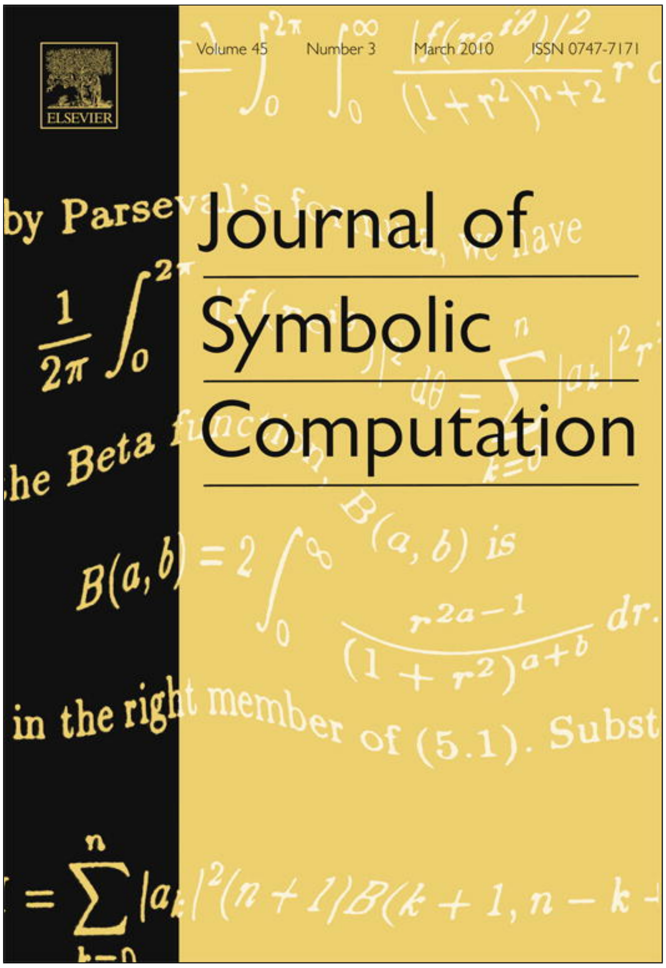
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An algorithm to compute Liouvillian solutions of prime order linear difference–differential equations

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ABSTRACT

A normal form is given for integrable linear difference–differential equations $\{\sigma(Y) = AY, \delta(Y) = BY\}$, which is irreducible over $\mathbb{C}(x, t)$ and solvable in terms of Liouvillian solutions. We refine this normal form and devise an algorithm to compute all Liouvillian solutions of such kinds of systems of prime order.

1. Introduction

Algorithms computing Liouvillian solutions of linear differential equations or difference equations have been well developed by Kovacic (1986), Singer (1981), Petkovšek (1992), Petkovšek and Salvy (1993), van Hoeij et al. (1999), Hendriks and Singer (1999), van Hoeij (1999) and Labahn and Li (2004). For a linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

over $\mathbb{C}(x)$, the differential Galois theory allows us to conclude that if $L(y) = 0$ has a Liouvillian solution, then it has a solution of the form $e^{\int f}$, where f is algebraic over $\mathbb{C}(x)$ (see Singer (1981)). Although one cannot deduce a similar conclusion from the difference Galois group, one can show that a linear difference equation will have a solution that is the interlacing of hypergeometric sequences if it has a solution in the ring of Liouvillian sequences (see Hendriks and Singer (1999); Bomboy (2002); Khmelnov (2008); Cha and van Hoeij (2009); Abramov et al. (2009)). Therefore, computing hyperexponential

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solutions (or hypergeometric sequences) is the basic step of the algorithm for finding Liouvillian solutions (or Liouvillian sequences); see Section 3 for further references.

In our previous paper Feng et al. (2009), we prove that if a linear difference–differential system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is irreducible over $\mathbb{C}(x, t)$ and solvable in terms of Liouvillian sequences, then there is some positive integer ℓ such that $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ is equivalent over a suitable algebraic extension of $\mathbb{C}(x, t)$ to a system of diagonal form (see Theorem 1 and Proposition 2 below). In other words, the solution space of the system $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ has a basis consisting of the interlacing of hypergeometric sequences (in the difference–differential sense). In this paper, we will devote ourselves to devising an algorithm to compute the above diagonal form in the case that the system is of prime order. Our algorithm will rely on the above known algorithms and the algorithms on computing rational solutions of linear difference equations.

The paper is organized as follows. In Section 2, based on some results in Feng et al. (2009) (see Theorem 1 and Proposition 2 below) and integrability conditions, we give a normal form for a linear difference–differential system of arbitrary order which is irreducible over $\mathbb{C}(x, t)$ and solvable in terms of Liouvillian sequences. We then further refine this normal form for systems of prime order. In Section 3, we give algorithms to compute all Liouvillian sequence solutions of systems which are irreducible over $\mathbb{C}(x, t)$ and of prime order. Two examples are given to illustrate our algorithms.

Throughout this paper, we will use the same notations as in Feng et al. (2009). We use k_0 to denote the difference–differential field $\mathbb{C}(x, t)$ with an automorphism $\sigma : x \mapsto x + 1$ and a derivation $\delta = \frac{\mathrm{d}}{\mathrm{d}t}$, and let k denote its extension field $\overline{\mathbb{C}(t)}(x)$. We use $(\cdot)^{\mathrm{T}}$ to denote the transpose of a vector or matrix and $\det(\cdot)$ to denote the determinant of a square matrix. The symbols $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ represent the set of non-negative integers and the set of positive integers, respectively. For a field k, denote by $\mathfrak{gl}_n(k)$ the set of $n \times n$ matrices over k and by $\mathrm{GL}_n(k)$ the set of $n \times n$ invertible matrices over k. All difference–differential systems of the form $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ with $A \in \mathrm{GL}_n(k)$ and $B \in \mathfrak{gl}_n(k)$ that are in discussion in the paper are assumed to be integrable, which means that $\sigma(B)A = \delta(A) + AB$. For any positive integer ℓ, the symbol A_ℓ denotes $\sigma^{\ell-1}(A)\sigma^{\ell-2}(A) \cdots \sigma(A)A$.

We would like to thank Reinhart Shaefke for supplying a simple proof of Lemma 20.

2. Normal forms for the system

Assume that a system $\{\sigma(Y) = AY,\ \delta(Y) = BY\}$ where $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over k_0 and that its Galois group over k_0 has solvable identity component. Let R and R_0 be the $\sigma\delta$-Picard–Vessiot extension of the system over k and k_0 respectively, and $\mathrm{Gal}(R/k)$ and $\mathrm{Gal}(R_0/k_0)$ be the Galois group of the system over k and k_0, respectively. Let $\mathcal{F}_0$ and $\mathcal{F}$ be the total ring of fractions of R_0 and R respectively. First we restate two results in Feng et al. (2009) here.

Theorem 1 (*Feng et al., 2009, Theorem 23*). *If a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ of order n is irreducible over k_0, then there exists a positive integer d such that the system is equivalent over $\hat{k}_0 := \mathcal{F}_0 \cap k$ to the system*

$$\sigma(Y) = \mathrm{diag}(A_1, A_2, \ldots, A_d)Y, \qquad \delta(Y) = \mathrm{diag}(B_1, B_2, \ldots, B_d)Y$$

where $A_i \in \mathrm{GL}_\ell(\hat{k}_0)$, $B_i \in \mathfrak{gl}_\ell(\hat{k}_0)$ and $\ell = \frac{n}{d}$ and the system $\{\sigma(Y) = A_iY, \delta(Y) = B_iY\}$ is irreducible over k for $i = 1, \ldots, d$. Moreover, there exists $g_i \in \mathrm{Gal}(R_0/k_0)$ such that $g_i(A_1) = A_i$ and $g_i(B_1) = B_i$.

Proposition 2 (*Feng et al., 2009, Proposition 37*). *If $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ is an irreducible system of order ℓ over k and its Galois group over k has solvable identity component, then $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to*

$$\sigma^\ell(Y) = \mathcal{D}Y, \qquad \delta(Y) = \mathrm{diag}(b_1, \ldots, b_\ell)Y$$

where $\mathcal{D} = \mathrm{diag}(a, \sigma(a), \ldots, \sigma^{\ell-1}(a))$ for some $a \in k \setminus \{0\}$ and $b_i \in k$ for $i = 1, \ldots, \ell$.

Theorem 1 and Proposition 2 imply that, for an irreducible system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ of order n over k_0, there exists $\ell \in \mathbb{Z}_{>0}$ with $\ell | n$ such that $\{\sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY\}$ is equivalent over k to a system of diagonal form. In this section, we will show further that the original system is equivalent over k to a more special form when its order n is prime.

2.1. Normal forms for a general system

Let us first review some notions and properties concerning rational solutions of difference equations.

Definition 3 (*Hardouin and Singer, 2008, Definition 6.1*). Let $f = \frac{P}{Q}$ with $P, Q \in \overline{\mathbb{C}(t)}[x]$ and $\gcd(P, Q) = 1$.

(1) The dispersion of Q, denoted by $\mathrm{disp}(Q)$ is

$$\max\{j \in \mathbb{Z}_{>0} | Q(\alpha) = Q(\alpha + j) = 0 \text{ for some } \alpha \in \overline{\mathbb{C}(t)}\}.$$

(2) The polar dispersion of f is the dispersion of Q and denoted $\mathrm{pdisp}(f)$.
(3) f is said to be standard with respect to σ^m, with $m \in \mathbb{Z}_{>0}$, if $\mathrm{disp}(P \cdot Q) < m$.

As in Hardouin and Singer (2008), we have the following

Lemma 4. *Assume that $f \in k \setminus \{0\}$, $a \in \overline{\mathbb{C}(t)} \setminus \{0\}$ and $m \in \mathbb{Z}_{>0}$.*

(1) *There exist $\tilde{f}, \tilde{g} \in k \setminus \{0\}$ such that $f = \frac{\sigma^m(\tilde{g})}{\tilde{g}}\tilde{f}$ where $\tilde{f}$ is standard with respect to σ^m.*
(2) *If f has a pole, then $\mathrm{pdisp}(\sigma^m(f) - af) \geq m$.*

Proof. The proof is similar to that of Lemma 6.2 in Hardouin and Singer (2008). □

Proposition 5. *Let $0 \neq a, b \in k$ satisfy $\sigma^m(b) - b = \frac{\delta(a)}{a}$ where $m \in \mathbb{Z}_{>0}$. Then*

$$a = \frac{\sigma^m(f)}{f}\alpha(x)\beta(t) \quad \textit{and} \quad b = \frac{\delta(f)}{f} + \frac{\delta(\beta(t))}{m\beta(t)}x + c$$

where $f \in k$, $c, \beta(t) \in \overline{\mathbb{C}(t)}$, and $\alpha(x) \in \mathbb{C}(x)$ is standard with respect to σ^m.

Proof. Let $a = \frac{\sigma^m(f)}{f}\hat{a}$ with $\hat{a}$ standard with respect to σ^m and $\hat{b} = b - \frac{\delta(f)}{f}$. Then $\sigma^m(\hat{b}) - \hat{b} = \frac{\delta(\hat{a})}{\hat{a}}$. View $\hat{a}$ and $\hat{b}$ as rational functions in x. Then $\mathrm{pdisp}(\frac{\delta(\hat{a})}{\hat{a}}) < m$. If $\frac{\delta(\hat{a})}{\hat{a}} \notin \overline{\mathbb{C}(t)}$, then $\frac{\delta(\hat{a})}{\hat{a}}$ has a pole and so does $\hat{b}$. By Lemma 4, $\mathrm{pdisp}(\sigma^m(\hat{b}) - \hat{b}) \geq m$, a contradiction. Hence $\frac{\delta(\hat{a})}{\hat{a}} = w(t) \in \overline{\mathbb{C}(t)}$, which means that $\hat{a} = \alpha(x)e^{\int w(t)\mathrm{d}t}$. Since $\hat{a} \in k$, $\hat{a}$ is of the form $\alpha(x)\beta(t)$ where $\alpha(x) \in \mathbb{C}(x)$ and $\beta(t) \in \overline{\mathbb{C}(t)}$. Then $\hat{b} \in \overline{\mathbb{C}(t)}[x]$. Suppose that $\hat{b} = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ where $c_i \in \overline{\mathbb{C}(t)}$ and $c_n \neq 0$. Then

$$\sigma^m(\hat{b}) - \hat{b} = nmc_n x^{n-1} + \cdots = \frac{\delta(\hat{a})}{\hat{a}} = \frac{\delta(\beta(t))}{\beta(t)}.$$

So $n = 1$ and $\hat{b} = \frac{\delta(\beta(t))}{m\beta(t)}x + c_0$. □

Now we proceed to give the normal form.

Theorem 6. *If $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over k_0 and its Galois group over k_0 has solvable identity component, then there exists $\ell \in \mathbb{Z}_{>0}$ with $\ell | n$ such that the system*

$$\sigma^\ell(Y) = A_\ell Y, \qquad \delta(Y) = BY$$

is equivalent over k to

$$\begin{cases} \sigma^\ell(Y) = \mathrm{diag}(\Lambda(x)\beta_1(t), \Lambda(x)\beta_2(t), \ldots, \Lambda(x)\beta_m(t))Y, \\ \delta(Y) = \mathrm{diag}\left(\frac{\delta(\beta_1(t))}{\ell\beta_1(t)}xI_\ell + C_1, \ldots, \frac{\delta(\beta_m(t))}{\ell\beta_m(t)}xI_\ell + C_m\right)Y \end{cases} \tag{1}$$

where $\Lambda(x) = \mathrm{diag}(\alpha(x), \ldots, \alpha(x + \ell - 1))$, $C_1 = \mathrm{diag}(c_1, \ldots, c_\ell)$ and $m\ell = n$. Moreover, $\alpha(x) \in \mathbb{C}(x)$ is standard with respect to σ^ℓ, $\beta_i(t), c_i \in \overline{\mathbb{C}(t)}$, and there exists g_i in the Galois group of the original system over k_0 such that $\beta_i(t) = g_i(\beta_1(t))$ and $C_i = g_i(C_1)$.

Proof. By Theorem 1, it suffices to prove the theorem for a factor over k of the given system. Let $\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}$ be such a factor with $\mathcal{A} \in \mathrm{GL}_\ell(k)$ and $\mathcal{B} \in \mathfrak{gl}_\ell(k)$. By Proposition 2, $\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}$ is equivalent over k to

$$\sigma^\ell(Y) = \mathcal{D}Y, \qquad \delta(Y) = \mathrm{diag}(b_1, \ldots, b_\ell)Y$$

where $\mathcal{D}$ is as in Proposition 2 and $b_i \in k$ for $i = 1, \ldots, \ell$. Since σ^ℓ and δ commute, we have $\sigma^\ell(b_1) - b_1 = \frac{\delta(a)}{a}$. By Proposition 5, we have

$$a = \frac{\sigma^\ell(f)}{f}\alpha(x)\beta_1(t) \quad \text{and} \quad b_1 = \frac{\delta(f)}{f} + \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}x + c_1$$

where $\alpha(x) \in \mathbb{C}(x)$ is standard with respect to σ^ℓ, $c_1, \beta_1(t) \in \overline{\mathbb{C}(t)}$ and $f \in k$. Then for $i = 1, \ldots, \ell$,

$$\sigma^{i-1}(a) = \frac{\sigma(\sigma^{i-1}(f))}{\sigma^{i-1}(f)}\alpha(x+i-1)\beta_1(t) \quad \text{and} \quad b_i = \frac{\delta(\sigma_{i-1}(f))}{\sigma^{i-1}(f)} + \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}x + c_i.$$

Let $F = \mathrm{diag}(f, \sigma(f), \ldots, \sigma^{\ell-1}(f))$. Then the system

$$\sigma^\ell(Y) = \mathcal{D}Y, \qquad \delta(Y) = \mathrm{diag}(b_1, \ldots, b_\ell)Y$$

is equivalent over k to

$$\sigma^\ell(Y) = \Lambda(x)\beta_1(t)Y, \qquad \delta(Y) = \left(\frac{\delta(\beta_1(t))}{\ell\beta_1(t)}xI_\ell + C_1\right)Y$$

under the transformation $Y \to FY$ where $\Lambda(x) = \mathrm{diag}(\alpha(x), \ldots, \alpha(x+\ell-1))$ and $C_1 = \mathrm{diag}(c_1, \ldots, c_\ell)$. $\square$

2.2. Normal forms for systems of prime order

If a difference–differential system is of prime order n, then the integer ℓ in Theorem 6 equals either 1 or n. For the case where the system is reducible over k, we can refine Theorem 6 further in the following

Proposition 7. *Assume that n is a prime number. Suppose that the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over k_0 and reducible over k and that its Galois group over k_0 has solvable identity component. Then the system is equivalent over k to*

$$\begin{cases} \sigma(Y) = \alpha(x)\mathrm{diag}(\beta_1(t), \beta_2(t), \ldots, \beta_n(t))Y, \\ \delta(Y) = \mathrm{diag}\left(\frac{\delta(\beta_1(t))}{\beta_1(t)}x + c_1, \ldots, \frac{\delta(\beta_n(t))}{\beta_n(t)}x + c_n\right)Y \end{cases} \tag{2}$$

where $\alpha(x) \in \mathbb{C}(x)$ is standard with respect to σ, $\beta_i(t) = g_i(\beta_1(t)) \in \overline{\mathbb{C}(t)}$ and $c_i = g_i(c_1) \in \overline{\mathbb{C}(t)}$ for some g_i in the Galois group of the original system over k_0.

Before discussing the other case where a difference–differential system is irreducible over k, let us look at the following

Lemma 8. *Assume that $\sigma(Y) = AY$ with $A \in \mathrm{GL}_n(k_0)$ is equivalent over k to $\sigma(Y) = \bar{A}Y$ where*

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta(t)\alpha(x) & 0 & 0 & \cdots & 0 \end{pmatrix},$$

with $\alpha(x) \in \mathbb{C}(x)$ and $\beta(t) \in \overline{\mathbb{C}(t)}$. Then $\beta(t) \in \mathbb{C}(t)$.

Proof. There exists $G \in GL_n(k)$ such that $\sigma(G)\bar{A}G^{-1} = A$. Then

$$\det(\sigma(G))\det(\bar{A})\det(G^{-1}) = \det(A).$$

Since $\det(\sigma(G)) = \sigma(\det(G))$ and $\det(G^{-1}) = \frac{1}{\det(G)}$, we have

$$(-1)^{n-1}\beta(t)\alpha(x)\frac{\sigma(\det(G))}{\det(G)} = \det(A).$$

Expand the rational functions in x in the above equation as series at $x = \infty$. Since $\frac{\sigma(\det(G))}{\det(G)} = 1 + \frac{1}{x}Q$ where $Q \in \overline{\mathbb{C}(t)}[[\frac{1}{x}]]$, one sees that $\beta(t) \in \mathbb{C}(t)$. $\square$

Proposition 9. *Let $A, \bar{A} \in \mathrm{GL}_n(k_0)$. If $\sigma(Y) = AY$ and $\sigma(Y) = \bar{A}Y$ are equivalent over k then they are equivalent over k_0.*

Proof. Suppose that there exists $G \in \mathrm{GL}_n(k)$ such that $\sigma(G)A = \bar{A}G$. Then there exists $\gamma(t) \in \overline{\mathbb{C}(t)}$ such that $G \in \mathrm{GL}_n(k_0(\gamma(t)))$. Let $m = [k_0(\gamma(t)) : k_0]$. Since $1, \gamma(t), \ldots, \gamma(t)^{m-1}$ is a basis of $k_0(\gamma(t))$ over k_0, we can write

$$G = G_0 + G_1\gamma(t) + \cdots + G_{m-1}\gamma(t)^{m-1}$$

where $G_i \in \mathfrak{gl}_n(k_0)$. From $\sigma(G)A = \bar{A}G$, it follows that $\sigma(G_i)A = \bar{A}G_i$ for $i = 0, \ldots, m-1$. Let λ be a parameter satisfying $\sigma(\lambda) = \lambda$ and let $H(\lambda) = \sum_{i=0}^{m-1}\lambda^i G_i$. Therefore, $\sigma(H(\lambda))A = \bar{A}H(\lambda)$. Since $\det(G) = \det(H(\gamma(t))) \neq 0$, $\det(H(\lambda))$ is a nonzero polynomial with coefficients in k_0. Hence there exists $c \in \mathbb{C}(t)$ such that $\det(H(c)) \neq 0$. So $\sigma(H(c))A = \bar{A}H(c)$ and $H(c) \in \mathrm{GL}_n(k_0)$. $\square$

We now turn to the case where a difference–differential system over k_0 is irreducible over k.

Proposition 10. *Suppose that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over k and that its Galois group over k_0 has solvable identity component. Then the system is equivalent over k_0 to*

$$\sigma(Y) = \bar{A}Y, \qquad \delta(Y) = \bar{B}Y$$

where $\bar{B} \in \mathfrak{gl}_n(k_0)$ and

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta(t)\alpha(x) & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(k_0)$$

with $\alpha(x) \in \mathbb{C}(x)$ standard with respect to σ^n and $\beta(t) \in \mathbb{C}(t)$. Moreover, $\frac{\alpha(x+1)}{\alpha(x)} \neq \frac{\sigma^n(b)}{b}$ for any $b \in \mathbb{C}(x)$.

Proof. By Proposition 32 in Feng et al. (2009), the given system is equivalent over k to the system $\{\sigma(Y) = \bar{\mathcal{A}}Y, \delta(Y) = \bar{\mathcal{B}}Y\}$ where $\bar{\mathcal{B}} \in \mathfrak{gl}_n(k)$ and

$$\bar{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(k).$$

Since σ and δ commute, we have $\sigma(\bar{\mathcal{B}})\bar{\mathcal{A}} = \delta(\bar{\mathcal{A}}) + \bar{\mathcal{A}}\bar{\mathcal{B}}$. Let $\bar{\mathcal{B}} = (\bar{b}_{ij})_{n\times n}$ where $\bar{b}_{i,j} \in k$. Then

$$\sigma(\bar{b}_{nn}) - \bar{b}_{11} = \frac{\delta(a)}{a}, \quad \bar{b}_{nn} = \sigma(\bar{b}_{n-1,n-1}), \ldots, \bar{b}_{22} = \sigma(\bar{b}_{11}).$$

Hence $\sigma^n(\bar{b}_{11}) - \bar{b}_{11} = \frac{\delta(a)}{a}$. By Proposition 5, we have $a = \frac{\sigma^n(f)}{f}\alpha(x)\beta(t)$ with $f \in k$, $\alpha(x) \in \mathbb{C}(x)$ standard with respect to σ^n and $\beta(t) \in \overline{\mathbb{C}(t)}$. Then $\{\sigma(Y) = \bar{\mathcal{A}}Y, \delta(Y) = \bar{\mathcal{B}}Y\}$ is equivalent over $\underline{k}$ to $\{\sigma(Y) = \bar{A}Y, \delta(Y) = \bar{B}Y\}$ under the transformation $Y \to \mathrm{diag}(f, \sigma(f), \ldots, \sigma^{n-1}(f))Y$, where $\bar{B} \in \mathfrak{gl}_n(k_0)$ and

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta(t)\alpha(x) & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha(x) \in \mathbb{C}(x)$ and $\beta(t) \in \overline{\mathbb{C}(t)}$. By Lemma 8 and Proposition 9, the original system is equivalent over k_0 to $\{\sigma(Y) = \bar{A}Y, \delta(Y) = \bar{B}Y\}$ with $\beta(t) \in \mathbb{C}(t)$. Assume that $\frac{\alpha(x+1)}{\alpha(x)} = \frac{\sigma^n(b)}{b}$ for some $b \in \mathbb{C}(x)$ and let $u = \sigma^{n-1}(b)\cdots\sigma(b)b$. We have $\frac{\alpha(x+1)}{\alpha(x)} = \frac{u(x+1)}{u(x)}$ thus $\alpha(x) = cu(x)$ for some constant c with respect to σ. Therefore $c \in \mathbb{C}$ since $\alpha(x)$ and $u(x)$ are both in $\mathbb{C}(x)$. Let $P \in \mathrm{GL}_n(\overline{\mathbb{C}(t)})$ be such that

$$P^{-1}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c\beta(t) & 0 & 0 & \cdots & 0 \end{pmatrix} P = \mathrm{diag}(\tilde{\beta}_1(t), \tilde{\beta}_2(t), \ldots, \tilde{\beta}_n(t))$$

where the $\tilde{\beta}_i(t)$'s are the roots of $Y^n - c\beta(t)$. Let

$$F = \mathrm{diag}(1, b, \sigma(b)b, \ldots, \sigma^{n-2}(b)\cdots\sigma(b)b)P$$

and $\tilde{B} = F^{-1}\bar{B}F - F^{-1}\delta(F)$. Then $\{\sigma(Y) = \bar{A}Y, \delta(Y) = \bar{B}Y\}$ is equivalent over k to

$$\sigma(Y) = b \cdot \mathrm{diag}(\tilde{\beta}_1(t), \tilde{\beta}_2(t), \ldots, \tilde{\beta}_n(t))Y, \qquad \delta(Y) = \tilde{B}Y$$

under the transformation $Y \to FY$. Assume $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$. Since σ and δ commute,

$$\sigma(\tilde{b}_{ij}) - \frac{\tilde{\beta}_i(t)}{\tilde{\beta}_j(t)}\tilde{b}_{ij} = 0,$$

for all i, j with $1 \le i \ne j \le n$. Hence $\tilde{b}_{ij} = 0$ if $i \ne j$. In other word, $\tilde{B}$ is of diagonal form. This contradicts to the irreducibility over k of the original system. $\square$

Lemma 11. *Let $a \in k_0 \setminus \{0\}$, n be a positive integer and $m > 0$ be the least integer such that $\frac{\sigma^m(a)}{a} = \frac{\sigma^n(b)}{b}$ for some $b \in k_0$. Then $m|n$.*

Proof. Suppose that $\frac{\sigma^m(a)}{a} = \frac{\sigma^n(b)}{b}$ with $b \in k_0$. Then for each $\ell > 0$,

$$\frac{\sigma^{\ell m}(a)}{a} = \frac{\sigma^n(c_\ell)}{c_\ell} \quad \text{with } c_\ell \in k_0.$$

Let $n = \ell_1 m + \ell_2$ where $0 \le \ell_2 \le m - 1$. Then

$$\frac{\sigma^{\ell_2}(a)}{a} = \frac{\sigma^{\ell_1 m + \ell_2}(a)}{a}\frac{\sigma^{\ell_2}(a)}{\sigma^{\ell_1 m + \ell_2}(a)} = \frac{\sigma^n(a)}{a}\frac{c}{\sigma^n(c)}$$

for some $c \in k_0$. Hence $\ell_2 = 0$ and so $m|n$. $\square$

Proposition 12. *Assume that n is a prime number, the system*

$$\sigma(Y) = AY, \qquad \delta(Y) = BY$$

with $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over k and its Galois group over k_0 has solvable identity component. Then $\{\sigma^n(Y) = A_n Y, \delta(Y) = BY\}$ is equivalent over k_0 to

$$\begin{cases} \sigma^n(Y) = \beta(t)\mathrm{diag}(\alpha(x), \ldots, \alpha(x+n-1))Y \\ \delta(Y) = \left(\frac{\delta(\beta(t))}{n\beta(t)} x I_n + \mathrm{diag}(\hat{b}_1, \ldots, \hat{b}_n)\right) Y \end{cases}$$

where $\alpha(x)$ and $\beta(t)$ are as in Proposition 10 and $\hat{b}_i \in \mathbb{C}(t)$ for $i = 1, \ldots, n$.

Proof. By Proposition 10, $\{\sigma^n(Y) = A_n Y, \delta(Y) = BY\}$ is equivalent over k_0 to the system

$$\sigma^n(Y) = \beta(t) \cdot \mathrm{diag}(\alpha(x), \ldots, \alpha(x+n-1))\, Y, \qquad \delta(Y) = \bar{B}Y$$

with $\alpha(x)$ and $\beta(t)$ as in Proposition 10 and $\bar{B} \in \mathfrak{gl}_n(k_0)$. Let $\bar{B}=(\bar{b}_{ij})_{n\times n}$. From $\sigma^n\delta = \delta\sigma^n$, we have

$$\begin{cases} \sigma^n(\bar{b}_{ii}) - \bar{b}_{ii} = \frac{\delta(\beta(t))}{\beta(t)}, & i = 1, \ldots, n, \\ \sigma^n(\bar{b}_{ij}) - \frac{\alpha(x+i)}{\alpha(x+j)}\bar{b}_{ij} = 0, & 1 \le i \ne j \le n. \end{cases}$$

Hence $\bar{b}_{ii} = \frac{\delta(\beta(t))}{n\beta(t)} x + \hat{b}_i$ with $\hat{b}_i \in \mathbb{C}(t)$. Note that n is prime and $\frac{\alpha(x+1)}{\alpha(x)} \ne \frac{\sigma^n(b)}{b}$ for any $b \in \mathbb{C}(x)$. Then by Lemma 11, $\frac{\alpha(x+i)}{\alpha(x)} \ne \frac{\sigma^n(b)}{b}$ for any $1 \le i \le n-1$ and $b \in \mathbb{C}(x)$. Hence $\bar{b}_{ij} = 0$ for $i \ne j$. This concludes the proposition. $\square$

3. A decision procedure for systems of prime order

Consider a system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ of order n over k_0. Assume that the order n is prime, the system is irreducible over k_0 and its Galois group over k_0 has solvable identity component (or, equivalently, the system has Liouvillian solutions). By Propositions 7 and 12, either the original system has hypergeometric solutions over k or the system $\{\sigma^n(Y) = A_n Y, \delta(Y) = BY\}$ has solutions which are the interlacing of hypergeometric solutions over k_0. In this section, we will give a decision procedure to find solutions of systems of both forms when the order n is prime. Our procedure relies on the following three facts in the ordinary cases:

(A_1) we can compute all rational solutions in k^n of an ordinary difference equation $\sigma(Y) = AY$ where $A \in \mathrm{GL}_n(k)$; ((Abramov and Barkatou , 1998; Abramov , 1995, 1989; van Hoeij, 1998));

(A_2) we can compute all hypergeometric solutions over $\mathbb{C}(x)$ of an ordinary difference equation $\sigma(Y) = \hat{A}Y$ where $\hat{A} \in \mathrm{GL}_n(\mathbb{C}(x))$ ((Hendriks and Singer, 1999; Labahn and Li, 2004; Wu, 2005; Li et al., 2006; Petkovšek, 1992; Petkovšek and Salvy, 1993; van Hoeij, 1999; Bomboy, 2002));

(A_3) we can compute all hyperexponential solutions over $\overline{\mathbb{C}(t)}$ of an ordinary differential equation $\delta(Y) = \hat{B}Y$ where $\hat{B} \in \mathrm{GL}_n(\mathbb{C}(t))$ ((Kovacic, 1986; Labahn and Li, 2004; Wu, 2005; Li et al., 2006; Singer, 1981; van Hoeij et al., 1999)).

In the following subsections, we will reduce the problem of finding solutions of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ or of $\{\sigma^n(Y) = A_n Y, \delta(Y) = BY\}$ to that in the ordinary cases as indicated above.

We have two case distinctions according to the reducibility of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over k.

3.1. The decision procedure for the reducible case

Assume that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is reducible over k. Proposition 7 implies that this system has hypergeometric solutions of the form $W_i h_i$ for $i = 1, \ldots, n$, where $W_i \in k^n$ and h_i satisfies

$$\sigma(h_i) = \alpha(x)\beta_i(t)h_i \quad \text{and} \quad \delta(h_i) = \left(\frac{\delta(\beta_i(t))}{\beta_i(t)} x + c_i\right) h_i$$

with $\alpha(x) \in \mathbb{C}(x)$ standard with respect to σ, $\beta_i(t) = g_i(\beta_1(t)) \in \overline{\mathbb{C}(t)}$ and $c_i = g_i(c_1) \in \overline{\mathbb{C}(t)}$ for some g_i in the Galois group of the original system over k_0. Substituting each $W_i h$ into the original system, we get

$$\sigma(W_i) = \frac{A}{\alpha(x)\beta_i(t)} W_i \quad \text{and} \quad \delta(W_i) = \left(B - \frac{\delta(\beta_i(t))}{\beta_i(t)} x - c_i\right) W_i. \tag{3}$$

So, to compute hypergeometric solutions of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ it suffices to find $\alpha(x)$, $\beta_i(t)$, c_i and W_i satisfying (3).

Remark 13. The equalities (3) still hold when replacing $\alpha(x)$ by $\frac{\sigma(g)\alpha(x)}{g}$ and W_i by $\frac{W_i}{g}$ for $g \in \mathbb{C}(x)$. So in the sequel, we will compute a suitable $\frac{\sigma(g)\alpha(x)}{g}$ instead of $\alpha(x)$.

Computing $\alpha(x)$: By Proposition 7, there exists $G \in \mathrm{GL}_n(k)$ such that

$$\frac{\sigma(\det(G))}{\det(G)} \alpha(x)^n \prod_{i=1}^{n} \beta_i(t) = \det(A).$$

Without loss of generality, we assume that the numerator and denominator of $\alpha(x)$ are monic. Expanding the functions in the above equality as series at $x = \infty$, one can compute $\prod_{i=1}^{n} \beta_i(t)$ from the series expansion of $\det(A)$ at $x = \infty$. Let $\tilde{a} = \frac{\det(A)}{\prod_{i=1}^{n} \beta_i(t)}$. Rewrite $\tilde{a} = \frac{\sigma(b)}{b} \bar{a}$ where $b, \bar{a} \in k_0$ and $\bar{a}$ is standard with respect to σ. Then

$$\bar{a} = \frac{\sigma(g)}{g} \alpha(x)^n \quad \text{for some } g \in k_0. \tag{4}$$

From Proposition 7, $\alpha(x)$ is standard with respect to σ and so is $\alpha(x)^n$. Proposition 15 below shows that $\frac{\sigma(g)}{g} \in \mathbb{C}(x)$ and thus $\bar{a} \in \mathbb{C}(x)$. Moreover, $\bar{a}$ has the form $\left(\frac{\sigma(\bar{g})}{\bar{g}}\alpha(x)\right)^n$ for some $\bar{g} \in \mathbb{C}(x)$. To prove Proposition 15, let us introduce a notation used in van der Put and Singer (1997, Section 2.1).

Definition 14. A divisor D on $\mathbb{P}^1(\overline{\mathbb{C}(t)})$ is defined to be a finite formal expression $\sum n_p[p]$ with $p \in \mathbb{P}^1(\overline{\mathbb{C}(t)})$ and $n_p \in \mathbb{Z}$. The support of a divisor D, denoted $\mathrm{supp}(D)$, is the finite set of all p with $n_p \neq 0$. Let $p \in \mathrm{supp}(D)$. The $\mathbb{Z}$-orbit E of p in $\mathrm{supp}(D)$ is defined to be

$$E(p, \mathrm{supp}(D)) = \{p + i | i \in \mathbb{Z} \text{ and } p + i \in \mathrm{supp}(D)\}.$$

As usual, the divisor $\mathrm{div}(f)$ of a rational function $f \in k \setminus \{0\}$ is given by $\mathrm{div}(f) = \sum \mathrm{ord}_p(f)[p]$, where $\mathrm{ord}_p(f)$ denotes the order of f at the point p. It is clear that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$. Moreover, if p is in $\mathrm{supp}(\mathrm{div}(f))$ but not in $\mathrm{supp}(\mathrm{div}(fg))$, then $p \in \mathrm{supp}(\mathrm{div}(f)) \cap \mathrm{supp}(\mathrm{div}(g))$. By Definition 14, if $f \in k \setminus \{0\}$ is standard with respect to σ, then $E(p, \mathrm{supp}(\mathrm{div}(f))) = \{p\}$ for each $p \in \mathrm{supp}(\mathrm{div}(f))$.

Proposition 15. *Assume that $f, g \in k \setminus \{0\}$ and f is standard with respect to σ. If $\sigma(g)g^{-1}f$ is standard with respect to σ, then*

$$\frac{\sigma(g)}{g} = \prod_i \frac{(x + k_i - c_i)^{m_i}}{(x - c_i)^{m_i}}$$

with $k_i \in \mathbb{Z}$, $m_i \in \mathbb{Z}_{>0}$, $c_i \in \overline{\mathbb{C}(t)}$ and $\mathrm{disp}(\prod_i (x - c_i)) = 0$. Moreover, for each i, either $\mathrm{ord}_{c_i}(f) = m_i$ or $\mathrm{ord}_{c_i - k_i}(f) = -m_i$.

Proof. Let $H = \sigma(g)g^{-1}f$, $S_1 = \mathrm{supp}(\mathrm{div}(f))$, $S_2 = \mathrm{supp}(\mathrm{div}(\sigma(g)g^{-1}))$ and $S_3 = \mathrm{supp}(\mathrm{div}(H))$. By Lemma 2.1 in van der Put and Singer (1997),

$$\sum_{q \in E(p, S_2)} \mathrm{ord}_q\left(\frac{\sigma(g)}{g}\right) = 0 \quad \text{for each } p \in S_2.$$

Then $|E(p, S_2)| \geq 2$ for each $p \in \mathrm{supp}(S_2)$. Since H and f are standard,

$$|E(p, S_2) \cap S_3| \leq 1 \quad \text{and} \quad |E(p, S_2) \cap S_1| \leq 1$$

thus $|E(p, S_2) \cap (S_1 \cup S_3)| \le 2$. From $S_2 \subseteq S_1 \cup S_3$, we have $|E(p, S_2)| \le 2$. Hence for each $p \in S_2$,

$$|E(p, S_2)| = 2, \qquad |E(p, S_2) \cap S_1| = 1 \quad \text{and} \quad |E(p, S_2) \cap S_3| = 1.$$

From $|E(p, S_2)| = 2$ and $|E(p, S_2) \cap S_3| = 1$, either

$$\mathrm{ord}_p(\sigma(g)g^{-1}) = -\mathrm{ord}_p(f) \quad \text{or} \quad \mathrm{ord}_{p+j_0}(\sigma(g)g^{-1}) = -\mathrm{ord}_{p+j_0}(f)$$

with $p + j_0 \in E(p, S_2)$. The proposition holds. $\square$

Let g be as in (4). Since $\alpha(x) \in \mathbb{C}(x)$, g can be chosen in $\mathbb{C}(x)$ according to Proposition 15. Then $\bar{a} \in \mathbb{C}(x)$. Moreover,

$$\frac{\sigma(g)}{g} = \prod_i \frac{(x + k_i - c_i)^{m_i}}{(x - c_i)^{m_i}}$$

where m_i has the form $\bar{m}_i n$ for some $\bar{m}_i \in \mathbb{Z}_{>0}$ since m_i is either $\mathrm{ord}_{c_i}(\alpha(x)^n)$ or $-\mathrm{ord}_{c_i-k_i}(\alpha(x)^n)$. Let $\bar{g} = \prod_i \prod_{j=0}^{k_i-1}(x + j - c_i)^{\bar{m}_i}$. Then

$$\left(\frac{\sigma(\bar{g})}{\bar{g}}\right)^n = \frac{\sigma(g)}{g} \quad \text{and} \quad \bar{a} = \left(\frac{\sigma(\bar{g})}{\bar{g}}\alpha(x)\right)^n.$$

Note that the numerator and the denominator of $\alpha(x)$ are monic, so we can compute $\frac{\sigma(\bar{g})}{\bar{g}}\alpha(x)$ from $\bar{a}$.

Example 16. Consider the integrable system

$$\sigma(Y) = AY, \qquad \delta(Y) = BY$$

where

$$A = \begin{pmatrix} -\frac{t(x^2+1)(t^2+1-x)}{t^2-x-1} & -\frac{x^2+1}{t^2-x-1} \\ \frac{(x^2+1)(t^4+t^2-x^2-x)}{t^2-x-1} & \frac{t(x^2+1)(t^2-x)}{t^2-x-1} \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac{-2xt^3-t^4-t^2+t^5+3t^3+2t+x^2t+t^2x+x}{(t^2-x)(t^2+1)} & -\frac{t^2}{(t^2-x)(t^2+1)} \\ \frac{-t^2x^2+t^6+2t^4+t^2-x^2+2t^2x+x}{(t^2-x)(t^2+1)} & \frac{-x^2t-t^2x-x+t^5+2t^3-xt+t^4+t^2}{(t^2-x)(t^2+1)} \end{pmatrix}.$$

We have

$$\det(A) = -\frac{(x^2+1)^2(t^4 - t^2x + t^2 - x)}{t^2 - x - 1} = -(t^2+1)x^4 + (t^2+1)x^3 + \cdots.$$

Thus $\beta_1(t)\beta_2(t) = -(t^2 + 1)$. Let $\tilde{a} = -\frac{\det(A)}{t^2+1}$ and write

$$\tilde{a} = \frac{t^2 - x}{t^2 - (x + 1)}(x^2 + 1)^2.$$

Then $\alpha(x) = x^2 + 1$.

Computing $\beta_i(t)$: We first prove the following

Lemma 17. *Either $\beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t)$ or $\beta_1(t), \ldots, \beta_n(t)$ are the conjugate roots of an irreducible polynomial of degree n with coefficients in $\mathbb{C}(t)$.*

Proof. Let R_0 be a $\sigma\delta$-PV extension of k_0 for $\{\sigma(Y) = AY, \delta(Y)=BY\}$ and $P = \prod_{i=1}^n (X - \beta_i(t))$. From the proof of Theorem 1, one sees that $\mathrm{Gal}(R_0/k_0)$ permutes the $\beta_i(t)$. Furthermore, the orbits of the $\beta_i(t)$ under this group action all have the same size. Therefore, P is a polynomial with coefficients in $\mathbb{C}(t)$. Since n is prime, either P is irreducible or all the factors of P in $\mathbb{C}(t)[X]$ are of degree one. This concludes the lemma. $\square$

The following two notions can be found in Barkatou and Chen (2001) and Barkatou (1991).

Definition 18. Let $H = (h_{ij})_{n\times n} \in \mathrm{GL}_n(k_0)$. The order of H at ∞ is defined as

$$\mathrm{ord}_\infty(H) = \min\{\mathrm{ord}_\infty(h_{ij})\}$$

where $\mathrm{ord}_\infty(h_{ij})$ is the order of h_{ij} at ∞.

We rewrite H into the form

$$H = \left(\frac{1}{x}\right)^{\mathrm{ord}_\infty(H)} \left(H_0 + H_1\frac{1}{x} + \cdots\right)$$

where $H_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ and $H_0 \neq 0$.

Definition 19. The rational number

$$m(H) = \begin{cases} -\mathrm{ord}_\infty(H) + \frac{\mathrm{rank}(H_0)}{n} & \mathrm{ord}_\infty(H) \leq 0, \\ 0 & \mathrm{ord}_\infty(H) > 0 \end{cases}$$

is called the first Moser order of H. And

$$\mu(H) = \min\{m(\sigma(G)HG^{-1}) | G \in \mathrm{GL}_n(k)\}$$

is called the Moser invariant of H. A matrix H is said to be irreducible if $m(H) = \mu(H)$, otherwise it is said to be reducible.

Given $H \in \mathrm{GL}_n(k_0)$, one can use the algorithm in Barkatou and Chen (2001) and Barkatou (1991) to compute $G \in \mathrm{GL}_n(k_0)$ such that $\tilde{H} := \sigma(G)HG^{-1}$ is irreducible. So we can assume that $\frac{A}{\alpha(x)}$ is irreducible where A and $\alpha(x)$ are as in (3). Under this assumption, we will show that

$$\frac{A}{\alpha(x)} = \tilde{A}_0 + \tilde{A}_1\frac{1}{x} + \cdots$$

with $\tilde{A}_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ for each i and that all the $\beta_i(t)$'s are eigenvalues of $\tilde{A}_0$. The following lemma can be deduced from the results in Barkatou (1991). We will present a self-contained proof due to Reinhart Shaefke.

Lemma 20. *Let $G \in \mathrm{GL}_n(k)$ and assume that $\mathrm{ord}_\infty(\sigma(G^{-1})G) = 0$. Then all the eigenvalues of $\sigma(G^{-1})G|_{x=\infty}$ are 1.*

Proof. Let $H = \sigma(G^{-1})G$. Then $H = H_0 + H_1\frac{1}{x} + \cdots$ with $H_i \in \mathfrak{gl}_n(\overline{\mathbb{C}(t)})$ and $H_0 \neq 0$. We now show that $H_0 - I_n$ is nilpotent. For a positive integer m, consider a map $L_m : \mathfrak{gl}_n(k) \to \mathfrak{gl}_n(k)$ given by $U \mapsto \sigma(U) - \sigma^m(H)U$ for any $U \in \mathfrak{gl}_n(k)$. Set $P_m = L_m \circ L_{m-1} \circ \cdots \circ L_0(I_n)$ where $\circ$ denotes the composition of maps. Then $P_m|_{x=\infty} = (I_n - H_0)^{m+1}$. On the other hand, $L_m(\sigma^m(G^{-1})V) = \sigma^{m+1}(G^{-1})\Delta(V)$ where $\Delta = \sigma - \mathbf{1}$ is a difference operator and $V \in \mathfrak{gl}_n(k)$. Hence $P_m = \sigma^{m+1}(G^{-1})\Delta^{m+1}(G)$. Note that when m increases, $\mathrm{ord}_\infty(\Delta^{m+1}(G))$ increases but $\mathrm{ord}_\infty(\sigma^{m+1}(G^{-1}))$ is invariant. Then for a sufficiently large m, $P_m|_{x=\infty} = 0$. This concludes the lemma. □

Now we can prove the following

Proposition 21. $\mathrm{ord}_\infty\left(\frac{A}{\alpha(x)}\right) = 0$ *and $\beta_1(t), \ldots, \beta_n(t)$ are eigenvalues of $\frac{A}{\alpha(x)}|_{x=\infty}$.*

Proof. By Proposition 7, there exists $G \in \mathrm{GL}_n(k)$ such that

$$\sigma(G)\frac{A}{\alpha(x)}G^{-1} = \mathrm{diag}(\beta_1(t), \ldots, \beta_n(t)).$$

This implies that $\mathrm{ord}_\infty\left(\det\left(\frac{A}{\alpha(x)}\right)\right) = 0$ and $m\left(\frac{A}{\alpha(x)}\right) = \mu\left(\frac{A}{\alpha(x)}\right) \leq 1$. By the property of orders,

$$\mathrm{ord}_\infty\left(\frac{A}{\alpha(x)}\right) \leq \frac{1}{n}\mathrm{ord}_\infty\left(\det\left(\frac{A}{\alpha(x)}\right)\right) = 0.$$

Since $m\left(\frac{A}{\alpha(x)}\right) \le 1$, $\mathrm{ord}_\infty\left(\frac{A}{\alpha(x)}\right) = 0$ by the definition of the first Moser orders. Therefore,

$$\frac{A}{\alpha(x)} = \tilde{A}_0 + \tilde{A}_1 \frac{1}{x} + \dots$$

where $\tilde{A}_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ and $\tilde{A}_0 \neq 0$. From (3), $\sigma(Y) = \frac{A}{\alpha(x)\beta_i(t)} Y$ has a rational solution W_i in k^n. Suppose that

$$W_i = \left(\frac{1}{x}\right)^{\mathrm{ord}_\infty(W_i)} \left(W_{i0} + \frac{1}{x} W_{i1} + \cdots\right)$$

where $W_{ij} \in \overline{\mathbb{C}(t)}^n$ and $W_{i0} \neq 0$. Then $W_{i0} = \frac{\tilde{A}_0}{\beta_i(t)} W_{i0}$. Since $W_{i0} \neq 0$, $\det\left(I_n - \frac{\tilde{A}_0}{\beta_i(t)}\right) = 0$. Hence all the $\beta_i(t)$ are the eigenvalues of $\tilde{A}_0$. If the $\beta_i(t)$ are the conjugate roots of some irreducible polynomial with degree n, then they are clearly eigenvalues of $\tilde{A}_0$. Thus by Lemma 17 we only need to consider the case $\beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(x)$. In this case, $\frac{A}{\alpha(x)} = \beta_1(t)\sigma(G^{-1})G$. Since $\mathrm{ord}_\infty\left(\frac{A}{\alpha(x)}\right) = 0$, we have $\mathrm{ord}_\infty(\sigma(G^{-1})G) = 0$. By Lemma 20, all the eigenvalues of $\sigma(G^{-1})G|_{x=\infty}$ equal 1. Hence all the eigenvalues of $\tilde{A}_0$ equal $\beta_1(t)$. $\square$

Example 22. (Continued) Let $\bar{A} = \frac{A}{x^2+1}$. From the process in Barkatou (1991), we can find an irreducible matrix of the form

$$\tilde{A} = \begin{pmatrix} -\frac{(x+1)t(t^2+1-x)}{(t^2-x-1)x} & -\frac{x+1}{t^2-x-1} \\ \frac{t^4+t^2-x^2-x}{(t^2-x-1)x} & \frac{t(t^2-x)}{t^2-x-1} \end{pmatrix}$$

which is equivalent to $\bar{A}$. Write $\tilde{A} = \tilde{A}_0 + \tilde{A}_1 \frac{1}{x} + \cdots$ where

$$\tilde{A}_0 = \begin{pmatrix} -t & 1 \\ 1 & t \end{pmatrix} \quad \text{and} \quad \tilde{A}_1 = \begin{pmatrix} t & t^2 \\ t^2 & -t \end{pmatrix}.$$

The eigenvalues of $\tilde{A}_0$ are $\pm\sqrt{t^2+1}$. So $\beta_1(t) = \sqrt{t^2+1}$ and $\beta_2(t) = -\sqrt{t^2+1}$.

Computing c_i and W_i: Let $\Lambda(t) = \mathrm{diag}(\beta_1(t), \ldots, \beta_n(t))$. From (A_1) we can find a matrix $G \in \mathrm{GL}_n(k)$ such that $\sigma(G)\alpha(x)\Lambda(t) = AG$. Let $\bar{B} = G^{-1}BG - G^{-1}\delta(G)$. Then $\bar{B} \in \mathfrak{gl}_n(k)$, and the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is equivalent over k to

$$\sigma(Y) = \alpha(x)\Lambda(t)Y, \qquad \delta(Y) = \bar{B}Y. \tag{5}$$

Note that G may not be the required transformation matrix in Proposition 7, so $\bar{B}$ may not be of diagonal form. Since $\sigma\delta = \delta\sigma$, the same argument as in the proof of Proposition 10 implies the following conclusions:

(i) If $\beta_i(t) \neq \beta_j(t)$ for all i, j with $1 \le i \neq j \le n$, then

$$\bar{B} = \mathrm{diag}\left(\frac{\delta(\beta_1(t))}{\beta_1(t)} x + c_1, \ldots, \frac{\delta(\beta_n(t))}{\beta_n(t)} x + c_n\right)$$

with $c_i \in \overline{\mathbb{C}(t)}$;

(ii) If $\beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t)$, then by Proposition 9, G can be chosen in $\mathrm{GL}_n(k_0)$. Thus

$$\bar{B} = \hat{B} + \frac{\delta(\beta_1(t))}{\beta_1(t)} x I_n$$

with $\hat{B} \in \mathfrak{gl}_n(\mathbb{C}(t))$.

In the case (i), the c_i's are obtained from $\bar{B}$ directly and the W_i's are just the columns of G. For the case (ii), since (5) is equivalent over k to (2), there exists $\hat{G} \in \mathrm{GL}_n(k)$ such that

$$\begin{cases} \sigma(\hat{G}) = \hat{G}, \\ \delta(\hat{G}) + \hat{G}\left(\frac{\delta(\beta_1(t))}{\beta_1(t)} xI_n + \mathrm{diag}(c_1, \ldots, c_n)\right) = \left(\hat{B} + \frac{\delta(\beta_1(t))}{\beta_1(t)} xI_n\right)\hat{G}. \end{cases}$$

Hence $\hat{G} \in \mathrm{GL}_n(\overline{\mathbb{C}(t)})$, and $\delta(Y) = \hat{B}Y$ is equivalent over $\overline{\mathbb{C}(t)}$ to

$$\delta(Y) = \mathrm{diag}(c_1, \ldots, c_n)Y.$$

Then the c_i's are obtained by solving the system $\delta(Y) = \hat{B}Y$ by (A_3) and the W_i are the columns of $G\hat{G}$.

Example 23. (Continued) Let $\Lambda(t) = \mathrm{diag}(\sqrt{t^2+1}, -\sqrt{t^2+1})$. From (A_1), we can obtain $G \in \mathrm{GL}_2(k)$ such that $\sigma(G)(x^2+1)\Lambda(t) = AG$ where

$$G = \begin{pmatrix} \frac{t-\sqrt{t^2+1}}{2(t^2-x)} & \frac{t+\sqrt{t^2+1}}{2(t^2-x)} \\ \frac{-x+t\sqrt{t^2+1}}{2(t^2-x)} & -\frac{x+t\sqrt{t^2+1}}{2(t^2-x)} \end{pmatrix}.$$

Then

$$\begin{aligned} \bar{B} &= G^{-1}BG - G^{-1}\delta(G) \\ &= \begin{pmatrix} \frac{xt}{t^2+1} + \sqrt{t^2+1} + 1 & 0 \\ 0 & \frac{xt}{t^2+1} - \sqrt{t^2+1} + 1 \end{pmatrix}. \end{aligned}$$

Hence $c_1 = \sqrt{t^2+1} + 1$, $c_2 = -\sqrt{t^2+1} + 1$ and W_i is the ith column of G for $i = 1, 2$. Furthermore, a basis of the solution space is

$$h(\sqrt{t^2+1})^x \mathrm{e}^{t+\int\sqrt{t^2+1}\mathrm{d}t} \begin{pmatrix} \frac{t-\sqrt{t^2+1}}{2(t^2-x)} \\ \frac{t\sqrt{t^2+1}-x}{2(t^2-x)} \end{pmatrix}, \qquad h(-\sqrt{t^2+1})^x \mathrm{e}^{t-\int\sqrt{t^2+1}\mathrm{d}t} \begin{pmatrix} \frac{t+\sqrt{t^2+1}}{2(t^2-x)} \\ \frac{x+t\sqrt{t^2+1}}{2(x-t^2)} \end{pmatrix}$$

where h satisfies that $\sigma(h) = (x^2+1)h$ and $\delta(h) = 0$.

3.2. *The decision procedure for the irreducible case*

Assume that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \mathrm{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is an irreducible system over k and its Galois group over k_0 has solvable identity component. By Proposition 12, the system $\{\sigma^n(Y) = A_nY, \delta(Y) = BY\}$ has solutions of the form W_ih_i for $i = 1, \ldots, n$, where $W_i \in k_0^n$ and h_i satisfies

$$\sigma^n(h_i) = \alpha(x+i-1)\beta(t)h_i, \qquad \delta(h_i) = \left(\frac{\delta(\beta(t))}{n\beta(t)}x + \hat{b}_i\right)h_i$$

with $\alpha(x)$, $\beta(t)$ and $\hat{b}_i$ as in Proposition 12. Substituting $Y = W_ih_i$ into $\{\sigma^n(Y) = A_nY, \delta(Y) = BY\}$, we have

$$\sigma^n(W_i) = \frac{A_n}{\beta(t)\alpha(x+i-1)}W_i \quad \text{and} \quad \delta(W_i) = \left(B - \frac{\delta(\beta(t))}{n\beta(t)}x - \hat{b}_i\right)W_i. \tag{6}$$

To compute W_ih_i, it suffices to compute $\alpha(x)$, $\beta(t)$, W_i and $\hat{b}_i$ which satisfy (6). Without loss of generality, we assume that the numerator and denominator of $\alpha(x)$ are monic. By Proposition 10, there exists $G \in \mathrm{GL}_n(k_0)$ such that

$$\frac{\sigma(\det(G))}{\det(G)}(-1)^{n-1}\alpha(x)\beta(t) = \det(A).$$

Expanding $\det(A)$ as a series in $\frac{1}{x}$, we get that $(-1)^{n-1}\beta(t)$ is the leading coefficient of the series. Hence we can obtain $\beta(t)$ from $\det(A)$. In this case, we cannot find $\alpha(x)$ by the method used in Section 3.1. However we can reduce this problem to working with difference equations over $\mathbb{C}(x)$. By Proposition 12, there exists $G \in \mathrm{GL}_n(k_0)$ (the same as that in Proposition 10) such that

$$\sigma^n(G) \cdot \mathrm{diag}(\alpha(x), \ldots, \alpha(x+n-1)) = \frac{A_n}{\beta(t)} G.$$

Assume that $t = p$ is not a pole of the entries of $\frac{A_n}{\beta(t)}$ and such that $\det\left(\frac{A_n}{\beta(t)}|_{t=p}\right) \neq 0$. Let $\frac{A_n}{\beta(t)} = \bar{A}_0 + (t-p)\bar{A}_1 + \cdots$ where $\bar{A}_i \in \mathfrak{gl}_n(\mathbb{C}(x))$. We will show that $\alpha(x)$ can be found by examining the hypergeometric solutions of $\sigma^n(Y) = \bar{A}_0 Y$. This will follow from the next proposition.

Proposition 24. *Some factor of $\sigma^n(Y) = \bar{A}_0 Y$ is equivalent over $\mathbb{C}(x)$ to some factor of $\sigma^n(Y) = \mathrm{diag}(\alpha(x), \alpha(x+1), \ldots, \alpha(x+n-1))Y$.*

Proof. Let G be as above and let $\Psi(x) = \mathrm{diag}(\alpha(x), \ldots, \alpha(x+n-1))$. We may multiply G by a power of $t-p$ and assume that $G = \bar{G}_0 + (t-p)\bar{G}_1 + \cdots$ where $\bar{G}_0 \neq 0$ and $\bar{G}_i \in \mathfrak{gl}_n(\mathbb{C}(x))$. Then

$$\sigma^n(\bar{G}_0 + (t-p)\bar{G}_1 + \cdots)\Psi(x) = (\bar{A}_0 + \cdots)(\bar{G}_0 + (t-p)\bar{G}_1 + \cdots).$$

Therefore $\sigma^n(\bar{G}_0)\Psi(x) = \bar{A}_0\bar{G}_0$. Let $r = \mathrm{rank}(\bar{G}_0)$. Then $r > 0$ because $\bar{G}_0 \neq 0$. There exist $P \in \mathrm{GL}_n(\mathbb{C}(x))$ and Q which is a product of some permutation matrices such that

$$\tilde{G} = P\bar{G}_0 Q = \begin{pmatrix} 0 & 0 \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix}$$

where $\tilde{G}_{22} \in \mathrm{GL}_r(\mathbb{C}(x))$. Then

$$\sigma^n(\tilde{G})\mathrm{diag}(\alpha(x+k_1), \ldots, \alpha(x+k_n)) = \sigma^n(P)\bar{A}_0 P^{-1}\tilde{G} \tag{7}$$

where $k_1, \ldots, k_n$ are a permutation of $\{0, 1, \ldots, n-1\}$. Now let

$$\tilde{A} = \sigma^n(P)\bar{A}_0 P^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \quad \text{where } \tilde{A}_{22} \in \mathfrak{gl}_r(\mathbb{C}(x)),$$

and $D_2 = \mathrm{diag}(\alpha(x+k_{n-r+1}), \ldots, \alpha(x+k_n))$. From (7), we have $\sigma^n(\tilde{G}_{22})D_2 = \tilde{A}_{22}\tilde{G}_{22}$ and $\tilde{A}_{12}\tilde{G}_{22} = 0$. Since $\tilde{G}_{22} \in \mathrm{GL}_r(\mathbb{C}(x))$, we have $\tilde{A}_{12} = 0$. Therefore $\sigma^n(Z) = \tilde{A}_{22}Z$ is a factor of $\sigma^n(Y) = \bar{A}_0 Y$, which is equivalent over $\mathbb{C}(x)$ to $\sigma^n(Z) = D_2 Z$. $\square$

Remark 25. For almost all of $p \in \mathbb{C}$, $\sigma^n(Y) = \frac{A_n}{\beta(t)}|_{t=p} Y$ is equivalent over $\mathbb{C}(x)$ to

$$\sigma^n(Y) = \mathrm{diag}(\alpha(x), \alpha(x+1), \ldots, \alpha(x+n-1))Y.$$

Since $G|_{t=p}$ is invertible.

The same argument as in Remark 13 implies that it suffices to compute $\frac{\sigma^n(g)\alpha(x)}{g}$ for some suitable $g \in \mathbb{C}(x)$ instead of $\alpha(x)$. We can use Proposition 24 to find $\frac{\sigma^n(g)\alpha(x+k)}{g}$ with $k \in \mathbb{Z}$ and $g \in \mathbb{C}(x)$ as follows. From Theorem 3 in Bronstein et al. (2005), if $(z_1, \ldots, z_r)^{\mathrm{T}}$ is a solution of $\sigma^n(Z) = \tilde{A}_{22}Z$, then $(0, \ldots, 0, z_1, \ldots, z_r)^{\mathrm{T}}$ is a solution of $\sigma^n(Y) = \tilde{A}Y$. So $\sigma^n(Y) = \bar{A}_0 Y$ has at least r solutions $\bar{W}_1\bar{h}_1, \ldots, \bar{W}_r\bar{h}_r$, where $\bar{W}_i \in \mathbb{C}(x)^n$ and $\bar{h}_i$ satisfies $\sigma^n(\bar{h}_i) = \alpha(x+k_{n-r+i})\bar{h}_i$. By (A_2), we can find all hypergeometric solutions of $\sigma(Z) = \bar{A}_0(nx)Z$ where $\bar{A}_0(nx)$ means replacing x by nx in $\bar{A}_0$. Then by interlacing, we can find all solutions of $\sigma^n(Y) = \bar{A}_0 Y$ of the form $\tilde{W}_j\tilde{h}_j$ where $\tilde{W}_j \in \mathbb{C}(x)^n$ and $\tilde{h}_j$ satisfies $\sigma^n(\tilde{h}_j) = \tilde{a}_j\tilde{h}_j$ for some $\tilde{a}_j \in \mathbb{C}(x)$. Then there exists $\tilde{h}_{j_0}$ such that $\tilde{h}_{j_0} = g\bar{h}_1$ for some $g \in \mathbb{C}(x)$ and

$$\hat{\alpha}(x+k_{n-r+1}) = \frac{\sigma^n(\tilde{h}_{j_0})}{\tilde{h}_{j_0}} = \frac{\sigma^n(g)}{g}\alpha(x+k_{n-r+1}).$$

After finding $\hat{\alpha}(x+k_{n-r+1})$, we can compute a matrix $\hat{G} \in \mathrm{GL}_n(k_0)$ in a finite number of steps by (A_1), such that

$$\sigma^n(\hat{G}^{-1})A_n\hat{G} = \beta(t)\mathrm{diag}(\hat{\alpha}(x), \dots, \hat{\alpha}(x+n-1)).$$

Let $\bar{B} = \hat{G}^{-1}B\hat{G} - \hat{G}^{-1}\delta(\hat{G})$. Then we get a new system

$$\sigma^n(Y) = \beta(t)\mathrm{diag}(\hat{\alpha}(x), \dots, \hat{\alpha}(x+n-1))Y, \qquad \delta(Y) = \bar{B}Y$$

which is equivalent to the original one under the transformation $Y \to \hat{G}^{-1}Y$. Since σ^n and δ commute and $\frac{\alpha(x+1)}{\alpha(x)} \neq \frac{\sigma^n(b)}{b}$ for any $b \in \mathbb{C}(x)$, the same argument as in the proof of Proposition 12 implies that $\bar{B}$ is of diagonal form, that is

$$\bar{B} = \mathrm{diag}\left(\frac{\delta(\beta(t))}{n\beta(t)}x + \hat{b}_1, \dots, \frac{\delta(\beta(t))}{n\beta(t)}x + \hat{b}_n\right).$$

We then get the $\hat{b}_i$, and the W_i are just the ith columns of $\hat{G}$.

Example 26. Consider an integrable system:

$$\sigma(Y) = AY, \qquad \delta(Y) = BY$$

where

$$A = \begin{pmatrix} \frac{x^3t^4+2x^2t^4+xt^4-x-1}{t^2+x+1} & \frac{t^2(tx^4+2tx^3+tx^2+1)}{t^2+x+1} & \frac{t(t-x-1)}{t^2+x+1} \\ -\frac{t(x^2t^4+xt^4-1)}{t^2+x+1} & -\frac{t(t^3x^3+t^3x^2-1)}{t^2+x+1} & \frac{t(1+t)}{t^2+x+1} \\ \frac{t^6x^2+t^6x+x+1}{t(t^2+x+1)} & \frac{t(t^3x^3+t^3x^2-1)}{t^2+x+1} & -\frac{t-x-1}{t^2+x+1} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{t^4+t^2x+x^2+t^4x-t^2}{t(t^2+x)} & -\frac{x(-t^2+t^3-1)}{t^2+x} & \frac{xt^3(-1+t)}{t^2+x} \\ -\frac{-t^2+t^4+1}{t^2+x} & \frac{2t^2x+x^2+t^5-t^2}{t(t^2+x)} & -\frac{t^4(-1+t)}{t^2+x} \\ \frac{-t^2+t^4+1}{t^2+x} & \frac{x(-t^2+t^3-1)}{t(t^2+x)} & \frac{x^2+xt^3+t^2x+t^6-x-t^2}{t(t^2+x)} \end{pmatrix}.$$

We have

$$\det(A) = \frac{xt^3(t^2x+t^2+x^2+x)}{x+1+t^2} = \frac{(x+1)(t^2+x)}{x(t^2+x+1)}x^2t^3.$$

By (4), if the Galois group over k_0 of the given system has solvable identity component, then this system have no hypergeometric solutions over k. Therefore we consider the system

$$\sigma^3(Y) = A_3Y, \qquad \delta(Y) = BY$$

where

$$A_3 = \begin{pmatrix} \frac{t^3(t^2x^2+t^2x+21x+x^3+8x^2+18)}{t^2+x+3} & -\frac{t^4(x+1)(5x+6)}{t^2+x+3} & \frac{2t^4(x+2)(x+3)}{t^2+x+3} \\ -\frac{2t^4(2x+3)}{t^2+x+3} & \frac{(x+1)t^3(x^2+t^2x+2t^2)}{t^2+x+3} & -\frac{2t^5(x+2)}{t^2+x+3} \\ \frac{2t^4(2x+3)}{t^2+x+3} & \frac{(x+1)t^3(5x+6)}{t^2+x+3} & \frac{(x+2)(x+3)t^3(x+1+t^2)}{t^2+x+3} \end{pmatrix}.$$

We can compute $\beta(t) = t^3$ from $\det(A)$. Let $\tilde{A} = \frac{A_3}{t^3}$. Then

$$\tilde{A}|_{t=0} = \begin{pmatrix} (x+2)(x+3) & 0 & 0 \\ 0 & \frac{(x+1)x^2}{x+3} & 0 \\ 0 & \frac{(x+1)(5x+6)}{x+3} & (x+1)(x+2) \end{pmatrix}.$$

By (A_2), all hypergeometric solutions of $\sigma^3(Y) = \tilde{A}|_{t=0}Y$ are

$$9^{\frac{x}{3}}\Gamma\left(\frac{x+2}{3}\right)\Gamma\left(\frac{x+3}{3}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad 9^{\frac{x}{3}}\Gamma\left(\frac{x+1}{3}\right)\Gamma\left(\frac{x+2}{3}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad 9^{\frac{x}{3}}\Gamma\left(\frac{x}{3}\right)\Gamma\left(\frac{x+1}{3}\right)\begin{pmatrix}0\\-\frac{3}{x}\\\frac{3}{x}\end{pmatrix},$$

where $\Gamma(x)$ satisfies $\Gamma(x+1) = x\Gamma(x)$. By (A_1), we can compute a rational solution of $\sigma^3(Y) = \frac{A_3}{x(x+1)t^3}Y$. Moreover, we can compute a matrix $G \in \mathrm{GL}_3(\mathbb{C}(x,t))$ such that

$$\sigma^3(G)\mathrm{diag}(x(x+1)t^3, (x+1)(x+2)t^3, (x+2)(x+3)t^3) = A_3G$$

where

$$G = \begin{pmatrix} \frac{t}{t^2+x} & -\frac{x}{t^2+x} & \frac{x}{t^2+x} \\ \frac{1}{t^2+x} & \frac{t}{t^2+x} & -\frac{t}{t^2+x} \\ -\frac{1}{t^2+x} & \frac{x}{t(t^2+x)} & \frac{t}{t^2+x} \end{pmatrix}.$$

Let $\bar{B} = G^{-1}BG - G^{-1}\delta(G)$. Then $\bar{B} = \mathrm{diag}\left(\frac{x}{t}+t, \frac{x}{t}+t^2, \frac{x}{t}+t^3\right)$. Hence a basis of the solution space of $\{\sigma^3(Y) = A_3Y, \delta(Y) = BY\}$ is

$$V_1(x) := 9^{\frac{x}{3}}\Gamma\left(\frac{x}{3}\right)\Gamma\left(\frac{x+1}{3}\right)t^x\mathrm{e}^{\frac{t^2}{2}}\begin{pmatrix}\frac{t}{t^2+x}\\\frac{1}{t^2+x}\\-\frac{1}{t^2+x}\end{pmatrix},$$

$$V_2(x) := 9^{\frac{x}{3}}\Gamma\left(\frac{x+1}{3}\right)\Gamma\left(\frac{x+2}{3}\right)t^x\mathrm{e}^{\frac{t^3}{3}}\begin{pmatrix}-\frac{x}{t^2+x}\\\frac{t}{t^2+x}\\\frac{x}{t(t^2+x)}\end{pmatrix},$$

$$V_3(x) := 9^{\frac{x}{3}}\Gamma\left(\frac{x+2}{3}\right)\Gamma\left(\frac{x+3}{3}\right)t^x\mathrm{e}^{\frac{t^4}{4}}\begin{pmatrix}\frac{x}{t^2+x}\\-\frac{t}{t^2+x}\\\frac{t}{t^2+x}\end{pmatrix}.$$

Clearly, $V_i(1) \neq 0$ for $i = 1, 2, 3$, $A(j)$ and $B(j)$ are well defined and $\det(A(j)) \neq 0$ for $j \geq 1$. By the results in Section 2 of Feng et al. (2009), we get a basis of the solution space of the original system:

$$W_1 = 9^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)t\mathrm{e}^{\frac{t^2}{2}}\begin{pmatrix}\left(0, \frac{t}{t^2+1}, \frac{4t^3}{t^2+2}, -\frac{6t^3}{t^2+3}, \cdots\right)\\\left(0, \frac{1}{t^2+1}, -\frac{2t^4}{t^2+2}, \frac{2t^4}{t^2+3}, \cdots\right)\\\left(0, -\frac{1}{t^2+1}, \frac{2t^4}{t^2+2}, \frac{6t^2}{t^2+3}, \cdots\right)\end{pmatrix},$$

$$W_2 = 9^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)\Gamma(1)t\mathrm{e}^{\frac{t^3}{3}}\begin{pmatrix}\left(0, -\frac{1}{t^2+1}, \frac{t}{t^2+2}, \frac{18t^3}{t^2+3}, \cdots\right)\\\left(0, \frac{t}{t^2+1}, \frac{1}{t^2+2}, -\frac{6t^4}{t^2+3}, \cdots\right)\\\left(0, \frac{1}{t(t^2+1)}, -\frac{1}{t^2+2}, \frac{6t^4}{t^2+3}, \cdots\right)\end{pmatrix}$$

and

$$W_3 = 9^{\frac{1}{3}}\Gamma(1)\Gamma\left(\frac{4}{3}\right)t\mathrm{e}^{\frac{t^4}{4}}\begin{pmatrix}\left(0, \frac{1}{t^2+1}, -\frac{2}{t^2+2}, \frac{t}{t^2+3}, \cdots\right)\\\left(0, -\frac{t}{t^2+1}, \frac{t}{t^2+2}, \frac{1}{t^2+3}, \cdots\right)\\\left(0, \frac{t}{t^2+1}, \frac{2}{t(t^2+2)}, -\frac{1}{t^2+3}, \cdots\right)\end{pmatrix}.$$

Note that all the W_i are Liouvillian.

3.3. Summary

Consider two systems

$$\sigma(Y) = AY, \qquad \delta(Y) = BY \tag{8}$$

and

$$\sigma^n(Y) = A_n Y, \qquad \delta(Y) = BY \tag{9}$$

where $A \in \mathrm{GL}_n(k_0)$, $B \in \mathfrak{gl}_n(k_0)$ and n is a prime number. Assume that (8) is irreducible over k_0. From the results in Sections 3.1 and 3.2, if (8) has a Liouvillian solution over k, then either the solution space of (8) has a basis consisting of hypergeometric solutions over k or the solution space of (9) has a basis each of whose members is the interlacing of hypergeometric vectors over k_0. Let us summarize the previous decision procedure as follows.

Decision Procedure 1 Compute a fundamental matrix of (8) whose entries are hypergeometric over k if it exists.

- (a) Write $\det(A) = \frac{\sigma(g)}{g}a$ where $g, a \in k_0$ and a is standard with respect to σ. If $a \neq \alpha(x)^n \beta(t)$ for any $\alpha(x) \in \mathbb{C}(x)$ and $\beta(t) \in \mathbb{C}(t)$, then by the results in Section 6.1, **exit** [(8) has no required fundamental matrix].
- (b) Assume that $a = \alpha(x)^n\beta(t)$ for some $\alpha(x) \in \mathbb{C}(x)$ and $\beta(t) \in \mathbb{C}(t)$. By the algorithms in Barkatou and Chen (2001) and Barkatou (1991), compute an irreducible matrix $\tilde{A}$ such that $\tilde{A} = \sigma(\tilde{G})\frac{A}{\alpha(x)}\tilde{G}^{-1}$ for some $\tilde{G} \in \mathrm{GL}_n(k_0)$. If $\mathrm{ord}_\infty(\tilde{A}) \neq 0$, then by Proposition 21, **exit** [(8) has no required fundamental matrix]. Otherwise, let $\tilde{A}_0 = \tilde{A}|_{x=\infty}$ and $\beta_1(t), \ldots, \beta_n(t)$ be the eigenvalues of $\tilde{A}_0$.
- (c) Goto Step (d_1) if the $\beta_i(t)$ are conjugate and goto Step (d_2) if $\beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t)$. In other cases, by Lemma 17 and Proposition 21, **exit** [(8) has no required fundamental matrix].
- (d_1) If (A_1) yields no rational solutions, then **exit** [(8) has no required fundamental matrix]. Otherwise, suppose that we find $G \in \mathrm{GL}_n(k)$ such that

  $$\sigma(G)\alpha(x)\mathrm{diag}(\beta_1(t), \ldots, \beta_n(t)) = AG.$$

  Then $\bar{B} := G^{-1}BG - G^{-1}\delta(G)$ is of diagonal form. Compute a fundamental matrix H of

  $$\sigma(Y) = \alpha(x)\mathrm{diag}(\beta_1(t), \ldots, \beta_n(t))Y, \qquad \delta(Y) = \bar{B}Y.$$

  Return [GH is a required fundamental matrix of (8)].
- (d_2) If we can compute a matrix $G \in \mathrm{GL}_n(k_0)$ such that $\sigma(G)\alpha(x)\beta_1(t) = AG$ then let

  $$\hat{B} = G^{-1}BG - G^{-1}\delta(G) - \frac{\delta(\beta_1(t))}{\beta_1(t)}xI_n \in \mathfrak{gl}_n(\mathbb{C}(t)),$$

  else **exit** [(8) has no required fundamental matrix]. If we can find a fundamental matrix H of $\delta(Y) = \hat{B}Y$ whose entries are hyperexponential over $\overline{\mathbb{C}(t)}$, then **return** [$GHh\beta_1(t)^x$ is a required fundamental matrix of (8)] where h satisfies $\sigma(h) = \alpha(x)h$ and $\delta(h) = 0$. Otherwise, **exit** [(8) has no required fundamental matrix].

Decision Procedure 2 Compute a fundamental matrix of (9) whose entries are the interlacing of hypergeometric vectors over k_0 if it exists.

- (a) If $\det(A) \neq (-1)^{n-1}\frac{\sigma(g)}{g}\alpha(x)\beta(t)$ holds for any $g \in k$, $\beta(t) \in \mathbb{C}(t)$ and $\alpha(x) \in \mathbb{C}(x)$ that is standard with respect to σ^n, then **exit** [(9) has no required fundamental matrix].
- (b) Expand $\det(A)$ as a series at $x = \infty$:

  $$\det(A) = (-1)^{n-1}\beta(t)x^m + \beta_1(t)x^{m-1} + \cdots$$

  where $\beta(t), \beta_i(t) \in \mathbb{C}(t)$ and $m \in \mathbb{Z}$. Suppose that $x = p$ is not a pole of the entries of $\frac{A}{\beta(t)}$ and that $\det(\tilde{A}_0) \neq 0$ where $\tilde{A}_0 = \frac{A}{\beta(t)}|_{x=p}$. Use (A_2) to find all hypergeometric solutions of $\sigma(Z) =$

$\tilde{A}_0(nx)Z$. By interlacing, we get all solutions of $\sigma^n(Y) = \tilde{A}_0 Y$ of the form $W_i h_i$. Denote these solutions by $W_1h_1, \ldots, W_dh_d$ where $W_i \in \mathbb{C}(x)^n$ and h_i satisfies $\sigma^n(h_i) = \tilde{a}_i h_i$ for some $\tilde{a}_i \in \mathbb{C}(x)$. If there is $i_0 \in \{1, \ldots, d\}$ such that $\sigma^n(Y) = \frac{h_{i_0}A}{\sigma^n(h_{i_0})\beta(t)}Y$ has a rational solution in k_0^n, then let $\lambda(x) = \frac{\sigma^n(h_{i_0})}{h_{i_0}}$, else **exit** [(9) has no required fundamental matrix]. Let j_0 be the least integer such that $\sigma^n(Y) = \frac{A}{\lambda(x+j_0)\beta(t)}Y$ has a rational solution in k_0^n. If we can compute $G \in \mathrm{GL}_n(k_0)$ such that

$$\sigma(Y)\beta(t)\mathrm{diag}(\lambda(x+j_0), \ldots, \lambda(x+j_0+n-1)) = AG,$$

then let $\bar{B} = G^{-1}BG - G^{-1}\delta(G)$. So $\bar{B}$ is of diagonal form and by the same process as in Step (d_1) of Decision Procedure 1, we can compute a required fundamental matrix of (9). Otherwise, by the results in Section 3.2, **exit** [(9) has no required fundamental matrix].

We can decide whether (8) has Liouvillian solutions or not as follows. If we can compute hypergeometric solutions over k of (8) by Decision Procedure 1, then we are done. Otherwise, consider the system (9). If we can compute Liouvillian solutions over k_0 of (9) by Decision Procedure 2, then by the results in Section 2 of Feng et al. (2009) we can compute Liouvillian solutions over k_0 of (8) and we are done. Otherwise (8) has no Liouvillian solutions.

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