\dots,x_d]]$ is a homomorphism. Then $\phi_p(Q\cdot F) = \phi_p(Q)\cdot\phi_p(F) = \phi_p(P)$, which then implies that the sequence $f : \mathbb{N}^d \to \{0,1\}$ has a rational generating function over any finite field $\mathbb{F}_p$, where p is a prime number. By Salon's theorem [30], which is a multi-dimensional extension of the theorem by Christol, Kamae, Mendès France, and Rauzy [8], the sequence $f : \mathbb{N}^d \to \{0,1\}$ is p-automatic for every prime number p. Then the Cobham-Semenov theorem [12] implies that the support set E of f is semilinear. □

We now use this result in the special case when $d = 2$.

Theorem 12. *Let $p(x,y) \in K[x,y]$ be a nonzero polynomial satisfying that the generating function*

$$F_p(x,y) := \sum_{\substack{(n,m)\in\mathbb{N}^2 \\ p(n,m)=0}} x^n y^m$$

is rational. Then $p = c \cdot f \cdot g$, where $c \in K$ is a constant, f is a product of linear polynomials in x and y with integer coefficients and g has only finite roots in $\mathbb{N}^2$.

Proof. Let $p = p_1 \cdots p_r$ with p_i irreducible over K. Assume that $p_1, \ldots, p_m$ have only finitely many zeros in $\mathbb{N}^2$ and that p_i with $i > m$ has infinitely many roots in $\mathbb{N}^2$. Then let $g = p_1 \cdots p_m$. We show that $p_{m+1}, \ldots, p_r$ are, up to scalar multiplication, polynomials of the form $ax + by + c$ with $a, b, c \in \mathbb{Z}$. By Proposition 11, the set E of all nonnegative points (n,m) on the curve $p(x,y) = 0$ is semilinear. Now suppose that E is infinite. Then if the subset V_i in (1) is not contained in a line in $\mathbb{Z}^2$ through the origin, then the set

$$b_i + \sum_{v\in V_i} v \cdot \mathbb{N}$$

is Zariski dense in the plane, which is impossible since E is contained in the zero set of a nonzero polynomial. Thus we see that after refining our decomposition of E if necessary, we may assume that each $|V_i| = 1$ for $i > 0$. Let q be any irreducible factor of p having infinitely many zeros in $\mathbb{N}^2$. Then there is some $V_i = \{v\} \subseteq \mathbb{N}^2$ with $i > 0$, such that $q(b_i + vn) = 0$ for infinitely many $n \in \mathbb{N}$. Write $b_i = (c,d)$ and $v = (a,b)$. Then $q(c + an, d + bn) = 0$ for infinitely many $n \in \mathbb{N}$ and so $q(c + at, d + bt) = 0$ for all $t \in K$. Hence the linear polynomial $ay - bx - (da - cb)$ divides q. Since q is irreducible over K, then $q = \lambda(ay - bx - (da - cb))$ for some constant $\lambda \in K$. This completes the proof. $\square$

The theorem as above cannot be extended to the case when $d > 2$ as shown in the following example.

Example 13. *Let $p = x - y + 2z^2 + zy^2$. We claim that $E := \{(n,n,0) \mid n \in \mathbb{N}\}$ is the set of all zeros of p in $\mathbb{N}^3$. Suppose that (a,b,c) is another $\mathbb{N}^3$-point with c nonzero. Then $a + 2c^2 + cb^2 = b$ and so $c(2c + b^2) = 2c^2 + cb^2 \le b$ since a is nonnegative. But if c is strictly positive then we must have $2c + b^2 \le c(2c + b^2) \le b$, which gives $c \le 0$, a contradiction.*

Now the corresponding generating function is equal to $1/((1-x)(1-y))$ which is rational, but the polynomial p is not of the integer-linear form up to scalar multiplication.

As in the first question, we can show that it is undecidable to test whether the generating function F_V for an arbitrary algebraic variety V is D-finite or not. Let $P \in \mathbb{Q}[x_1, \ldots, x_d]$ be any polynomials over $\mathbb{Q}$ in $x_1, \ldots, x_d$ and let V be the algebraic variety defined by

$$V := \{(a_1, \ldots, a_d, b, c) \in \overline{\mathbb{Q}} \mid P(a_1, \ldots, a_d)^2 + (b - c^2)^2 = 0\}.$$

The undecidability follows from the equivalence that the generating function F_V is D-finite if and only if P has no root in $\mathbb{N}^d$. Clearly, $F_V = 0$ if P has no root in $\mathbb{N}^d$ and then it is D-finite. Now suppose that P has at least one root in $\mathbb{N}^d$. Then the generating function F_V is of the form

$$F_V = \sum_{\substack{(n_1,\ldots,n_d,m)\in\mathbb{N}^{d+1}\\ P(n_1,\ldots,n_d)=0}} x_1^{n_1}\cdots x_d^{n_d} y^{m^2} z^m.$$

It is sufficient to show that $G_V(x_1,\ldots,x_d,y) := F_V(x_1,\ldots,x_d,y,1)$ is not D-finite. Clearly, the set

$$\{m \mid \exists (n_1,\ldots,n_d)\in\mathbb{N}^d \text{ such that } g(n_1,\ldots,n_d,m)\neq 0\}$$

is the set of square numbers, which is neither finite nor syndetic. Thus G_V is not D-finite by Lemma 7.

Example 14. *Let $p = x^2 - y \in K[x,y]$. Then the associated generating function is $F(x,y) = \sum_{m\geq 0} x^m y^{m^2}$. Since p is not of the integer-linear form, $F(x,y)$ is not D-finite by Theorem 12. Actually, we can show that $F(x,y)$ is transcendentally transcendental. Suppose that $F(x,y)$ is differentially algebraic. Then it satisfies a nontrivial algebraic differential equation $Q(x,y,F,D_x(F),\ldots,D_x^r(F)) = 0$, where $r\in\mathbb{N}$ and $Q\in K[z_1,z_2,\ldots,z_{r+3}]$. Note that the evaluation of a power series at $y=2$ gives a ring homomorphism $e_2 : K[[x,y]]\to K[[x]]$ and we have a commuting square*

$$\begin{array}{ccc} K[[x,y]] & \xrightarrow{e_2} & K[[x]] \\ \downarrow & & \downarrow \\ K[[x,y]] & \xrightarrow{e_2} & K[[x]], \end{array}$$

where both vertical maps are differentiation with respect to x. It follows that $F(x,2) = \sum_{m\geq 0} 2^{m^2} x^m$ is also differentially algebraic. This leads to a contradiction with the fact proved by Mahler in [21, p. 200, Theorem 16] on the rate of coefficient growth of a differentially algebraic power series, since $2^{m^2} \gg (m!)^c$ for any positive constant c.

This example motivates us to formulate the following conjecture, which can be viewed as an analogue of the Pólya-Carlson theorem in the context of algebraic geometry and differential algebra.

Conjecture 15. *Let $V\subseteq \mathbb{A}_K^d$ be an affine variety over an algebraically closed field K of characteristic zero. Then the power series*

$$F_V(x_1,\ldots,x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is either rational or transcendentally transcendental.

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