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Computing the intersections of three conics according to their Jacobian curve

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ABSTRACT

In this study, we describe the intersection of three conics based on the singularities of their corresponding Jacobian curve. In particular, we show that certain singular points and sub-lines of the Jacobian curve are the precise common points and common tangent lines of the conics, respectively. Based on our results, these points or the tangent line can be computed as the singularities of the Jacobian curve. These results facilitate investigations of the relationships between a net of conics and their Jacobian curve.

1. Introduction

Computing the intersection of given curves and surfaces is an important problem in geometric modeling, computer-aided geometric design, and computer-aided design (Hoschek and Lasser, 1993; Patrikalakis and Maekawa, 2002). In general, the problem is transformed into the solution of a polynomial system using elimination methods. It is well known that nonlinear polynomial systems are difficult to solve efficiently. For some typical curves and surfaces, there are special methods for computing the intersection or collision conditions, such as conics (Liu and Chen, 2004; Briand, 2007), quadrics (Wang et al., 2001; Wang, 2002; Lazard et al., 2006; Dupont et al., 2008; Jia et al., 2011), and ruled surfaces (Chen et al., 2011; Shen et al., 2012). Previous studies (Liu and Chen, 2004; Etayo et al., 2006) gave the algebraic conditions for classifying the positional relationships between two conics by checking the roots of the so-called Segre's characteristic polynomial of two quadrics. For two ruled surfaces, Chen et al. (2011) provided a collision condition by introduc-

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ing the bracket method and the intersection formula was given using the brackets. Previous studies have focused on two aspects: determining the collision (e.g., Wang et al., 2001; Liu and Chen, 2004; Briand, 2007) and finding the intersection (e.g., Wang, 2002). Some studies have also focused on both of these two aspects (e.g., Chen et al., 2011).

All previous studies have considered the intersections or collision conditions for a pair of geometric objects, but more than two geometric elements often exist even in a simple scene. In the present study, we consider the intersection problem for three conics. We determine the conditions that allow us to characterize the intersections of three conics, which are easily computed. To achieve this, it is natural to generalize Segre's characteristic polynomial, as presented in Liu and Chen (2004), Wang et al. (2001), to the case of three conics. In Liu and Chen (2004), Wang et al. (2001), Segre's characteristic polynomial is a polynomial with one variable. By checking the roots of Segre's characteristic polynomial, we can determine the positional relations between two geometric objects such as conics and ellipsoids. In the case of three conics, Segre's characteristic polynomial defines a planar algebraic cubic curve and we may expect that the singularities of this curve will allow the classification of the intersection property of the given three conics. In addition, Segre's characteristic polynomial is defined as the determinant of a matrix. Representing a planar algebraic curve as the determinant of a matrix is a classical problem in so-called determinantal representations. Determinantal representations of curves have been studied widely in Barth (1977), Wall (1977), Piontkowski (2006), Vinnikov (1984, 1993), Shen (2012). In particular, Wall (1977), Piontkowski (2006) gave complete lists for all typical cubic curves, each of which was generated by a triple of conics.

In this study, we employ a different approach. Instead of using Segre's characteristic polynomial, we apply the Jacobian curve of conics (see Definition 2.1), which is also a planar cubic curve. From the viewpoint of geometry, the Jacobian curve of three conics is formed by the points for which the polars with respect to these conics are concurrent. Let P be a point on the Jacobian curve and P' is the concurrence of the polars of P. Then, Theorem 8 on p. 171 of Semple and Kneebone (1952) implies that P' also lies on the Jacobian curve. We observe that if $P = P'$, then the polar of P is a tangent line of the corresponding conic. In this case, P is an intersect point of the conics. This observation suggests that all of the intersections of the conics are the singularities of the associated Jacobian curve. This fact is established in Proposition 3.4. In practical computer design, it is often desirable to know the existence of the intersections of conics but also to represent these intersections explicitly. The singularities of the curve defined by Segre's characteristic polynomial may imply the existence of intersections of the corresponding conics, but they cannot give the intersections without further calculations. Compared with the determinantal curve, the triple of conics and their Jacobian curve are on the same space. Based on this fact, we construct an explicit structural connection between the intersections of three conics and the singularities of their associated Jacobian curve. This connection allows us to obtain the common points of three conics from the singularities of their Jacobian curve without further computations. The Jacobian curve was mentioned previously in Salmon (1895). Wall also discussed the Jacobian curve in Wall (1977), where he classified the correspondences of the net of conics and the associated Jacobian curve by certain parametric transformations.

It should be noted that other methods are available for finding the intersections of a triple of conics. The first method is using the resultant for three conics, as described in Gelfand et al. (1994) (Chapter 3, Section 4). The second method is parameterizing one conic and then solving the two univariate polynomials by substituting it into the other two conics. Our approach appears to be simpler when the intersections of the conics comprise more than one point (counting multiplicities). In this case, the Jacobian curve is reducible and its singularities are the intersection points of a line and a conic or lines. When the Jacobian curve is irreducible, methods can be used to find its singularities, such as (Paluszny et al., 2002; Eigenwillig et al., 2006). In particular, in Paluszny et al. (2002), a formula of the singular point was given for a singular irreducible algebraic cubic curve. However, our approach can find more information than the common points, e.g., the tangent line can be obtained if the conics are tangent at a common point, where this can be shown if the conics share the same center or symmetry axis even without common points.

The remainder of this paper is organized as follows. In Section 2, we introduce the Jacobian curve of three conics, before providing the main results and examples. In Section 3, we give detailed discussions and the proofs. In Section 4, we present an algorithm and we give our conclusion.

2. Main results

Let K be $\mathbb{R}$ or $\mathbb{C}$. We use $\mathbb{P}^2(K)$ to denote the projective space of dimension two over the field K. $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ are used to denote the homogeneous coordinates of $\mathbb{P}^2(K)$. The letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are used to denote the points in $\mathbb{P}^2(K)$. We also use $\text{Zero}(P_1, P_2, \cdots, P_n)$ to denote the set of common points of $P_1 = 0, P_2 = 0, \cdots, P_n = 0$ in $\mathbb{P}^2(\mathbb{C})$, where $P_i \in K[x_0, x_1, x_2]$.

Let $M \in \mathbb{R}^{3\times 3}$ be a symmetric matrix. Then, $\mathbf{x}M\mathbf{x}^T = 0$ defines an algebraic conic in $\mathbb{P}^2(\mathbb{R})$ with respect to $\mathbf{x}$. However, given a conic $H = 0$, up to a non-zero scalar multiple, a unique symmetric matrix M_H exists such that $H = \mathbf{x}M_H\mathbf{x}^T$. For convenience, the corresponding matrix of a conic $H = 0$ is always denoted as M_H. It is well known that a conic $H = 0$ is non-degenerate if and only if M_H is invertible. Let

$$A \triangleq \mathbf{x}M_A\mathbf{x}^T = 0,\ \ B \triangleq \mathbf{x}M_B\mathbf{x}^T = 0,\ \ C \triangleq \mathbf{x}M_C\mathbf{x}^T = 0 \tag{1}$$

be three conics defined over $\mathbb{R}$. If all of A, B, C are reducible, then their intersections can be computed easily. Now, assume that one of them is irreducible, say A. Then, it is easy to see that $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ exist such that $A + \alpha B$ and $A + \beta C$ are irreducible. Obviously, the intersections of A, B, C coincide with those of $A, A + \alpha B, A + \beta C$. Therefore, in the following, unless stated specifically, we always assume that A, B and C are irreducible, i.e., M_A, M_B, and M_C are always assumed to be invertible.

Definition 2.1. The *Jacobian determinant* of A, B, and C is defined as

$$J(x_0, x_1, x_2) \triangleq \left| \frac{\partial(A, B, C)}{\partial(x_0, x_1, x_2)} \right| = \begin{vmatrix} \frac{\partial A}{\partial x_0} & \frac{\partial B}{\partial x_0} & \frac{\partial C}{\partial x_0} \\ \frac{\partial A}{\partial x_1} & \frac{\partial B}{\partial x_1} & \frac{\partial C}{\partial x_1} \\ \frac{\partial A}{\partial x_2} & \frac{\partial B}{\partial x_2} & \frac{\partial C}{\partial x_2} \end{vmatrix}. \tag{2}$$

If $J \neq 0$ over $\mathbb{P}^2(\mathbb{R})$, then we refer to the curve defined by $J = 0$ as the Jacobian curve associated with A, B, C.

We can find that $J(\mathbf{x}) = 8\left|M_A\mathbf{x}^T, M_B\mathbf{x}^T, M_C\mathbf{x}^T\right|$. Obviously, A, B, and C are linearly dependent over $\mathbb{R}$ if and only if J vanishes on $\mathbb{P}^2(\mathbb{R})$. In the following, we assume that J does not vanish on $\mathbb{P}^2(\mathbb{R})$. We discuss the relationship between the intersections of the triple conics and the singularities of their Jacobian curve.

It is well known that the singular points of a projective curve defined by a homogeneous polynomial $f(x_0, x_1, x_2)$ are given by the solutions of the following system

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0.$$

Now, we can give the main theorem of the paper.

Theorem 2.2. *Let A, B, and C be three conics defined over $\mathbb{R}$ and $J = 0$ is their Jacobian curve.*

1. *If J is irreducible, then A, B, and C have a common point if and only if $J = 0$ has a singular point.*
2. *If $J = LQ$, where L is linear and Q is quadric, then:*
   (a) *A, B, and C have a common point if and only if $J = 0$ has a unique singular point;*
   (b) *A, B, and C either have two common points, which are two singular points of $J = 0$, or no common point.*
3. *If $J = L_1L_2L_3$, where $L_1 = 0$, $L_2 = 0$, and $L_3 = 0$ are three distinct lines, then:*
   (a) *A, B, and C are tangent to one line from $\{L_1 = 0, L_2 = 0, L_3 = 0\}$ at $\mathbf{u}$ if $\text{Zero}(L_1, L_2, L_3) = \mathbf{u}$;*
   (b) *A, B, and C have either of three common points at $L_1 \cap L_2$, $L_1 \cap L_3$, and $L_2 \cap L_3$ if $J = 0$ has no reduced point (see Definition 3.1) and $\text{Zero}(L_1, L_2, L_3) = \emptyset$;*
   (c) *A, B, and C have no common points but they share the same center or symmetry axis if $J = 0$ has reduced points and $\text{Zero}(L_1, L_2, L_3) = \emptyset$.*

4. *If $J = L_1 L_2^2$, where $L_1 = 0$ and $L_2 = 0$ are two distinct lines, let $\mathbf{u} = L_1 \cap L_2$, and then:*
   (a) *A, B, and C have two distinct common points (none of them equals $\mathbf{u}$) on $L_2 = 0$ if $\mathbf{u}$ is not a reduced point of $J = 0$;*
   (b) *A, B, and C are tangent to $L_1 = 0$ at $\mathbf{u}$ and have another common point on $L_2 = 0$ if $\mathbf{u}$ is a reduced point of $J = 0$ and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_1)$;*
   (c) *A, B, and C are tangent to $L_2 = 0$ at $\mathbf{u}$ if $\mathbf{u}$ is a reduced point of $J = 0$ and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_2)$.*
5. *If $J = \lambda L^3$, where $L = 0$ is a line and $\lambda \in \mathbb{R}\{0\}$, then A, B, and C are tangent to $L = 0$ at a point with order three.*

Proof. This theorem is proved in later sections.

Statements 1 and 2.a can be proved by Proposition 3.4 and Proposition 3.6. Statement 2.b is followed by Proposition 3.14. Statements 3, 4, and 5 are summarized by Proposition 3.17, Proposition 3.18, and Proposition 3.19, respectively. □

The results are illustrated in Fig. 1, Fig. 2, and Fig. 3, which show that the common points of the conics are simply the singular points of their Jacobian curve, where the singular points, reduced points, and common points are abbreviated as SP, RP, and CP, respectively. Based on the singularities of the Jacobian curve, we can determine whether the three conics have intersections, while the intersections can also be obtained directly from these singularities if they exist. We also give explicit examples in Tables 1 and 2. Note that the figures shown are not those for the explicit examples.

In addition, we can design cubic curves with expected singularities by setting certain intersections for a triple of conics, as shown by the following example.

Example 2.3. To obtain a projective cubic with a singular point $(1, 1, 1)$, we start with the following

$$A = (x_0 - x_2)(x_1 - x_2),$$
$$B = (x_0 - x_1)x_0,$$
$$C = x_1(x_0 + x_1 - 2x_2).$$

Then, we can verify that $(1, 1, 1)$ is the only common point of A, B, C and that A, B, C are not tangent at this point. Furthermore, we can also verify that $A + B$ is irreducible. Then, we compute their Jacobian curve

$$J(x_0, x_1, x_2) = \frac{{x_1}^3 - x_0{x_1}^2 - {x_0}^2 x_1 - {x_0}^3}{4} + \frac{x_0 x_1 x_2 - x_2 {x_1}^2 + x_1 {x_2}^2}{2} + {x_0}^2 x_2 - x_0 {x_2}^2,$$

which is an irreducible curve with only one singular point $(1, 1, 1)$.

3. Proofs

In this section, we always assume that A, B, C are linearly independent over $\mathbb{R}$. The coefficients of A, B, C are in $\mathbb{R}$, so A, B, C are linearly independent over $\mathbb{R}$ if and only if they are linearly independent over $\mathbb{C}$. Consider the following system

$$\mathbf{x} M_A \mathbf{y}^T = 0,\ \mathbf{x} M_B \mathbf{y}^T = 0,\ \mathbf{x} M_C \mathbf{y}^T = 0. \tag{3}$$

Since M_A, M_B, M_C are symmetric, then the equations above can be rewritten in matrix form as

$$\mathbf{y}\frac{\partial(A, B, C)}{\partial(x_0, x_1, x_2)} = 0.$$

The equations (3) define a projective variety in $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$. We denote this variety by $\mathcal{P}$. Given a point $\mathbf{u} \in \mathbb{P}^2(\mathbb{C})$, a point $\mathbf{v} \in \mathbb{P}^2(\mathbb{C})$ exists such that $(\mathbf{u}, \mathbf{v}) \in \mathcal{P}$ if and only if $J(\mathbf{u}) = 0$. A similar statement holds if $\mathbf{u}$ and $\mathbf{v}$ are exchanged. From the viewpoint of geometry, for each $M \in \{M_A, M_B, M_C\}$,

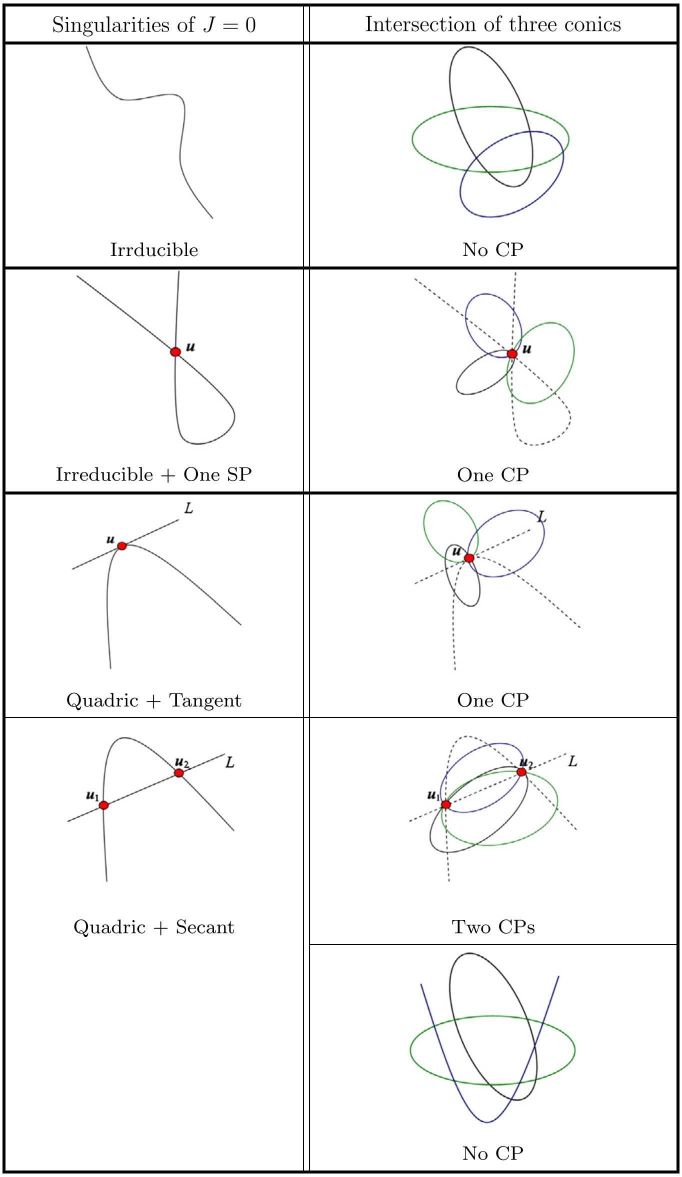

Fig. 1. Singularities of $J = 0$ v.s. Intersections of three conics.

$\mathbf{u}M\mathbf{y}^T = 0$ is the polar line of the conic $\mathbf{y}M\mathbf{y}^T = 0$ at $\mathbf{u}$. The points $\mathbf{v}$ such that $(\mathbf{u}, \mathbf{v}) \in \mathcal{P}$ are the precise intersection points of the polar lines of the conics $A = 0$, $B = 0$, $C = 0$ at $\mathbf{u}$. Please refer to Semple and Kneebone (1952) for more details.

Let V be a subvariety of $J = 0$. Denote

$$\mathcal{I}(V) = \left\{ \mathbf{v} \in \mathbb{P}^2(\mathbb{C}) \middle| \exists \mathbf{u} \in V,\ \text{s.t.}\ (\mathbf{u}, \mathbf{v}) \in \mathcal{P} \right\}.$$

If V is defined by the equation where $L = 0$, we can also use $\mathcal{I}(L)$ to denote $\mathcal{I}(V)$.

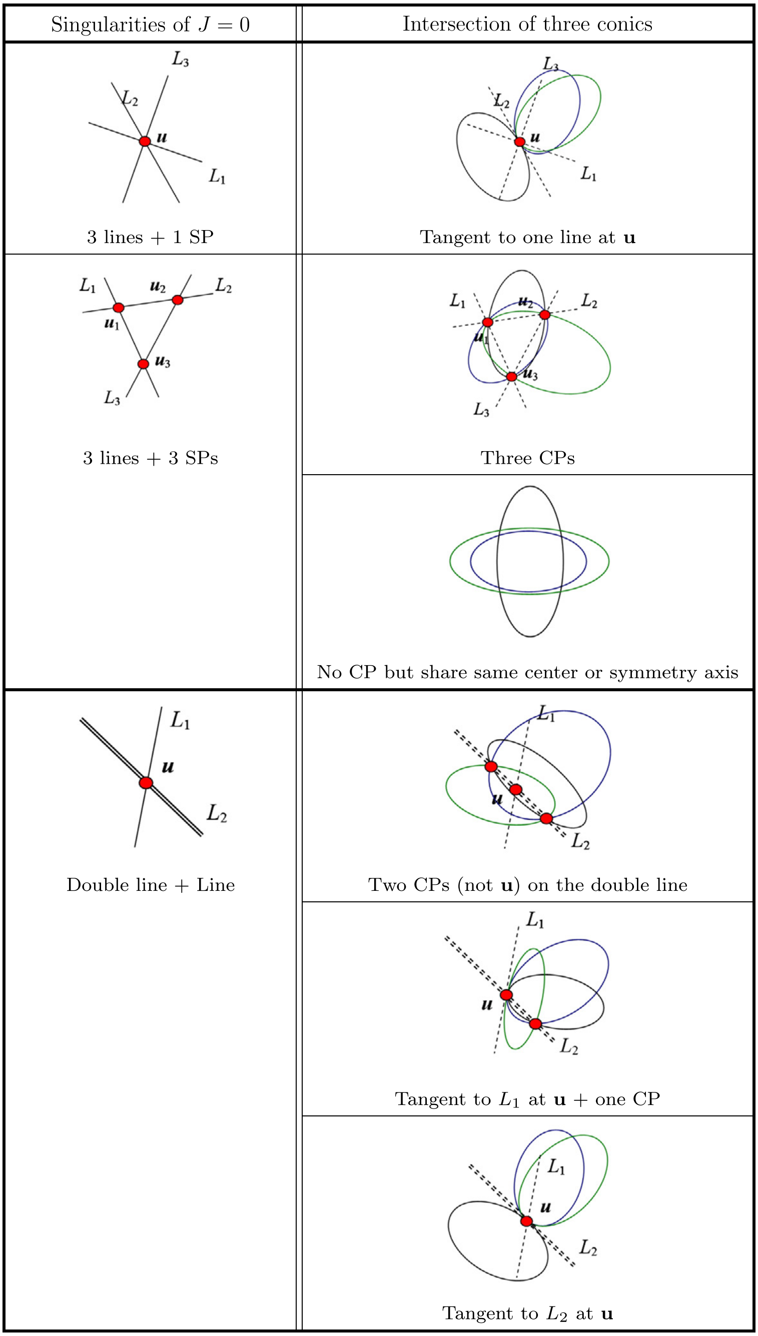

Fig. 2. Singularities of $J = 0$ v.s. Intersections of three conics (continued).

Definition 3.1. Let $\mathbf{u} \in \mathbb{P}^2(\mathbb{C})$ such that $J(\mathbf{u}) = 0$. Obviously, $\mathcal{I}(\mathbf{u})$ is either a projective line or a point. $\mathbf{u}$ is said to be a reduced point of $J = 0$ if $\mathcal{I}(\mathbf{u})$ is a projective line.

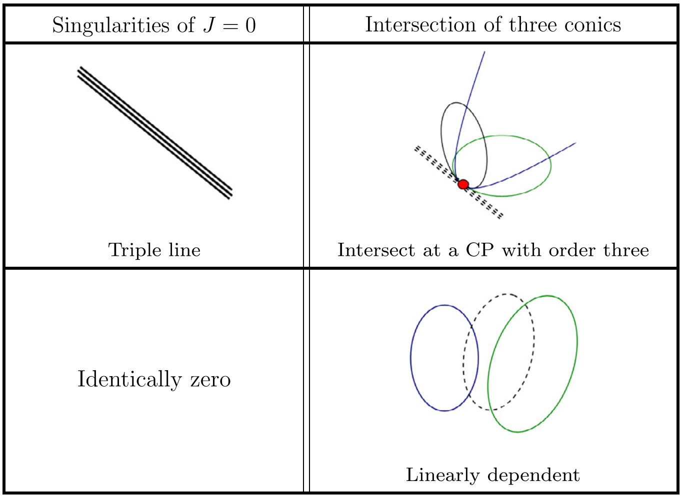

Fig. 3. Singularities of $J = 0$ v.s. Intersections of three conics (continued).

From the symmetry of (3), if $\mathbf{u} \in \mathcal{I}(\mathbf{v})$, then $\mathbf{v} \in \mathcal{I}(\mathbf{u})$. The following lemmas are used frequently in the remainder of this study.

Lemma 3.2. *Let* $\mathbf{u} \in \mathbb{P}^2(\mathbb{C})$ *such that* $J(\mathbf{u}) = 0$. *Then,*

(a) $\mathbf{u} \in \mathrm{Zero}(A, B, C)$ *if and only if* $\mathbf{u} \in \mathcal{I}(\mathbf{u})$.
(b) $\mathbf{u}$ *is a reduced point of* $J = 0$ *if and only if* $\mathrm{rank}\left(\left.\frac{\partial(A,B,C)}{\partial(x_0,x_1,x_2)}\right|_{\mathbf{x}=\mathbf{u}}\right) = 1$.
(c) *Denote* $\mathcal{R}$ *as the set of reduced points of* $J = 0$. *Then,* $\mathbf{u} \in \mathcal{R} \cap \mathrm{Zero}(A, B, C)$ *if and only if* $A = 0$, $B = 0$, $C = 0$ *are tangent at* $\mathbf{u}$. *Furthermore, in the case where* $A = 0$, $B = 0$, $C = 0$ *are tangent at* $\mathbf{u}$, $\mathcal{I}(\mathbf{u})$ *is their common tangent line.*

Proof. (a) and (b) follow from the definition.

(c) ($\Rightarrow$) Note that $\mathcal{I}(\mathbf{u})$ is the set of intersection points of the polars of $\mathbf{u}$ with respect to $A = 0$, $B = 0$, $C = 0$. $\mathcal{I}(\mathbf{u})$ is a line, so these polars of $\mathbf{u}$ coincide. In addition, since $\mathbf{u} \in \mathrm{Zero}(A, B, C)$, then these polars of $\mathbf{u}$ are the exact corresponding tangent lines of $A = 0$, $B = 0$, $C = 0$ at $\mathbf{u}$. Hence, $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{u}$ and $\mathcal{I}(\mathbf{u})$ is their common tangent line.

($\Leftarrow$) Since $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{u}$, $\mathbf{u} \in \mathrm{Zero}(A, B, C)$, then the assertion follows from the fact that the polars of $\mathbf{u}$ with respect to $A = 0$, $B = 0$, $C = 0$ are the corresponding tangent lines at $\mathbf{u}$. $\square$

Lemma 3.3. *Assume that* $(\mathbf{u}, \mathbf{v}) \in \mathcal{P}$. *Then, for all* i *with* $0 \le i \le 2$,

$$u_{k_0} \frac{\partial J}{\partial x_i}(\mathbf{v}) = (-1)^{\epsilon} \begin{vmatrix} \frac{\partial A}{\partial x_i}(\mathbf{u}) & \frac{\partial B}{\partial x_i}(\mathbf{u}) & \frac{\partial C}{\partial x_i}(\mathbf{u}) \\ \frac{\partial A}{\partial x_{k_1}}(\mathbf{v}) & \frac{\partial B}{\partial x_{k_1}}(\mathbf{v}) & \frac{\partial C}{\partial x_{k_1}}(\mathbf{v}) \\ \frac{\partial A}{\partial x_{k_2}}(\mathbf{v}) & \frac{\partial B}{\partial x_{k_2}}(\mathbf{v}) & \frac{\partial C}{\partial x_{k_2}}(\mathbf{v}) \end{vmatrix},$$

where $\{k_0, k_1, k_2\}$ *is a permutation of* $\{0, 1, 2\}$ *and if the permutation* $\binom{0,\ 1,\ 2}{k_0, k_1, k_2}$ *is even, then* $\epsilon = 0$, *otherwise* $\epsilon = 1$.

Table 1
Examples.

Singularities of $J = 0$	Intersection of three conics
$J(x_0, x_1, x_2) = -3/2\,x_0{}^2x_1 - 1/2\,x_0{}^2x_2$ $- x_0x_1{}^2 - 2\,x_0x_1x_2 - 1/2\,x_0{}^3 - 1/2\,x_0x_2{}^2$ $- 3/4\,x_2x_1{}^2 - x_1x_2{}^2 + 1/4\,x_1{}^3$ Irreducible	$A = x_0{}^2 + 2\,x_0x_2 + 1/2\,x_1{}^2$ $B = 2\,x_0{}^2 + x_0x_1 + 4\,x_0x_2 + 2\,x_1{}^2$ $C = x_0{}^2 + x_0x_1 + x_1{}^2 - 2\,x_1x_2 - x_2{}^2$ No CP
$J(x_0, x_1, x_2) = x_0^3/2 - x_0^2x_1 + 12x_0x_1^2$ $- 5x_0^2x_2/2 + 2x_1^3 + 5x_1^2x_2$ Irreducible + One SP $(0, 0, 1)$	$A = 5x_0^2 + x_1^2 + 2x_0x_1 + 5x_0x_2$ $B = x_0^2 + 2x_1^2 + x_0x_1 - x_1x_2$ $C = x_0^2 + x_1^2 + x_0x_2 + 2x_1x_2$ One CP $(0, 0, 1)$
$J\,(x_0, x_1, x_2) = 1/4\,x_0(x_0{}^2 + 2\,x_0x_1 - 2\,x_0x_2 + x_1{}^2)$ Quadric ∩ Tangent $= \{(0, 0, 1)\}$	$A = x_0{}^2 + 2\,x_0x_2 + 1/2\,x_1{}^2$ $B = x_0{}^2 + x_0x_2 + 1/4\,x_1{}^2$ $C = x_0{}^2 + x_0x_1 + x_1{}^2 - 2\,x_1x_2$ One CP $(0, 0, 1)$
$J\,(x_0, x_1, x_2) = -3/4\,(-x_2 + 2\,x_1 + 3\,x_0)$ $\left(x_0{}^2 + 4\,x_0x_1 - 5\,x_0x_2 - 2\,x_1x_2 + 6\,x_2{}^2\right)$ Quadric ∩ Secant $= \{(-1, 2, 1), (1, -1, 1)\}$	$A = 4\,x_0{}^2 + 3\,x_0x_2 + 2\,x_1{}^2 - 9\,x_2{}^2$ $B = x_0{}^2 + 3\,x_0x_2 + 2\,x_1{}^2 - 6\,x_2{}^2$ $C = x_0{}^2 + x_0x_1 - 2\,x_0x_2 + x_1{}^2 - 2\,x_1x_2 - x_2{}^2$ Two CPs $(-1, 2, 1), (1, -1, 1)$
$J\,(x_0, x_1, x_2) = 1/4\,(x_0 - x_1)$ $\left(3\,x_0{}^2 + 5\,x_1x_0 - 20\,x_1x_2 + 8\,x_2{}^2\right)$ Quadric ∩ Secant $= \{(2, 2, 1), (1/2, 1/2, 1)\}$	$A = x_0{}^2 - x_1x_0 - x_0x_2 + 2\,x_1{}^2 + x_1x_2 - 2\,x_2{}^2$ $B = x_0{}^2 - 2\,x_0x_2 + 3\,x_1{}^2 + 2\,x_1x_2 - 4\,x_2{}^2$ $C = x_0{}^2 - x_1x_0 + 2\,x_0x_2 + 1/2\,x_1{}^2 - 3\,x_2{}^2$ No CP
$J\,(x_0, x_1, x_2) = -1/4\,x_0x_1\,(2\,x_0 + x_1)$ 3 lines + one CP $(0, 0, 1)$	$A = x_0{}^2 + x_0x_1 + 2\,x_0x_2 + x_1{}^2$ $B = 2\,x_0x_2 + x_1{}^2$ $C = x_0{}^2 + x_0x_1 + 3\,x_0x_2 + 2\,x_1{}^2$ Tangent to one line $x_0 = 0$ at $(0, 0, 1)$
$J\,(x_0, x_1, x_2) = \frac{(63\,x_1 - 63\,x_2)(x_0+x_2)(x_0-x_1-x_2)}{2}$ 3 lines + 3 SPs $\{(-1, -2, 1), (-1, 1, 1), (2, 1, 1)\}$, No RP	$A = x_0{}^2 - x_0x_2 + 5\,x_1{}^2 + 5\,x_1x_2 - 12\,x_2{}^2$ $B = 4\,x_0{}^2 - 7\,x_0x_1 + 3\,x_0x_2 + 6\,x_1{}^2 - x_1x_2 - 13\,x_2{}^2$ $C = x_0{}^2 + x_0x_1 - 2\,x_0x_2 + x_1{}^2 + 2\,x_1x_2 - 5\,x_2{}^2$ 3 CPs $\{(-1, -2, 1), (-1, 1, 1), (2, 1, 1)\}$
$J\,(x_0, x_1, x_2) = \frac{7\,x_1(x_0+2\,x_2)(2\,x_0+x_2)}{8}$ 3 lines + 3 SPs $\{(-2, 0, 1), (0, 1, 0), (-1/2, 0, 1)\}$ and $(0, 1, 0)$ is a RP	$A = x_0{}^2 + 1/2\,x_1{}^2 - 3/2\,x_2{}^2$ $B = 1/4\,x_0{}^2 + x_1{}^2 - 5/4\,x_2{}^2$ $C = x_0{}^2 + 4\,x_0x_2 + x_1{}^2 + 3\,x_2{}^2$ No CP but with symmetry axis of $x_1 = 0$

Proof. We only prove the case where $(k_0, k_1, k_2) = (0, 1, 2)$. The other cases can be proved in a similar manner. For all i with $0 \le i \le 2$,

$$\frac{\partial J}{\partial x_i} = 2\left(\begin{vmatrix} a_{i,0} & \frac{\partial B}{\partial x_0} & \frac{\partial C}{\partial x_0} \\ a_{i,1} & \frac{\partial B}{\partial x_1} & \frac{\partial C}{\partial x_1} \\ a_{i,2} & \frac{\partial B}{\partial x_2} & \frac{\partial C}{\partial x_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial A}{\partial x_0} & b_{i,0} & \frac{\partial C}{\partial x_0} \\ \frac{\partial A}{\partial x_1} & b_{i,1} & \frac{\partial C}{\partial x_1} \\ \frac{\partial A}{\partial x_2} & b_{i,2} & \frac{\partial C}{\partial x_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial A}{\partial x_0} & \frac{\partial B}{\partial x_0} & c_{i,0} \\ \frac{\partial A}{\partial x_1} & \frac{\partial B}{\partial x_1} & c_{i,1} \\ \frac{\partial A}{\partial x_2} & \frac{\partial B}{\partial x_2} & c_{i,2} \end{vmatrix}\right).$$

Table 2
Examples (continued).

Singularities of $J=0$	Intersection of three conics
$J(x_0,x_1,x_2)=1/2\,(x_0-x_1)^2\,(2x_0+3x_1+3x_2)$ Double line$(L_1)\cap$Line$(L_2)=\{(-3,-3,5)\}$, No RP	$A={x_0}^2-x_1x_0-x_0x_2+2{x_1}^2+x_1x_2-2{x_2}^2$ $B={x_0}^2-2x_0x_2+3{x_1}^2+2x_1x_2-4{x_2}^2$ $C={x_0}^2+x_1x_0+x_0x_2+{x_1}^2-x_1x_2-3{x_2}^2$ Two CPs $\{(-1,-1,1),(1,1,1)\}$ on $(L_1=0)$
$J(x_0,x_1,x_2)=3/4\,(x_0-x_1)^2\,(x_0+x_1+2x_2)$ Double line$(L_1)\cap$Line$(L_2)=\{(-1,-1,1)\}$ and $\mathcal{I}((-1,-1,1))=$Zero(L_2)	$A={x_0}^2-x_1x_0-x_0x_2+2{x_1}^2+x_1x_2-2{x_2}^2$ $B={x_0}^2-2x_0x_2+3{x_1}^2+2x_1x_2-4{x_2}^2$ $C={x_0}^2+x_1x_0+{x_1}^2-3{x_2}^2$ Tangent to $L_2=0$ at $(-1,-1,1)$, with one CP $(1,1,1)$ on $L_1=0$
$J(x_0,x_1,x_2)=1/4{x_0}^2\,(2x_1+x_0)$ Double line$(L_1)\cap$Line$(L_2)=\{(0,0,1)\}$ and $\mathcal{I}((0,0,1))=$Zero(L_1)	$A={x_0}^2+2x_0x_2+1/2{x_1}^2$ $B={x_0}^2+x_0x_2+1/4{x_1}^2$ $C=2{x_0}^2+x_0x_1+4x_0x_2+2{x_1}^2$ Tangent to $L_1=0$ at $(0,0,1)i$
$J(x_0,x_1,x_2)=x_0^3/4$ Triple line(L)	$A=x_0^2+x_1^2/2+x_0x_2$ $B=x_0^2+x_1^2/4+x_0x_2$ $C=x_0^2+x_1^2/2+x_0x_1+2x_0x_2$ Tangent to $L=0$ at $(0,0,1)$
$J(x_0,x_1,x_2)\equiv 0$ Identically zero	$A={x_0}^2+2x_0x_2+1/2{x_1}^2$ $B={x_0}^2+x_0x_2+1/4{x_1}^2$ $C=\alpha A+\beta B,\alpha,\beta\in\mathbb{R}\setminus\{0\}$ Linearly dependent

Note that $\sum_{j=0}^{2} u_j \frac{\partial H}{\partial x_j}(\mathbf{v})=0$ for all $H\in\{A,B,C\}$. A direct calculation implies that

$$2u_0\begin{vmatrix} a_{i,0} & \frac{\partial B}{\partial x_0}(\mathbf{v}) & \frac{\partial C}{\partial x_0}(\mathbf{v}) \\ a_{i,1} & \frac{\partial B}{\partial x_1}(\mathbf{v}) & \frac{\partial C}{\partial x_1}(\mathbf{v}) \\ a_{i,2} & \frac{\partial B}{\partial x_2}(\mathbf{v}) & \frac{\partial C}{\partial x_2}(\mathbf{v}) \end{vmatrix} = \begin{vmatrix} \frac{\partial A}{\partial x_i}(\mathbf{u}) & 0 & 0 \\ 2a_{i,1} & \frac{\partial B}{\partial x_1}(\mathbf{v}) & \frac{\partial C}{\partial x_1}(\mathbf{v}) \\ 2a_{i,2} & \frac{\partial B}{\partial x_2}(\mathbf{v}) & \frac{\partial C}{\partial x_2}(\mathbf{v}) \end{vmatrix} = \begin{vmatrix} \frac{\partial A}{\partial x_i}(\mathbf{u}) & 0 & 0 \\ \frac{\partial A}{\partial x_1}(\mathbf{v}) & \frac{\partial B}{\partial x_1}(\mathbf{v}) & \frac{\partial C}{\partial x_1}(\mathbf{v}) \\ \frac{\partial A}{\partial x_2}(\mathbf{v}) & \frac{\partial B}{\partial x_2}(\mathbf{v}) & \frac{\partial C}{\partial x_2}(\mathbf{v}) \end{vmatrix}.$$

A similar result holds for the last two determinants in the above expression of $\frac{\partial J}{\partial x_i}$. Therefore, we have

$$u_0\frac{\partial J}{\partial x_i}(\mathbf{v}) = \begin{vmatrix} \frac{\partial A}{\partial x_i}(\mathbf{u}) & \frac{\partial B}{\partial x_i}(\mathbf{u}) & \frac{\partial C}{\partial x_i}(\mathbf{u}) \\ \frac{\partial A}{\partial x_1}(\mathbf{v}) & \frac{\partial B}{\partial x_1}(\mathbf{v}) & \frac{\partial C}{\partial x_1}(\mathbf{v}) \\ \frac{\partial A}{\partial x_2}(\mathbf{v}) & \frac{\partial B}{\partial x_2}(\mathbf{v}) & \frac{\partial C}{\partial x_2}(\mathbf{v}) \end{vmatrix}. \qquad \square$$

3.1. *Singularities of $J=0$*

In this subsection, we present some results based on the relation between the intersection points of $A=0$, $B=0$, $C=0$ and the singular points of $J=0$. The former comprises a subset of the latter and this subset may be proper. We also give some conditions under which the latter implies the former.

Proposition 3.4.

(*a*) *If* $\mathbf{u} \in \mathrm{Zero}(A, B, C)$*, then* $\mathbf{u}$ *is a singular point of* $J = 0$.
(*b*) *If* $\mathbf{u}$ *is a reduced point of* $J = 0$*, then* $\mathbf{u}$ *is a singular point of* $J = 0$.
(*c*) *If we assume that* $\mathbf{u}$ *is a singular point but not a reduced point of* $J = 0$*, then* $\mathcal{I}(\mathbf{u})$ *is also a singular point of* $J = 0$.

Proof. For convenience, we denote the matrix $\left.\frac{\partial(A,B,C)}{\partial(x_0,x_1,x_2)}\right|_{\mathbf{x}=\mathbf{u}}$ by M and we assume that $\mathbf{u} = (u_0, u_1, u_2)$.

(*a*) Since $\mathbf{u}$ is a common point of the three conics, we have $(\mathbf{u}, \mathbf{u}) \in \mathcal{P}$. Without loss of generality, we assume that $u_0 \neq 0$. Then, by Lemma 3.3, we can see that $u_0 \frac{\partial J}{\partial x_i}(\mathbf{u}) = 0$ for all $i = 0, 1, 2$. Hence, $\mathbf{u}$ is a singular point.

(*b*) By the hypothesis, $\mathrm{rank}(M) = 1$. Then, the conclusion can be easily deduced from Lemma 3.3.

(*c*) By the hypothesis, $\mathrm{rank}(M) = 2$. Without loss of generality, assume that the last two rows of M, say $\mathbf{w}_1, \mathbf{w}_2$, are linearly independent over $\mathbb{C}$. Let $\mathbf{v} = \mathcal{I}(\mathbf{u}) = (v_0, v_1, v_2)$. Since $\mathbf{u}$ is a singular point of $J = 0$, $v_0 \frac{\partial J}{\partial x_i}(\mathbf{u}) = 0$ for all $i = 0, 1, 2$. By Lemma 3.3, for each $i = 0, 1, 2$, the vector $(\frac{\partial A}{\partial x_i}(\mathbf{v}), \frac{\partial B}{\partial x_i}(\mathbf{v}), \frac{\partial C}{\partial x_i}(\mathbf{v}))$ is a $\mathbb{C}$-linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. It is clear that all rows of M are also $\mathbb{C}$-linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$. Then, again by Lemma 3.3, for all i, j with $0 \le i, j \le 2$, $u_j \frac{\partial J}{\partial x_i}(\mathbf{v}) = 0$. Consequently, $\mathbf{v}$ is a singular point of $J = 0$. $\square$

The converse of Proposition 3.4 (*a*) may not be true, as shown by the following example.

Example 3.5. Let

$$A = 2x_0x_1 + x_2^2, \ B = x_2^2 + 2x_0x_1 + 2x_1x_2, \ C = x_1^2 + 2x_2x_0 + x_0^2.$$

Then, $\mathrm{Zero}(A, B, C) = \emptyset$, but

$$J = 8(x_0 + x_1 + x_2)(x_1^2 + x_2^2 - x_0x_1 - x_1x_2) = 0$$

has two singular points.

The following proposition indicates when the singular points of $J = 0$ are the common points of the conics.

Proposition 3.6. *If* $\mathbf{u}$ *is the unique singular point of* $J = 0$*, then* $\mathbf{u} \in \mathrm{Zero}(A, B, C)$.

Proof. By Lemma 3.2 (*a*), it suffices to prove that $\mathbf{u} \in \mathcal{I}(\mathbf{u})$. We divide the proof into two cases. (1) $\mathcal{I}(\mathbf{u})$ is a point. By Proposition 3.4, $\mathcal{I}(\mathbf{u})$ is also a singular point of $J = 0$, so $\mathcal{I}(\mathbf{u}) = \mathbf{u}$. (2) $\mathcal{I}(\mathbf{u})$ is a projective line. Suppose that $\mathcal{I}(\mathbf{u})$ is defined by $L = 0$. Then, $L|J$ by Lemma 3.2. Let $T = J/L$. Since $\mathrm{Zero}(L, T)$ is a nonempty subset of singular points of $J = 0$, $\mathbf{u} \in \mathcal{I}(\mathbf{u})$; thus, $\mathbf{u} \in \mathrm{Zero}(A, B, C)$. $\square$

Proposition 3.7. *Assume that* $\mathbf{u} \in \mathrm{Zero}(A, B, C)$*. Then,* $A = 0$, $B = 0$, $C = 0$ *are tangent at* $\mathbf{u}$ *if and only if* $\mathbf{u}$ *is a threefold singular point of* $J = 0$.

Proof. By a suitable projective transformation, we may assume that $\mathbf{u} = (1, 0, 0)$ and the line $x_1 = 0$ is the tangent line of $A = 0$ at $\mathbf{u}$. Then, we have

$$\begin{aligned}
A &= 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2, \\
B &= 2b_{01}x_0x_1 + 2b_{02}x_0x_2 + b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2, \\
C &= 2c_{01}x_0x_1 + 2c_{02}x_0x_2 + c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2.
\end{aligned}$$

A straightforward calculation yields

$$\begin{aligned}
&\frac{\partial^2 J}{\partial x_0 \partial x_0}(\mathbf{u}) = \frac{\partial^2 J}{\partial x_0 \partial x_1}(\mathbf{u}) = \frac{\partial^2 J}{\partial x_0 \partial x_2}(\mathbf{u}) = 0,\\
&\frac{\partial^2 J}{\partial x_1 \partial x_1}(\mathbf{u}) = 16 b_{02}(a_{11}c_{01} - a_{01}c_{11}) - 16 c_{02}(a_{11}b_{01} - a_{01}b_{11}),\\
&\frac{\partial^2 J}{\partial x_1 \partial x_2}(\mathbf{u}) = 16 b_{02}(a_{12}c_{01} - a_{01}c_{12}) - 16 c_{02}(a_{12}b_{01} - a_{01}b_{12}),\\
&\frac{\partial^2 J}{\partial x_2 \partial x_2}(\mathbf{u}) = 16 b_{02}(a_{22}c_{01} - a_{01}c_{22}) - 16 c_{02}(a_{22}b_{01} - a_{01}b_{22}).
\end{aligned} \tag{4}$$

($\Rightarrow$) Assume that $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{u}$. Then, it is easy to see that $b_{02} = c_{02} = 0$. We can find that $\mathbf{u}$ is a threefold singular point of $J = 0$ from (4).

($\Leftarrow$) Assume that either $b_{02} \neq 0$ or $c_{02} \neq 0$. Then, (a_{01}, b_{01}, c_{01}) and $(0, b_{02}, c_{02})$ are linearly independent over $\mathbb{C}$ since $a_{01} \neq 0$; otherwise, A is reducible. It follows from (4) that

$$\operatorname{rank}\left(\begin{pmatrix} a_{01} & 0 & a_{11} & a_{12} & a_{22} \\ b_{01} & b_{02} & b_{11} & b_{12} & b_{22} \\ c_{01} & c_{02} & c_{11} & c_{12} & c_{22} \end{pmatrix}\right) = 2.$$

Hence, A, B, C are linearly dependent over $\mathbb{C}$, which is a contradiction. Therefore, we have $b_{02} = c_{02} = 0$, which implies that $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{u}$. □

3.2. *Reduced points of $J = 0$*

In this subsection, we present some intersection properties of the conics $A = 0$, $B = 0$, $C = 0$ in the case where $J = 0$ has reduced points. We begin with two lemmas.

Lemma 3.8. *Assume that $\mathbf{u}, \mathbf{v}$ are two distinct reduced points of $J = 0$. Then, $\mathcal{I}(\mathbf{u}) \neq \mathcal{I}(\mathbf{v})$.*

Proof. Let $L = 0$ be the line passing through $\mathbf{u}$ and $\mathbf{v}$. Assume that $\mathcal{I}(\mathbf{u}) = \mathcal{I}(\mathbf{v})$. For each point $\mathbf{w}$ on $L = 0$, $\mathbf{w}$ is also a reduced point of $J = 0$ and $\mathcal{I}(\mathbf{w}) = \mathcal{I}(\mathbf{u})$. By Proposition 3.4 and Lemma 3.2, the line $L = 0$ is a singular line of $J = 0$. However, for each point $\mathbf{w}' \in \mathcal{I}(\mathbf{u})$, it is easy to see that $\mathcal{I}(\mathbf{w}')$ coincides with $L = 0$. This implies that $\mathbf{w}'$ is also a reduced point, and thus $\mathcal{I}(\mathbf{u})$ is also a singular line of $J = 0$. Note that $J = 0$ has at most one singular line. Hence, $\mathcal{I}(\mathbf{u})$ and $L = 0$ coincide. Now, for each point $\mathbf{w}$ on $L = 0$, $\mathbf{w} \in \mathcal{I}(\mathbf{u}) = \mathcal{I}(\mathbf{w})$, which means that $\mathbf{w} \in \mathrm{Zero}(A, B, C)$ by Lemma 3.2. This contradicts the fact that $\mathrm{Zero}(A, B, C)$ is finite. Therefore, $\mathcal{I}(\mathbf{u}) \neq \mathcal{I}(\mathbf{v})$. □

Corollary 3.9. *There are at most three reduced points of $J = 0$.*

Proof. From Lemmas 3.2 and 3.8, the distinct reduced points correspond to the distinct irreducible components of $J = 0$. Since $J = 0$ has at most three irreducible components, then there are at most three reduced points. □

Lemma 3.10. *Assume that $\mathbf{u}$ is a reduced point of $J = 0$ and $\mathbf{u} \in \mathcal{I}(\mathbf{u})$. If there is another singular point of $J = 0$ on $\mathcal{I}(\mathbf{u})$, then $\mathcal{I}(\mathbf{u})$ is a singular line of $J = 0$.*

Proof. By Lemma 3.2, $\mathbf{u} \in \mathrm{Zero}(A, B, C)$ and $A = 0$, $B = 0$, $C = 0$ are tangent to $\mathcal{I}(\mathbf{u})$ at $\mathbf{u}$. By Proposition 3.7, $\mathbf{u}$ is a threefold singular point of $J = 0$. Then, $J = 0$ is degenerated into three lines. If these three lines are distinct, then since $J = 0$ has a threefold singular point, it has only one singular point, which is a contradiction. Thus, $J = 0$ contains a singular line. Since $\mathcal{I}(\mathbf{u})$ contains at least two singular points of $J = 0$, then $\mathcal{I}(\mathbf{u})$ is a singular line. □

Proposition 3.11. *Assume that* $\mathbf{u}$ *is a reduced point of* $J = 0$ *and* $\mathbf{u} \notin \mathcal{I}(\mathbf{u})$*, then* A, B *and* C *can be transformed into the form* $ax_0^2 + bx_1^2 + cx_1x_2 + dx_2^2$ *under a suitable projective transformation over* $\mathbb{C}$*, where* $a, b, c, d \in \mathbb{C}$.

Proof. By a suitable transformation, we may assume that $\mathbf{u} = (1, 0, 0)$ and $\mathcal{I}(\mathbf{u})$ is defined by $x_0 = 0$. Assume that A is of the following form under the transformation,

$$A = a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

By an easy calculation, we have $a_{01} = a_{02} = 0$. Therefore, A has the required form. By a similar argument, B and C are of the required form, as well as A. $\square$

Now, let us consider the case where $J = 0$ has more than one reduced point.

Proposition 3.12. *Assume that* $\mathbf{u}, \mathbf{v}$ *are two distinct reduced points of* $J = 0$*. If* $\mathbf{u}, \mathbf{v} \notin \mathrm{Zero}(A, B, C)$*, then* $J = 0$ *has three reduced points.*

Proof. According to Lemma 3.8, $\mathcal{I}(\mathbf{u}) \neq \mathcal{I}(\mathbf{v})$. Let $\mathbf{w}$ be the intersection point of $\mathcal{I}(\mathbf{u})$ and $\mathcal{I}(\mathbf{v})$. We have $\mathbf{w} \neq \mathbf{u}$ and $\mathbf{w} \neq \mathbf{v}$. Otherwise, either $\mathbf{u} \in \mathcal{I}(\mathbf{u})$ or $\mathbf{v} \in \mathcal{I}(\mathbf{v})$, which implies that either $\mathbf{u} \in \mathrm{Zero}(A, B, C)$ or $\mathbf{v} \in \mathrm{Zero}(A, B, C)$ by Lemma 3.2, which is a contradiction. We can see that $\mathcal{I}(\mathbf{w})$ coincides with the line passing through $\mathbf{u}$ and $\mathbf{v}$. Therefore, $\mathbf{w}$ is also a reduced point of $J = 0$. $\square$

Proposition 3.13. *Assume that* $J = 0$ *has three reduced points. Then,*

(a) $\mathrm{Zero}(A, B, C) = \emptyset$;
(b) *A, B, C can be transformed into the form* $ax_0^2 + bx_1^2 + cx_2^2$ *under a suitable projective transformation over* $\mathbb{C}$*, where* $a, b, c \in \mathbb{C}$.

Proof. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct reduced points of $J = 0$.

(a) It follows from Lemma 3.8 that $\mathcal{I}(\mathbf{u}), \mathcal{I}(\mathbf{v}), \mathcal{I}(\mathbf{w})$ are three distinct components of $J = 0$. Hence, $J = 0$ has at most three singular points, which are the precise points $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Assume that $\mathrm{Zero}(A, B, C) \neq \emptyset$. Then, at least one of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ will be in $\mathrm{Zero}(A, B, C)$. Without loss of generality, assume that $\mathbf{u} \in \mathrm{Zero}(A, B, C)$. By Lemma 3.2, $\mathbf{u} \in \mathcal{I}(\mathbf{u})$. $\mathcal{I}(\mathbf{u})$ must contain either $\mathbf{v}$ or $\mathbf{w}$ because $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the intersection points of $\mathcal{I}(\mathbf{u}), \mathcal{I}(\mathbf{v}), \mathcal{I}(\mathbf{w})$ in pairs. By Lemma 3.10, $\mathcal{I}(\mathbf{u})$ is a singular line of $J = 0$, which is a contradiction. Hence, $\mathrm{Zero}(A, B, C) = \emptyset$.

(b) By a suitable projective transformation over $\mathbb{C}$, we may assume that $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, and $\mathbf{w} = (0, 0, 1)$. Then, $J = \lambda x_0x_1x_2$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mathcal{I}(\mathbf{u}), \mathcal{I}(\mathbf{v})$, and $\mathcal{I}(\mathbf{w})$ coincide with the lines defined by $x_0 = 0, x_1 = 0$, and $x_2 = 0$, respectively. Assume that A is of the following form under the transformation,

$$A = a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

An easy calculation shows that $a_{01} = a_{02} = a_{12} = 0$. Therefore, A has the required form. By a similar argument, B and C are of the required form, as well as A. $\square$

3.3. Algebraic properties of intersections

The results in Section 3.1 and Section 3.2 allow us to find the correspondence between the singularities of $J = 0$ and the intersections of the conics $A = 0$, $B = 0$, $C = 0$. If the cubic curve $J = 0$ is irreducible, then according to Propositions 3.4 (a) and 3.6, $\mathrm{Zero}(A, B, C) \neq \emptyset$ if and only if $J = 0$ has a singular point. In the following paragraph, we assume that $J = 0$ is reducible. Then, there are four cases to consider: (1) $J = LQ$, where $L = 0$ is a projective line and $Q = 0$ is an irreducible conic; (2) $J = L_1L_2L_3$, where $L_1 = 0, L_2 = 0, L_3 = 0$ are three different projective lines; (3) $J = L_1L_2^2$, where $L_1 = 0, L_2 = 0$ are two distinct projective lines; and (4) $J = L^3$, where $L = 0$ is a projective line.

Proposition 3.14. *Assume that $J = LQ$, where $L = 0$ is a projective line and $Q = 0$ is an irreducible conic. Then, we have:*

(*a*) *If* $\mathrm{Zero}(L, Q) = \{\mathbf{u}\}$, *then* $\mathrm{Zero}(A, B, C) = \{\mathbf{u}\}$. *In particular, $A = 0$, $B = 0$, $C = 0$ are not tangent at* $\mathbf{u}$.
(*b*) *If* $\mathrm{Zero}(L, Q) = \{\mathbf{u}, \mathbf{v}\}$, *where* $\mathbf{u} \neq \mathbf{v}$, *then either* $\mathrm{Zero}(A, B, C) = \{\mathbf{u}, \mathbf{v}\}$ *or* $\mathrm{Zero}(A, B, C) = \emptyset$.

Proof. (*a*) Since $\mathbf{u}$ is the unique singular point of $J = 0$, then by Proposition 3.6 and 3.4, $\mathrm{Zero}(A, B, C) = \{\mathbf{u}\}$. Suppose that $A = 0$, $B = 0$, and $C = 0$ are tangent at $\mathbf{u}$. By Proposition 3.7, $\mathbf{u}$ is a threefold singular point of $J = 0$. Obviously, $\mathbf{u} \in \mathbb{P}^2(\mathbb{R})$. By a suitable projective transformation over $\mathbb{R}$ if necessary, we may assume that $\mathbf{u} = (1, 0, 0)$ and $L = \lambda x_2$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Thus, we have $J = \lambda x_2 Q$, where

$$Q = a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_0 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2, \ a_i \in \mathbb{R}.$$

Since $x_2 = 0$ and $Q = 0$ are tangent at $\mathbf{u}$, then $a_0 = 0$ and $a_1 = 0$. We have $a_2 = 0$ because $\frac{\partial^2 J}{\partial x_2^2}(\mathbf{u}) = 2\lambda a_2 = 0$. Therefore Q is reducible, which is a contradiction.

(*b*) If $\mathcal{I}(\mathbf{u})$ is a projective line, then it will coincide with $L = 0$. By Lemma 3.10, $\mathcal{I}(\mathbf{u})$ is a singular line of $J = 0$, which is a contradiction. Hence, $\mathcal{I}(\mathbf{u})$ is a point and thus it is $\mathcal{I}(\mathbf{v})$. Since $\mathbf{u}$ and $\mathbf{v}$ are all singular points of $J = 0$, then by Proposition 3.4 (c), either $\mathcal{I}(\mathbf{u}) = \mathbf{u}$ or $\mathcal{I}(\mathbf{u}) = \mathbf{v}$. If $\mathcal{I}(\mathbf{u}) = \mathbf{u}$, then $\mathcal{I}(\mathbf{v}) = \mathbf{v}$, which implies that $\{\mathbf{u}, \mathbf{v}\} \subseteq \mathrm{Zero}(A, B, C)$, and thus $\{\mathbf{u}, \mathbf{v}\} = \mathrm{Zero}(A, B, C)$. If $\mathcal{I}(\mathbf{u}) = \mathbf{v}$, then $\mathrm{Zero}(A, B, C) = \emptyset$. □

Lemma 3.15. *Assume that $P = 0$ is an irreducible component of $J = 0$, which does not contain reduced points of $J = 0$. Then, $\mathcal{I}(P)$ is an irreducible subvariety of $J = 0$.*

Proof. Note that the equations given by $P = 0$ and (3) define an algebraic correspondence (see p. 100, Hodge and Pedoe, 1994). Since $P = 0$ does not contain reduced points of $J = 0$,

$$\mathrm{rank}\left(\begin{pmatrix} \mathbf{u}M_A \\ \mathbf{u}M_B \\ \mathbf{u}M_C \end{pmatrix}\right) = 1, \ \forall\, \mathbf{u} \in \mathrm{Zero}(P).$$

By Theorem I in Hodge and Pedoe (1994), p. 108, this algebraic correspondence is irreducible. The lemma follows from the property of irreducible algebraic correspondence. □

Lemma 3.16. *Assume that the line $L = 0$ is a component of $J = 0$. If* $\mathrm{Zero}(L) \subseteq \mathcal{I}(L)$, *then $A = 0$, $B = 0$, $C = 0$ have two common points on $L = 0$ (by counting the multiplicities).*

Proof. Let $\mathbf{u}$ be a point on $L = 0$, which is neither a reduced point of $J = 0$ nor in $\mathrm{Zero}(A, B, C)$. Since $\mathbf{u} \in \mathrm{Zero}(L) \subset \mathcal{I}(L)$, then $\mathbf{v} \in \mathrm{Zero}(L)$ exists such that $\mathbf{u} \in \mathcal{I}(\mathbf{v})$. In other words, $\mathbf{v} = \mathcal{I}(\mathbf{u})$. Obviously, $\mathbf{u} \neq \mathbf{v}$. Let $\beta \in \mathbb{C} \setminus \{0\}$. Then, $\mathbf{u} + \beta\mathbf{v} \in \mathrm{Zero}(L) \subseteq \mathcal{I}(L)$ and $\mathbf{w} \in \mathrm{Zero}(L)$ exists such that $\mathbf{u} + \beta\mathbf{v} \in \mathcal{I}(\mathbf{w})$. In other words,

$$(\mathbf{u} + \beta\mathbf{v})M\mathbf{w} = 0, \ \forall\, M \in \{M_A, M_B, M_C\}.$$

If $\mathbf{w} = \mathbf{v}$, then $\mathbf{v}$ is not only a reduced point of $J = 0$ but it is also a common point of $A = 0$, $B = 0$, $C = 0$. Lemma 3.2 implies that $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{v}$. Now, assume that $\mathbf{w} \neq \mathbf{v}$. Then, $\mathbf{w} = \alpha\mathbf{u} + \mathbf{v}$ with $\alpha \in \mathbb{C} \setminus \{0\}$. An easy calculation shows that

$$\left(\sqrt{\alpha}\mathbf{u} \pm \sqrt{\beta}\mathbf{v}\right) M \left(\sqrt{\alpha}\mathbf{u} \pm \sqrt{\beta}\mathbf{v}\right) = (\mathbf{u} + \beta\mathbf{v})M\mathbf{w} = 0, \ \forall\, M \in \{M_A, M_B, M_C\}.$$

This implies that $\sqrt{\alpha}\mathbf{u} \pm \sqrt{\beta}\mathbf{v} \in \mathrm{Zero}(A, B, C)$. □

Proposition 3.17. *Assume that $J = L_1 L_2 L_3$, where $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are three distinct lines. Then,*

(*a*) *If* $\mathrm{Zero}(L_1, L_2, L_3) \neq \emptyset$, *then* $\mathrm{Zero}(A, B, C) \neq \emptyset$ *and* $A = 0$, $B = 0$, $C = 0$ *are tangent at a point* $\mathbf{u}$ *in* $\mathrm{Zero}(L_1, L_2, L_3)$, *while their common tangent line is one of* $L_1 = 0$, $L_2 = 0$, $L_3 = 0$.
(*b*) *If* $\mathrm{Zero}(L_1, L_2, L_3) = \emptyset$ *and* $J = 0$ *has no reduced point, then* $A = 0$, $B = 0$, $C = 0$ *have three common points.*
(*c*) *If* $\mathrm{Zero}(L_1, L_2, L_3) = \emptyset$ *and* $J = 0$ *has reduced points, then* $\mathrm{Zero}(A, B, C) = \emptyset$ *and* A, B, C *can be transformed into the form* $ax_0^2 + bx_1^2 + cx_1x_2 + dx_2^2$ *under a suitable transformation, where* $a, b, c, d \in \mathbb{C}$.

Proof. (*a*) It follows from Propositions 3.6 and 3.7 that $\mathrm{Zero}(A, B, C) \neq \emptyset$ and $A = 0$, $B = 0$, $C = 0$ are tangent at a point $\mathbf{u}$ in $\mathrm{Zero}(L_1, L_2, L_3)$. Since their common tangent line coincides with $\mathcal{I}(\mathbf{u})$, which is an irreducible component of $J = 0$, then the tangent line should be one of $L_1 = 0, L_2 = 0, L_3 = 0$.

(*b*) According to the hypothesis, $J = 0$ has three distinct singular points, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since $J = 0$ has no reduced point, Proposition 3.4 implies that

$$\{\mathcal{I}(\mathbf{v}_1), \mathcal{I}(\mathbf{v}_2), \mathcal{I}(\mathbf{v}_3)\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Assume that $\mathcal{I}(\mathbf{v}_1) \neq \mathbf{v}_1$. Then, either $\mathcal{I}(\mathbf{v}_1) = \mathbf{v}_2$ or $\mathcal{I}(\mathbf{v}_1) = \mathbf{v}_3$. Hence, either $\mathcal{I}(\mathbf{v}_3) = \mathbf{v}_3$ or $\mathcal{I}(\mathbf{v}_2) = \mathbf{v}_2$. This implies that $\mathrm{Zero}(A, B, C) \neq \emptyset$. Without loss of generality, suppose that $\mathbf{v}_1 \in \mathrm{Zero}(A, B, C)$. Let $L_3 = 0$ be the line passing through $\mathbf{v}_2$ and $\mathbf{v}_3$. Then, by Lemma 3.15, $\mathcal{I}(L_3)$ is a line and since $\mathbf{v}_1 \notin \mathrm{Zero}(L_3)$, $\mathcal{I}(L_3) = \mathrm{Zero}(L_3)$. Lemma 3.16 implies that $A = 0$, $B = 0$, $C = 0$ has two common points on $L_3 = 0$. These common points must be $\mathbf{v}_2$ and $\mathbf{v}_3$. This proves (b).

(*c*) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the singular points of $J = 0$. Without loss of generality, assume that $\mathbf{v}_1$ is a reduced point of $J = 0$. Lemma 3.10 implies that $\mathbf{v}_1 \notin \mathcal{I}(\mathbf{v}_1)$; otherwise, $\mathcal{I}(\mathbf{v}_1)$ is a singular line of $L = 0$, which is impossible. Therefore, $\mathbf{v}_1 \notin \mathrm{Zero}(A, B, C)$. Furthermore, we find that $\mathbf{v}_2, \mathbf{v}_3 \in \mathcal{I}(\mathbf{v}_1)$. Suppose that $\mathbf{v}_2 \in \mathrm{Zero}(A, B, C)$. Then, $\mathbf{v}_2 \in \mathcal{I}(\mathbf{v}_2)$ and $\mathbf{v}_2 \in \mathcal{I}(\mathbf{v}_1)$, so $\mathbf{v}_2$ is also a reduced point of $J = 0$. By Lemma 3.10, $\mathcal{I}(\mathbf{v}_2)$ is a singular line of $J = 0$, which is a contradiction. A similar argument implies that $\mathbf{v}_3 \notin \mathrm{Zero}(A, B, C)$. Because $\mathrm{Zero}(A, B, C) \subseteq \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\mathrm{Zero}(A, B, C) = \emptyset$. This proves the first assertion. The second assertion follows from Proposition 3.11. □

Proposition 3.18. *Assume that $J = L_1 L_2^2$, where $L_1 = 0$, $L_2 = 0$ are two distinct lines. Assume that $\{\mathbf{u}\} = \mathrm{Zero}(L_1, L_2)$. Then,*

(*a*) *If $\mathbf{u}$ is not a reduced point of $J = 0$, then $A = 0$, $B = 0$, $C = 0$ have two distinct common points on $L_2 = 0$. Furthermore, neither of them is equal to $\mathbf{u}$.*
(*b*) *If $\mathbf{u}$ is a reduced point of $J = 0$ and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_1)$, then $A = 0$, $B = 0$, $C = 0$ are tangent to $L_1 = 0$ at $\mathbf{u}$ and they have another common point on $L_2 = 0$.*
(*c*) *If $\mathbf{u}$ is a reduced point of $J = 0$ and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_2)$, then $A = 0$, $B = 0$, $C = 0$ are tangent to $L_2 = 0$ at $\mathbf{u}$.*

Proof. First, we claim that $\mathrm{Zero}(L_2) \subseteq \mathcal{I}(L_2)$. Since $\mathbf{w} \in \mathcal{I}(\mathbf{v})$ implies that $\mathbf{v} \in \mathcal{I}(\mathbf{w})$, it suffices to prove that $\mathcal{I}(\mathbf{v}) \cap \mathrm{Zero}(L_2) \neq \emptyset$ for all $\mathbf{v} \in \mathrm{Zero}(L_2)$. Let $\mathbf{v} \in \mathrm{Zero}(L_2)$. If $\mathbf{v}$ is a reduced point, then $\mathcal{I}(\mathbf{v}) = \mathrm{Zero}(L_1)$ or $\mathcal{I}(\mathbf{v}) = \mathrm{Zero}(L_2)$. In either of these cases, $\mathbf{u} \in \mathcal{I}(\mathbf{v}) \cap \mathrm{Zero}(L_2)$. If $\mathbf{v}$ is not a reduced point, then by Proposition 3.4, $\mathcal{I}(\mathbf{v})$ is a singular point, which means that $\mathcal{I}(\mathbf{v}) \in \mathrm{Zero}(L_2)$. This proves the claim. Now, by Lemma 3.16, $A = 0$, $B = 0$, $C = 0$ have two common points on $L_2 = 0$, say $\mathbf{v}_1, \mathbf{v}_2$ ($\mathbf{v}_1$ may be equal to $\mathbf{v}_2$).

(*a*) If $\mathbf{u} \in \mathrm{Zero}(A, B, C)$, then by Proposition 3.7, $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{u}$. Lemma 3.2 implies that $\mathbf{u}$ is a reduced point, which is a contradiction. Therefore, neither $\mathbf{u} = \mathbf{v}_1$ nor $\mathbf{u} = \mathbf{v}_2$. It remains to show that $\mathbf{v}_1 \neq \mathbf{v}_2$. By contrast, suppose that $\mathbf{v}_1 = \mathbf{v}_2$. Then, $A = 0$, $B = 0$, $C = 0$ are tangent at $\mathbf{v}_1$. Due to Proposition 3.7, $\mathbf{v}_1$ is a threefold singular point of $J = 0$. However, $\mathbf{u}$ is the unique threefold singular point of $J = 0$. Hence $\mathbf{v}_1 = \mathbf{u}$, which is a contradiction. In the sequel, $\mathbf{v}_1 \neq \mathbf{v}_2$.

(b) We have $\mathbf{u} \in \mathcal{I}(\mathbf{u})$. Due to Lemma 3.2, $A=0$, $B=0$, $C=0$ are tangent at $\mathbf{u}$ and $\mathcal{I}(\mathbf{u})$ is the common tangent line. It remains to prove that $A=0$, $B=0$, $C=0$ have one more common point on $L_2=0$. Let $\mathbf{w}$ be a point on $L_2=0$, which is not a reduced point. Then, by Proposition 3.4, $\mathcal{I}(\mathbf{w})$ is a singular point, and thus it is on $L_2=0$. In other words, $\beta \in \mathbb{C}$ exists such that $\mathcal{I}(\mathbf{w}) = \beta\mathbf{u} + \mathbf{w}$, i.e.,

$$\mathbf{w}M(\beta\mathbf{u}+\mathbf{w}) = 0, \ \forall\, M \in \{M_A, M_B, M_C\}.$$

This implies that

$$\left(\frac{\beta}{2}\mathbf{u}+\mathbf{w}\right) M \left(\frac{\beta}{2}\mathbf{u}+\mathbf{w}\right) = \mathbf{w}M(\beta\mathbf{u}+\mathbf{w}) = 0, \ \forall\, M \in \{M_A, M_B, M_C\}.$$

Hence, $\mathbf{w} + \beta\mathbf{u}/2 \in \mathrm{Zero}(A, B, C)$.

(c) As in (b), we have $\mathbf{u} \in \mathcal{I}(\mathbf{u})$. Due to Lemma 3.2, $A=0$, $B=0$, $C=0$ are tangent at $\mathbf{u}$ and $\mathcal{I}(\mathbf{u})$ is the common tangent line. $\square$

Proposition 3.19. *Assume that $J = \lambda L^3$, where $L=0$ is a projective line and $\lambda \in \mathbb{R}\setminus\{0\}$. Then, $A=0$, $B=0$, $C=0$ intersect at a point on $L=0$ with order three.*

Proof. Obviously, $\mathrm{Zero}(L) = \mathcal{I}(L)$. By Lemma 3.16, $A=0$, $B=0$, $C=0$ have two common points $\mathbf{v}_1, \mathbf{v}_2$ on $L=0$. By Proposition 3.7, $A=0$, $B=0$, $C=0$ are tangent to $L=0$ at $\mathbf{v}_i$ for $i=1,2$. Hence, $\mathbf{v}_1 = \mathbf{v}_2$. Now, by a suitable projective transformation, we may assume that $\mathbf{v}_1 = (1,0,0)$ and $L = x_1$. Under this assumption, the terms x_0^2, x_0x_2 do not appear in A, B, C but the term x_0x_1 does appear because all of A, B, C are irreducible. Hence, we can also assume that the coefficients of x_0x_1 in A, B, C equal two. Then, A, B, C will have the following forms:

$$\begin{aligned}
A &= 2x_0x_1 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2,\\
B &= 2x_0x_1 + b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2,\\
C &= 2x_0x_1 + c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2.
\end{aligned}$$

Since $J = \lambda x_1^3$, we have

$$(b_{12}-a_{12})(c_{22}-a_{22}) - (c_{12}-a_{12})(b_{22}-a_{22}) = 0, \tag{5}$$

$$(b_{11}-a_{11})(c_{22}-a_{22}) - (c_{11}-a_{11})(b_{22}-a_{22}) = 0, \tag{6}$$

$$(b_{11}-a_{11})(c_{12}-a_{12}) - (c_{11}-a_{11})(b_{12}-a_{12}) \neq 0. \tag{7}$$

From the equalities (5), (6), and (7), $c_{22}-a_{22} = b_{22}-a_{22} = 0$. Let $B_1 = B - A$ and $C_1 = C - A$. Now, let us consider the affine space by assigning $x_0 = 1$ in A, B_1, C_1. Let

$$\bar{A} = A(1, x_1, x_2), \bar{B_1} = B_1(1, x_1, x_2), \bar{C_1} = C_1(1, x_1, x_2).$$

We have $\langle \bar{B_1}, \bar{C_1}\rangle = \langle x_1^2, x_1x_2\rangle$ by (7), where $\langle\cdot\rangle$ denotes the ideal generated by $\cdot$ in $\mathbb{C}[x_1, x_2]$. Furthermore, we have

$$\langle \bar{A}, \bar{B_1}, \bar{C_1}\rangle = \langle 2x_1 + a_{22}x_2^2,\ x_1^2,\ x_1x_2\rangle.$$

Note that $a_{22} \neq 0$. Since $(0,0)$ is the unique common point of $\bar{A}=0, \bar{B_1}=0, \bar{C_1}=0$, the intersection number of $\bar{A}, \bar{B_1}, \bar{C_1}$ at $(0,0)$ is equal to

$$\dim_{\mathbb{C}}(\mathbb{C}[x_1, x_2]/\langle \bar{A}, \bar{B_1}, \bar{C_1}\rangle)$$

(see p. 81, Fulton, 1969). We can easily verify that $\dim_{\mathbb{C}}\left(\mathbb{C}[x_1, x_2]/\langle \bar{A}, \bar{B_1}, \bar{C_1}\rangle\right) = 3$. Therefore, A, B_1, C_1 intersect at $\mathbf{v}_1$ with order three and so do the conics $A=0$, $B=0$, $C=0$. $\square$

Algorithm: Intersection of three conics.

INPUT: Three irreducible conics A, B, C.
OUTPUT: The intersections of $A = 0$, $B = 0$, $C = 0$.

1. Compute the Jacobian curve $J(x_0, x_1, x_2) = 0$ associated with A, B, C.
2. If $J(x_0, x_1, x_2)$ is irreducible, using the formulae in Paluszny et al. (2002), check the singularity and compute the singular point $\mathbf{u}$ if it exists and RETURN $\mathbf{u}$; otherwise, RETURN $\emptyset$.
3. If $J(x_0, x_1, x_2)$ is reducible, then $J = LQ$ or $J = L_1L_2L_3$, where L, L_1, L_2, L_3 are linear and Q is quadric. The singularities of $J = 0$ can be computed as the intersections of the line and conic (or lines).
   (a) If $J = LQ$ and L is tangent to Q at a point $\mathbf{u}$, RETURN $\mathbf{u}$.
   (b) If $J = LQ$ and $\mathrm{Zero}(L) \cap \mathrm{Zero}(Q) = \{\mathbf{u}_1, \mathbf{u}_2\}$,
      b1 if $\{\mathbf{u}_1, \mathbf{u}_2\}$ belongs to $\mathrm{Zero}(A, B, C)$, then RETURN $\{\mathbf{u}_1, \mathbf{u}_2\}$;
      b2 otherwise, RETURN $\emptyset$.
   (c) If $J = L_1L_2L_3$ and $\mathrm{Zero}(L_1) \cap \mathrm{Zero}(L_2) \cap \mathrm{Zero}(L_3) = \mathbf{u}$, find the common tangent line T for the conics from $\{L_1, L_2, L_3\}$ and RETURN "Tangent to T at $\mathbf{u}$."
   (d) If $J = L_1L_2L_3$ and $\mathrm{Zero}(L_1) \cap \mathrm{Zero}(L_2) = \mathbf{u}_1$, $\mathrm{Zero}(L_1) \cap \mathrm{Zero}(L_2) = \mathbf{u}_2$ and $\mathrm{Zero}(L_3) \cap \mathrm{Zero}(L_1) = \mathbf{u}_3$,
      d1 if $J = 0$ has no reduced point, then RETURN $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$;
      d2 otherwise, RETURN $\emptyset$ and "Share same center or symmetry axis."
   (e) If $J = L_1L_2^2$ and $\mathrm{Zero}(L_1) \cap \mathrm{Zero}(L_2) = \mathbf{u}$,
      e1 if $\mathbf{u}$ is not a reduced point, then compute $\{\mathbf{u}_1, \mathbf{u}_2\} = \mathrm{Zero}(A) \cap \mathrm{Zero}(L_2)$ and RETURN $\{\mathbf{u}_1, \mathbf{u}_2\}$;
      e2 if $\mathbf{u}$ is a reduced point and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_1)$, then compute another point $\mathbf{v} \in \mathrm{Zero}(A) \cap \mathrm{Zero}(L_2)$ and RETURN $\{\mathbf{u}, \mathbf{v}\}$ and "Tangent to L_1 at $\mathbf{u}$;"
      e3 if $\mathbf{u}$ is a reduced point and $\mathcal{I}(\mathbf{u}) = \mathrm{Zero}(L_2)$, RETURN "Tangent to L_2 at $\mathbf{u}$."
   (f) If $J = L^3$, then RETURN "Tangent to L at $\mathbf{u}$ with order three."
   (g) If $J \equiv 0$, then RETURN "Linearly dependent."

4. Algorithm and conclusion

We provide an algorithm for computing the intersections based on our discussions.

Remark 4.1. In our algorithm, to check the tangent line in step (c), we can use the polar line formula of $\mathbf{u}$ with respect to A, B, C.

In this study, we considered the singularities of the Jacobian curve associated with the intersections of its determining conics. As shown in the classification table and examples, we can determine the intersection of three conics by computing the singularity cubic algebraic curve. In addition, this discussion yielded a method for designing cubic curves with expected singularities from a triple of conics by setting certain intersections.

Two interesting problems should be addressed in future research. First, it will be necessary to find the algebraic conditions for all the positional relationships of three conics as well as their intersection. Second, the algebraic conditions should be determined for the intersection of a bundle of more than three conics.

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