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Rational solutions of ordinary difference equations[☆]

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ABSTRACT

In this paper, we generalize the results of Feng and Gao [Feng, R., Gao, X.S., 2006. A polynomial time algorithm to find rational general solutions of first order autonomous ODEs. J. Symbolic Comput., 41(7), 735–762] to the case of difference equations. We construct two classes of ordinary difference equations (OΔEs) whose solutions are exactly the univariate polynomial and rational functions respectively. On the basis of these OΔEs and the difference characteristic set method, we give a criterion for an OΔE with any order and nonconstant coefficients to have a rational type general solution. For the first-order autonomous (constant coefficient) OΔE, we give a polynomial time algorithm for finding the polynomial solutions and an algorithm for finding the rational solutions for a given degree.

1. Introduction

Difference equations are a very important class of equations for describing special functions and number sequences. Finding closed form solutions for difference equations is one of the main research topics for difference equations. Most existing results on symbolic solutions are limited to the linear case. Algorithms for computing the rational or hypergeometric solutions for linear OΔEs have been proposed in Abramov (1989, 1995, 1998), Abramov et al. (1998, 1995), Böing and Koepf (1999), Gosper (1978), Koepf (1995), Paule (1995), Paule and Riese (1997), Paule and Schorn (1995), Petkovšek (1992) and van Hoeij (1998, 1999). In Karr (1981, 1985), Karr introduced $\Pi\Sigma$-fields and used them to find closed form formulas for finite sums or disproving the existence of such formulas. In Bronstein (2000), Bronstein extended the notion of monomial extensions of differential fields to difference fields and described an algorithm for finding the solutions of parameterized linear difference equations in a

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Table 1
Running time for OΔEs with form (7)

	tdeg	Term	Time (s)	Solution		tdeg	Term	Time(s)	Solution
F_1	11	78	0.375	Yes	G_1	8	30	1.875	No
F_2	12	91	1.093	Yes	G_2	9	36	5.655	No
F_3	13	105	3.204	Yes	G_3	10	40	13.702	No
F_4	14	120	5.047	Yes	G_4	11	44	22.297	No
F_5	15	136	9.094	Yes	G_5	12	56	55.329	No
F_6	16	153	13.985	Yes	G_6	13	58	70.641	No
F_7	17	171	23.282	Yes	G_7	14	69	156.031	No
F_8	18	190	40.875	Yes	G_8	15	76	297.282	No

subclass of monomial extensions. In Kauers and Schneider (2006), Kauers and Schneider extended Karr's algorithm to a general summation framework and presented a new algorithm for indefinite nested summation. In Schneider (2005), the author developed the algorithms to find all solutions of parameterized linear difference equations within $\Pi\Sigma$-field. Hendriks and Singer gave a procedure for determining the Liouvillian solutions of linear OΔEs (Hendriks and Singer, 1999). Wolfram gave a formula for the general solution of a linear OΔE with constant coefficients (Wolfram, 2000). On the other hand, *difference algebra* founded by Cohn provides a general algebraic setting in which to study the structure of the solutions of OΔEs (Cohn, 1965).

In this paper, rational solutions of non-linear OΔEs are considered. We consider rational general solutions of the OΔEs with any order and nonconstant coefficients. For two non-negative integers n and m, we construct two classes of OΔEs $\mathcal{P}_n$ and $\mathcal{R}_{n,m}$ in one variable y with rational numbers as coefficients. $\mathcal{P}_n$ is a linear equation whose solutions are exactly the univariate polynomial functions with degree less than or equal to n. $\mathcal{R}_{n,m}$ is a nonlinear equation whose solutions are exactly the univariate rational functions with degree (n, m). Here, a rational function with degree (n, m) means that its numerator is of degree $\le n$ and its denominator is of degree $\le m$. As a consequence, we give a *difference equation description* for univariate polynomial and rational functions. On the basis of $\mathcal{R}_{n,m}$, we give an equivalent condition for an OΔE to have a rational type general solution.

For a first-order autonomous OΔE $F(y, y_1) = 0$, where y_1 is the shift of y, we give a polynomial time algorithm for finding its polynomial solutions. This is mainly due to a detailed analysis for the structural shape of the OΔE which has polynomial solutions. We try to generalize the method of finding polynomial solutions to the method of finding rational solutions. It is unfortunate that the difference version of Theorem 3.7 in Feng and Gao (2006) is not always true (see Example 4.1), which means that we cannot bound the degree of rational solutions through the parametrization and the algorithm based on parametrization does not work here. However, for a given degree (n, n), or for short n, we can give a polynomial time algorithm for deciding whether $F(y, y_1) = 0$ has a rational solution with degree not greater than n and find one if it has. This algorithm is based on the algorithm for finding a Laurent series solution of $F(y, y_1) = 0$ and Padé approximation.

The above results can be considered as a difference analogue of the results in Feng and Gao (2006). But, the techniques used here are different from those for the differential case due to the difference between differential and difference operators. One of the major differences between the difference case and the differential case is that the degree of the rational solutions is not always equal to the degree of $F(y, y_1) = 0$ with respect to the variable y. Another major difference is in the algorithm for finding a rational function solution of a first-order autonomous OΔE $F(y, y_1) = 0$. In the differential case, it is relatively easy to find power series solutions of $F = 0$. But such a method cannot be extended to the difference case. Instead, we give an algorithm for computing Laurent series solutions in $\frac{1}{x}$ of $F = 0$ and use it for finding the rational solutions of $F = 0$.

The algorithm for finding polynomial solutions is implemented in Maple. The algorithm is very efficient in that our program can find the solutions of OΔEs with high degrees and hundreds of terms (see Table 1).

The paper is organized as follows. In Section 2, we define $\mathcal{P}_n$ and $\mathcal{R}_{n,m}$ and give an equivalent condition for an OΔE to have a rational general solution. In Section 3, we give a polynomial time algorithm for finding polynomial solutions of first-order autonomous OΔEs. In Section 4, for the first-order autonomous O$\triangle$E, we give an algorithm for finding its Laurent series solution and an algorithm for finding its possible rational solution with given degree based on Padé approximation.

2. Rational general solutions of O$\triangle$E

2.1. Some concepts of difference algebra

In this subsection, we will introduce some concepts of difference algebra, which can be found in Cohn (1965, Chapters 2, 4, and 6). A field with an automorphism σ is called a *difference field* and σ is called a *difference operator*. The difference field considered in this paper is the rational function field $\mathbf{Q}(x)$ in the variable x with the difference operator $\sigma(x) = x + 1$. If a difference field $\mathcal{F}$ with difference operator $\bar{\sigma}$ satisfies $\mathcal{F} \supseteq \mathbf{Q}(x)$ and $\bar{\sigma}|_{\mathbf{Q}(x)} = \sigma$, then $\mathcal{F}$ is called the *difference extension field* of $\mathbf{Q}(x)$. Let y be an indeterminate over $\mathbf{Q}(x)$. We use y_i to denote the ith transformation $\sigma^i(y)$ of y. Assume that $y, y_1, y_2, \ldots$ are algebraically independent over $\mathbf{Q}(x)$. Denote $\mathbf{Q}(x)[y, y_1, y_2, \ldots]$ by $\mathbf{Q}(x)\{y\}$. We call elements in $\mathbf{Q}(x)\{y\}$ *difference polynomials*. Then $\mathbf{Q}(x)\{y\}$ is a difference polynomial ring (Cohn, 1965, p.64). Let $F \in \mathbf{Q}(x)\{y\} \setminus \mathbf{Q}(x)$. The *order* of F is the largest k such that y_k appears in F, denoted by $\mathrm{ord}(F)$. Let $\mathrm{ord}(F)$ be o. We can also regard F as an algebraic polynomial in $y, y_1, \ldots, y_o$ with coefficients in $\mathbf{Q}(x)$. Then $\deg(F, y_i)$ denotes the degree of F with respect to y_i. When we say that F is an irreducible difference polynomial, we mean that F is irreducible over $\bar{\mathbf{Q}}(x)$ as an algebraic polynomial where $\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$.

Let $F \in \mathbf{Q}(x)\{y\}$ and $o = \mathrm{ord}(F)$. Write F as a polynomial in y_o:

$$F := I_n y_o^n + I_{n-1} y_o^{n-1} + \cdots + I_0$$

where $I_i \in \mathbf{Q}(x)[y, y_1, \ldots, y_{o-1}]$. Then I_n is called the *initial* of F, $\frac{\partial F}{\partial y_o}$ is called the *separant* of F. Let Σ be an ideal of algebraic polynomial ring $\mathbf{Q}(x)[y, y_1, y_2, \cdots]$. Σ is called a *difference ideal* if $P \in \Sigma$ implies that $\sigma(P) \in \Sigma$. A difference ideal Σ is *reflexive* if $\sigma(P) \in \Sigma$ implies that $P \in \Sigma$. A difference ideal Σ is *prime* if $PQ \in \Sigma$ implies that $P \in \Sigma$ or $Q \in \Sigma$. A difference ideal Σ is *perfect* if

$$(\sigma^{k_1}(P)\sigma^{k_2}(P) \cdots \sigma^{k_n}(P))^m \in \Sigma \Rightarrow P \in \Sigma$$

where $\sigma^i(P)$ is the ith shift of P and the k_i are nonnegative integers (Cohn, 1965, p. 76). We will use $\{F\}$ to denote the perfect difference ideal generated by F. A perfect difference ideal $\{F\}$ can be decomposed into the intersection of the reflexive prime difference ideals, which are called *irreducible components* of $F = 0$ (Cohn, 1965, p. 88, Theorem 4). An irreducible component Λ of $F = 0$ is *principal* if the dimension of Λ (in the sense of algebraic ideal) equals $\mathrm{ord}(F) - k$ where k is the smallest integer i such that y_i appears in F (Cohn, 1965, p. 161).

Let $F \in \mathbf{Q}(x)\{y\} \setminus \mathbf{Q}(x)$ be an irreducible difference polynomial and

$$\Sigma_F = \{A \in \mathbf{Q}(x)\{y\} | SA \equiv 0 \bmod \{F\}\} \tag{1}$$

where S is the separant of F. Cohn proved that Σ_F is a perfect difference ideal and it can be decomposed into the intersection of the principal components of F (Cohn, 1965, p. 192). Now we give a simple example to explain the above definitions.

Example 2.1. Let $F = (y_1 - y)^2 - 2(y_1 + y) + 1$. Then $\mathrm{ord}(F) = 1$, the initial of F is 1 and the separant of F is $2y_1 - 2y - 2$. Moreover

$$\Sigma_F = \{F\} = \underbrace{\{F,\ y_2 - 2y_1 + y - 2\}}_{\text{prime difference ideal}} \bigcap \underbrace{\{F,\ y_2 - y\}}_{\text{prime difference ideal}}.$$

Both of these prime difference ideals are the principal components of $F = 0$.

Let $\mathcal{U}$ be the complete system of difference extension fields of $\mathbf{Q}(x)$, which is the collection of difference extension fields of $\mathbf{Q}(x)$ (Cohn, 1965, p. 238). For simplification, when we say that an element w is in $\mathcal{U}$, we mean that w belongs to some difference fields in $\mathcal{U}$. The elements in $\mathcal{U}$ which annul each element in a difference ideal are called *zeros* of this difference ideal. A *generic zero* $\eta(x)$ of a prime difference ideal Λ is a zero of Λ such that for any $Q \in \mathbf{Q}(x)\{y\}$, $Q(\eta(x)) = 0 \Leftrightarrow Q \in \Lambda$ where $Q(\eta(x))$ is defined by replacing any occurrence of y_i by $\eta(x+i)$. The complete system has the property that two distinct perfect ideals in $\mathbf{Q}(x)\{y\}$ have at least one different zero in $\mathcal{U}$ (Cohn, 1965, p. 238). An element in $\mathcal{U}$ which is invariant under the difference operators is called a *constant*.

Definition 2.2. Let $F \in \mathbf{Q}(x)\{y\} \setminus \mathbf{Q}(x)$ be an irreducible difference polynomial. A general solution of $F = 0$ is defined as a generic zero of one of the principal components of Σ_F. A rational general solution of $F(y) = 0$ is defined as a general solution of $F = 0$ with the following form:

$$\hat{y}(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{2}$$

where the a_i, b_j are constants. In particular, if $m = 0$, we call $\hat{y}(x)$ a polynomial general solution.

Example 2.3. The difference equation $(y - y_1)^2 - 2(y + y_1) + 1 = 0$ has two general solutions: $y(x) = (x + c)^2$ and $y(x) = (c\mathrm{e}^{\mathrm{i}\pi x} + \frac{1}{2})^2$ where c is an arbitrary constant.

2.2. Difference equation description for univariate rational functions

In this subsection, we will construct a class of difference equations whose solutions are rational functions. Moreover, each rational function is a solution of some difference equation in this class. This result is significant, because it gives a difference equation description for univariate rational functions. First let us look at the polynomial case.

Let $\mathcal{P} = y - y_1$. Then

$$\mathcal{P}_n = \underbrace{\mathcal{P} \circ \cdots \circ \mathcal{P}}_{n+1} = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} y_i$$

is a linear difference polynomial with order $n + 1$, where $\circ$ means the composition of two difference polynomials. Then we get

Theorem 2.4. *For each $n \geq 0$ and $y(x) \in \mathcal{U}$,*

$$y(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \Leftrightarrow \mathcal{P}_n(y(x)) = 0$$

where the a_i are constants.

Proof. We will first apply induction on n to prove that $\mathcal{P}_n(x^k) = 0$ for $0 \leq k \leq n$. When $n = 0$, it is obvious that $\mathcal{P}_0(x^0) = \mathcal{P}(1) = 0$. Now assume that it is true for $n < N$. We have that $\mathcal{P}(x^l) = \sum_{i=0}^{l-1} \binom{l}{i} x^i$ for any integer l. For $l \leq N$,

$$\mathcal{P}_N(x^l) = \mathcal{P}_{N-1} \circ \mathcal{P}(x^l) = \sum_{i=0}^{l-1} \binom{l}{i} \mathcal{P}_{N-1}(x^i) = 0.$$

This implies that $1, x, x^2, \ldots, x^n$ form a basis of the solution space of $\mathcal{P}_n(y) = 0$. Hence the lemma holds. $\square$

We can generalize the above result to the rational functions. We start with a lemma. Let $M(x, k)$ be the following matrices:

$$M(x, k) := \begin{pmatrix} 1 & x+k & \ldots & (x+k)^m \\ 1 & x+k+1 & \ldots & (x+k+1)^m \\ \vdots & \vdots & \ldots & \vdots \\ 1 & x+k+m & \ldots & (x+k+m)^m \end{pmatrix}.$$

It is easy to see that $M(x, k)$ is always invertible for every $x \in \mathbf{Q},\ k \in \mathbf{Z}$. Moreover, we have the following lemma:

Lemma 2.5. *$M(x,k)M^{-1}(x,0) \in \mathbf{Q}^{(m+1)\times(m+1)}$ for each $k \in \mathbf{Z}$, where $M^{-1}(x,0)$ is the inverse of the matrix $M(x,0)$.*

Proof. Assume that $M^{-1}(x,0) = (c_{i,j}(x))_{(m+1)\times(m+1)}$. By $M(x,0)M^{-1}(x,0) = I_{m+1}$, we have that

$$\sum_{i=0}^{m}(x+l)^i c_{i,j}(x) = \delta_{l,j} \quad \text{for } l, j = 0, \ldots, m \tag{3}$$

where $\delta_{l,j}$ is the Kronecker δ function. Suppose that

$$M(x,k)M^{-1}(x,0) = (d_{l,j}(x,k))_{(m+1)\times(m+1)}.$$

That is,

$$d_{l,j}(x,k) = \sum_{i=0}^{m}(x+k+l)^i c_{i,j}(x).$$

Regarding $d_{l,j}(x,k)$ as polynomials in k, then the degree of $d_{l,j}(x,k)$ is not greater than m. From (3), we get

$$d_{l,j}(x, j-l) = 1 \quad \text{and} \quad d_{l,j}(x, s-l) = 0 \quad \text{for } s = 0, 1, \ldots, j-1, j+1, \ldots, m$$

which implies that

$$d_{l,j}(x,k) = \frac{\prod_{i=0, i\neq j}^{m}(k-i+l)}{\prod_{i=0, i\neq j}^{m}(j-i+l)}.$$

Hence the lemma holds. □

Since $M(x,k) = M(x+i, k-i)$ for all integers i, the above lemma implies that $M(x,k)M^{-1}(x,l) \in \mathbf{Q}^{(m+1)\times(m+1)}$ for every $k, l \in \mathbf{Z}$.

In the following, we describe the idea of how we construct difference equations which have the property that all rational functions are their solutions. Let

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + 1}$$

be a generic rational function with degree (n,m) where the a_i, b_j are arbitrary constants. By Theorem 2.4, we have that

$$\sigma^k(\mathcal{P}_n(r(x)(b_m x^m + \cdots + 1))) = 0, \quad k = 0, 1, \ldots, m.$$

Since the b_j are arbitrary constants and $\mathcal{P}_n$ are linear, we get

$$b_m\sigma^k(\mathcal{P}_n(r(x)x^m)) + b_{m-1}\sigma^k(\mathcal{P}_n(r(x)x^{m-1})) + \cdots + \sigma^k(\mathcal{P}_n(r(x))) = 0 \tag{4}$$

where $k = 0, 1, \ldots, m$. Because the coefficients of $\mathcal{P}_n$ are constants, $\sigma(\mathcal{P}_n) = \mathcal{P}_n(\sigma)$. Rewrite (4) in matrix form:

$$A_{n,m}(r(x))\begin{pmatrix}1\\ b_1\\ \vdots\\ b_m\end{pmatrix} = \begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix} \tag{5}$$

where

$$A_{n,m}(y) = \begin{pmatrix} \mathcal{P}_n(y) & \mathcal{P}_n(yx) & \ldots & \mathcal{P}_n(yx^m) \\ \mathcal{P}_n(y_1) & \mathcal{P}_n(y_1(x+1)) & \ldots & \mathcal{P}_n(y_1(x+1)^m) \\ \vdots & \vdots & \ldots & \vdots \\ \mathcal{P}_n(y_m) & \mathcal{P}_n(y_m(x+m)) & \ldots & \mathcal{P}_n(y_m(x+m)^m) \end{pmatrix}.$$

From (5), we have that $|A_{n,m}(r(x))| = 0$ where $|\ |$ is used to denote the determinant of a matrix. Hence $|A_{n,m}(y)|$ satisfies the property that each rational function is a solution of $|A_{k,l}(y)| = 0$ for some nonnegative integers k, l. In general, the variable x appears in the coefficients of $|A_{n,m}(y)| = 0$. In the following, from $|A_{n,m}(y)|$, we construct new difference polynomials $\mathcal{R}_{n,m}$ whose coefficients are independent of x. We rewrite $A_{n,m}(y)$ in the form

$$A_{n,m}(y) = \mathcal{P}_n\left(\begin{pmatrix} y & yx & \dots & yx^m \\ y_1 & y_1(x+1) & \dots & y_1(x+1)^m \\ \vdots & \vdots & \dots & \vdots \\ y_m & y_m(x+m) & \dots & y_m(x+m)^m \end{pmatrix}\right) = \mathcal{P}_n(Y_0 M(x,0))$$

where $Y_i = \mathrm{diag}(y_i, y_{i+1} \dots, y_{i+m})$ are the diagonal matrices. Therefore

$$\begin{aligned} A_{n,m}(y) &= \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(x,i) \\ &= \left(\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(x,i)M^{-1}(x,0)\right)M(x,0). \end{aligned}$$

By Lemma 2.5 and the remark below, $|\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(x,i)M^{-1}(x,0)|$ is independent of x. This implies that

$$\left|\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(x,i)M^{-1}(x,0)\right| = \left|\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(0,i)M^{-1}(0,0)\right|.$$

Now we set

$$\mathcal{R}_{n,m} = \left|\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Y_i M(0,i)\right|.$$

Remark 2.6. In fact, $|A_{n,m}(y)|$ is the Casoratian determinant of $\mathcal{P}_n(y), \mathcal{P}_n(xy), \dots, \mathcal{P}_n(x^m y)$ (Cohn, 1965, p. 271). It is similar to the Wronskian determinant in the differential case.

When $m = 0$, $\mathcal{R}_{n,0} = \mathcal{P}_n$. By induction on m, we know that $\mathcal{R}_{n,m}$ is an ordinary difference polynomial with order $n+m+1$.

Theorem 2.7. *For every $n \geq 0$, $m \geq 0$ and $y(x) \in \mathcal{U}$,*

$$y(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} \Leftrightarrow \mathcal{R}_{n,m}(y(x)) = 0$$

where the a_i, b_j are constants.

Proof. Since $|M(x,0)| \neq 0$ for any $x \in \mathbf{Q}$, we have that

$$\mathcal{R}_{n,m}(y(x)) = 0 \Leftrightarrow |A_{n,m}(y(x))| = 0.$$

Now by Cohn (1965, p. 271, Lemma 2), for some function $y(x)$, $|A_{n,m}(y(x))| = 0$ if and only if there exist $b_m, b_{m-1}, \dots, b_0$ which are constants and not all 0 such that $\sum_{j=0}^m b_j \mathcal{P}_n(x^j y(x)) = 0$. Moreover, we have that

$$\sum_{j=0}^m b_j \mathcal{P}_n(x^j y(x)) = 0 \Leftrightarrow \mathcal{P}_n\left(\left(\sum_{j=0}^m b_j x^j\right)y(x)\right) = 0 \Leftrightarrow \left(\sum_{j=0}^m b_j x^j\right)y(x) = \sum_{i=0}^n a_i x^i$$

where the a_i are constants. The last step is due to Theorem 2.4. Hence the theorem holds. □

2.3. A criterion for an O$\triangle$E to have rational general solutions

On the basis of the property of $\mathcal{R}_{n,m}$, we give an equivalent condition for an O$\triangle$E to have a rational general solution.

Theorem 2.8. *Let $F = 0$ be an irreducible polynomial in $\mathbf{Q}(x)\{y\}$. Then $F = 0$ has a rational general solution iff there exist n and m such that $\mathcal{R}_{n,m}$ belongs to one of the components of Σ_F where Σ_F is defined by (1).*

Proof ($\Rightarrow$). Suppose that $F = 0$ has a rational general solution $\hat{y}(x)$ of the form (2), which is a generic zero of Λ where Λ is one of the components of Σ_F. Then $\mathcal{R}_{n,m}(\hat{y}(x)) = 0$ for some non-negative integers n, m. By the definition of the generic zero, we have that $\mathcal{R}_{n,m} \in \Lambda$.

($\Leftarrow$) Assume that $\mathcal{R}_{n,m}$ belongs to one of the components Λ of Σ_F. Let $\hat{y}(x)$ be a generic zero of Λ. Then $\mathcal{R}_{n,m}(\hat{y}(x)) = 0$. Hence $\hat{y}(x)$ has the form (2) by Theorem 2.7. So $F = 0$ has a rational general solution. $\square$

Corollary 2.9. *Let $F = 0$ be an irreducible O$\triangle$E in $\mathbf{Q}(x)\{y\}$. Then $F = 0$ has a polynomial general solution iff there exists an n such that $\mathcal{P}_n$ belongs to one of the components of Σ_F where Σ_F is defined by (1).*

Remark 2.10. Given n and m, the property of whether $\mathcal{R}_{n,m}$ belongs to one of the components of Σ_F can be decided by the difference characteristic set method (Gao and Luo, 2004).

Now we give an example to explain the above results.

Example 2.11. Consider the difference polynomial: $F = (y_1 - y)^2 - 2(y_1 + y) + 1$. From Example 2.1, $F = 0$ has two principal components:

$$\{F, y_2 - 2y_1 + y - 2\} \quad \text{and} \quad \{F, y_2 - y\}.$$

Since $\mathcal{P}_3 = y_3 - 3y_2 + 3y_1 - y = \sigma(y_2 - 2y_1 + y - 2) - (y_2 - 2y_1 + y - 2)$, $\mathcal{P}_3 \in \{F,\ y_2 - 2y_1 + y - 2\}$ which implies that $F = 0$ has a polynomial general solution with degree 2 on the first component. In fact, $y(x) = (x + c)^2$ is its polynomial general solution.

In order to use the equivalent condition given in Theorem 2.8, we need to know the degree bound n and m of the possible rational general solutions of $F = 0$. In general, we do not know this bound.

Example 2.12. Consider $F = x(y_1 - y) - (n + 1)y$. Then $y(x) = cx(x + 1)\cdots(x + n)$ is a polynomial general solution of $F = 0$ where c is an arbitrary constant. Here we cannot bound the degree of $y(x)$ from the degree of F.

3. A polynomial time algorithm for the polynomial solutions

In this section, F will always be an absolutely irreducible first-order autonomous difference polynomial with coefficients in $\mathbf{Q}$, i.e. $F \in \mathbf{Q}[y, y_1]$ and F is irreducible over $\bar{\mathbf{Q}}$ where $\mathbf{Q}$ is the rational number field and $\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$. We will describe a method for computing polynomial solutions of a first-order autonomous difference equation $F(y, y_1) = 0$. Here an element in $\bar{\mathbf{Q}}$ will not be considered as a polynomial solution.

We will need some basic facts on the parametrization of a plane algebraic curve. Let $F(x, y)$ be a polynomial in $\mathbf{Q}[x, y]$ which is irreducible over $\bar{\mathbf{Q}}[x, y]$.

Definition 3.1. Assume that $r(x) = r_1(x)/r_2(x) \in \bar{\mathbf{Q}}(x)$ where $r_1(x), r_2(x) \in \bar{\mathbf{Q}}[x]$ and $\gcd(r_1(x), r_2(x)) = 1$. Then $\deg(r(x))$ is defined as $\max\{\deg(r_1(x)), \deg(r_2(x))\}$. Assume that

$$P = a_0 x^{i_0} y^{j_0} + \cdots + a_n x^{i_n} y^{j_n} \in \bar{\mathbf{Q}}[x, y]$$

where $a_k \neq 0$ and $(i_k, j_k) \neq (i_l, j_l)$ if $k \neq l$. Then the total degree of P with respect to x, y is defined as $\max\{i_k + j_k | 0 \le k \le n\}$, denoted by $\operatorname{tdeg}(P)$.

Definition 3.2. $(r(t), s(t))$ is called a *parametrization* of $F(x, y) = 0$ if $F(r(t), s(t)) = 0$ where $r(t), s(t) \in \bar{\mathbf{Q}}(t)$ and not all of them are in $\bar{\mathbf{Q}}$. A parametrization $(r(t), s(t))$ is called proper if $\bar{\mathbf{Q}}(r(t), s(t)) = \bar{\mathbf{Q}}(t)$.

Lüroth's Theorem guarantees that there always exists a proper parametrization if the parametrization exists (Walker, 1950, p. 151, Theorem 7.3). A proper parametrization has the following properties (Sendra and Winkler, 2001):

Proposition 3.3. *Let $(r(t), s(t))$ be a proper parametrization of $F(x, y) = 0$. Then*

(1) $\deg(r(t)) = \deg(F, y)$, $\deg(s(t)) = \deg(F, x)$.
(2) *Assume that* $r(t) = r_1(t)/r_2(t)$ *and* $s(t) = s_1(t)/s_2(t)$ *where* $r_i(t), s_i(t) \in \bar{\mathbf{Q}}[t]$. *Then* $\mathrm{res}(r_2(t)x - r_1(t), s_2(t)y - s_1(t), t) = \lambda F$ *where* res *is the Sylvester resultant and* $\lambda \in \bar{\mathbf{Q}} \setminus \{0\}$.

The following is a key lemma for determining the degree of the polynomial solutions of $F = 0$.

Lemma 3.4. *Let $p(x) \in \bar{\mathbf{Q}}[x] \setminus \bar{\mathbf{Q}}$. Then $\bar{\mathbf{Q}}(p(x), p(x+1)) = \bar{\mathbf{Q}}(x)$.*

Proof. By Lüroth's Theorem, there exists $g(x) \in \bar{\mathbf{Q}}(x)$ such that

$$\bar{\mathbf{Q}}(p(x), p(x+1)) = \bar{\mathbf{Q}}(g(x)).$$

By Schinzel (1982, p. 10, Theorem 4), $g(x)$ can be chosen to be in $\bar{\mathbf{Q}}[x]$. Hence there are $h_1(x), h_2(x) \in \bar{\mathbf{Q}}[x]$ such that $p(x) = h_1(g(x))$ and $p(x+1) = h_2(g(x))$. It is clear that $\deg(h_1(x)) = \deg(h_2(x))$. Assume that

$$h_1(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots, \qquad h_2(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots$$

and $g(x) = x^m + \cdots + c_0$ with $m > 1$. Then we have

$$\begin{aligned} p(x+1) &= h_1(g(x+1)) = a_n x^{nm} + n(m + c_{m-1})a_n x^{nm-1} + r_1(x) \\ &= h_2(g(x)) = b_n x^{nm} + nc_{m-1}b_n x^{nm-1} + r_2(x) \end{aligned}$$

where $\deg(r_1(x)) < nm - 1$ and $\deg(r_2(x)) < nm - 1$, which implies that $a_n = b_n$ and $n = 0$, a contradiction. Therefore $m = 1$. The lemma holds. □

Since F is first order and autonomous, we can view $F(y, y_1) = 0$ as a plane algebraic curve. The above lemma means that $y(x) = p(x), y_1(x) = p(x+1)$ is a proper parametrization of $F(y, y_1) = 0$. Hence by Proposition 3.3, we have the following theorem.

Theorem 3.5. *If $y(x) = p(x)$ is a polynomial solution of $F(y) = 0$, then*

$$\deg(p(x)) = \deg(F, y) = \deg(F, y_1).$$

This theorem gives the exact degree of polynomial solutions and we can compute the polynomial solutions by solving algebraic equations. However, the difference equations with polynomial solutions have special structure, which provides a more efficient method.

Definition 3.6. Let P be a nonzero monomial (term) in $\mathbf{Q}[x, t, y, y_1, a_1, \ldots, a_n]$ where $t, a_1, \ldots, a_n$ are independent indeterminates. A weight of P is defined as

$$w(P) = \deg(P, x) + \deg(P, t) + n(\deg(P, y) + \deg(P, y_1)) + \sum_{i=1}^{n}(n - i)\deg(P, a_i).$$

A polynomial $G \in \mathbf{Q}[x, t, y, y_1, a_1, \ldots, a_n]$ is said to be *isobaric* if all monomials in G have the same weight.

Theorem 3.7. *Let $F = 0$ be an irreducible first-order autonomous OΔE in $\mathbf{Q}[y, y_1]$ and $\bar{y}(x) = a_n x^n + \cdots + a_0$ be its polynomial solution with $\deg(\bar{y}(x)) = n > 0$. Then F must have the following form:*

$$F = a(y - y_1)^n + G(y, y_1) \tag{6}$$

where a is a nonzero element in $\mathbf{Q}$, $G \in \mathbf{Q}[y, y_1]$ and $\mathrm{tdeg}(G) \le n - 1$.

Proof. Consider the following resultant:

$$R(y, y_1, t) = \mathrm{res}(a_n x^n + \cdots + a_0 - y, a_n(x+t)^n + \cdots + a_0 - y_1, x).$$

Here we regard a_i and t as indeterminates. By Definition 3.6, $a_nx^n+\cdots+a_0-y$ and $a_n(x+t)^n+\cdots+a_0-y_1$ are isobaric polynomials with the weight n. We know that $R(y, y_1, 0) = a_n^n(y-y_1)^n$. Hence

$$R(y, y_1, t) = a_n^n(y-y_1)^n + tG(y, y_1, t)$$

for some $G \in \mathbf{Q}[y, y_1, t]$. Because $R(y, y_1, t)$ is an isobaric polynomial with the weight n^2, the total degrees of the terms in $G(y, y_1, t)$ with respect to y, y_1 are less than n. By Proposition 3.3 , we have that $F = \lambda R(y, y_1, 1)$. Hence F has the form (6). $\square$

Assume that $\bar{y}(x)$ is a polynomial solution of $F = 0$ and $\deg(\bar{y}(x)) = n > 0$. Since $F = 0$ is autonomous, $\bar{y}(x+c)$ is still a solution of $F = 0$ for any $c \in \mathbf{Q}$. Hence if $\bar{y}(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial solution of $F = 0$, we can always let $a_{n-1} = 0$ by a translation. The following theorems show that we can compute the coefficients of $\bar{y}(x)$ only by rational operations over $\mathbf{Q}$. Now assume that F has the following form:

$$F = \lambda(y-y_1)^n + \sum_{i=0}^{n-1} b_i y^i y_1^{n-1-i} + H(y, y_1) \tag{7}$$

where λ is a nonzero element in $\mathbf{Q}$, $b_i \in \mathbf{Q}$ and $\mathrm{tdeg}(H) \le n-2$.

Theorem 3.8. *Let $F = 0$ be of the form (7) and $\bar{y}(x) = a_nx^n + \cdots + a_0$ be its polynomial solution with $\deg(\bar{y}(x)) = n > 0$. Then we have that $\sum_{i=0}^{n-1} b_i \neq 0$ and $a_n = \frac{\sum_{i=0}^{n-1} b_i}{(-1)^{n+1}n^n\lambda}$.*

Proof. Substitute $\bar{y}(x)$ into $F = 0$. Then in $F(\bar{y}(x), \bar{y}(x+1))$, the highest possible degree of x equals $n(n-1)$. It is easy to compute the coefficient of $x^{n(n-1)}$ in $F(\bar{y}(x), \bar{y}(x+1))$ which equals $(-1)^n n^n \lambda a_n^n + \sum_{i=0}^{n-1} b_i a_n^{n-1}$. Because $F(\bar{y}(x), \bar{y}(x+1)) \equiv 0$ and $a_n \neq 0$, we have that $\sum_{i=0}^{n-1} b_i \neq 0$ and $a_n = \frac{\sum_{i=0}^{n-1} b_i}{(-1)^{n+1}n^n\lambda}$. $\square$

Theorem 3.9. *Assume that $F = 0$ has the form (7). Let $z(x) = \beta x^n + a_{n-2}x^{n-2} + \cdots + a_0$ where $\beta = \frac{\sum_{i=0}^{n-1} b_i}{(-1)^{n+1}n^n\lambda}$ and the a_i are indeterminates. Then for $i = n-2, \ldots, 0$, the coefficients C_i of $x^{(n-1)^2+i-1}$ in $F(z(x), z(x+1))$ are*

$$\left(\beta^{n-2}(n-1-i)\sum_{j=0}^{n-1} b_j\right) a_i + h_i(a_{n-2}, \ldots, a_{i+1}) \tag{8}$$

where h_i are polynomials in $a_{n-2}, \ldots, a_{i+1}$.

Proof. By Definition 3.6, $z(x)$ is an isobaric polynomial with the weight n and the highest weight of the terms in $z(x+1)$ equals n. Hence the weight of every term in $z(x)^i z(x+1)^j$ is not greater than $n(i+j)$. Since the highest weight of the terms in $z(x) - z(x+1)$ equals $n-1$, the highest weight of the terms in $F(z(x), z(x+1))$ equals $n(n-1)$. So if $k < i$, a_k cannot appear in C_i and if a_i appears in C_i then it can only appear linearly. It is not difficult to see that the coefficient of $x^{(n-1)^2+i-1}$ in $\lambda(z(x) - z(x+1))^n$ which includes a_i is $\lambda(-1)^n n^n i\beta^{n-1} a_i$ and the coefficient in $\sum_{j=0}^{n-1} b_j z(x)^j z(x+1)^{n-1-j}$ which includes a_i is $(n-1)\sum_{j=0}^{n-1} b_j \beta^{n-2} a_i$. Because the highest weight of the terms in $H(z(x), z(x+1))$ is not greater than $n(n-2)$, a_i cannot appear in the coefficient of $x^{(n-1)^2+i-1}$ in $H(z(x), z(x+1))$. Since $\beta = \frac{\sum_{i=0}^{n-1} b_i}{(-1)^{n+1}n^n\lambda}$,

$$\left(\lambda(-1)^n n^n i\beta^{n-1} + (n-1)\sum_{j=0}^{n-1} b_j\beta^{n-2}\right) a_i = \left(\beta^{n-2}(n-1-i)\sum_{j=0}^{n-1} b_j\right) a_i.$$

Hence C_i has the form (8). $\square$

Table 2
Timings for randomly generated first-order autonomous ODEs

$\deg(F, y_1)$	10	11	12	13	14	15
$\mathrm{tdeg}(F)$	10	11	12	13	14	15
Average time	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
Polynomial solution	No	No	No	No	No	No

In Theorem 3.9, since $h_i(a_{n-2}, \ldots, a_{i+1})$ is independent of $a_i, \ldots, a_0$, we can compute $h_i(a_{n-2}, \ldots, a_{i+1})$ as follows: if one substitutes $u(x) = a_n x^n + \cdots + a_{i+1}x^{i+1}$ into F then $h_i(a_{n-2}, \ldots, a_{i+1})$ is the coefficient of $x^{(n-1)^2+i-1}$ in $F(u(x), u(x+1))$. Now we can give the algorithm for finding a polynomial solution.

Theorems 3.7–3.9 provide an almost explicit solution for the polynomial solution of a first-order autonomous difference equation.

Algorithm 3.10. Input: An absolutely irreducible first-order autonomous OΔE $F = 0$.
Output: All polynomial solutions of $F = 0$ if there is one; otherwise, a message "NULL" if $F = 0$ has no polynomial solution.

(1) Rewrite F in the form (7). If F cannot be rewritten in the form (7) or in the form (7) $\sum_{i=0}^{n-1} b_i = 0$, then by Theorems 3.7 and 3.8 $F = 0$ has no polynomial solutions. Then return (NULL) and the algorithm terminates.
(2) Let $n = \deg(F, y) > 0$. Let $\bar{a}_n = \frac{\sum_{i=0}^{n-1} b_i}{(-1)^{n+1} n^n \lambda}$, $\bar{a}_{n-1} = 0$ and $z(x) = \bar{a}_n x^n$.
(3) For i from $n-2$ to 0 do
  (a) Substitute $z(x)$ to F and let $C = \mathrm{coeff}(F(z(x), z(x+1)), x, (n-1)^2 + i - 1)$.
  (b) Let $\bar{a}_i = -\frac{C}{\bar{a}_n^{n-2}(n-1-i)\sum_{j=0}^{n-1} b_j}$.
  (c) Let $z(x) = z(x) + \bar{a}_i x^i$.
(4) If $F(z(x), z(x+1)) = 0$, then $z(x)$ is a polynomial solution of $F = 0$. Then return $(z(x))$ and the algorithm terminates. Otherwise, $F = 0$ has no polynomial solutions, then return (NULL) and the algorithm terminates.

The complexity of the above algorithm is polynomial in the number of the multiplications (or divisions) over $\mathbf{Q}$. The dominating step for the complexity comes from the Step (3) and Step (4). At these steps, we need only to substitute a polynomial in x with degree n to a polynomial in y, y_1 with total degree n. It is not difficult to show that the total number of the multiplications and divisions in these steps is polynomial in n.

We implemented the above algorithm in Maple. Tables 1 and 2 show the experiment results. In Table 1, all difference polynomials F_i, G_i have the form (7). Here the coefficients of F_i, G_i are integers. The column of "tdeg" means the total degrees of the difference polynomials; "term" means the total numbers of the monomials; "solution" means whether the difference equations have polynomial solutions. The running time is in seconds and is collected on a computer with Pentium 4, 2.99 GHz CPU and 760M memory.

From Table 2, we can see that for randomly given OΔEs, our program can identify almost immediately that such equations have no polynomial solutions. This is because the OΔEs with polynomial solutions have very special structure as shown by Theorems 3.7 and 3.8. From Table 1, we can see that for OΔEs with the "correct" form (7), our program solves very large OΔEs. Also, the most difficult case is that the given OΔE has the "correct" form (7) but does not have a polynomial solution, which is shown on the right part of Table 1.

4. Rational solutions for first-order autonomous OΔEs

The idea introduced in Feng and Gao (2006) for finding rational solutions for $F = 0$ is first finding a Laurent series solution in $x = 0$ and from it constructing a Padé approximation to this Laurent series solution. If the degree of the Padé approximation is high enough, it will be the rational solution of $F = 0$. But Laurent series solutions in $x = 0$ for difference equations are meaningless ($\bar{\mathbf{Q}}((x))$ is not a difference field under the difference operation $\sigma(x) = x + 1$). Instead, we give an algorithm for computing Laurent series solutions in $\frac{1}{x}$ and use it to find the rational solutions of $F = 0$.

As in Section 3, F will always be an absolutely irreducible first-order autonomous difference polynomial with coefficients in $\mathbf{Q}$, i.e. $F \in \mathbf{Q}[y, y_1]$ and an element in $\bar{\mathbf{Q}}$ will not be considered as a rational solution. The idea for computing rational solutions with a given degree includes two ingredients: (1) find a formal series solution of $F = 0$ which is a series expansion of a rational solution at $x = \infty$; (2) recover the rational solution by Padé approximation. First we give an example to show that Lemma 3.4 cannot be generalized to the rational case.

Example 4.1. Let $g_1(x) = \frac{(x-2)^2}{x-\frac{3}{2}}$, $g_2(x) = \frac{x^2-2}{x+\frac{3}{2}}$ and $h(x) = x + \frac{1}{2x}$. Then we have that

$$g_1(h(x+1)) = \frac{(2x^2-1)^2}{2(2x+1)(x+1)x} = g_2(h(x)).$$

Hence $\bar{\mathbf{Q}}(g_1(h(x)), g_1(h(x+1))) = \bar{\mathbf{Q}}(h(x)) \neq \bar{\mathbf{Q}}(x)$.

Lemma 4.2. *If $F = 0$ has a rational solution, then* $\deg(F, y) = \deg(F, y_1)$.

Proof. Assume that $r(x)$ is a rational solution of $F = 0$. by Lüroth's Theorem, there exists $h(x) \in \bar{\mathbf{Q}}(x)$ such that

$$r(x) = r_1(h(x)), \quad r(x+1) = r_2(h(x)) \quad \text{and} \quad \bar{\mathbf{Q}}(r_1(x), r_2(x)) = \bar{\mathbf{Q}}(x).$$

Then by Definition 3.2, $(r_1(x), r_2(x))$ is a proper parametrization of $F = 0$. By Proposition 3.3,

$$\deg(F, y) = \deg(r_2(x)), \quad \deg(F, y_1) = \deg(r_1(x)). \tag{9}$$

By Binder (1996, 1.2. Proposition), we have that

$$\deg(r(x)) = \deg(r_1(x))\deg(h(x)) = \deg(r(x+1)) = \deg(r_2(x))\deg(h(x)). \tag{10}$$

Then (9) and (10) imply the conclusion. □

4.1. An algorithm for finding Laurent series solutions

Since each element in $\bar{\mathbf{Q}}(x)$ can be expanded as Laurent series at $x = \infty$, we can view the difference field $\bar{\mathbf{Q}}(x)$ as a subfield of the difference field $\bar{\mathbf{Q}}((\frac{1}{x}))$ where $\bar{\mathbf{Q}}((\frac{1}{x}))$ is the field of the Laurent series in $\frac{1}{x}$. Hence, it is helpful to find a solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$. In this subsection, we will describe an algorithm for computing Laurent series solutions in $\frac{1}{x}$ of $F = 0$.

Lemma 4.3. *Let $P(x)/Q(x)$ be a rational solution of an OΔE $F = 0$. If $\deg(P(x))$ is not less than $\deg(Q(x))$, we can obtain a new OΔE $\bar{F} = 0$ which has a rational solution $\bar{P}(x)/\bar{Q}(x)$ with $\deg(\bar{P}(x)) < \deg(\bar{Q}(x))$. Moreover,*

(1) *if $\deg(P(x)) > \deg(Q(x))$, $P(x)/Q(x) = \bar{Q}(x)/\bar{P}(x)$;*
(2) *if $\deg(P(x)) = \deg(Q(x))$, $P(x)/Q(x) = \bar{P}(x)/\bar{Q}(x) + \alpha$ where α is one of the nonzero solutions of $F(y, y) = 0$ in $\bar{\mathbf{Q}}$.*

Proof. If $\deg(P(x)) = \deg(Q(x))$, let $\bar{P}(x) = P(x) - (p/q)Q(x)$ and $\bar{Q}(x) = Q(x)$ where p, q are the leading coefficients of $P(x)$ and $Q(x)$ respectively. Then $\deg(\bar{P}(x))$ is less than $\deg(\bar{Q}(x))$ and $\bar{P}(x)/\bar{Q}(x)$ is a rational solution of $\bar{F} = F(y+p/q, y_1+p/q) = 0$. Note that in this case p/q is a solution of $F(y, y) = 0$. If $\deg(P(x)) > \deg(Q(x))$, let $\bar{F} = y^n y_1^n F(\frac{1}{y}, \frac{1}{y_1})$ where $n = \deg(F, y)$. Then $\bar{F}$ is an irreducible polynomial in y, y_1 by Lemma 4.2 and $Q(x)/P(x)$ is a solution of $\bar{F} = 0$. The second statement is obvious. □

Remark 4.4. The new OΔE $\bar{F} = 0$ in Lemma 4.3 may be not in $\mathbf{Q}[y, y_1]$. However, from the proof of Lemma 4.3, there exists $\alpha \in \bar{\mathbf{Q}}$ with $[\mathbf{Q}(\alpha) : \mathbf{Q}] \leq \text{tdeg}(F)$ such that $\bar{F} \in \mathbf{Q}(\alpha)[y, y_1]$.

Remark 4.5. By Lemma 4.3, we need only to consider the case that $F = 0$ has a rational solution $P(x)/Q(x)$ in $\bar{\mathbf{Q}}(x)$ with $\deg(P(x)) < \deg(Q(x))$. In this case, $F = 0$ has a Laurent series solution $y(x) = c_k \frac{1}{x^k} + c_{k+1} \frac{1}{x^{k+1}} + \cdots$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$ with $k > 0$. Moreover, if $c = -\frac{c_{k+1}}{k}$, we have that $y(x - c) = c_k \frac{1}{x^k} + \bar{c}_{k+2} \frac{1}{x^{k+2}} + \cdots$ and $y(x - c)$ is still a solution of $F = 0$ because F is autonomous.

Lemma 4.6. *Let $\bar{y}(x) = c_k \frac{1}{x^k} + c_{k+2} \frac{1}{x^{k+2}} + \cdots$ be a Laurent series solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$ where $k > 0$ and $c_k \neq 0$. Then regarding y_1 as an algebraic function in y defined by $F = 0$, there is a Puiseux series expansion of y_1 at $y = 0$ which has the form*

$$y_1 = y + a_1 y^{\frac{k+1}{k}} + a_2 y^{\frac{k+2}{k}} + \cdots \tag{11}$$

where $a_1 \neq 0$ and $a_2 = \frac{k^2+k}{2k^2} a_1^2$.

Proof. Let $\bar{\mathbf{Q}}((\frac{1}{x}))$ be the field consisting of the Laurent series in $\frac{1}{x}$. Then $\bar{y}(x) \in \bar{\mathbf{Q}}((\frac{1}{x}))$ and since

$$\bar{y}(x+1) = c_k \frac{1}{x^k} - kc_k \frac{1}{x^{k+1}} + \left(\frac{k^2+k}{2} c_k + c_{k+2}\right) \frac{1}{x^{k+2}} + \cdots,$$

$\bar{y}(x+1)$ is also in $\bar{\mathbf{Q}}((\frac{1}{x}))$. Let $\varphi(x) = x(b_0 + b_1 \frac{1}{x} + b_2 \frac{1}{x^2} + \cdots)$ where the b_i are indeterminates. Then

$$\frac{1}{\varphi(x)} = \frac{1}{b_0 x} - \frac{b_1}{b_0^2 x^2} + \left(\frac{b_1^2}{b_0^3} - \frac{b_2}{b_0^2}\right) \frac{1}{x^3} + \cdots$$

and

$$\begin{aligned}
\bar{y}(\varphi(x)) &= c_k \left(\frac{1}{b_0 x} - \frac{b_1}{b_0^2 x^2} + \cdots\right)^k + c_{k+2} \left(\frac{1}{b_0 x} - \frac{b_1}{b_0^2 x^2} + \cdots\right)^{k+2} + \cdots \\
&= \frac{c_k}{(b_0 x)^k} - \frac{kc_k b_1}{(b_0 x)^{k+1}} + \left(kc_k b_1^2 - kc_k b_0 b_2 + c_{k+2}\right) \frac{1}{(b_0 x)^{k+2}} + \cdots \\
&\quad + (-kc_k b_0 b_i + H_i(c_k, \ldots, c_{k+i}, b_0, \ldots, b_{i-1})) \frac{1}{(b_0 x)^{k+i}} + \cdots
\end{aligned}$$

where H_i is a polynomial in its arguments. Let $b_0 = c_k^{\frac{1}{k}}$, $b_1 = 0$, $b_2 = \frac{c_{k+2}}{k} c_k^{-\frac{k+1}{k}}$ and $b_i = \frac{H_i}{k} c_k^{-\frac{k+1}{k}}$ for $i = 3, 4, \ldots$. Then we have that

$$\varphi(x) = x \left(c_k^{\frac{1}{k}} + \frac{c_{k+2} c_k^{-\frac{k+1}{k}}}{k} \frac{1}{x^2} + \cdots\right)$$

and

$$\begin{aligned}
\bar{y}(\varphi(x)) &= \frac{1}{x^k}, \\
\bar{y}(\varphi(x)+1) &= \frac{1}{x^k} - kc_k^{-\frac{1}{k}} \frac{1}{x^{k+1}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} \frac{1}{x^{k+2}} + \cdots \\
&= \bar{y}(\varphi(x)) - kc_k^{-\frac{1}{k}} \bar{y}(\varphi(x))^{\frac{k+1}{k}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} \bar{y}(\varphi(x))^{\frac{k+2}{k}} + \cdots.
\end{aligned}$$

By Theorem 1.5 on page 92 of Walker (1950), the map $\varphi(x) : g(x) \to g(\varphi(x))$ is an automorphism of $\bar{\mathbf{Q}}((\frac{1}{x}))$. Applying this automorphism to $F(\bar{y}(x), \bar{y}(x+1)) = 0$ yields $F(\bar{y}(\varphi(x)), \bar{y}(\varphi(x)+1)) = 0$. Hence

$$F\left(\bar{y}(\varphi(x)), \bar{y}(\varphi(x)) - kc_k^{-\frac{1}{k}} \bar{y}(\varphi(x))^{\frac{k+1}{k}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} \bar{y}(\varphi(x))^{\frac{k+2}{k}} + \cdots\right) = 0. \tag{12}$$

Applying the inverse automorphism of $\varphi(x)$ to (12), we have that

$$F\left(\bar{y}(x), \bar{y}(x) - kc_k^{-\frac{1}{k}} \bar{y}(x)^{\frac{k+1}{k}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} \bar{y}(x)^{\frac{k+2}{k}} + \cdots\right) = 0. \tag{13}$$

Since $\bar{y}(x)$ is transcendental over $\bar{\mathbf{Q}}$, so is $\bar{y}(x)^{\frac{1}{k}}$. Then the map $\bar{y}(x)^{\frac{1}{k}} \to y^{\frac{1}{k}}$ gives rise to an isomorphism between $\bar{\mathbf{Q}}((\bar{y}(x)^{\frac{1}{k}}))$ and $\bar{\mathbf{Q}}((y^{\frac{1}{k}}))$ where y is an indeterminate. Applying this isomorphism to (13) yields

$$F\left(y, y - kc_k^{-\frac{1}{k}} y^{\frac{k+1}{k}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} y^{\frac{k+2}{k}} + \cdots\right) = 0.$$

Hence

$$y_1 = y - kc_k^{-\frac{1}{k}} y^{\frac{k+1}{k}} + \frac{k^2+k}{2} c_k^{-\frac{2}{k}} y^{\frac{k+2}{k}} + \cdots$$

is a Puiseux series expansion of the algebraic function defined by $F(y, y_1) = 0$ at $y = 0$. Let $a_1 = -kc_k^{-\frac{1}{k}}$ and $a_2 = \frac{k^2+k}{2} c_k^{-\frac{2}{k}}$. Then $a_2 = \frac{k^2+k}{2k^2} a_1^2$. The lemma holds. $\square$

From the proof of Lemma 4.6, we know how to construct a Puiseux series expansion of $F = 0$ at $y = 0$ from a solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$. Conversely, we can find a solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$ from a Puiseux series which has the form (11). We describe this process as the following algorithm.

Algorithm 4.7. Input: The first $N+1$ terms of a Puiseux series expansion of $F = 0$, which has the form (11).
Output: The first N terms of a solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$, which must be of the following form: $b_k \frac{1}{x^k} + b_{k+2} \frac{1}{x^{k+2}} + \cdots + b_{k+N-1} \frac{1}{x^{k+N-1}}$.

(1) Let $c_1 = -\frac{k}{a_1}$.
(2) Let $c_2 = 0$.
(3) Let $i = 2$ and $\phi_2(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^2}$.
(4) while $i < N$ do
  (a) Let r_{i+1} be the coefficient of $\frac{1}{x^{k+i+1}}$ of the polynomial
  $$P_i(x) = \phi_i(x)^k + a_1 \phi_i(x)^{k+1} + \cdots + a_{i+1} \phi_i(x)^{k+i+1} - \phi_i(x+1)^k$$
  and $c_{i+1} = -\frac{r_{i+1}}{(i-1)kc_1^{k-1}}$.
  (b) Let $\phi_{i+1}(x) = \phi_i(x) + c_{i+1} \frac{1}{x^{i+1}}$.
  (c) Let $i = i + 1$.
(5) return $(\phi_N(x)^k \bmod \frac{1}{x^{k+N}})$.

Theorem 4.8. *Algorithm 4.7 is correct.*

Proof. Let c_i, a_i, $\phi_i(x)$ and $P_i(x)$ be as those in Algorithm 4.7. Let

$$\phi(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^2} + c_3 \frac{1}{x^3} + \cdots.$$

We need only to prove that

$$F(\phi(x)^k, \phi(x+1)^k) = 0.$$

Since $F(y, y + a_1 y^{\frac{k+1}{k}} + \cdots) = 0$ and y is transcendental over $\bar{\mathbf{Q}}$, applying the isomorphism $y^{\frac{1}{k}} \to y$ between $\bar{\mathbf{Q}}((y^{\frac{1}{k}}))$ and $\bar{\mathbf{Q}}((y))$ and the isomorphism $y \to \phi(x)$ between $\bar{\mathbf{Q}}((y))$ and $\bar{\mathbf{Q}}((\phi(x)))$,

$$\begin{aligned} F(y, y + a_1 y^{\frac{k+1}{k}} + \cdots) = 0 &\Rightarrow F(y^k, y^k + a_1 y^{k+1} + \cdots) = 0 \\ &\Rightarrow F(\phi(x)^k, \phi(x)^k + a_1 \phi(x)^{k+1} + \cdots) = 0. \end{aligned}$$

Hence we need only to prove

$$\phi(x)^k + \sum_{j=1}^{\infty} a_j \phi(x)^{k+j} - \phi(x+1)^k = 0.$$

For this, we need only to show

$$P_i(x) \equiv 0 \bmod \frac{1}{x^{k+i+1}}, \quad i = 1, 2, \ldots. \tag{14}$$

We will prove (14) by induction. Because $c_1 = -k/a_1$, $c_2 = 0$ and $a_2 = \frac{k^2+k}{2k^2}a_1^2$,

$$\begin{aligned} P_2(x) &\equiv \frac{c_1^k}{x^k} + a_1\frac{c_1^{k+1}}{x^{k+1}} + a_2\frac{c_1^{k+2}}{x^{k+2}} - c_1^k\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)^k \\ &\equiv c_1^k(a_1c_1 + k)\frac{1}{x^{k+1}} + c_1^k\left(c_1^2a_2 - \frac{k(k+1)}{2}\right)\frac{1}{x^{k+2}} \equiv 0 \bmod \frac{1}{x^{k+3}}. \end{aligned}$$

Now we assume that (14) is true for $i < m$ where $m > 2$.

$$\begin{aligned} P_m(x) &\equiv \phi_m(x)^k + \sum_{j=1}^{m+1} a_j\phi_m(x)^{k+j} - \phi_m(x+1)^k \\ &\equiv \left(\phi_{m-1}(x) + c_m\frac{1}{x^m}\right)^k + a_1\left(\phi_{m-1}(x) + c_m\frac{1}{x^m}\right)^{k+1} + \sum_{j=2}^{m} a_j\phi_{m-1}(x)^{k+j} \\ &\quad - \left(\phi_{m-1}(x+1) + c_m\frac{1}{(x+1)^m}\right)^k \\ &\equiv P_{m-1}(x) + \frac{kc_m}{x^m}\phi_{m-1}(x)^{k-1} + \frac{(k+1)a_1c_m}{x^m}\phi_{m-1}(x)^k \\ &\quad - \phi_{m-1}(x+1)^{k-1}\frac{kc_m}{(x+1)^m} \\ &\equiv P_{m-1}(x) + \frac{kc_1^{k-1}c_m}{x^{k+m-1}} + \frac{(k+1)a_1c_1^kc_m}{x^{k+m}} \\ &\quad - kc_m\left(\frac{c_1^{k-1}}{x^{k-1}} - \frac{(k-1)c_1^{k-1}}{x^k}\right)\left(\frac{1}{x^m} - \frac{m}{x^{m+1}}\right) \\ &\equiv P_{m-1}(x) + \frac{a_1(k+1)c_1^kc_m}{x^{m+k}} + \frac{k(m+k-1)c_1^{k-1}c_m}{x^{k+m}} \\ &\equiv P_{m-1}(x) + \frac{k(m-2)c_1^{k-1}c_m}{x^{k+m}} \equiv 0 \bmod \frac{1}{x^{k+m+1}}. \end{aligned}$$

The last equality is true because $a_1 = -k/c_1$ and c_m equals $-\frac{r_m}{k(m-2)c_1^{k-1}}$ where r_m is the coefficient of $\frac{1}{x^{k+m}}$ in $P_{m-1}(x)$. Hence the theorem holds. $\square$

Lemma 4.9. *Let $y_1 = y + a_1y^{\frac{k+1}{k}} + a_2y^{\frac{k+2}{k}} + \cdots$ have the form (11). The Laurent series $\phi(x)$ satisfying*

(1) *$\phi(x)$ has the form $c_1\frac{1}{x} + c_3\frac{1}{x^3} + \cdots$;*
(2) *$\phi(x)^k + a_1\phi(x)^{k+1} + \cdots - \phi(x+1)^k = 0$ is unique.*

Proof. Substitute $c_1\frac{1}{x} + c_3\frac{1}{x^3} + \cdots$ into $y^k + a_1y^{k+1} + \cdots - (\sigma(y))^k = 0$. We have that $c_1 = -k/a_1$. By Algorithm 4.7, we know that c_{m+1} uniquely depends on $c_1, \ldots, c_m$ for $m > 1$. The lemma holds. $\square$

4.2. An algorithm based on Padé approximation

Padé approximation is a particular type of rational fraction approximation to the value of a function. It constructs the rational fraction from Taylor series expansion of the original function. Its definition is given below (George and Baker, 1975, p. 5).

Definition 4.10. For the formal power series $A(x) = \sum_0^\infty a_jx^j$ and two non-negative integers L and M, the (L, M) Padé approximation to $A(x)$ is the rational fraction $\frac{P_L(x)}{Q_M(x)}$ such that

$$A(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1})$$

where $P_L(x)$ is a polynomial with degree not greater than L and $Q_M(x)$ is a polynomial with degree not greater than M. Moreover, $P_L(x)$ and $Q_M(x)$ are relatively prime and $Q_M(0) = 1$.

Let $P_L(x) = \sum_0^L p_i x^i$ and $Q_M(x) = \sum_0^M q_i x^i$. We can compute $P_L(x)$ and $Q_M(x)$ by solving linear systems, but there are more efficient algorithms for computing Padé approximation (Beckermann et al., 1997; Brent et al., 1980; Cabay and Choi, 1986; Cabay and Labahn, 1992; von zur Gathen and Gerhard, 1999). We can see that in order to compute the (L, M) Padé approximation to $A(x)$, we need to know $a_0, \ldots, a_{L+M}$.

For Padé approximation, the following results will be used in this paper (George and Baker, 1975, Theorem 1.1 and Theorem 2.2).

Theorem 4.11 (*Frobenius and Padé*). *When it exists, the Padé approximation to any formal power series $A(x)$ is unique.*

Theorem 4.12 (*Padé*). *The function $f(x)$ is a rational function of the following form:*

$$f(x) = \frac{p_l x^l + p_{l-1}x^{l-1} + \cdots + p_0}{q_m x^m + q_{m-1}x^{m-1} + \cdots + 1}$$

iff the (L, M) Padé approximation to $f(x)$ equals $f(x)$ itself for all $L \geq l$ and $M \geq m$.

We first give an algorithm for finding a rational solution $P(x)/Q(x)$ of $F = 0$ satisfying $\deg(P(x)) < \deg(Q(x))$.

Algorithm 4.13. Input: An absolutely irreducible first-order autonomous OΔE $F = 0$ and a non-negative integer n.
Output: A rational solution $y(x) = P(x)/Q(x)$ of $F = 0$ satisfying $\deg(y(x)) \leq n$ and $\deg(P(x)) < \deg(Q(x))$ if there is one, or a message "NULL" if $F = 0$ has no such rational solutions.

(1) Compute the first $2n + 2$ terms of all the Puiseux series expansions of the algebraic function y_1 defined by $F(y, y_1) = 0$ at $(0, 0)$ which have the form (11) (see Cano (1993), Duval (1989) and Walsh (2000)); denote all these series by $p_1(y), \ldots, p_d(y)$.
(2) Let $i = 1$. While $i \leq d$ do
  (a) By Algorithm 4.7, compute the first $2n + 1$ terms $\varphi(x)$ of a solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$ from $p_i(y)$.
  (b) Let $\bar{\varphi}(x) = \varphi(\frac{1}{x})$. Compute the (n, n) Padé approximation to $\bar{\varphi}(x)$, denoted by $r(x)$.
  (c) Let $y(x) = r(\frac{1}{x})$.
  (d) If $F(y(x), y(x + 1)) \equiv 0$, then return $y(x)$ and terminate the algorithm.
  (e) Let $i = i + 1$.
(3) Return (NULL).

Theorem 4.14. *Algorithm 4.13 is correct.*

Proof. Assume that $F = 0$ has a rational solution $P(x)/Q(x)$ with $\deg(P(x)/Q(x)) \leq n$ and $\deg(P(x)) < \deg(Q(x))$. By Remark 4.5, $F = 0$ has a solution in $\bar{\mathbf{Q}}((\frac{1}{x}))$ which has the form: $y(x) = c_k \frac{1}{x^k} + c_{k+2}\frac{1}{x^{k+2}} + \cdots$ where $k > 0$ and $c_k \neq 0$. By Lemma 4.6, there exists a Puiseux series expansion of $F = 0$ which has the form (11). By Algorithm 4.7, we can find the first $2n + 1$ terms of a Laurent series solution from this Puiseux series expansion of $F = 0$. By Lemma 4.9, the Laurent series solution computed by Algorithm 4.7 should equal $y(x)$. By Theorem 4.12, if $y(x)$ is the Laurent series expansion of $P(x)/Q(x)$ at $x = \infty$, then the (n, n) Padé approximation to $y(\frac{1}{x})$ must be $P(\frac{1}{x})/Q(\frac{1}{x})$. Here we use the fact that $\deg(P(\frac{1}{x})/Q(\frac{1}{x})) = \deg(P(x)/Q(x))$. $\square$

The complexity of Algorithm 4.13 is polynomial in terms of the number of multiplications (or divisions) in $\mathbf{Q}$. From Walsh (2000), we know that the computation complexity of the first $2n + 2$ terms of a Puiseux series expansion of $F = 0$ is polynomial in n and $\mathrm{tdeg}(F)$. The coefficients of the Puiseux series are in $\mathbf{Q}(\alpha)$ where $[\mathbf{Q}(\alpha) : \mathbf{Q}]$ is a polynomial in $\mathrm{tdeg}(F)$. In Algorithm 4.7, we need to compute $(c_1 x + \cdots + c_i x^i)^{k+i+1}$ where $c_i \in \mathbf{Q}(\alpha)$ and $i \leq 2n + 1$, which can be computed polynomially in n and $\mathrm{tdeg}(F)$ (Feng and Gao, 2004). It is not difficult to check that all other steps are of polynomial complexity.

Example 4.15. Consider the difference equation

$$F = (-2y + 1 - 3y^2)y_1^2 - (2y^2 + 2y)y_1 + y^2 = 0.$$

We try to find a rational solution with degree not greater than 2.

(1) Compute the first six terms of the Puiseux series expansion of $F = 0$ at $y = 0$ which has the form (11):

$$p_1 = y + 2y^{\frac{3}{2}} + 3y^2 + 5y^{\frac{5}{2}} + 9y^3 + \frac{63}{4}y^{\frac{7}{2}}.$$

(2) From the above Puiseux series, we have $k = 2$. Compute the first five terms of the solution of $F = 0$ in $\bar{\mathbf{Q}}((\frac{1}{x}))$:

$$\varphi(x) = \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6}.$$

(3) Let $\bar{\varphi}(x) = \varphi(\frac{1}{x}) = x^2 + x^4 + x^6$. Compute a $(2, 2)$ Padé approximation $y(x)$ to $\bar{\varphi}(x)$. We have that $y(x) = \frac{x^2}{1-x^2}$.

(4) Let $y(x) = y(\frac{1}{x}) = \frac{1}{x^2-1}$. Since $F(\frac{1}{x^2-1}, \frac{1}{x^2+2x}) \equiv 0$, $y(x)$ is a rational solution of $F = 0$.

Now we can give the algorithm, which is clearly true by Lemma 4.3.

Algorithm 4.16. Input: An absolutely irreducible first-order autonomous O$\triangle$E $F = 0$ and a non-negative integer n.

Output: a rational solution $P(x)/Q(x)$ with $\deg(P(x)/Q(x)) \le n$ of $F = 0$ if there is one, or a message "NULL" if $F = 0$ has no such rational solutions.

(1) If $\deg(F, y) \ne \deg(F, y_1)$, then by Lemma 4.2, $F = 0$ has no rational solutions. Then return(NULL) and terminate the algorithm.

(2) Case 1 ($\deg(P(x)) < \deg(Q(x))$):
Find a rational solution $y(x)$ of $F = 0$ with Algorithm 4.13. If $y(x)$ exists, then return $y(x)$ and terminate the algorithm. Otherwise $F = 0$ has no rational solutions $P(x)/Q(x)$ with $\deg(P(x)) < \deg(Q(x))$ and $\deg(P(x)/Q(x)) \le n$.

(3) Case 2 ($\deg(P(x)) > \deg(Q(x))$):
Let $G = y^N y_1^N F(\frac{1}{y}, \frac{1}{y_1})$ where $N = \deg(F, y)$. Find a rational solution $y(x)$ of $G = 0$ with Algorithm 4.13. If $y(x)$ exists, then by Lemma 4.3, $\frac{1}{y(x)}$ is one of the solutions of $F = 0$. Return $\frac{1}{y(x)}$ and terminate the algorithm. Otherwise $F = 0$ has no rational solutions $P(x)/Q(x)$ with $\deg(P(x)) > \deg(Q(x))$ and $\deg(P(x)/Q(x)) \le n$.

(4) Case 3 ($\deg(P(x)) = \deg(Q(x))$):
Solve $F(y, y) = 0$. Assume that the nonzero roots of $F(y, y) = 0$ are $a_1, \ldots, a_s$. Let $i = 1$. While $i \le s$ do
  (a) Let $G = F(y - a_i, y_1 - a_i)$. Find a rational solution $y(x)$ of $G = 0$ with Algorithm 4.13.
  (b) If we find a solution $y(x)$ of $G = 0$, then by Lemma 4.3, $y(x) + a_i$ is a solution of $F = 0$. Return $y(x) + a_i$ and terminate the algorithm.
  (c) Let $i = i + 1$.

If we cannot find any solutions for $i = 1, 2, \ldots, s$, $F = 0$ has no rational solutions $P(x)/Q(x)$ with $\deg(P(x)) = \deg(Q(x))$ and $\deg(P(x)/Q(x)) \le n$.

(5) If we cannot find any solutions in Steps (2), (3) or (4), then $F = 0$ has no rational solutions $P(x)/Q(x)$ with $\deg(P(x)/Q(x)) \le n$. Then return (NULL) and the algorithm terminates.

5. Conclusion

In this paper, we study OΔEs with rational and polynomial solutions. We give a difference equation description for univariate polynomials and rational functions by giving OΔEs whose solutions are exactly polynomials and rational functions respectively. On the basis of these OΔEs, we give a criterion for an OΔE to have a rational type general solution.

For the first-order autonomous OΔEs, we give a polynomial time algorithm for computing polynomial solutions if they exist. As shown in Example 4.1, in the difference case, the rational solution with its shift is not always a proper parametrization of the original equation (considered as a plane algebraic curve). Hence we cannot bound the degree of the rational solutions by the method used in Feng and Gao (2006). However, for a given degree, we can still give an algorithm for finding a rational solution if one exists, based on the Padé approximation and an algorithm for computing the Laurent series solutions of difference equations.

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References

Abramov, S.A., 1989. Rational solutions of linear differential and difference equations with polynomial coefficients. U.S.S.R. Comput. Maths. Math. Phys. 29, 7–12.
Abramov, S.A., 1995. Rational solutions of linear difference and q-difference equations with polynomial coefficients. In: Proc. ISSAC 1995. ACM Press, pp. 285–289.
Abramov, S.A., Paule, P., Petkovšek, M., 1998. q-Hypergeometric solutions of q-difference equations. Discrete Math. 180, 3–22.
Abramov, S.A., 1998. Rational solutions of first order difference systems. In: Proc. ISSAC 1998. ACM Press, pp. 124–131.
Abramov, S.A., Bronstein, M., Petkovšek, M., 1995. On polynomial solutions of linear operators equations. In: Proc. ISSAC 1995. ACM Press, pp. 290–295.
Beckermann, B., Cabay, S., Labahn, G., 1997. Fraction-free computation of matrix Padé systems. In: Proc. ISSAC 1997. ACM Press, pp. 125–132.
Binder, F., 1996. Fast computations in the lattice of polynomial rational function fields. In: Proc. ISSAC 1996. ACM Press, pp. 43–48.
Böing, H., Koepf, W., 1999. Algorithms for q-Hypergeometric summation in computer algebra. J. Symbolic Comput. 28 (6), 777–799.
Brent, R., Gustavson, F.G., Yun, D.Y.Y., 1980. Fast solution of Toeplitz systems of equations and computation of Padé approximants. J. Algorithms 1, 259–295.
Bronstein, M., 2000. On solutions of linear ordinary difference equations in their coefficient field. J. Symbolic Comput. 29 (6), 841–877.
Cabay, S., Choi, D.K., 1986. Algebraic computations of scaled Padé fractions. SIAM J. Comput. 15, 243–270.
Cabay, S., Labahn, G., 1992. A superfast algorithm for multidimensional Padé systems. Numer. Algorithms 2 (2), 201–224.
Cano, J., 1993. On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations. Analysis 13, 103–119.
Cluzeau, T., van Hoeij, M., 2006. Computing hypergeometric solutions of linear recurrence equations. AAECC 17 (2), 83–115.
Cohn, R.M., 1965. Difference Algebra. In: Tracts in Mathematics, vol. 17. Interscience, New York.
Duval, D., 1989. Rational Puiseux expansions. Compositio Math. 70, 119–154.
Feng, R., Gao, X.S., 2004. Rational general solutions of algebraic ordinary differential equations. In: Proc. ISSAC 2004. ACM Press, pp. 155–162.
Feng, R., Gao, X.S., 2006. A polynomial time algorithm to find rational general solutions of first order autonomous ODEs. J. Symbolic Comput. 41 (7), 735–762.
Gao, X.S., Luo, Y., 2004. A Characteristic set method for finitely generated difference polynomial systems. MM Research Preprint, no. 23, 66-91, MMRC, CAS; presented at ICPSS, Paris, 2004.
George, A., Baker, J.R., 1975. Essentials of Padé Approximants. ACM Press, New York.
Gosper, R.W., 1978. Decision procedure for indefinite hypergeometric summation. Proc. Nat. Acad. Sci. U.S.A. 75, 40–42.
Hendriks, P., Singer, M., 1999. Solving difference equations in finite terms. J. Symbolic Comput. 27 (3), 239–259.
Karr, M., 1981. Summation in finite terms. J. ACM 28 (2), 305–350.
Karr, M., 1985. Theory of summation in finite terms. J. Symbolic Comput. 1 (3), 303–315.
Kauers, M., Schneider, C., 2006. Indefinite summation with unspecified summands. Discrete Math. 306 (17), 2073–2083.
Koepf, W., 1995. Algorithms for m-fold Hypergeometric summation. J. Symbolic Comput. 20 (4), 399–417.
Paule, P., 1995. Greatest factorial factorization and symbolic summation. J. Symbolic Comput. 20 (3), 235–268.
Paule, P., Riese, A., 1997. A Mathematica q-analogue of Zeilberger's Algorithm based on an algebraically motivated approach to q-Hypergeometric telescoping. In: Ismail, M.E.H., Rahman, M. (Eds.), Special Functions, q-Series and Related Topics. In: Fields Inst. Commun., vol. 14. Amer. Math. Soc., pp. 179–210.

Paule, P., Schorn, M., 1995. A Mathematica version of Zeilberger's Algorithm for proving binomial coefficient identities. J. Symbolic Comput. 20 (5-6), 673–698.
Petkovšek, M., 1992. Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. 14 (2–3), 243–264.
Schinzel, A., 1982. Selected Topics on Polynomials. University of Michigan Press, Ann Arbor, the University of Michigan Press.
Schneider, C., 2005. Solving parameterized linear difference equations in terms of indefinite nested sums and products. J. Difference Equ. Appl. 11 (9), 799–821.
Sendra, J.R., Winkler, F., 2001. Tracing index of rational curve parametrization. Comp. Aided Geo. Design 18 (8), 771–795.
van Hoeij, M., 1998. Rational solutions of linear difference equations. In: Proc. ISSAC 1998. ACM Press, pp. 120–123.
van Hoeij, M., 1999. Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra 139, 109–131.
von zur Gathen, J., Gerhard, J., 1999. Modern Computer Algebra. Cambridge University Press.
Walsh, P.G., 2000. A polynomial-time complexity bound for the computation of the singular part of a Puiseux expansion of an algebraic function. Math. Comput. 69 (231), 1167–1182.
Walker, R.J., 1950. Algebraic Curves. Princeton University Press, New Jersey.
Wolfram, D.A., 2000. A formula for the general solution of a constant-coefficient difference equations. J. Symbolic Comput. 29 (1), 79–82.