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A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs☆

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Abstract

We give a necessary and sufficient condition for an algebraic ODE to have a rational type general solution. For a first order autonomous ODE $F = 0$, we give an exact degree bound for its rational solutions, based on the connection between rational solutions of $F = 0$ and rational parametrizations of the plane algebraic curve defined by $F = 0$.

For a first order autonomous ODE, we further give a polynomial time algorithm for computing a rational general solution if it exists based on the computation of Laurent series solutions and Padé approximants. Experimental results show that the algorithm is quite efficient.

Keywords: Rational general solution; First order autonomous ODE; Rational parametrizations; Laurent series; Padé approximants; Polynomial time algorithm

1. Introduction

In the pioneering papers (Risch, 1969, 1970), Risch described a method for finding the elementary integral of $\int u dx$ where u is an elementary function. In Kovacic (1986), Kovacic presented an effective method for finding Liouvillian solutions for second order linear homogeneous differential equations and Riccati equations. In Singer (1981), Singer established the general framework for finding Liouvillian solutions for general linear homogeneous ODEs. Many other interesting results on finding Liouvillian solutions of linear ODEs were reported

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in Abramov and Kvashenko (1991), Bronstein and Lafaille (2002), Cormier (2001), Ulmer and Calmet (1990), van Hoeij et al. (1999) and Van der Put and Singer (2003). In Li and Schwarz (2001), Li and Schwarz gave the first method for finding rational solutions for a class of partial differential equations.

Most of these results are limited to the linear case or some special type of nonlinear equations. There seem to exist no general methods for finding closed form solutions for nonlinear differential equations. With respect to the particular ODEs of the form $y' = R(x, y)$ where $R(x, y)$ is a rational function, Darboux and Poincaré made important contributions (Poincaré, 1897). More recently, Cerveau, Neto and Carnicer also made important progress (Cerveau and Lins Neto, 1991; Carnicer, 1994). In particular, Carnicer gave the degree bound of algebraic solutions in the nondicritical case. In Cano (2003), Cano proposed an algorithm for finding their polynomial solutions. In Singer (1992), Singer studied the Liouvillian first integrals of differential equations. In Bronstein (1992), Bronstein gave an effective method for computing rational solutions of the Riccati equations. For a general first order differential equation, Eremenko proved that there exists a degree bound of rational solutions in Eremenko (1998), but the proof is not constructive. In Hubert (1996), Hubert gave a method for computing a basis of the general solutions of first order ODEs and applied it to study the local behavior of the solutions.

In this paper, we try to find rational type general solutions to algebraic ODEs. For example, the general solution for $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0$ is $y = \frac{1}{x+c}$, where c is an arbitrary constant. The motivation for finding the rational general solutions to algebraic ODEs is as follows. Converting between implicit representation and parametric representation of (differential) varieties is one of the basic topics in (differential) algebraic geometry. For the differential case, implicitization algorithms were given in Gao (2003). As far as we know, there exist no general results on parametrization of differential varieties. The results in this paper could be considered as a first step to the rational parametrization problem for differential varieties.

Three main results are given in this paper. In Section 2, for non-negative integers n and m, we define a differential polynomial $\mathcal{D}_{n,m}$ in variable y such that the solutions of the ODE $\mathcal{D}_{n,m} = 0$ are rational solutions whose numerator and denominator are of degrees less than n and m respectively. On the basis of this, we give a sufficient and necessary condition for an algebraic ODE to have a rational general solution.

By treating the variable and its derivative as independent variables, a first order autonomous (constant coefficients) ODE defines an algebraic plane curve. In Section 3, we show that a nontrivial rational solution of a first order autonomous ODE and its derivative provides a proper parametrization of its corresponding algebraic curve. From this result, we may obtain an exact degree bound for its rational solutions. In Section 4, we give a detailed analysis of the structural properties of a first order autonomous ODE with a rational solution. These properties give necessary conditions for a first order autonomous ODE to have rational solutions. We also present a polynomial time algorithm for computing the first $2n + 1$ terms of a Laurent series solution to a first order autonomous ODE in a certain case. These results and Padé approximants are finally used to give a polynomial time algorithm for finding a rational general solution for a first order autonomous ODE.

For the first order autonomous ODE, finding the solutions is equivalent to finding the integration of an algebraic function, because $F(\frac{\mathrm{d}y}{\mathrm{d}x}, y) = 0$ implying that there exists a $G(z_1, z_2)$ such that $G(\frac{\mathrm{d}x}{\mathrm{d}y}, y) = 0$. In Davenport (1981), Trager (1984), Davenport and Trager gave an algorithm for finding the integration of algebraic functions. In Bronstein (1990), Bronstein generalized Trager's algorithm to elementary functions. Their algorithms can compute the elementary integration but the complexity in the worst case is exponential. Here, our algorithm is equivalent to an algorithm for finding a special algebraic integration for algebraic functions and the complexity is polynomial.

The algorithm is implemented in Maple and experimental results show that the algorithm is very efficient. For two hundred randomly generated first order autonomous ODEs, the algorithm can immediately (without computation) decide that the ODEs do not have rational general solutions using the necessary conditions presented in Section 4. For large first order autonomous ODEs with rational general solutions, the algorithm can find their rational solutions efficiently.

This paper is an essential improvement of our ISSAC2004 paper (Feng and Gao, 2004). The main improvement is in Section 4, where we present a polynomial time algorithm for finding rational general solutions to first order autonomous ODEs by proving structural properties of the ODEs and a new algorithm for computing the Laurent series solutions. While, the algorithm in Feng and Gao (2004) is exponential. Experimentally, the average running time of our new algorithm for the same set of ODEs with rational solutions is about thirty times faster and the new algorithm can solve much larger problems. Due to the structural properties, the new algorithm gives an immediate negative answer for almost all randomly generated ODEs. Another advantage of the new algorithm is that we do not need to work over the fields of algebraic numbers. In Sections 2 and 3, Lemmas 2.9 and 3.2, Theorems 2.5, 2.10 and 3.8 are also new results.

2. Rational general solutions of algebraic ODEs

2.1. *Definition of rational general solutions*

In the following, let $\mathbf{K} = \mathbf{Q}(x)$ be the differential field of rational functions in x with differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ and y an indeterminate over $\mathbf{K}$. Let $\bar{\mathbf{Q}}$ be the algebraic closure of the rational number field $\mathbf{Q}$. We denote by y_i the i-th derivative of y. We use $\mathbf{K}\{y\}$ to denote the ring of differential polynomials over the differential field $\mathbf{K}$, which consists of the polynomials in the y_i with coefficients in $\mathbf{K}$. All differential polynomials in this paper are in $\mathbf{K}\{y\}$. Let Σ be a system of differential polynomials in $\mathbf{K}\{y\}$. A *zero* of Σ is an element in a universal extension field of $\mathbf{K}$, which vanishes for every differential polynomial in Σ (Ritt, 1950). In this paper, we also assume that the universal extension field of $\mathbf{K}$ contains an infinite number of arbitrary constants. The totality of the zeros in $\mathbf{K}$ is denoted by $\mathrm{Zero}(\Sigma)$.

Let $P \in \mathbf{K}\{y\}/\mathbf{K}$. We denote by $\mathrm{ord}(P)$ the highest derivative of y in P, called the *order* of P. Let $o = \mathrm{ord}(P) > 0$. We may write P as follows

$$P = a_d y_o^d + a_{d-1} y_o^{d-1} + \cdots + a_0$$

where a_i are polynomials in $y, y_1, \ldots, y_{o-1}$ and $a_d \neq 0$. a_d is called the *initial* of P and $S = \frac{\partial P}{\partial y_o}$ is called the *separant* of P. The k-th derivative of P is denoted by $P^{(k)}$. Let S be the separant of P, $o = \mathrm{ord}(P)$ and an integer $k > 0$. Then we have

$$P^{(k)} = S y_{o+k} - R_k \tag{2.1}$$

where R_k is of lower order than $o + k$.

Let P be a differential polynomial of order o. A differential polynomial Q is said to be *reduced* with respect to P if $\mathrm{ord}(Q) < o$ or $\mathrm{ord}(Q) = o$ and $\deg(Q, y_o) < \deg(P, y_o)$. For two differential polynomials P and Q, let $R = \mathrm{prem}(P, Q)$ be the differential pseudo-remainder of P with respect to Q. We have the following *differential remainder formula* for R (Kolchin, 1973; Ritt, 1950)

$$JP = \sum_i B_i Q^{(i)} + R$$

where J is a product of certain powers of the initial and separant of Q and B_i, R are differential polynomials. Moreover, R is reduced with respect to Q. For a differential polynomial P with order o, we say that P is *irreducible* if P is irreducible when P is treated as a polynomial in $\mathbf{K}[y, y_1, \ldots, y_o]$. In this paper, when we say that a differential polynomial is irreducible, we always mean that it is irreducible over $\bar{\mathbf{Q}}(x)[y, y_1, \ldots, y_o]$.

Let $P \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial and

$$\Sigma_P = \{A \in \mathbf{K}\{y\} | SA \equiv 0 \bmod \{P\}\} \tag{2.2}$$

where $\{P\}$ is the perfect differential ideal generated by P (Kolchin, 1973; Ritt, 1950). In Ritt (1950), Ritt proved:

Lemma 2.1. *Σ_P is a prime differential ideal and a differential polynomial Q belongs to Σ_P iff* $\mathrm{prem}(Q, P) = 0$.

Let Σ be a nontrivial prime ideal in $\mathbf{K}\{y\}$. A zero η of Σ is called *a generic zero* of Σ if for any differential polynomial P, $P(\eta) = 0$ implies that $P \in \Sigma$. It is well known that an ideal Σ is prime iff it has a generic zero (Ritt, 1950).

When we say a *constant*, we mean it is in the constant field of the universal extension field of $\mathbf{K}$. The following definition of a general solution is due to Ritt.

Definition 2.2. Let $F \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial. A *general solution* of $F = 0$ is defined as a generic zero of Σ_F. A *rational general solution* of $F = 0$ is defined as a general solution of the form

$$\hat{y} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{x^m + b_{m-1} x^{m-1} + \cdots + b_0} \tag{2.3}$$

where a_i, b_j are constants. When $m = 0$, $\hat{y}$ is called a *polynomial general solution* of $F = 0$.

Notation 2.3. $\deg_x(\hat{y}) := \max\{n, m\}$ *where $\hat{y}$ is as in* (2.3) *and* $a_n \neq 0$.

As a consequence of Lemma 2.1, we have

Lemma 2.4. *Let $F \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial with a generic solution η. Then for a differential polynomial P we have $P(\eta) = 0$ iff* $\mathrm{prem}(P, F) = 0$.

A general solution of $F = 0$ is usually defined as a family of solutions with o independent parameters in a loose sense where $o = \mathrm{ord}(F)$. The definition given by Ritt is more precise. Theorem 6 in Section 12, Chapter 2 in Kolchin (1973) tells us that Ritt's definition of general solution is equivalent to the definition in the classical literature.

The *universal constant extension* of $\mathbf{Q}$ is obtained by first adding an infinite number of arbitrary constants to $\mathbf{Q}$ and then taking the algebraic closure.

2.2. A criterion for existence of rational general solutions

For non-negative integers n and m, let $\mathcal{D}_{n,m}$ be the following differential polynomial in y:

$$\begin{vmatrix} \binom{n+1}{0}y_{n+1} & \binom{n+1}{1}y_n & \dots & \binom{n+1}{m}y_{n+1-m} \\ \binom{n+2}{0}y_{n+2} & \binom{n+2}{1}y_{n+1} & \dots & \binom{n+2}{m}y_{n+2-m} \\ \vdots & \vdots & \dots & \vdots \\ \binom{n+m+1}{0}y_{n+m+1} & \binom{n+m+1}{1}y_{n+m} & \dots & \binom{n+m+1}{m}y_{n+1} \end{vmatrix}$$

where $\binom{n}{k}$ are binomial coefficients and $\binom{n}{k} = 0$ for $k > n$. First, we can prove that $\mathcal{D}_{n,m}$ is irreducible.

Theorem 2.5. *$\mathcal{D}_{n,m}$ is an irreducible differential polynomial.*

Proof. Since $\mathcal{D}_{n,m}$ is a homogeneous polynomial with degree $m+1$ and includes a term y_{n+1}^{m+1}, we need only to prove the result in the case $y_{n+2} = y_{n+3} = \cdots = y_{n+m} = 0$. We will use $\bar{\mathcal{D}}_{n,m}$ to denote the polynomial obtained by replacing y_i $(n+1 < i < n+1+m)$ with 0 in $\mathcal{D}_{n,m}$. By the computation process, we have $\bar{\mathcal{D}}_{n,m} = y_{n+1}^{m+1} + (-1)^m y_{n+1+m} D$ where D is a polynomial including a term $n^m y_n^m$ and with total degree not greater than m. Because y_{n+1+m} is linear in $\bar{\mathcal{D}}_{n,m}$, by Eisenstein's Criterion (Van der Waerden, 1970), $\bar{\mathcal{D}}_{n,m}$ is irreducible. $\square$

Note that when $m = 0$, $\mathcal{D}_{n,0} = y_{n+1}$, whose solutions are $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ where c_i are constants. In the general case, we have

Lemma 2.6. *The solutions of $\mathcal{D}_{n,m} = 0$ have the following form:*

$$\hat{y} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

where a_i, b_j are constants.

Proof. Let

$$B = \begin{pmatrix} x^m & x^{m-1} & \dots & 1 \\ mx^{m-1} & (m-1)x^{m-1} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ m!x & (m-1)! & \dots & 0 \\ m! & 0 & \dots & 0 \end{pmatrix}.$$

Then we have

$$\mathcal{D}_{n,m} * |B| = \begin{vmatrix} (x^m y)^{(n+1)} & (x^{m-1}y)^{(n+1)} & \dots & y^{(n+1)} \\ (x^m y)^{(n+2)} & (x^{m-1}y)^{(n+2)} & \dots & y^{(n+2)} \\ \vdots & \vdots & \dots & \vdots \\ (x^m y)^{(2n)} & (x^{m-1}y)^{(2n)} & \dots & y^{(2n)} \end{vmatrix}$$

which is a Wronskian determinant for $(x^m y)^{(n+1)}, (x^{m-1}y)^{(n+1)}, y^{(n+1)}$ (Ritt, 1950). Hence we have $\mathcal{D}_{n,m}(\hat{y}) * |B| = 0$ if and only if there exist constants $b_m, b_{m-1}, \dots, b_0$, not all of them equal to 0, such that

$$b_m(x^m\hat{y})^{(n+1)} + b_{m-1}(x^{m-1}\hat{y})^{(n+1)} + \cdots + b_0\hat{y}^{(n+1)} = 0.$$

Since $|B| \neq 0$, $\mathcal{D}_{n,m}(\hat{y}) * |B| = 0 \Longleftrightarrow \mathcal{D}_{n,m}(\hat{y}) = 0$. As a consequence,

$$\mathcal{D}_{n,m}(\hat{y}) = 0 \Longleftrightarrow ((b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0)\hat{y})^{(n+1)} = 0$$
$$\Longleftrightarrow \hat{y} = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$$

where a_i are constants. □

By Lemma 2.6, we can prove the following theorem easily.

Theorem 2.7. *Let F be an irreducible differential polynomial. Then the differential equation $F = 0$ has a rational general solution $\hat{y}$ iff there exist non-negative integers n and m such that* $\mathrm{prem}(\mathcal{D}_{n,m}, F) = 0$.

Proof. ($\Rightarrow$) Let $\hat{y} = \frac{P(x)}{Q(x)}$ be a rational general solution of $F = 0$. Let $n \geq \deg(P(x))$ and $m \geq \deg(Q(x))$. Then from Lemmas 2.4 and 2.6

$$\mathcal{D}_{n,m}(\hat{y}) = 0 \Rightarrow \mathcal{D}_{n,m} \in \Sigma_F \Rightarrow \mathrm{prem}(\mathcal{D}_{n,m}, F) = 0.$$

($\Leftarrow$) By Lemma 2.1, $\mathrm{prem}(\mathcal{D}_{n,m}, F) = 0$ implies that $\mathcal{D}_{n,m} \in \Sigma_F$. Assume that m is the least integer such that $\mathcal{D}_{n,m} \in \Sigma_F$. Then all the zeros of Σ_F must have the form

$$\bar{y} = \frac{\bar{a}_n x^n + \bar{a}_{n-1}x^{n-1} + \cdots + \bar{a}_0}{\bar{b}_m x^m + \bar{b}_{m-1}x^{m-1} + \cdots + \bar{b}_0}.$$

In particular, the generic zero of Σ_F has the following form

$$\hat{y} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}.$$

Moreover, $b_m \neq 0$. Otherwise, we would have $\mathcal{D}_{n,m-1}(\hat{y}) = 0$ which implies that $\mathcal{D}_{n,m-1} \in \Sigma_F$, a contradiction. So the generic zero has the form (2.3). □

In the above theorem, let $m = 0$. Then we have the following corollary:

Corollary 2.8. *Let F be an irreducible differential polynomial. Then the differential equation $F = 0$ has a polynomial general solution $\hat{y}$ iff there exists a non-negative integer n such that* $\mathrm{prem}(y_n, F) = 0$.

Given a differential equation $F = 0$, if we know the degree bound N of the rational general solution of it, then we can decide whether it has a rational general solution or not by computing $\mathrm{prem}(D_{n,m}, F)$ for $n, m = 1, \ldots, N$. However, for higher order ODEs or ODEs with variate coefficients, we do not know this degree bound. In the following, we will show that a special case can be solved elegantly. In Section 3, for the first order autonomous ODEs, we will give an exact degree bound for its rational solutions.

Lemma 2.9. *Let $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where a_i are arbitrary constants. Let y_i be the i-th derivative of y with respect to x and $y_0 = y$. Then for $i = 0, \ldots, n$, we have*

$$a_i = (-1)^{n-i} \frac{x^{n-i+1}}{i!(n-i)!} \left(\frac{y_i}{x}\right)^{(n-i)}.$$

Proof. We note that $(\frac{y_i}{x})^{(n-i)} = (\frac{i!a_i}{x})^{(n-i)}$. Then by the computation directly, we can prove the lemma. □

Theorem 2.10. *Let F be an irreducible differential polynomial and $n = \operatorname{ord}(F)$. Then $F = 0$ has a polynomial general solution with degree n iff F can be rewritten as the following form:*

$$F = p(x)\left(\sum c_{i_0,i_1,\ldots,i_n} P_0^{i_0} P_1^{i_1} \cdots P_n^{i_n}\right) \tag{2.4}$$

where $p(x) \in \mathbf{K}$, $P_i = (-1)^{n-i} \frac{x^{n-i+1}}{i!(n-i)!}(\frac{y_i}{x})^{(n-i)}$ and $c_{i_0,i_1,\ldots,i_n} \in \mathbf{Q}$.

Proof. ($\Leftarrow$) Suppose that $\hat{y} = \hat{a}_n x^n + \hat{a}_{n-1}x^{n-1} + \cdots + \hat{a}_0$ is a polynomial general solution of $F = 0$ with degree n. Then $(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n)$ satisfies an algebraic equation: $G(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n) = 0$ where $G \in \mathbf{Q}[z_0, \ldots, z_n]$ and is irreducible. Let $\tilde{G} = G(P_0, P_1, \ldots, P_n)$ where $P_i = (-1)^{n-i} \frac{x^{n-i+1}}{i!(n-i)!}(\frac{y_i}{x})^{(n-i)}$. It is easy to establish that $P_i \in \mathbf{Q}[x, y_0, y_1, \ldots, y_n]$, y_k $(k \geq i)$ appears linearly in P_i, and the coefficient of y_i is a nonzero rational number. Hence $\tilde{G} \in \mathbf{Q}[x, y_0, \ldots, y_n]$ is an irreducible polynomial. By Lemma 2.9, we have that $\hat{a}_i = P_i(\hat{y})$ for $i = 0, \ldots, n$. So if we regard $\tilde{G}$ as a differential polynomial, we have that $\tilde{G}(\hat{y}) = G(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n) = 0$. Because F and $\tilde{G}$ are all irreducible and with order n, $F = p(x)\tilde{G}$ where $p(x) \in \mathbf{K}$.

($\Rightarrow$) Suppose that F has the form (2.4). Let

$$G = \sum c_{i_0,i_1,\ldots,i_n} z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n} \in \mathbf{Q}[z_0, \ldots, z_n],$$

then $p(x)G(P_0, P_1, \ldots, P_n) = F$. Let $\hat{y} = \hat{a}_n x^n + \hat{a}_{n-1}x^{n-1} + \cdots + \hat{a}_0$ where $(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n)$ is a generic zero of $G = 0$ and $\hat{a}_n \neq 0$, $\hat{a}_i$ are arbitrary constants. We will prove that $\hat{y}$ is a polynomial general solution. By Lemma 2.9, we know that

$$F(\hat{y}) = p(x)G(P_0(\hat{y}), \ldots, P_n(\hat{y})) = p(x)G(\hat{a}_0, \ldots, \hat{a}_n) = 0.$$

Assume that $H \in \mathbf{K}\{y\}$ satisfies $H(\hat{y}) = 0$. Let $R = \operatorname{prem}(H, F)$. Then $R(\hat{y}) = 0$. Assume that $R \neq 0$, we will get a contradiction. Since y_k $(k \geq i)$ appear linearly in P_i and the coefficient of y_i is a nonzero rational number, we can rewrite R as the form

$$R = \sum b_{j_0,j_1,\ldots,j_n} P_0^{j_0} P_1^{j_1} \cdots P_n^{j_n} + h(x)$$

where $b_{j_0,j_1,\ldots,j_n}, h(x) \in \mathbf{K}$. Let $\tilde{R} = \sum b_{j_0,j_1,\ldots,j_n} z_0^{j_0} z_1^{j_1} \cdots z_n^{j_n} + h(x)$. Then by Lemma 2.9, $\tilde{R}(\hat{a}_0, \ldots, \hat{a}_n) = R(\hat{y}) = 0$. Hence $\tilde{R} = M * G$ where $M \in \mathbf{K}[z_0, \ldots, z_n]$ because $(\hat{a}_0, \ldots, \hat{a}_n)$ is a generic zero of $G = 0$. That is, $p(x)R = M(P_0, \ldots, P_n) * F$, this is impossible. Hence $R \equiv 0$ which implies that $H \in \Sigma_F$ where Σ_F as in (2.2). So $\hat{y}$ is a generic zero of Σ_F. From the definition of the general solution, $\hat{y}$ is a polynomial general solution. □

3. Rational general solution of first order autonomous ODE

In the following sections, F will always be a nonzero first order autonomous differential polynomial with coefficients in $\mathbf{Q}$ and irreducible in the polynomial ring $\bar{\mathbf{Q}}[y, y_1]$. We call a rational solution $\bar{y}$ of $F = 0$ *nontrivial* if $\deg_x(\bar{y}) > 0$.

It is a trivial fact that for an autonomous ODE, the solution set is invariant by a translation of the independent variable x. Moreover, we have the following fact.

Lemma 3.1. *Let $\bar{y} = \frac{\bar{a}_n x^n + \cdots + \bar{a}_0}{x^m + \cdots + \bar{b}_0}$ be a nontrivial solution of $F = 0$, where $\bar{a}_i$, $\bar{b}_j$ are constants and $\bar{a}_n \neq 0$. Then*

$$\hat{y} = \frac{\bar{a}_n (x + c)^n + \cdots + \bar{a}_0}{(x + c)^m + \cdots + \bar{b}_0}$$

is a rational general solution of $F = 0$, where c is an arbitrary constant.

Proof. It is easy to show that $\hat{y}$ is still a zero of Σ_F. For any $G \in \mathbf{K}\{y\}$ satisfying $G(\hat{y}) = 0$, let $R = \mathrm{prem}(G, F)$. Then $R(\hat{y}) = 0$. Suppose that $R \neq 0$. Since F is irreducible and $\deg(R, y_1) < \deg(F, y_1)$, there are two differential polynomials $P, Q \in \mathbf{K}\{y\}$ such that $PF + QR \in \mathbf{K}[y]$ and $PF + QR \neq 0$. Thus $(PF + QR)(\hat{y}) = 0$. Because c is an arbitrary constant which is transcendental over $\mathbf{K}$, we have $PF + QR = 0$, a contradiction. Hence $R = 0$ which means that $G \in \Sigma_F$. So $\hat{y}$ is a generic zero of Σ_F. □

Note that when we say *constant*, we mean a constant in the constant field of universal extension field of $\mathbf{K}$. We do not know whether its constant field is exactly the universal constant extension of $\mathbf{Q}$. The following lemma shows that for first order autonomous ODE, the constants in its rational general solution can be chosen in the universal constant extension of $\mathbf{Q}$.

Lemma 3.2. *The constants in a rational general solution of a first order autonomous algebraic ODE can be chosen in the universal constant extension of* $\mathbf{Q}$.

Proof. From Lemma 3.1, we need only to prove $\bar{a}_i, \bar{b}_j$ can be chosen in $\bar{\mathbf{Q}}$. Substituting an arbitrary rational function (2.3) into $F = 0$, we have $F = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials in x whose coefficients are polynomials in a_i, b_j. Not that $Q(x)$ do not always vanish because the rational function is of the form (2.3). Let PS be the coefficients of $P(x)$. Then $(\bar{a}_0, \ldots, \bar{a}_n, \bar{b}_0, \ldots, \bar{b}_{m-1})$ must be a zero of PS with $a_n \neq 0$. Since $\bar{\mathbf{Q}}$ is algebraical closure, we can get a zero of PS in $\bar{\mathbf{Q}}$ with $a_n \neq 0$. Hence $\bar{a}_i, \bar{b}_j$ can be chosen in $\bar{\mathbf{Q}}$. □

Lemma 3.1 reduces the problem of finding a rational general solution to the problem of finding a nontrivial rational solution. In what appears below, we will show how to find a nontrivial rational solution. First of all, we decide the degree of a nontrivial rational solution.

3.1. Parametrization of algebraic curves

In this subsection, we will introduce some basic concepts on the parametrization of an algebraic plane curve. Let $F(x, y)$ be a polynomial in $\mathbf{Q}[x, y]$ and irreducible over $\bar{\mathbf{Q}}$.

Definition 3.3. $(r(t), s(t))$ is called a parametrization of $F(x, y) = 0$ if $F(r(t), s(t)) \equiv 0$ where $r(t), s(t) \in \bar{\mathbf{Q}}(t)$ and not all of them are in $\bar{\mathbf{Q}}$. A parametrization $(r(t), s(t))$ is called proper if $\bar{\mathbf{Q}}(r(t), s(t)) = \bar{\mathbf{Q}}(t)$.

Lüroth's Theorem guarantees that we can always obtain a proper parametrization from an arbitrary rational parametrization (Van der Waerden, 1970; Gao and Chou, 1991).

Lemma 3.4. *A proper parametrization has the following properties (Sendra and Winkler, 2001):*

(1) $\deg_t(r(t)) = \deg(F, y)$.
(2) $\deg_t(s(t)) = \deg(F, x)$.
(3) *If $(p(t), q(t))$ is another proper parametrization of $F(x, y)$, then there exists $f(t) = \frac{at+b}{ct+d}$ such that $p(t) = r(f(t))$, $q(t) = s(f(t))$ where a, d, b, c are elements in $\bar{\mathbf{Q}}$ satisfying $ad \neq bc$.*
(4) *Assume that $r(t) = \frac{r_1(t)}{r_2(t)}$ and $s(t) = \frac{s_1(t)}{s_2(t)}$ where $r_i(t), s_i(t) \in \bar{\mathbf{Q}}[t]$. Let $R(x, y)$ be the Sylvester-resultant of $r_2(t)x - r_1(t)$ and $s_2(t)y - s_1(t)$ with respect to t. Then $R(x, y) = \lambda F$ where $\lambda \in \bar{\mathbf{Q}}$.*

3.2. Degree for rational solutions of first order autonomous ODEs

Since F has order one and constant coefficients, we can regard it as an algebraic polynomial in y, y_1.

Notation 3.5. *We use $F(y, y_1)$ to denote F as an algebraic polynomial in y and y_1 which defines an algebraic curve.*

If $\bar{y} = r(x)$ is a nontrivial rational solution of $F = 0$, then $(r(x), r'(x))$ can be regarded as a parametrization of $F(y, y_1) = 0$. Moreover, we will show that $(r(x), r'(x))$ is a proper parametrization of $F(y, y_1) = 0$.

Lemma 3.6. *Let $f(x) = \frac{p(x)}{q(x)} \notin \bar{\mathbf{Q}}$ be a rational function in x such that* $\gcd(p(x), q(x)) = 1$. *Then $\bar{\mathbf{Q}}(f(x)) \neq \bar{\mathbf{Q}}(f'(x))$.*

Proof. If $f'(x) \in \bar{\mathbf{Q}}$ then the result is clearly true. Otherwise, $f(x), f'(x)$ are transcendental over $\bar{\mathbf{Q}}$. If $\bar{\mathbf{Q}}(f(x)) = \bar{\mathbf{Q}}(f'(x))$, from the Theorem in Section 63 of Van der Waerden (1970), we have

$$f(x) = \frac{af'(x) + b}{cf'(x) + d}$$

where $a, b, c, d \in \bar{\mathbf{Q}}$. Then

$$\frac{p(x)}{q(x)} = \frac{a(p'(x)q(x) - p(x)q'(x)) + bq(x)^2}{c(p'(x)q(x) - p(x)q'(x)) + dq(x)^2}$$

which implies that $q(x)|cp(x)q'(x)$ because $\gcd(p(x), q(x)) = 1$. So $c = 0$ or $q'(x) = 0$ which implies that $f(x) = (\frac{a}{d})f'(x) + \frac{b}{d}$ or $p(x) = \frac{c_1 p'(x)+c_2}{c_3 p'(x)+c_4}$ where $c_i \in \bar{\mathbf{Q}}$. This is impossible, because $f(x)$ is a rational function and $p(x)$ is a nonconstant polynomial if $q(x) \in \bar{\mathbf{Q}}$. □

Theorem 3.7. *Let $f(x)$ be the same as in Lemma 3.6. Then $\bar{\mathbf{Q}}(f(x), f'(x)) = \bar{\mathbf{Q}}(x)$.*

Proof. From Lüroth's Theorem (Van der Waerden, 1970), there exists $g(x) = \frac{u(x)}{v(x)}$ such that $\bar{\mathbf{Q}}(f(x), f'(x)) = \bar{\mathbf{Q}}(g(x))$, where $u(x), v(x) \in \bar{\mathbf{Q}}[x]$, $\gcd(u(x), v(x)) = 1$. We may assume that $\deg(u) > \deg(v)$. Otherwise, we have $\frac{u}{v} = c + \frac{w}{v}$ where $c \in \bar{\mathbf{Q}}$ and $\deg(w) < \deg(v)$, and $\frac{v}{w}$ is also a generator of $\bar{\mathbf{Q}}(g(x))$. Then we have

$$f(x) = \frac{p_1(g(x))}{q_1(g(x))}, \quad f'(x) = \frac{p_2(g(x))}{q_2(g(x))} = \frac{g'(x)(p_1' q_1 - p_1 q_1')}{q_1^2}$$

which implies that $g'(x) \in \bar{\mathbf{Q}}(g(x))$. If $g'(x) \notin \bar{\mathbf{Q}}$, we have

$$[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))] = [\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))][\bar{\mathbf{Q}}(g(x)) : \bar{\mathbf{Q}}(g'(x))].$$

However, we have $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))] = \deg(u)$ and $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))] \leq 2\deg(u) - 1$. Hence $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))] = [\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))]$. That is, $\bar{\mathbf{Q}}(g'(x)) = \bar{\mathbf{Q}}(g(x))$, a contradiction by Lemma 3.6. Hence, $g'(x) \in \bar{\mathbf{Q}}$ which implies that $g(x) = ax + b$. □

The above theorem implies that $(\bar{y}, \bar{y}_1)$ is a proper parametrization of $F(y, y_1) = 0$ if $\bar{y}$ is a nontrivial rational solution of $F = 0$. By Lemma 3.4, we have

Theorem 3.8. *Assume that $\bar{y} = \frac{P(x)}{Q(x)}$ is a nontrivial rational solution of $F = 0$. Let $U = P(x) - yQ(x)$, $V = P'(x) - yQ'(x) - y_1Q(x)$ and R be the Sylvester-resultant of U and V with respect to x. Then $R = \lambda F$ where λ is a nonzero element in $\bar{\mathbf{Q}}$.*

The above theorem proves the intuition that if $y = f(x)$ is a rational solution of $F = 0$ then F may be obtained by eliminating x from $y = f(x)$ and $y_1 = f'(x)$. It is easy to show that this result is not valid any longer for algebraic solutions.

Lemma 3.9. *Let $f(x)$ be the same as in Lemma 3.6. Then* $\deg_x(f(x)) - 1 \le \deg_x(f'(x)) \le 2\deg_x(f(x))$.

Proof. Since $f(x)$ is rational, it is clear that $\deg_x(f'(x)) \le 2\deg_x(f(x))$. If $q(x) \in \bar{\mathbf{Q}}$, then $\deg_x((\frac{p(x)}{q(x)})') = \deg_x(\frac{p(x)}{q(x)}) - 1$. Assume that $q(x) \notin \bar{\mathbf{Q}}$. Let $q(x) = (x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\ldots(x-a_r)^{\alpha_r}$. Then

$$\left(\frac{p(x)}{q(x)}\right)' = \frac{p'\prod(x-a_i) - p\left(\sum_{i=1}^{r}\prod_{j\neq i}\alpha_i(x-a_j)\right)}{(x-a_1)^{\alpha_1+1}(x-a_2)^{\alpha_2+1}\ldots(x-a_r)^{\alpha_r+1}}.$$

Since $p'\prod(x-a_i) - p(\sum_{i=1}^{r}\prod_{j\neq i}\alpha_i(x-a_j))$ and $(x-a_1)^{\alpha_1+1}(x-a_2)^{\alpha_2+1}\ldots(x-a_r)^{\alpha_r+1}$ have no common divisors, we have $\deg_x((\frac{p(x)}{q(x)})') = \max\{\deg(p) + r - 1, \deg(q) + r\} > \deg_x(\frac{p(x)}{q(x)}) - 1$. $\square$

From Lemmas 3.4 and 3.9, we have proved the following key theorem.

Theorem 3.10. *If $F = 0$ has a rational general solution $\hat{y}$, then we have*

$$\begin{cases}\deg_x(\hat{y}) = \deg(F, y_1)\\ \deg(F, y_1) - 1 \le \deg(F, y) \le 2\deg(F, y_1).\end{cases}$$

From Theorems 2.7 and 3.10, we have the following corollary:

Corollary 3.11. *Let $d = \deg(F, y_1)$. Then $F = 0$ has a rational general solution iff* $\mathrm{prem}(\mathcal{D}_{d,d}, F) = 0$.

Remark 3.12. In chapter X, vol 2 (Forsyth, 1959), there is a necessary and sufficient condition for a first order autonomous ODE to have a uniform solution by analytical consideration. However, the condition given here is simpler.

Remark 3.13. We may find a rational solution to $F = 0$ as follows. Let $d = \deg(F, y_1)$. As in Lemma 3.2, substituting an arbitrary rational function (2.3) of degree d into $F = 0$, we have $F = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials in x whose coefficients are polynomials in a_i, b_j. Note that $Q(x)$ does not always vanish because the rational function is of the form (2.3). Let PS be the coefficients of $P(x)$ as a polynomial in x. Then (2.3) is a rational solution to $F = 0$ iff a_i, b_j are zeros of the polynomial equations in PS. This method is not efficient for large d since it involves the solution of a nonlinear algebraic equation system in $2d$ variables. We will give more efficient algorithms in Section 4.4.

In the paper (Sendra and Winkler, 1997), Sendra and Winkler proved that for a rational algebraic curve defined by a polynomial over $\mathbf{Q}$ which is irreducible over $\bar{\mathbf{Q}}$, it can be parametrized over an extension field of $\mathbf{Q}$ with degree at most two. Theorem 3.14 will tell us that $F(y, y_1) = 0$ can always be parametrized over $\mathbf{Q}$.

Theorem 3.14. *If $F = 0$ has a nontrivial rational solution, then the coefficients of the nontrivial rational solution can be chosen in $\mathbf{Q}$.*

Proof. From Theorem 3.1 in Sendra and Winkler (1997), we know that there exists a nontrivial rational solution $r(x)$ of $F = 0$ whose coefficients belong to $\mathbf{Q}(\alpha)$ where $\alpha^2 \in \mathbf{Q}$. From Lemma 3.1, we can assume that $r(x) = \frac{\alpha p_1(x)+p_2(x)}{x^m+\alpha q_1(x)+q_2(x)}$ where $p_i(x), q_j(x) \in \mathbf{Q}(x)$. Assume that $\alpha p_1(x) + p_2(x)$ and $x^m + \alpha q_1(x) + q_2(x)$ have no common divisors over $\mathbf{Q}(\alpha)[x]$. We further assume that $\deg(q_j(x)) \leq m - 2$, which may be achieved by a proper linear transformation. It is easy to check that $\bar{r}(x) = \frac{-\alpha p_1(x)+p_2(x)}{x^m-\alpha q_1(x)+q_2(x)}$ is also a nontrivial rational solution of $F = 0$. Since both $(r(x), r'(x))$ and $(\bar{r}(x), \bar{r}'(x))$ are proper parametrizations of $F(y, y_1) = 0$, there exists an $f(x)$ such that $r(x) = \bar{r}(f(x))$ and $r'(x) = \bar{r}'(f(x))$. Since $r'(x) = f'(x)\bar{r}'(f(x))$, we have $f'(x) = 1$ which implies that $f(x) = x + c$ where $c \in \mathbf{Q}(\alpha)$. Thus

$$\frac{\alpha p_1(x) + p_2(x)}{x^m + \alpha q_1(x) + q_2(x)} = \frac{-\alpha p_1(x + c) + p_2(x + c)}{(x + c)^m - \alpha q_1(x + c) + q_2(x + c)}.$$

Since $\alpha p_1(x) + p_2(x)$ and $x^m + \alpha q_1(x) + q_2(x)$ have no common divisors, we have

$$x^m + \alpha q_1(x) + q_2(x) = (x + c)^m - \alpha q_1(x + c) + q_2(x + c).$$

If $m > 0$, we have $c = 0$ because $\deg(q_j(x)) \leq m - 2$, which implies that $p_1(x) = q_1(x) = 0$. If $m = 0$, then $r(x)$ is a polynomial. We can assume that $r(x) = (a_n\alpha + \tilde{a}_n)x^n + \alpha p_1(x) + p_2(x)$ where $p_i(x) \in \mathbf{Q}(x)$, $\deg(p_i(x)) \leq n - 2$ and $a_n, \tilde{a}_n \in \mathbf{Q}$, where at least one of a_n and $\tilde{a}_n$ is not 0. In a similar way, we have $a_n = 0$ and $p_1(x) = 0$. $\square$

4. A polynomial time algorithm for first order autonomous ODEs

In this section, we will give an effective method with polynomial complexity.

4.1. Structure of first order autonomous ODEs with rational solutions

In order to present the algorithm, we need to analyze the structure of a first order autonomous ODE with nontrivial rational solutions. In this section, $F = 0$ is always a first order autonomous ODE. We write F as the following forms:

$$F = A_d(y)y_1^d + A_{d-1}(y)y_1^{d-1} + \cdots + A_0(y) \tag{4.1}$$

$$F = F_{\bar{d}}(y, y_1) + F_{\bar{d}-1}(y, y_1) + \cdots + F_{\underline{d}}(y, y_1) \tag{4.2}$$

where $A_i(y)$ are polynomials in y, $F_i(y, y_1)$ is the homogeneous part of F with total degree i, and $\bar{d} \geq \underline{d}$. It is clear that $\bar{d} = \mathrm{tdeg}(F)$ is the total degree of F. We are going to assume that $F = 0$ has a nontrivial rational solution of the form

$$\bar{y} = \frac{P(x)}{Q(x)}, \quad \text{where } n = \deg(P(x)), m = \deg(Q(x)). \tag{4.3}$$

By Theorem 3.10, we have $d = \max\{n, m\}$. As a corollary of the theorem on page 311 of Forsyth (1959), we have

Lemma 4.1. *If $F = 0$ of the form (4.1) has a nontrivial rational solution, then*

$$\deg(A_i(y)) \le 2(d-i) \quad \text{for } i = 0, \ldots, d.$$

In fact, the above lemma is still true for a first order ODE with variate coefficients which has no movable singularities (Matsuda, 1980).

Theorem 4.2. *Assume that $F = 0$ of the form (4.2) has a nontrivial rational solution of form (4.3). We have*

(1) *If $n > m$, then $\bar{d} = \deg(F, y) + 1$.*
(2) *If $n \le m$, then $\bar{d} = \deg(F, y)$.*

Proof. In this proof, we use $F(y, y_1)$ to denote F. Replacing y_1 by $ty + z$,

$$F_{\bar{d}}(y, ty+z) = \sum_{j=0}^{\bar{d}} C_{\bar{d},j} y^{\bar{d}-j}(ty+z)^j = \left(\sum_{j=0}^{\bar{d}} C_{\bar{d},j} t^j\right) y^{\bar{d}} + B$$

where B does not contain the term $y^{\bar{d}}$. Because $\sum_{j=0}^{\bar{d}} C_{\bar{d},j} t^j \ne 0$, there exists a nonzero number $\bar{t} \in \mathbf{Q}$ such that $\sum_{j=0}^{\bar{d}} C_{\bar{d},j} \bar{t}^j \ne 0$. Then $\bar{d} = \mathrm{tdeg}(F(y, \bar{t}y+z)) = \deg(F(y, \bar{t}y+z), y)$. It is not difficult to verify that $F(y, \bar{t}y+z)$ is still an irreducible polynomial and

$$\bar{y} = \frac{P(x)}{Q(x)}, \quad \bar{z} = \frac{P'(x)Q(x) - P(x)Q'(x) + \bar{t}P(x)Q(x)}{Q(x)^2}$$

is a proper parametrization of $F(y, \bar{t}y+z) = 0$. So $\bar{d} = \deg(F(y, \bar{t}y+z), y) = \deg(\bar{z})$. By Lemma 3.4 and the same analysis as in the proof of Lemma 3.9, we get

(1) If $n > m$, then $\bar{d} = \deg(F(y, \bar{t}y+z), y) = \deg(\bar{z}) = \deg(\bar{y}_1) + 1 = \deg(F(y, y_1), y) + 1$.
(2) If $n \le m$, then $\bar{d} = \deg(F(y, \bar{t}y+z), y) = \deg(\bar{z}) = \deg(\bar{y}_1) = \deg(F(y, y_1), y)$. □

Furthermore, we have

Theorem 4.3. *Assume that $F = 0$ of the forms (4.1) and (4.2) has a nontrivial rational solution of form (4.3). Further assume that*

$$\begin{aligned} F_{\bar{d}}(y, y_1) &= C_{\bar{d},\bar{k}} y^{\bar{d}-\bar{k}} y_1^{\bar{k}} + \cdots + C_{\bar{d},\underline{k}} y^{\bar{d}-\underline{k}} y_1^{\underline{k}}, \\ F_{\underline{d}}(y, y_1) &= C_{\underline{d},\underline{d}} y_1^{\underline{d}} + \cdots + C_{\underline{d},\underline{l}} y^{\underline{d}-\underline{l}} y_1^{\underline{l}}, \\ A_0(y) &= C_{\bar{p},0} y^{\bar{p}} + \cdots + C_{\underline{p},0} y^{\underline{p}} \end{aligned} \tag{4.4}$$

where $C_{\bar{d},\underline{k}} C_{\underline{d},\underline{l}} C_{\bar{p},0} C_{\underline{p},0} \ne 0$. Then

(1) *$n > m$ iff $\bar{d} = \bar{p} + 1$. Moreover if $n > m$, then $\underline{k} = n - m$.*
(2) *$n < m$ iff $\underline{d} = \underline{p} - 1$. Moreover if $n < m$, then $\underline{l} = m - n$.*
(3) *$n = m$ iff $\bar{d} = \bar{p}$ and $\underline{d} = \underline{p}$.*

Proof. Case 1. Assume that $n > m$, then the Laurent series expansion of the solution $\bar{y}(x) = \frac{P(x)}{Q(x)}$ at $x = \infty$ has the form: $\bar{y}(x) = a_{n-m}x^{n-m} + \cdots + \sum_{j=0}^{\infty} a_{-j}x^{-j}$ (see Section 4.2). Substituting $\bar{y}(x)$ into F, for each monomial in F, the highest degree of $C_{\alpha_i,\beta_i}\bar{y}(x)^{\alpha_i}\bar{y}_1(x)^{\beta_i}$ is equal to $(n-m)(\alpha_i+\beta_i) - \beta_i$. Let $N = \max\{(n-m)(\alpha_i+\beta_i) - \beta_i \mid y^{\alpha_i} y_1^{\beta_i}$ appears in $F\}$. Since $F(\bar{y}(x)) = 0$, there exist at least two monomials such that their highest degrees equal N. Because

$\bar{p} \le \deg(F, y)$, by Theorem 4.2, $\bar{p} \le \bar{d} - 1$. Suppose that $\bar{p} < \bar{d} - 1$, then $y^{\bar{d}-1}y_1$ must appear in F. However, $N = (n - m)\bar{d} - 1 > (n - m)(\alpha_i + \beta_i) - \beta_i$ for all $y^{\alpha_i}y_1^{\beta_i}$ except that $y^{\bar{d}-1}y_1$, a contradiction. Hence $\bar{p} = \bar{d} - 1$. From (4.4), we have $(n - m)(\bar{d} - 1) \ge (n - m)(\alpha_i + \beta_i) - \beta_i$ for all $\alpha_i + \beta_i \le \bar{d} - 1$ and $(n - m)\bar{d} - \underline{k} \ge (n - m)\bar{d} - \beta_i$ for all $\alpha_i + \beta_i = \bar{d}$ which imply $(n - m)(\bar{d} - 1) = (n - m)\bar{d} - \underline{k}$. Hence $\underline{k} = n - m$. Now assume that $\bar{d} = \bar{p} + 1$. We will prove $\bar{p} = \deg(F, y)$. Suppose that $\bar{p} \ne \deg(F, y)$ which implies that $\deg(F, y) > \bar{p}$. That is, in the form (4.1), there exists $i_0 > 0$ such that $\deg(A_{i_0}(y)) > \deg(A_0(y)) = \bar{p}$. So we have $\bar{d} \ge \deg(A_{i_0}(y)) + 1 > \bar{p} + 1$, a contradiction. Hence $\bar{p} = \deg(F, y)$ which implies that $\bar{d} = \deg(F, y) + 1$. From Theorem 4.2, we have $n > m$.

Case 2. Replace y by $\frac{1}{z}$ and y_1 by $-\frac{z_1}{z^2}$ in F and multiply $F(\frac{1}{z})$ by z^{2d}. By Lemma 4.1, we get an irreducible polynomial $G(z)$. In $G(z)$, the highest degree of the monomials is $2d - \underline{d}$ and the lowest degree of the monomials is $2d - \bar{d}$. Corresponding to (4.4), we have

$$\begin{aligned}
G(z) &= G_{2d-\underline{d}}(z) + G_{2d-\underline{d}-1}(z) + \cdots + G_{2d-\bar{d}}(z)\\
G_{2d-\underline{d}}(z) &= C_{\underline{d},\underline{d}} z^{2(d-\underline{d})} z_1^{\underline{d}} + \cdots + C_{\underline{d},\underline{l}} z^{2d-\underline{d}-\underline{l}} z_1^{\underline{l}}\\
G_{2d-\bar{d}}(z) &= C_{\bar{d},\bar{k}} z^{2d-\bar{d}-\bar{k}} z_1^{\bar{k}} + \cdots + C_{\bar{d},\underline{k}} z^{2d-\bar{d}-\underline{k}} z_1^{\underline{k}}\\
\bar{A}_0(z) &= C_{\underline{p},0} z^{2d-\underline{p}} + \cdots + C_{\bar{p},0} z^{2d-\bar{p}}.
\end{aligned}$$

Moreover, $\bar{z} = \frac{Q(x)}{P(x)}$ is a rational solution of $G(z) = 0$. By the first case, $m > n$ iff $2d - \underline{p} = 2d - \underline{d} - 1$ and if $m > n$, then $\underline{l} = m - n$. In other words, $n < m$ iff $\underline{d} = \underline{p} - 1$ and if $m > n$, then $\underline{l} = m - n$.

Case 3. Assume $n = m$. By Theorem 4.2, $\bar{d} = \deg(F, y) = \max\{\deg(A_i(y))\}$. Since $\bar{d} = \max\{\deg(A_i(y)) + i\}$, $\bar{d} = \deg(A_0(y)) = \bar{p}$. As in the second case, we can obtain $G(z)$, $G_{2d-\underline{d}}(z)$, $G_{2d-\bar{d}}(z)$ and $\bar{A}_0(z)$. For the same reason, $2d - \underline{d} = 2d - \underline{p}$ which implies that $\underline{d} = \underline{p}$. The sufficiency is clear from the first case and the second case. □

In the special case $m = 0$, F will have the following particular type.

Theorem 4.4. *If $F = 0$ has a nontrivial rational solution of the form (4.3), then the nontrivial rational solution is polynomial iff F has the following form:*

$$F = ay_1^n + by^{n-1} + G(y, y_1)$$

where $n = \deg(F, y_1)$, $a, b \in \mathbf{Q}$ are not zero, $\mathrm{tdeg}(G(y, y_1)) \le n - 1$ *and G does not contain the term y^{n-1}.*

Proof. ($\Longrightarrow$) Since $m = 0$, from Theorem 4.3, $\underline{k} = n = \deg(F, y_1)$. Hence $F_{\bar{d}}(y) = C_{\bar{d},n} y^{\bar{d}-n} y_1^n$. From Lemma 4.1, $\bar{d} = n$. By Theorem 4.3 again, $\bar{p} = n - 1$. So F has the above form.

($\Longleftarrow$) From Theorem 4.2, $n > m$. Since $F_{\bar{d}}(y) = ay_1^n$, we have $\underline{k} = n - m = n$ which implies $m = 0$. □

4.2. Computing the Laurent series solution of $F = 0$

The first step of our algorithm for finding the rational solutions to $F = 0$ is finding its Laurent series solutions. We consider a Laurent series of the following form

$$y(x) = \sum_{i=k}^{-\infty} a_i x^i \tag{4.5}$$

where k is an integer and the a_i are in $\bar{\mathbf{Q}}$. Assuming that $F = 0$ has a nontrivial rational solution of the form (4.3) the above theorem provides a method for computing the Laurent series expansion of some solution of $F = 0$ at $x = \infty$.

Theorem 4.5. *Use the notation in* (4.1), (4.2) *and* (4.4). *Assume that* $F = 0$ *has a nontrivial rational solution of form* (4.3). *Substituting* (4.5) *into* F, *we obtain a new Laurent series*

$$F(y(x)) = \sum_{i=m}^{-\infty} L_i x^i$$

where L_i *are polynomials in* a_j. *We have*

(1) *If* $\bar{d} = \bar{p} + 1$, *then in* (4.5) *let* $k = \underline{k}$, $a_{\underline{k}} = -\frac{C_{\bar{p},0}}{\underline{k}^{\underline{k}} C_{\bar{d},\underline{k}}}$. *We have*

$$L_{\underline{k}(\bar{d}-2)+i} = C_{\bar{p},0} a_{\underline{k}}^{\bar{d}-2} (\underline{k} - 1 - i) a_i + H_i(a_{\underline{k}-1}, \ldots, a_{i+1}) \tag{4.6}$$

where $i = \underline{k} - 1, \underline{k} - 2, \ldots$, *and* $H_i \in \mathbf{Q}[a_{\underline{k}-1,\ldots,a_{i+1}}]$. *Moreover,* $H_{\underline{k}-1}(a_{\underline{k}}) = 0$.

(2) *If* $\bar{d} = \bar{p}$ *and* $\underline{d} = \underline{p} - 1$, *then in* (4.5) *let* $k = -\underline{l}$, $a_{-\underline{l}} = -\frac{(-\underline{l})^{\underline{l}} C_{\underline{d},\underline{l}}}{C_{\underline{p},0}}$. *We have*

$$L_{-\underline{l}\underline{d}+i} = C_{\underline{p},0} a_{-\underline{l}}^{\underline{d}} (\underline{l} + 1 + i) a_i + K_i(a_{-\underline{l}-1}, \ldots, a_{i+1}) \tag{4.7}$$

where $i = -\underline{l} - 1, -\underline{l} - 2, \ldots$, *and* $K_i \in \mathbf{Q}[a_{-\underline{l}-1,\ldots,a_{i+1}}]$. *Moreover,* $K_{-\underline{l}-1}(a_{-\underline{l}}) = 0$.

(3) *If* $\bar{d} = \bar{p}$ *and* $\underline{d} = \underline{p}$, *we can find a* $\bar{c} \in \mathbf{Q}$ *such that* $F(y + \bar{c})$ *satisfies the condition in case 2.*

Proof. We will prove the first case. The second case can be proved in the same way. Note that $\bar{p} = \deg(F, y)$. First, we introduce some notation. Let $A = C_{\bar{d},\underline{k}} y(x)^{\bar{d}-\underline{k}} y_1(x)^{\underline{k}} + C_{\bar{p},0} y(x)^{\bar{d}-1}$, $B = F(y(x)) - A$. We will define a weight

$$w : \mathbf{Z}[x, a_i] \to \mathbf{Z}$$

which satisfies $w(st) = w(s) + w(t)$ and $w(x) = 1$, $w(a_i) = \underline{k} - i$. Then $y(x)$ is an isobaric polynomial with the weight $\underline{k}$ and $y_1(x)$ is an isobaric polynomial with the weight $\underline{k} - 1$. Hence A is an isobaric polynomial with the weight $\underline{k}(\bar{d} - 1)$ and the highest weight in B is less than $\underline{k}(\bar{d} - 1)$. Then the weight of $L_{\underline{k}(\bar{d}-2)+i}$ is less than or equal to $\underline{k} - i$. Therefore, a_j cannot appear in $L_{\underline{k}(\bar{d}-2)+i}$ for $j < i$ and a_i can only appear linearly in the coefficient of $x^{\underline{k}(\bar{d}-2)+i}$ in A. By the computation process, in the coefficient of $x^{\underline{k}(\bar{d}-2)+i}$ in A, the terms containing a_i are $C_{\bar{p},0} a_{\underline{k}}^{\bar{d}-2} (\underline{k} - 1 - i) a_i$. Since a_i cannot appear in the coefficient of $x^{\underline{k}(\bar{d}-2)+i}$ in B, $L_{\underline{k}(\bar{d}-2)+i}$ has the form (4.6). In the following, we prove $H_{\underline{k}-1}(a_{\underline{k}}) = 0$. From Theorem 4.3, we have that $n > m$. Then the Laurent series expansion of $\bar{y}(x)$ at $x = \infty$ will have the form

$$\bar{y}(x) = \bar{a}_{n-m} x^{n-m} + \cdots + \sum_{j=0}^{\infty} \bar{a}_{-j} x^{-j}.$$

Moreover, by computation, we have $\bar{a}_{n-m} = -\frac{C_{\bar{p},0}}{\underline{k}^{\underline{k}} C_{\bar{d},\underline{k}}} = a_{\underline{k}}$. Now substituting $\bar{y}(x)$ into F, we have $\bar{L}_i = 0$ for all $i \leq \underline{k}(\bar{d} - 1)$ where $\bar{L}_i$ are obtained by replacing a_i with $\bar{a}_i$ in L_i in (4.6). In particular, $\bar{L}_{\underline{k}(\bar{d}-1)-1} = H_{\underline{k}-1}(a_{\underline{k}}) = 0$.

We now prove the third case. From Theorem 4.3, we have $n = m$. Then the Laurent series expansion of the solution $\bar{y}(x)$ at $x = \infty$ will have the form

$$\bar{y}(x) = \bar{a}_0 + \bar{a}_{-1}\frac{1}{x} + \bar{a}_{-2}\left(\frac{1}{x}\right)^2 + \cdots.$$

Then $(\bar{a}_0, 0)$ will be a zero of $F = 0$ if we regard F as an algebraic polynomial. Hence $A_0(\bar{a}_0) = 0$. From Theorem 3.14, there is a rational solution of $F = 0$ whose coefficients are in $\mathbf{Q}$, assume that this solution is $\bar{y}(x)$. Then $\bar{a}_0$ must be a rational number. Hence $\bar{a}_0$ is a rational root of $A_0(y) = 0$. It is easy to establish that $F(y + \bar{a}_0)$ is still irreducible and $\bar{y}(x) - \bar{a}_0$ is one of the solutions of $F(y + \bar{a}_0) = 0$. Because $\bar{y}(\infty) - \bar{a}_0 = 0$, the degree of the numerator of $\bar{y}(x) - \bar{a}_0$ is less than that of its denominator. Hence $F(y + \bar{a}_0)$ should satisfy the condition of case 2. □

From Lemma 4.1, Theorems 4.2 and 4.5, we have an algorithm for computing the first m terms of a Laurent series solution of $F = 0$ in some special case.

Algorithm 4.6. Input: F and a positive integer m. Output: The first m terms of the Laurent series solution of $F = 0$ of form (4.5) or a message: $F = 0$ has no nontrivial rational solution.

1 Rewrite F as the form (4.1), (4.2) and (4.4).
2 Let $d_i = \deg(A_i(y))$. For all $i = 0, \ldots, d$, if $d_i \le 2(d-i)$, then go to the next step. Otherwise by Lemma 4.1, $F = 0$ has no nontrivial rational solution; the algorithm terminates.
3 If $\bar{d} = \bar{p} + 1$, let $k := \underline{k}$, $a_{\underline{k}} := -\frac{C_{\bar{p},0}}{\underline{k}^{\underline{k}} C_{\bar{d},\underline{k}}}$, and $\bar{y} := a_{\underline{k}} x^{\underline{k}}$.
   (a) $C :=$ the coefficient of $x^{\underline{k}(\bar{d}-1)-1}$ in $F(\bar{y})$. If $C \neq 0$ then by (4.6), $F = 0$ has no rational solutions and the algorithm terminates, else $a_{\underline{k}-1} := 0$.
   (b) $i := \underline{k} - 2$.
   while $i \ge \underline{k} - m + 1$ do
      i $C :=$ the coefficient of $x^{\underline{k}(\bar{d}-2)+i}$ in $F(\bar{y})$.
      ii $a_i := -\frac{C}{C_{\bar{p},0} a_{\underline{k}}^{\bar{d}-2}(\underline{k}-1-i)}$.
      iii $\bar{y} := \bar{y} + a_i x^i$.
      iv $i := i - 1$.
   (c) return $(\bar{y})$.
4 If $\bar{d} = \bar{p}$ and $\underline{d} = \underline{p} - 1$, let $k := -\underline{l}$, $a_{-\underline{l}} := -\frac{(-\underline{l})^{\underline{l}} C_{\underline{d},\underline{l}}}{C_{\underline{p},0}}$, and $\bar{y} := a_{-\underline{l}} x^{-\underline{l}}$.
   (a) $C :=$ the coefficient of $x^{-\underline{l}(\underline{d}+1)-1}$ in $F(\bar{y})$. If $C \neq 0$ then by (4.7), $F = 0$ has no rational solutions and the algorithm terminates, else $a_{-\underline{l}-1} := 0$.
   (b) $i := -\underline{l} - 2$.
   while $i \ge -\underline{l} - m + 1$ do
      i $C :=$ the coefficient of $x^{-\underline{l}\underline{d}+i}$ in $F(\bar{y})$.
      ii $a_i := -\frac{C}{C_{\underline{p},0} a_{-\underline{l}}^{\underline{d}}(\underline{l}+1+i)}$.
      iii $\bar{y} := \bar{y} + a_i x^i$.
      iv $i := i - 1$.
   (c) return $(\bar{y})$.

5 If $\bar{d} = \bar{p}$ and $\underline{p} = \underline{d}$, then let $r_1, \ldots, r_k$ be all solutions of $A_0(y) = 0$ in $\mathbf{Q}$. Let $i := 1$.
while $i \le k$ do
  (a) $F := F(y + r_i)$. Rewrite F as the form (4.1), (4.2) and (4.4).
  (b) If F satisfies the assumption of step 2 and step 4, then go to step 4. In step (a) of step 4, if $C = 0$, then run the steps (b) and (c) and return $(\bar{y} + r_i)$, else go to the following step.
  (c) $i := i + 1$
  If for all r_i, we cannot compute the first m terms of the Laurent series solution for $F(y+r_i) = 0$ in step 4, then $F = 0$ has no nontrivial rational solution by Theorem 4.5 and the algorithm terminates.
6 In all other cases, F has no nontrivial rational solutions and the algorithm terminates.

The complexity of computing $\bar{y}^d$ where $\bar{y}$ is a polynomial in $\mathbf{Q}[x]$ with degree m is $O(m^2d^2)$. By Lemma 4.1, the total degree of F is at most $2d$ and the number of recurrences is at most $2md$. The complexity of factorization of a polynomial with degree d in $\mathbf{Q}[x]$ is $O(d^3)$(p. 411 (von Zur Gathen and Gerhard, 1999)). Hence Algorithm 4.6 is a polynomial time complexity algorithm. Here we only consider the number of multiplications (or divisions) in the algorithm.

Remark 4.7. There is an algorithm based on the Newton Polygon method for computing Puiseux series solutions of differential equations (Cano, 1993; Duval, 1989; Grigor'ev and Singer, 1991). In general, we need to solve the high degree algebraic equations to find the Puiseux series solutions by the Newton Polygon method. Here, the differential equations which we consider are special. By the analysis of the structures of these special differential equations, we can determine the first term of one of its Laurent series solutions from the degrees and the coefficients of the original equation. Then we need only to use rational operations in $\mathbf{Q}$ to find all of the coefficients of its Laurent series solutions.

4.3. Padé approximants

The Padé approximants are a particular type of rational fraction approximation to the value of a function. It constructs the rational fraction from the Taylor series expansion of the original function. Its definition is given below (George and Baker, 1975).

Definition 4.8. For the formal power series $A(x) = \sum_0^\infty a_j x^j$ and two non-negative integers L and M, the (L, M) Padé approximant to $A(x)$ is the rational fraction

$$[L \setminus M] = \frac{P_L(x)}{Q_M(x)}$$

such that

$$A(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1})$$

where $P_L(x)$ is a polynomial with degree not greater than L and $Q_M(x)$ is a polynomial with degree not greater than M. Moreover, $P_L(x)$ and $Q_M(x)$ are relatively prime and $Q_M(0) = 1$.

Let $P_L(x) = \sum_0^L p_i x^i$ and $Q_M(x) = \sum_0^M q_i x^i$. We can compute $P_L(x)$ and $Q_M(x)$ with the following linear equations in p_i and q_i:

$$\begin{aligned} a_0 &= p_0 \\ a_1 + a_0 q_1 &= p_1 \\ &\cdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L &= p_L \\ a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M &= 0 \\ &\cdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M &= 0 \end{aligned} \tag{4.8}$$

where $a_n = 0$ if $n < 0$ and $q_j = 0$ if $j > M$.

For the Padé approximation, we have the following theorems (George and Baker, 1975).

Theorem 4.9 (*Frobenius and Padé*). *When it exists, the Padé approximant* $[L \setminus M]$ *to any formal power series* $A(x)$ *is unique.*

Theorem 4.10 (*Padé*). *The function* $f(x)$ *is a rational function of the following form*

$$f(x) = \frac{p_l x^l + p_{l-1} x^{l-1} + \cdots + p_0}{q_m x^m + q_{m-1} x^{m-1} + \cdots + 1}$$

iff the Padé approximants are given by $[L \setminus M] = f(x)$ *for all* $L \geq l$ *and* $M \geq m$.

4.4. A polynomial time algorithm

Let $f(x)$ be a rational function. Rewrite $f(x)$ as the form: $f(x) = x^k \frac{P(x)}{Q(x)}$ where $k \in \mathbf{Z}$ and $P(0) \neq 0$, $Q(0) \neq 0$. Suppose that $a_0 + a_1 x + \cdots$ is the Taylor series expansion of $\frac{P(x)}{Q(x)}$ at $x = 0$. Then by the uniqueness of the Laurent series expansion, $x^k(a_0 + a_1 x + \cdots)$ is the Laurent series expansion of $f(x)$ at $x = 0$. So for a rational function $g(x)$, if $\sum_{i=k}^{\infty} a_i x^i$ is its Laurent series expansion at $x = 0$, then $\sum_{i=k}^{\infty} a_i x^{i-k}$ will be the Taylor series expansion of $x^{-k} g(x)$ at $x = 0$. Then we can find $x^{-k} g(x)$ by constructing Padé approximants to $\sum_{i=k}^{\infty} a_i x^{i-k}$. This means that to find a rational function, we need only to know its Laurent series expansion at $x = 0$. Since the Laurent series expansion of $g(x)$ at $x = \infty$ is equivalent to the Laurent series expansion of $g(\frac{1}{t})$ at $t = 0$, in order to find $g(x)$, we need only to find the Laurent series expansion of $g(x)$ at $x = \infty$. In Section 4.2, we presented a method for computing a Laurent series solution of $F = 0$ at $x = \infty$.

Now, we are ready to give the main algorithm.

Algorithm 4.11. The input is F. The output is a rational general solution of $F = 0$ if it exists. Also, if we find such a solution, it is of the following form

$$\hat{y} = \frac{a_n (x+c)^n + a_{n-1} (x+c)^{n-1} + \cdots + a_0}{b_m (x+c)^m + b_{m-1} (x+c)^{m-1} + \cdots + b_0}$$

where a_i, b_j are in $\mathbf{Q}$ and c is an arbitrary constant.

1 $d := \deg(F, y_1)$. Compute the first $2d + 1$ terms of the Laurent series solution of $F = 0$ by Algorithm 4.6. If it returns a series $\bar{y}(x)$, then go to the next step, else the algorithm terminates.
2 Select an integer k such that $z(t) := t^k \bar{y}(\frac{1}{t})$ is a polynomial and the first term of $z(t)$ is a nonzero constant.
3 $a_i :=$ the coefficient of t^i in $z(t)$ for $i = 0, \ldots, 2d$. In (4.8), let $L = M = d$. Then we can find q_i by solving the following linear equations (note that we have $q_0 = 1$):

$$\begin{pmatrix} a_1 & a_2 & \dots & a_d \\ a_2 & a_3 & \dots & a_{d+1} \\ \vdots & \vdots & \dots & \vdots \\ a_d & a_{d+1} & \dots & a_{2d-1} \end{pmatrix} \begin{pmatrix} q_d \\ q_{d-1} \\ \vdots \\ q_1 \end{pmatrix} = - \begin{pmatrix} a_{d+1} \\ a_{d+2} \\ \vdots \\ a_{2d} \end{pmatrix}.$$

If the above linear equation has no solutions, then the algorithm terminates. Otherwise, if the matrix A is singular, from Theorem 4.9, we need only to select one of the solutions of the above linear equations.

4 $p_i := a_0 q_i + a_1 q_{i-1} + \cdots + a_i q_0$ for $i = 0, \dots, d$ and

$$r(t) := \frac{p_d t^d + p_{d-1} t^{d-1} + \cdots + p_0}{q_d t^d + q_{d-1} t^{d-1} + \cdots + 1}.$$

5 $\bar{y}(x) := x^k r(\frac{1}{x})$. Substituting $\bar{y}(x)$ in F, if $F(\bar{y}) = 0$ then return $\hat{y} = (x + c)^k r(\frac{1}{x+c})$. Otherwise, $F = 0$ has no rational general solution.

By Lemma 2.6 and Theorem 2.7, we know that if $F = 0$ has a nontrivial rational solution, then every nontrivial formal Laurent series solution of $F = 0$ must be the Laurent series of the rational solution. From Algorithm 4.6, we know that the Laurent series is nontrivial. By Theorem 3.10 and the discussion at the beginning of this subsection, the above algorithm is correct.

Now we give an example to show how Algorithm 4.11 works.

Example 4.12. Consider the differential equation:

$$F = y_1^3 + 4y_1^2 + (27y^2 + 4)y_1 + 27y^4 + 4y^2 = 0.$$

(1) Rewrite F as the form (4.2)

$$F = 27y^4 + y_1^3 + 27y^2 y + 4y_1^2 + 4y^2 + 4y_1 \quad \text{and} \quad A_0(y) = 27y^4 + 4y^2.$$

(2) Use the notation in Theorem 4.5. Since $\bar{d} = \bar{p} = 4$ and $\underline{d} = 1 = \underline{p} - 1$, F is in case 2 of Theorem 4.5.

(3) $d := \deg(F, y_1) = 3$. By Algorithm 4.6, compute the first seven terms of the Laurent series solution $\bar{y}(x)$ of $F = 0$:

$$\bar{y}(x) = \frac{1}{x} + \frac{1}{x^3}.$$

(4) $k := -1$ and $z(t) := \frac{\bar{y}(\frac{1}{t})}{t} = 1 + t^2$.

(5) $a_0 := 1, : a_2 := 1$ and $a_i := 0$ for $i = 1, \dots, 7$ and $i \neq 2$. Solve the linear equations:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Select one of the solutions as $(q_3, q_2, q_1) := (0, 0, 0)$.

(6) $p_0 := 1$, $p_1 := 0$, $p_2 := 1$, $p_3 := 0$ and $r(t) = t^2 + 1$.

(7) $\bar{y}(x) := \frac{r(\frac{1}{x})}{x} = \frac{x^2+1}{x^3}$. Substituting $\bar{y}(x)$ in F, $F(\bar{y}(x)) \equiv 0$. Hence $\hat{y}(x) = \frac{(x+c)^2+1}{(x+c)^3}$ is a rational general solution.

The complexity of the algorithm is clearly polynomial in terms of the number of multiplications (or divisions) in $\mathbf{Q}$, since all the operations involved have polynomial complexity.

Table 1
Timings for randomly generated first order autonomous ODEs

$\deg(F, y_1)$	12	13	14	15	16
$\mathrm{tdeg}(F)$	≤ 24	≤ 26	≤ 28	≤ 30	≤ 32
Average time	0.026	0.039	0.065	0.049	0.073
Rational solutions	no	no	no	no	no

Table 2
Timings for solving first order autonomous ODEs

	deg	tdeg	Term	Time (s)	sol		deg	tdeg	Term	Time (s)	sol
F_1	7	9	35	0.771	yes	F_4	10	16	87	14.611	yes
F_2	8	15	77	6.580	yes	F_5	11	16	99	20.288	yes
F_3	9	18	94	20.678	yes	F_6	12	16	97	20.629	yes

In Step 1, the complexity is polynomial. In Step 3, we need to solve a linear equation system with rational coefficients and in $2n$ variables, which has polynomial complexity. In Step 5, we need to replace y and y_1 in $F(y, y_1)$ by two rational functions. From Theorem 3.10, $\deg(F, y) \le 2n$. Then this step also has polynomial complexity. It is easy to check that all other steps are of polynomial complexity too.

4.5. Experimental results

We implement Algorithm 4.11 in Maple. Two sets of experiments are performed.

In the first experiment, we randomly generate two hundred first order autonomous ODEs of $\deg(F, y_1) = d$ and $\mathrm{tdeg}(F) \le 2d$ for each d and compute the rational solutions for these equations. The result of the experiment shows that almost all of first order autonomous ODEs have no rational general solutions. The average running time is given in Table 1. Times are collected on a PC with a 2.66G CPU and 256M memory and are given in seconds. "no" in the last row of the table means there are no ODEs which have a rational general solution among two hundred ODEs which we compute. From Table 1, we can see that the program gives a negative answer immediately. The reason is that we can decide that the ODEs do not have rational general solutions using the necessary conditions presented in Section 4 without computation.

In the second experiment, we generate first order autonomous ODEs having rational solutions based on Theorem 3.8 and then compute their rational solutions. Table 2 shows the computing times of the program for six examples. All of these examples have rational general solutions. Times are collected on a PC with a 2.66G CPU and 256M memory and are given in seconds. We can see that the algorithm is generally very fast. In the table, "deg" means $\deg(F_i, y_1)$, "tdeg" means $\mathrm{tdeg}(F_i)$, "term" means the number of terms in F_i, "sol" means whether F_i has rational general solutions or not. The differential equations $F_i = 0$ are given in the Appendix of this paper.

From Table 1, we can see that the new algorithm gives an immediate negative answer to almost all randomly generated ODEs. This is due to the structural properties in Theorem 4.5. From Table 2, we can see that the average running time of the new algorithm for first order autonomous ODEs with rational solutions is about thirty times faster than that of the old version in Feng and Gao (2004).

5. Conclusion

In this paper, we give a necessary and sufficient condition for an ODE to have a rational general solution and a polynomial algorithm for computing the rational general solution of a first order autonomous ODE if it exists.

As mentioned in Section 1, this work is motivated by the parametrization of differential algebraic varieties, which is still wide open. A problem of particular interest is finding conditions for a differential curve $f(y, z) = 0$ to have rational differential parametrizations. Developing effective algorithms for computing rational solutions for ODEs of the form $y' = R(x, y)$ is also very interesting. We may further ask whether we can define a differential genus for a differential curve similar to the genus of algebraic curves.

Appendix

$$\begin{aligned}
F_1 = {} & -870199 + 48y_1^6y + y_1^7 + 256y_1^3y^6 + 3336568y^5 - 924496y^6 + 339557y^2y_1^3 \\
& - 55752y^4y_1^2 - 18527499y^4 + 140154y_1^2y^3 + 38016y^7 + 3660594y \\
& + 457074y_1^2 - 16729917y^2 - 1033424yy_1^2 + 231921y_1^3 - 70101y^2y_1^2 \\
& - 405468y_1^3y + 30410226y^3 + 76914y_1^4 - 1536y_1^2y^6 - 1408y_1^3y^5 + 768y_1^4y^4 \\
& + 32y_1^5y^3 - 70744y_1^3y^3 + 7584y_1^2y^5 + 22512y_1^3y^4 - 6912y^8 + 27109y^2y_1^4 \\
& + 14238y_1^5 - 60660yy_1^4 - 2046y_1^5y - 3904y_1^4y^3 + 504y_1^5y^2 - 10y_1^6 \\
F_2 = {} & -15953673 + 1370992y_1^4y^3 + 24525130y_1^3y^2 - 4760y_1^6y^3 \\
& - 5477446y_1^3y^6 - 403850y_1^4y^5 + 1418136y_1^3y^4 + 17298y_1^6y - 520y_1^6y^4 \\
& + 6037404y_1^3y^5 - 3394159y_1^4y^2 - 4496y_1^6y^2 + 2231856y_1^4y - 830455y_1y^{14} \\
& + 6366870y_1y^{13} - 18943599y_1y^{12} + 28262808y_1y^{11} - 35708488y_1y^{10} \\
& + 100967244y_1y^9 - 261613864y_1y^8 + 376852194y_1y^7 - 299932766y_1y^6 \\
& + 97568944y_1y^5 + 129489688y_1y^4 - 270400620y_1y^3 - 94155264y_1y \\
& + 226173416y_1y^2 + 621180y_1^3y^9 + 4528440y_1^3y^7 + 29376y_1^5y^3 + 616965y_1^4y^4 \\
& - 9600y_1^3y^10 + 12435y_1^4y^8 + 15953673y_1 - 12105024y_1^3y + 42670992yy_1^2 \\
& + 94155264y - 7159296y_1^2 - 226173416y^2 + 2138496y_1^3 + 270400620y^3 \\
& - 469248y_1^4 - 42659760y_1^2y^4 - 1200y_1^5y^6 - 60353y_1^5y^4 - 255900y_1^5y \\
& - 5248y_1^4y^6 - 17080y_1^4y^7 - 19404672y_1^3y^3 + 230800y_1^5y^2 + 109228644y_1^2y^3 \\
& + 59506976y_1^2y^8 - 19721790y_1^2y^9 - 89299784y_1^2y^7 + 2500965y_1^2y^{10} \\
& - 30252096y_1^2y^5 - 100447788y_1^2y^2 + 76098138y_1^2y^6 - 4450y_1^5y^5 - 200y_1^7y^2 \\
& - 480y_1^7y - 2688614y_1^3y^8 - 129489688y^4 - 97568944y^5 + 299932766y^6 \\
& - 376852194y^7 + 261613864y^8 - 100967244y^9 + 35708488y^{10} - 28262808y^{11} \\
& + 18943599y^{12} - 6366870y^{13} + 830455y^{14} + 70356y_1^5 - 7212y_1^6 \\
& + 462y_1^7 - 15y_1^8 \\
F_3 = {} & -54186088y_1^4y^3 + 4560724y_1^3y^2 + 867258y_1^5y^8 - 4026120y_1^5y^7 - 13y_1^8y^2 \\
& + 48y_1^8y + 5773194y_1^4y^{10} - 41319768y_1^4y^9 - 598230y_1^6y^3 + 1041256552y_1^3y^6
\end{aligned}$$

$$
\begin{aligned}
&+ 34639164y_1^3y^12 - 308944440y_1^3y^11 + 390714661y^{18} + 11690852213y^{16}\\
&- 3323199538y^{17} - 22370573820y^{15} - 222955974y_1^4y^5 + 109689272y_1^3y^4\\
&+ 3y_1^9 - 19614840322y_1y^{11} + 9482373824y_1y^{10} - 2423208852y_1y^9\\
&+ 65755075y_1y^8 + 29215306y_1y^7 + 124774832y_1y^6 - 380708216y_1y^5\\
&+ 600288010y_1y^4 - 565689600y_1y^3 + 343380212y_1y^2 - 124028856y_1y\\
&- 276960y_1^6y + 517539y_1^6y^4 + 390714661y_1y^{16} + 11690852213y_1y^{14}\\
&- 3323199538y_1y^{15} - 22370573820y_1y^{13} + 26055647226y_1y^{12}\\
&- 398929188y_1^3y^5 + 18827534y_1^4y^2 - 332024y_1^6y^5 + 85148y_1^6y^6\\
&+ 4917702402y_1^2y^{12} - 9351799012y_1^2y^{11} + 560215y_1^6y^2 - 15016y_1^7y^3\\
&+ 3116y_1^7y^4 - 6130320y_1^4y - 2109932188y_1^3y^9 - 1869195748y_1^3y^7\\
&- 13184686y_1^5y^3 + 126787014y_1^4y^4 + 1115205672y_1^3y^{10} + 143581542y_1^4y^8\\
&+ 15953673y_1 - 1522920y_1^3y + 15953673y^2 + 2160900y_1^3 - 124028856y^3\\
&+ 1940400y_1^4 + 2620324y_1^2y^4 + 10541784y_1^5y^6 + 20050378y_1^5y^4\\
&- 2415240y_1^5y + 300166232y_1^4y^6 - 271532970y_1^4y^7 - 25416440y_1^3y^3\\
&+ 6570588y_1^5y^2 + 158081009y_1^2y^{14} - 1385906760y_1^2y^{13} - 1522920y_1^2y^3\\
&+ 4316853258y_1^2y^8 - 8366016396y_1^2y^9 - 1342989716y_1^2y^7 + 10870365853y_1^2y^{10}\\
&- 19286120y_1^2y^5 + 2160900y_1^2y^2 + 203148999y_1^2y^6 - 18777470y_1^5y^5\\
&+ 21758y_1^7y^2 - 11322y_1^7y + 2408378252y_1^3y^8 + 343380212y^4 - 565689600y^5\\
&+ 600288010y^6 - 380708216y^7 + 124774832y^8 + 29215306y^9 + 65755075y^{10}\\
&- 2423208852y^{11} + 9482373824y^{12} - 19614840322y^{13} + 26055647226y^{14}\\
&+ 611550y_1^5 + 79200y_1^6 + 4500y_1^7 + 111y_1^8
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & 455957330y_1^4y^3 - 811745184y_1^3y^2 - 2914249105y_1^2 - 565440y_1^5y^8\\
&+ 744320y_1^5y^7 - 26340y_1^8y^2 + 12400y_1^8y + 6074064y_1^4y^9 + 1550476y_1^6y^3\\
&+ 29930752y_1^3y^6 + 15745024y^{15} + 55395922y_1^4y^5 - 3217657856y_1^3y^4\\
&+ 4150552y_1^6y - 686934y_1^6y^4 + 1567668240y_1^3y^5 - 390197131y_1^4y^2\\
&+ 638368y_1^6y^5 - 315904y_1^6y^6 - 3406497y_1^6y^2 + 128192y_1^7y^3 + 29544y_1^7y^4\\
&+ 284881446y_1^4y + 8335459190y_1^2y + 15959296y_1^3y^9 - 259116544y_1^3y^7\\
&+ 33229166y_1^5y^3 - 1696y_1^9y - 1552y_1^9y^2 - 219083953y_1^4y^4 - 216y_1^{10} - 4024y_1^9\\
&+ 3433884y_1^4y^8 + 634095480y_1^3y + 36576y_1^6y^7 + 23808y_1^6y^8 + 100865672100y\\
&- 353800922580y^2 - 926372697y_1^3 + 765414525680y^3 - 290699351y_1^4\\
&+ 32544y_1^7y^5 + 3968y_1^7y^6 - 30370466816y_1^2y^4 + 4912y_1^8y^3 - 3280y_1^8y^4\\
&+ 3840672y_1^5y^6 - 18125700y_1^5y^4 + 20656460y_1^5y + 40303141y_1^4y^6\\
&- 29987684y_1^4y^7 + 2837793532y_1^3y^3 - 17897913y_1^5y^2 + 37830481110y_1^2y^3\\
&+ 1786117568y_1^2y^8 - 361845376y_1^2y^9 - 2332658560y_1^2y^7 - 115417088y_1^2y^{10}\\
&+ 11337914328y_1^2y^5 - 22117225058y_1^2y^2 - 635847984y_1^2y^6 - 1343000y_1^5y^5
\end{aligned}
$$

$$
\begin{aligned}
&- 159008y_1^7y^2 + 121480y_1^7y - 498432y_1^3y^8 - 1104141752960y^4\\
&+ 1095737343616y^5 - 850150659264y^6 + 541847855360y^7 - 225768623872y^8\\
&+ 28569503488y^9 + 18122338304y^{10} - 6476303360y^{11} - 1621618688y^{12}\\
&+ 601063424y^{13} + 190464000y^{14} - 49898171y_1^5 - 7716083y_1^6 - 781167y_1^7\\
&- 73066y_1^8 - 123008y_1^5y^9 - 9612579511 - 915456y_1^4y^{10} + 12094464y_1^2y^{11}\\
&- 559488y_1^4y^1 1 - 61504y_1^4y^{12} + 4920320y_1^2y^{12}\\
F_5 \ =\ & 19260688272y_1^4y^2 - 512282880y^{14} + 157743936y^{15} - 189610662y_1^5y^3\\
&+ 13538806068y_1^4y^4 + 18580853003062y^9 + 284436821213y_1^2y^6\\
&- 33075926861y^2y_1^3 + 1893549346980y^{11} + 729262257148y^3y_1^2\\
&+ 100188088729596y^5 - 59320y_1^9y - 107966118993369y^6 - 258793418880y^{12}\\
&+ 15348y_1^9 - 2275y_1^{10} - 286861352567 - 36514800y_1^2y^{12} + 21535450y_1^6y\\
&+ 2332913328590y - 11881809417y_1^3 - 1367916650y_1^4 - 63441670746876y^4\\
&- 36653364y_1^5 - 2219012y_1^6 - 26428836087y_1^2 - 542710538691y^2y_1^2\\
&+ 37995589044yy_1^3 - 15337494172y^3y_1^3 - 474869441275y^4y_1^2\\
&- 3996857908yy_1^4 - 955091004yy_1^5 - 381116y_1^7 + 68355993176y^5y_1^2\\
&+ 22626669575y^4y_1^3 + 1316530323y_1^5y^2 - 22862044694y^3y_1^4 + 8226914y_1^7y\\
&+ 177124y_1^8 + 80118214755842y^7 - 207855967y_1^6y^2 - 12708135y_1^6y^6\\
&+ 568852y_1^8y^2 + 53594y_1^9y^2 + 543y_1^{10}y^2 + 8238816y_1^7y^3 + 27052562y_1^6y^5\\
&- 1132879392y_1^4y^7 + 14076333y_1^5y^6 + 15747321y_1^4y^8 + 24272298y_1^5y^7\\
&- 208174y_1^8y^3 + 2085724y_1^7y^5 - 4630014y_1^7y^4 - 12054y_1^9y^3 - 15518591y^2y_1^7\\
&+ 187244142y^3y_1^6 - 447966452y^5y_1^5 - 22181722906y^7y_1^3 + 5769616550y^6y_1^4\\
&+ 4224197920y^8y_1^3 - 52667091y^4y_1^6 - 6460554711y^{10}y_1^2 - 472608y^6y_1^7\\
&+ 6999766404y^9y_1^2 - 316645476y^9y_1^3 + 17568y_1^8y^5 - 180y_1^{10}y - 76634y_1^8y^4\\
&+ 2543028y_1^6y^7 + 126y_1^9y^4 - 43504981131485y^8 + 27766350y^9y_1^4\\
&+ 774095616y^{11}y_1^2 + 139891704y^{10}y_1^3 - 5009592y^8y_1^5 - 1152y_1^8y^6\\
&+ 501014756y^4y_1^5 - 313659411866y_1^2y^7 + 10019013120y^{13}\\
&- 9704409417819y^2 + 188254886364y_1^2y + 28616409083310y^3\\
&+ 76035036070y^8y_1^2 + 54096y_1^7y^7 - 159394y^8y_1^6 - 22395744y^{11}y_1^3\\
&- 3042009y^{10}y_1^4 - 6555317414763y^{10} - 12036060664y^5y_1^4\\
&+ 49884303363y^6y_1^3 + 365148y^{11}y_1^4 - 23602681204y_1^3y^5 + 10980y^9y_1^6\\
&- 466380y^9y_1^5 - 3381y^{10}y_1^6 + 81144y^{10}y_1^5 - 177y_1^{11} - 487640y_1^8y\\
F_6 \ :=\ & 866940720y_1^8y^3 - 37322752y_1^7y^6 + 880902144y_1^7y^5 + 3366912y_1^6y^7\\
&+ 129892117469650944y^5 - 671088640y_1^3y^{11} - 41943040y_1^5y^7\\
&- 425564133y_1^8y^4 + 50331648y_1^4y^{11} - 6273432y_1^{10} - 6867973480118y^2y_1^4\\
&- 263171173137776640y^6 + 169439554301952y^4y_1^2 - 64902184314y_1^6
\end{aligned}
$$

$$
\begin{aligned}
&-51363377657856y^3y_1^2+2889198780672yy_1^4-290564505792y_1^5\\
&-1494528y_1^9y^2-934293634632y_1^4-39011227336704y_1^3y^4\\
&+384112613400772608y^7-4093640704y_1^4y^{10}-111654416y_1^9\\
&-215927732568y^2y_1^6+340041240494080y^6y_1^2-13801483146240y^4y_1^4\\
&-1297293192y_1^8-10683609840y_1^7+96476135280yy_1^6\\
&-312909562363904y^5y_1^2+9595662312960y^3y_1^4+29346222440448y_1^3y^3\\
&-1382798575136y_1^5y^2-5494272y_1^8y^6+499273980y_1^8y+1123465052160y_1^5y^3\\
&+28196455219200y_1^3y^5+8739700320y_1^7y+647856664752y_1^5y\\
&-397012558990540800y^8+118707191808y_1^4y^9-211474120704y^4y_1^5\\
&-18759607736y^2y_1^7+9025165721600y^6y_1^3+279845295410380800y^9\\
&-185421640237056y_1^2y^7+205533344924y_1^6y^3+18483974350848y_1^4y^5\\
&+23367840y_1^9y-787047512y_1^8y^2-1941650143928y_1^3-3762391351296y^{14}\\
&+31227280827088896y^{11}-3240505580191744y^{12}+159592394784768y^{13}\\
&-1816307035224y_1^2-14346470039552y_1^4y^6-21180672704512y_1^3y^7\\
&-18584759771136y_1^3y^2+7418594944464y_1^3y+9170577260544y_1^3y^8\\
&-46200177624152064y^4-125436481203339264y^{10}-8630788777645\\
&-2403y_1^{12}+200599155618048y-1971424141125120y^2\\
&+11626423971499008y^3+6683437056y_1^6y^5-70884903936y_1^6y^4\\
&-1066997383168y_1^3y^9-7807868928y_1^7y^4+19134438224y_1^7y^3\\
&+2147483648y_1^2y^{11}-233197568y_1^6y^6-1428412170240y_1^4y^8\\
&+46103789568y_1^3y^{10}-2048y_1^8y^8+1354956800y_1^5y^6+6367216001024y_1^4y^7\\
&-134882y_1^{10}y^2-804454400y_1^5y^5+26624y_1^9y^3-21978152960y_1^2y^{10}\\
&-2206304567296y_1^2y^9+41159862779904y_1^2y^8+254675874816y_1^2y^2\\
&+5445572592812y_1^2y+75706624y_1^8y^5+175104y_1^8y^7-128y_1^{10}y^4\\
&-1568y^2y_1^{11}+34888y_1^{11}y+5920y_1^{10}y^3-192863y_1^{11}+1508016y_1^{10}y\\
&+34359738368y^{15}+557056y_1^7y^7
\end{aligned}
$$

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