On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras

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To Manuel Bronstein, in memoriam

Abstract

A Laurent-Ore algebra $\mathcal{L}$ over a field F is a mathematical abstraction of common properties of linear partial differential and difference operators. A linear (partial) functional system is of the form $A(\mathbf{z}) = 0$ where A is a matrix over $\mathcal{L}$ and $\mathbf{z}$ is a vector of unknowns. Typically, it is a system consisting of linear partial differential, shift and q-shift operators, or any mixture thereof.

We associate to a linear functional system $A(\mathbf{z}) = 0$ an $\mathcal{L}$-module M_A, which is called the module of formal solutions. For our purpose, the dimension of an $\mathcal{L}$-module is defined to be the dimension of the module as a vector space over F. A system $A(\mathbf{z}) = 0$ is said to be ∂-finite if M_A has finite dimension.

A Picard-Vessiot extension for a ∂-finite system $A(\mathbf{z}) = 0$ is a ring containing "all" solutions of $A(\mathbf{z}) = 0$. We prove the existence of Picard-Vessiot extensions for all ∂-finite linear functional systems and show that the dimension of the solution space of a ∂-finite system equals the dimension of its module of formal solutions.

The Gröbner basis techniques for left ideals in Ore algebras are extended to left submodules over Laurent-Ore algebras. This extension enables us to determine whether a linear functional system is ∂-finite.

We present an algorithm for finding all submodules of an $\mathcal{L}$-module with finite dimension. This algorithm allows us to find all "subsystems" whose solution spaces are contained in that of a given ∂-finite linear functional system.

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Chapter 1

Introduction

1.1 Motivation

A (partial) linear functional system consists of linear partial differential, shift, and q-shift operators. The following is an example:

$$\left\{ \begin{array}{lcl} P''(x,\, k) - \frac{2x}{1-x^2} P'(x,\, k) + \frac{k(k+1)}{1-x^2} P(x,\, k) & = & 0 \\ P(x,\, k+2) - \frac{(2k+3)x}{k+2} P(x,\, k+1) + \frac{k+1}{k+2} P(x,\, k) & = & 0\,. \end{array} \right. \tag{1.1}$$

The sequence of the Legendre polynomials $\{P(x,\, k)\}_{k=1}^{\infty}$ is a solution of (1.1) with the initial conditions:

$$P(0,0) = 0, \quad P'(0,0) = 0, \quad P(0,1) = 0, \quad P'(0,1) = 1.$$

Given a linear functional system L, we are interested in the following questions: (i) Does L have a nonzero solution? (ii) Is there a ring containing "all" the solutions of L? (iii) How does one compute the dimension of the solution space of L? (iv) How does one find (if it exists) a "subsystem" whose solution space is properly contained in that of L? (v) Determine whether the solution space of L can be written as a direct sum of those of its subsystems?

This thesis is intended for answering these questions algorithmically for ∂-finite linear functional systems.

In terms of Picard-Vessiot extensions (Definition 2.4.13) and modules of formal solutions (Definition 2.4.5), the above questions translate respectively to: (i) Is a module M of formal solutions trivial? (ii) Does there exist a Picard-Vessiot extension for a given system? (iii) How does one compute the dimension of M? (iv) How does one find a nontrivial submodule of M? (v) Is M decomposable?

We now briefly review some related work.

A Picard-Vessiot extension for a linear ordinary differential (resp. difference) equation is a field (resp. ring) that contains "all" solutions of the equation. For the historical development of Picard-Vessiot extensions and the associated Galois theory, please see [36, 37, 52, 53] and the references therein. Picard-Vessiot fields for integrable systems of PDEs have been studied by Kolchin [38] who proved their existence and developed the associated Galois theory. Cassidy and Singer generalize Kolchin's method in [17].

In [15, 16], Buchberger introduces the notion of Gröbner bases for ideals of commutative polynomials and designs an algorithm for computing Gröbner bases. The theory and applications of Gröbner bases are described in two excellent books [23, 24]. Buchberger's algorithm has been extended to a class of polynomial rings [35] intermediate between the commutative and the most general noncommutative case. An extension [19] of these results shows that for a large class of Ore algebras, (left) Gröbner bases can be computed by a noncommutative version of Buchberger's algorithm. Algorithms are presented in [27, 42] for noncommutative Gröbner bases in Poincaré-Birkhoff-Witt extensions.

The problem of factoring linear ODEs was first studied by Beke [9] and Schlesinger [57] who brought forward the associated equations method. Schwarz [58] presents an algorithmic description of their method, and Bronstein [12] proposes an efficient way to the actual generation of the associated equations. In [61], Tsarev describes the Plücker relation among the factors of linear ODEs. Van Hoeij [34] develops an efficient algorithm, partially based on the associated equations method, to factor linear ODEs. His algorithm has been implemented in MAPLE.

The associated equations method also carries over to the difference case. Based on this method, Bronstein and Petkovšek develop a unified approach to factoring both linear differential and difference equations in [14].

In [44, 45], Li, Schwarz and Tsarev generalize the associated equations method to factor linear PDEs with finite-dimensional solution spaces. Their work motivates us to search for an algorithm to factor linear (partial) functional systems.

The problem of factoring linear ODEs in positive characteristic has been studied in [51] and [53, Ch.13]. One of the motivations for this study lies in the observation that, for the factorization of differential operators over $\mathbb{Q}(x)$, the reduction modulo prime numbers provides useful information. Giesbrecht proposes a factorization algorithm for skew polynomials over finite fields in [26]. This algorithm has been extended in [28] by him and Zhang to factor Ore polynomials in positive characteristic. At the same time, Cluzeau presents an algorithm for the factorization of differential systems in positive characteristic [20]. Recently, Barkatou, Cluzeau and Weil propose in [6, 21] a generalized algorithm for factoring linear PDEs in positive characteristic with finite-dimensional solution spaces.

The associated equations method has been formulated in terms of (partial) differential modules in [53], which translates the problem of factoring linear ODEs or PDEs into that of finding submodules of their associated differential modules. The advantages of this module-theoretic formulation are the following: (i) it is more concise since the bases of vector spaces are used instead of bases of ideals; (ii) it is more powerful because of the convenience for using multi-linear algebra and module theory. (iii) it is intrinsic in the sense that differential modules for equivalent systems are isomorphic to each other. These advantages inspire us to write our factorization algorithm in a module-theoretic setting.

The main results in this thesis include three aspects: a natural generalization of Picard-Vessiot extensions for (∂-finite) linear functional systems; an algorithm for computing Gröbner bases in finitely generated free modules over Laurent-Ore algebras; and an algorithm for finding all submodules of a finite-dimensional module over a Laurent-Ore algebra.

1.2 Outline of the Thesis

The outline of the thesis is as follows.

Chapter 2. The notion of (∂-finite) linear functional systems is defined. We consider the following question. Given a linear functional system $A(\mathbf{z}) = 0$, does there exist an extension that contains "all" solutions of the system? This question can be answered by generalizing Picard-Vessiot extensions of linear ordinary differential (difference) equations. We describe a proper setting for studying solutions of $A(\mathbf{z}) = 0$ by extending Ore algebras to Laurent-Ore algebras $\mathcal{L}$ (Definition 2.4.2) and by associating an $\mathcal{L}$-module M (Definition 2.4.5) to $A(\mathbf{z}) = 0$.

The main results of this chapter include the existence of fundamental matrices and Picard-Vessiot extensions for ∂-finite linear functional systems (Theorem 2.4.13), an approach to completing partial solutions of a fully integrable system and an algorithm for computing the linear dimension of an integrable system.

Chapter 3. We extend the classical Gröbner basis techniques in the usual commutative case to finitely generated free modules over Ore algebras. Based on this extension, we present an algorithm for computing Gröbner bases of submodules over Laurent-Ore Algebras. This algorithm allows us to determine whether a linear functional system is ∂-finite.

Chapter 4. We present an algorithm **FactorModule** for finding all "submodules" of a finite-dimensional module M over a Laurent-Ore algebra. This algorithm has two building blocks: finding one-dimensional submodules of the exterior power $\wedge^d M$ and deciding whether a one-dimensional submodule is generated by a decomposable element. In addition, we present an algorithm for determining the eigenring of M.

Chapter 5. We conclude our contribution and propose some research topics.

Many results in this thesis can be viewed as natural generalizations of their ordinary counterparts of linear differential or difference equations. These generalizations are however necessary in view of their wider applicability and the complications caused by the appearance of several differential and difference operators.

1.3 Notation and Abbreviations

Throughout the thesis, rings are not necessarily commutative and have arbitrary characteristic, unless otherwise specified. Ideals, modules and vector spaces are left ideals, left modules and left vector spaces. For a ring R, the commutator of two elements $a, b \in R$ is $[a, b] = ab - ba$. We write $\mathbf{1}_R$ for the identity map on R and $\mathbf{0}_R$ for the zero map on R, and we omit the subscripts when the context is clear. The notation $\cong_R$ means "isomorphic as R-modules".

Fields are always assumed to be commutative. For a field F, let $F^* = F \setminus \{0\}$.

Denote by $R^{p\times q}$ the set of all $p \times q$ matrices with entries in R, and by $\mathbf{e}_{in}$, for $1 \le i \le n$, the unit vector in $R^{1\times n}$ with 1 in the ith position and 0 elsewhere. For a field F, we write $\mathrm{GL}_n(F)$ for the set of all invertible matrices of size n with entries in F. The notation $(\cdot)^\tau$ denotes the transpose of a vector or matrix, and $\mathbf{1}_n$ is the identity matrix of size n. Vectors are represented by the boldfaced letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$, etc. Vector of unknowns are denoted $\mathbf{x}, \mathbf{y}, \mathbf{z}$, etc.

The symbols $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the complex numbers, the nonnegative integers, the rational numbers and the integers, respectively.

In all examples throughout this thesis, the ground field is $F = \mathbb{C}(x, k)$, and δ_x and σ_k denote respectively the ordinary differentiation with respect to x and the shift operator with respect to k, unless otherwise specified.

Chapter 2

Picard-Vessiot Extensions for Linear Functional Systems

The goal of this chapter is to generalize fundamental matrices and Picard-Vessiot extensions to linear functional systems. The notion of Picard-Vessiot extension fields was first defined in [10] for fields with operators, which are more general fields where the operators do not necessarily commute. While the associated Galois theory was developed there, the existence of Picard-Vessiot extensions was not shown. Indeed, with automorphisms allowed, there are fully integrable systems for which no Picard-Vessiot field exists. Picard-Vessiot fields for integrable systems of partial differential equations have been studied by Kolchin who proved their existence and developed the associated Galois theory [17, §2][38]. In [52, 53], Picard-Vessiot rings for linear ordinary differential and difference systems are defined. Generalizing the definition of Picard-Vessiot rings used for difference equations [52, (Errata)], we obtain Picard-Vessiot rings together with a construction proving their existence. Our definition is compatible with the previous ones: for differential systems, Picard-Vessiot rings turn out to be integral domains, and the Picard-Vessiot fields of [38] are their fields of fractions; for Δ-rings, the Picard-Vessiot rings are generated by elements satisfying linear scalar operator equations, which is the defining property of the Picard-Vessiot fields of [10].

This chapter is organized as follows. The notion of (∂-finite) linear functional systems is introduced in Section 2.1. In Section 2.2, we construct Picard-Vessiot extensions for fully integrable systems, which is a common special case of linear functional systems. In Section 2.3, we apply Picard-Vessiot extensions to show that all the solutions of a factor of a fully integrable system can be completed to solutions of the original system. In Section 2.4, Picard-Vessiot extensions are generalized to ∂-finite linear functional systems. Some techniques are presented in Section 2.5 to determine in practice whether a linear functional system is ∂-finite.

Many of the results in this chapter are from [13]. New additions include Lemma 2.4.5, Proposition 2.4.6 and the results in Sections 2.4.4 and 2.5.

2.1 Orthogonal Δ-Rings

In this section, we describe a general setting for linear functional systems.

Let R be a ring. A *derivation* on R is an additive map $\delta : R \to R$ satisfying

$$\delta(ab) = \delta(a)b + a\delta(b), \qquad \text{for } a, b \in R.$$

Let σ be an endomorphism of R. A σ*-derivation* ([19]) on R is an additive map $\delta : R \to R$ satisfying the Leibniz rule: $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Obviously, derivations are $\mathbf{1}_R$-derivations.

Definition 2.1.1 *A* Δ-ring (R, Φ) *is a ring* R *endowed with a set* $\Phi = \{(\sigma_1, \delta_1), \ldots, (\sigma_m, \delta_m)\}$ *where each* σ_i *is an automorphism of* R*,* δ_i *is a* σ_i*-derivation on* R *and the commutators* $[\sigma_i, \sigma_j] = [\delta_i, \delta_j] = [\sigma_i, \delta_j] = 0$ *for all* $i \neq j$*. When* R *is a field,* (R, Φ) *is called a* Δ-field*.*

Note that a Δ-ring is a (partial) differential ring if $\sigma_i = \mathbf{1}$ for all i, and a (partial) difference ring if $\delta_i = \mathbf{0}$ for all i.

Definition 2.1.2 *We say that a* Δ*-ring* (R, Φ) *is* orthogonal *if* $\delta_i = \mathbf{0}$ *for each* i *such that* $\sigma_i \neq \mathbf{1}$*. By reordering the indices, we can assume that there exists an integer* $\ell \geq 0$

such that $\sigma_i = \mathbf{1}$ for $1 \le i \le \ell$ and $\delta_i = \mathbf{0}$ for $\ell < i \le m$. We write (R, Φ, ℓ) for such an orthogonal Δ-ring.

Clearly, all the δ_i in an orthogonal Δ-ring are usual derivations.

Let (F, Φ) be a Δ-field where $\Phi = \{(\sigma_1, \delta_1), \ldots, (\sigma_m, \delta_m)\}$. The *Ore algebra* [19] over F is the ring $\mathcal{S} := F[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$ of polynomials in ∂_i with coefficients in F, with the usual addition and a multiplication defined by the following rule: $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i a = \sigma_i(a)\partial_i + \delta_i(a)$ for $a \in F$ and $1 \le i \le m$. Suppose that for each i such that $\sigma_i \neq \mathbf{1}$, there exists $a_i \in F$ such that $\sigma_i(a_i) \neq a_i$ and $\sigma_j(a_i) - a_i = \delta_j(a_i) = 0$ for all $j \neq i$. Replacing the x_i by the a_i in the proof of Theorem 1 in [39], one sees that the Ore algebra $\mathcal{S}$ is isomorphic to an Ore algebra $B = F[\Delta_1, \ldots, \Delta_n]$ whose multiplicative rules are:

$$(i)\ \Delta_i \Delta_j = \Delta_j \Delta_i, \quad (ii)\ \Delta_i a = a\Delta_i + \delta_i(a) \quad \text{if } \sigma_i = \mathbf{1}, \quad (iii)\ \Delta_i a = \sigma_i(a)\Delta_i \quad \text{if } \sigma_i \neq \mathbf{1}.$$

We call B an *orthogonal Ore algebra.* There are however orthogonal Δ-fields that do not contain such a_i's, for example, the field $F = \mathbb{C}(x)$ together with $\Phi = \{(\mathbf{1}, \frac{d}{dx}), (\sigma_x, \mathbf{0})\}$ where σ_x is the automorphism of F over $\mathbb{C}$ that sends x to $x - 1$. This field is used in modelling differential-delay equations, and does not match the definition of orthogonality given in [39].

Definition 2.1.3 *Let (F, Φ, ℓ) be an orthogonal Δ-field. A* linear functional system *over F is a system of the form $A(\mathbf{z}) = 0$ where A is a $p \times q$ matrix with entries in the Ore algebra $\mathcal{S}$ and $\mathbf{z}$ is a column vector of q unknowns. Here the action of ∂_i is meant to be δ_i for $i \le \ell$ and to be σ_i for $i > \ell$.*

Let us look at some examples for linear functional systems.

Example 2.1.4 *Let F be an orthogonal Δ-field and $\mathcal{S}$ be the Ore algebra over F. A system of the form $\{L_1(z) = 0, \ldots, L_p(z) = 0\}$ with $L_i \in \mathcal{S}$ can be rewritten as $A(z) = 0$ where $A = (L_1, \ldots, L_p)^\tau \in \mathcal{S}^{p \times 1}$.*

Similarly, the system (1.1), satisfied by the Legendre polynomials, can be rewritten as the linear functional system $A(z) = 0$ *where*

$$A = \left(\partial_x^2 - \frac{2x}{1-x^2}\partial_x + \frac{k(k+1)}{1-x^2}, \quad \partial_k^2 - \frac{(2k+3)x}{k+2}\partial_k + \frac{k+1}{k+2} \right)^{\tau},$$

with ∂_x *the differentiation with respect to* x *and* ∂_k *the shift operator with respect to* k. □

Definition 2.1.5 *Let* (F, Φ, ℓ) *be an orthogonal* Δ*-field. A commutative ring* $E \supseteq F$ *is called an* orthogonal Δ-extension *of* (F, Φ, ℓ) *if the* σ_i *and* δ_i *can be extended to automorphisms and derivations of* E *satisfying: (i) the commutators* $[\sigma_i, \sigma_j] = [\delta_i, \delta_j] = [\sigma_i, \delta_j] = 0$ *on* E *for all* $i \neq j$*; (ii)* $\sigma_i = \mathbf{1}_E$ *for* $i \leq \ell$ *and* $\delta_i = \mathbf{0}_E$ *for* $j > \ell$.

We remark that, although Δ-rings are not necessarily commutative, all orthogonal Δ-extensions are commutative by Definition 2.1.5.

Let E and $\tilde{E}$ be two orthogonal Δ-extensions of F. A map $\phi : E \to \tilde{E}$ is called a *morphism* if ϕ is a ring homomorphism leaving F fixed and commutes with all the δ_i and σ_i. Two orthogonal Δ-extensions of F are said to be *isomorphic* if there exists a bijective morphism between them.

2.2 Picard-Vessiot Extensions of Fully Integrable Systems

A common special case of linear functional systems consists of *fully integrable systems*, which are of the form $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$, and correspond to the linear functional system $A(\mathbf{z}) = 0$ where the matrix A is given by the stacking of blocks of the form $(\partial_i - A_i)$. Fully integrable systems are of interest to our study, since to every ∂-finite linear functional system, we can associate a fully integrable system whose solution space is isomorphic to that of the original system (See Proposition 2.4.12).

In this section, we generalize fundamental matrices and Picard-Vessiot extensions of linear ordinary differential and difference equations to fully integrable systems. In addition, if the field of coefficients has characteristic 0 and has an algebraically closed field of constants, then Picard-Vessiot extensions for such systems contain no new constants.

2.2.1 Fully Integrable Systems

Definition 2.2.1 *A system of the form*

$$\delta_i(\mathbf{z}) = A_i\mathbf{z}, \quad 1 \leq i \leq \ell\,, \qquad \sigma_i(\mathbf{z}) = A_i\mathbf{z}, \quad \ell+1 \leq i \leq m\,, \tag{2.1}$$

where $A_i \in F^{n\times n}$ *and* $\mathbf{z}$ *is a column vector of* n *unknowns, is called an* integrable system *of size* n *over* F *if the following compatibility conditions are satisfied:*

$$\sigma_i(A_j)A_i + \delta_i(A_j) = \sigma_j(A_i)A_j + \delta_j(A_i), \quad \text{for all } i, j. \tag{2.2}$$

The integrable system (2.1) is said to be fully integrable *if the matrices* $A_{\ell+1}, \ldots, A_m$ *are invertible.*

Using Ore algebra notation, we write $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ for the system (2.1) where the action of ∂_i is meant to be δ_i for $i \leq \ell$ and to be σ_i for $i > \ell$, as given in Definition 2.1.3. Observe that the conditions (2.2) are derived from the condition $\partial_i(\partial_j(\mathbf{z})) = \partial_j(\partial_i(\mathbf{z}))$ and are exactly the matrix-analogues of the compatibility conditions for first-order scalar equations in [39].

Example 2.2.2 *Let* $F = \mathbb{C}(x, k)$*. The system* $\mathcal{A} : \{\,\delta_x(\mathbf{z}) = A_x\mathbf{z}, \sigma_k(\mathbf{z}) = A_k\mathbf{z}\,\}$ *is a fully integrable system where*

$$A_x = \begin{pmatrix} \frac{x^2-kx-k}{x(x-k)(x-1)} & \frac{x^2-kx+3k-2x}{kx(x-k)(x-1)} \\ \frac{k(kx+x-x^2-2k)}{(x-k)(x-1)} & \frac{x^3+x^2-kx^2-2x+2k}{x(x-k)(x-1)} \end{pmatrix} \text{ and } A_k = \begin{pmatrix} \frac{k+1+kx^2-xk^2-x}{(x-k)(x-1)} & -\frac{k+1+kx-k^2-x}{k(x-k)(x-1)} \\ \frac{x(k+1)(k+1+kx-k^2-x)}{(x-k)(x-1)} & \frac{(k+1)(x^2-2kx-x+k^2)}{k(x-k)(x-1)} \end{pmatrix}.$$

□

2.2.2 Fundamental Matrices and Picard-Vessiot Extensions

A square matrix with entries in a commutative ring is said to be *invertible* if its determinant is a unit in that ring. As in Chapter 1 and Appendix D of [53] for the purely differential case and in [52] for the ordinary difference case, we define

Definition 2.2.3 *Let (F, Φ, ℓ) be an orthogonal Δ-field and $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ be a fully integrable system of size n over F. An $n \times n$ matrix U with entries in an orthogonal Δ-extension E of F is a* fundamental matrix *for the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ if U is invertible and $\partial_i(U) = A_iU$ for each i, i.e., each column of U is a solution of the system.*

A (two-sided) ideal I of a commutative Δ-ring R is said to be *invariant* if $\delta_i(I) \subset I$ and $\sigma_i(I) \subset I$ for all $1 \leq i \leq m$. The ring R is said to be *simple* if its only invariant ideals are (0) and R.

Lemma 2.2.1 *Let E be an orthogonal Δ-extension of F and I a maximal invariant ideal in E. Then*

(i) *$R := E/I$ is a simple orthogonal Δ-extension of F.*

(ii) *C_R is a field.*

Proof. (i) Let $\bar{I} = \left\{\sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(a) \mid a \in I \text{ and } k_{\ell+1}, \ldots, k_m \in \mathbb{Z}\right\}$. Clearly, $I \subseteq \bar{I}$. Let r_1 and r_2 be in $\bar{I}$. Then $r_1 = \sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(a_1)$ and $r_1 = \sigma_{\ell+1}^{h_{\ell+1}} \cdots \sigma_m^{h_m}(a_2)$ for $a_1, a_2 \in I$ and $k_i, h_i \in \mathbb{Z}$. Set $g_i = \min(k_i, h_i)$ for $i = \ell+1, \ldots, m$. Then $r_1 + r_2 = \sigma_{\ell+1}^{g_{\ell+1}} \cdots \sigma_m^{g_m}(a)$ for some $a \in I$ and thus $r_1 + r_2 \in \bar{I}$. Let $r = \sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(a) \in \bar{I}$ where $a \in I$ and $k_i \in \mathbb{Z}$. For any $b \in E$, there exists $b' \in E$ such that $b = \sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(b')$ since the σ_i are automorphisms. Thus $br = \sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(b'a)$ belongs to $\bar{I}$ because $b'a \in I$. So $\bar{I}$ is an ideal of E. Moreover, the commutativity of the σ_i and δ_j implies that $\bar{I}$ is invariant. Clearly, $\bar{I}$ contains I and 1 is not in $\bar{I}$, for otherwise I would be E. The maximality of I then implies that $\bar{I} = I$.

The δ_i and σ_j can be viewed respectively as derivations and surjective endomorphisms on $R = E/I$ via the formulas $\delta_i(a + I) = \delta_i(a) + I$ and $\sigma_j(a + I) = \sigma_j(a) + I$ for all $a \in E$. If $\sigma_j(a + I) = I$ then $\sigma_j(a) \in I = \bar{I}$. Therefore $\sigma_j(a) = \sigma_{\ell+1}^{k_{\ell+1}} \cdots \sigma_m^{k_m}(b)$ with $b \in I$ and the $k_s \in \mathbb{Z}$. By applying σ_j^{-1}, we have $a \in \bar{I} = I$. So, the σ_j are automorphisms on R and R is an orthogonal Δ-extension of F.

Let $\bar{J}$ be a nonzero invariant ideal of R. Then $\bar{J} = J/I$ where J is an ideal of E containing I. For any $r \in J$, we have $r + I \in \bar{J}$ and $\delta_i(r) + I = \delta_i(r + I) \in \bar{J}$ for $i \leq \ell$, since $\bar{J}$ is invariant. Thus $\delta_i(r) + I = r' + I$ for some $r' \in J$ and $\delta_i(r) - r' \in I \subseteq J$, which implies that $\delta_i(J) \subset J$ for $i \leq \ell$. Similarly, $\sigma_j(J) \subset J$ for $j > \ell$ and therefore J is invariant. The maximality of I then implies $J = E$ and $\bar{J} = R$. So R is simple.

(ii) Let c be a nonzero constant of R. Then the (two-sided) ideal (c) of R generated by c is invariant. Since R is simple, (c) contains 1 and c is invertible. □

The existence of fundamental matrices is stated in the next theorem.

Theorem 2.2.2 *Every fully integrable system has a fundamental matrix whose entries lie in a simple orthogonal* Δ*-extension of* F.

Proof. Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ be a fully integrable system of size n over F, $U = (u_{st})$ be a matrix of n^2 distinct indeterminates and $E = F[u_{11}, \ldots, u_{1n}, \ldots, u_{n1}, \ldots, u_{nn}]$. For $i \leq \ell$, the δ_i are extended to derivations of E via $\delta_i(U) = A_iU$ and for $j > \ell$, the σ_j are extended to automorphisms of E via $\sigma_j(U) = A_jU$ (σ_j is bijective because A_j is invertible, and the action of σ_j^{-1} is given by $\sigma_j^{-1}(U) = \sigma_j^{-1}(A_j^{-1})U$.) It follows from the conditions (2.2) that these extended maps turn E into a well-defined orthogonal Δ-extension of F and that $\partial_i(U) = A_iU$ for $1 \leq i \leq m$. Let $D = \det(U)$ and $\overline{E}$ be the localization of E with respect to D. Extend the δ_i and σ_j via the formulas $\delta_i\left(\frac{1}{D}\right) = -\frac{\delta_i(D)}{D^2}$ and $\sigma_j\left(\frac{1}{D}\right) = \frac{1}{\sigma_j(D)}$ (note that $\sigma_j(D) = \det(A_j)D$ for $j > \ell$), respectively. Then $\overline{E}$ becomes an orthogonal Δ-extension of F. Let I be a maximal invariant ideal of $\overline{E}$ and $R = \overline{E}/I$. By Lemma 2.2.1, R is a simple orthogonal Δ-extension of F. Moreover, the image of U in $R^{n\times n}$ is a fundamental matrix for the system. □

The following proposition reveals that any two fundamental matrices differ by a constant matrix.

Proposition 2.2.3 *Let* $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ *be a fully integrable system of size* n *over* F *and* $U \in E^{n\times n}$ *be its fundamental matrix where* E *is an orthogonal* Δ*-extension of* F. *If* $V \in E^{n\times d}$ *with* $d \geq 1$ *is a matrix whose columns are solutions of the system then* $V = UT$ *for some* $T \in C_E^{n\times d}$. *In particular, any solution of* $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ *in* E^n *is a linear combination of the columns of* U *over* C_E.

Proof. Let $T = U^{-1}V$. We have $\partial_i(V) = A_iV$ for each i, since each column of V is a solution of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$. Thus $\delta_i(T) = -U^{-1}\delta_i(U)U^{-1} + U^{-1}A_iV = 0$ for $i \le \ell$ and $\sigma_j(T) = \sigma_j(U^{-1})\sigma_j(V) = (A_jU)^{-1} A_jV = T$ for $j > \ell$. This implies that all the entries of T belong to C_E. The second statement follows by taking $d = 1$. □

Definition 2.2.4 *Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ be a fully integrable system over F. A* Picard-Vessiot ring *for this system is a (commutative) ring E such that:*

(i) *E is a simple orthogonal Δ-extension of F.*

(ii) *There exists some fundamental matrix U with entries in E for the system such that E is generated by the entries of U and $\det(U)^{-1}$ over F.*

We now construct Picard-Vessiot rings by the same approach used in [52, 53].

Lemma 2.2.4 *If F has characteristic 0, C_F is algebraically closed and R is a finitely generated algebra over F then $C_R = C_F$.*

Proof. Suppose that $b \in C_R$ but $b \notin C_F$. By the argument used in the proof of Lemma 1.8 in [52], there exists a nonzero monic polynomial g over F with minimal degree d such that $g(b) = \left(b^d + \sum_{k=0}^{d-1} g_k b^k\right) = 0$. Apply the δ_i and σ_j to $g(b)$, respectively, we obtain $\left(\sum_{k=0}^{d-1} \delta_i(g_k)b^k\right) = 0$ for $i \le \ell$ and $\left(\sum_{k=0}^{d-1}(\sigma_j(g_k) - g_k)b^k\right) = 0$ for $j > \ell$. The minimality of d then implies $g_k \in C_F$ for $0 \le k < d$. So $b \in C_F$ since C_F is algebraically closed, a contradiction. □

The existence of the Picard-Vessiot extensions is stated in the next theorem.

Theorem 2.2.5 *Every fully integrable system over F has a Picard-Vessiot ring R. If F has characteristic 0 and C_F is algebraically closed, then $C_R = C_F$. Furthermore, that extension is minimal, meaning that no proper subring of R satisfies condition (ii) of Definition 2.2.4.*

Proof. Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ be a fully integrable system of size n over F. From the proof of Theorem 2.2.2, it has a fundamental matrix $U = (u_{st})$ with entries in the simple orthogonal Δ-extension $R = F[u_{11}, \ldots, u_{nn}, \det(U)^{-1}]$. By Definition 2.2.4, R is a Picard-Vessiot ring for the system.

Assume further that F has characteristic 0 and C_F is algebraically closed. Then C_R equals C_F by Lemma 2.2.4. Let $S = F[V, \det(V)^{-1}]$ be a subring of E where V is some fundamental matrix of the system. By Proposition 2.2.3, there exists $T \in C_R^{n\times n}$ such that $V = UT$. Since U and V are invertible, so is T. Since $C_R = C_F$, all the entries of U and the inverse of $\det(U)$ are contained in S. Hence $S = R$ and R is minimal. □

From Theorem 2.2.5, if the field of coefficients has characteristic 0 and has an algebraically closed field of constants, then Picard-Vessiot extensions for fully integrable systems contain no new constants.

As a direct generalization of Proposition 1.20 in [53] and Proposition 1.9 in [52], we have

Proposition 2.2.6 *Suppose that F has characteristic 0 and C_F is algebraically closed. Then any two Picard-Vessiot rings for a fully integrable system over F are isomorphic as orthogonal Δ-extensions.*

Proof. Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ be a fully integrable system over F. Suppose that R_1 and R_2 are two Picard-Vessiot rings for the system, with the respective fundamental matrices U_1 and U_2. Then $R_1 \otimes_F R_2$ can be viewed as an orthogonal Δ-ring extension of F via $\delta_i(r_1 \otimes r_2) = \delta_i(r_1) \otimes r_2 + r_1 \otimes \delta_i(r_2)$ and $\sigma_j(r_1 \otimes r_2) = \sigma_j(r_1) \otimes \sigma_j(r_2)$ for all $r_1 \in R_1, r_2 \in R_2$, $i \leq \ell$ and $j > \ell$. Choose a maximal invariant ideal I of $R_1 \otimes_F R_2$ and define $R_3 := (R_1 \otimes_F R_2)/I$. Since R_1 and R_2 are simple, the canonical maps $\phi_i : R_i \to R_3$ are injective morphisms for $i = 1, 2$. Hence $\phi_1(U_1)$ and $\phi_2(U_2)$ are invertible matrices and moreover fundamental matrices over the ring R_3. From the construction of Picard-Vessiot rings, R_i is generated over F by the entries of U_i and $\det(U_i)^{-1}$, therefore $\phi_i(R_i)$ is generated over F by the entries of $\phi_i(U_i)$ and $\phi_i(\det(U_i)^{-1})$, for $i = 1, 2$. From Proposition 2.2.3, $\phi_1(U_1)$ and $\phi_2(U_2)$ differ by a matrix with entries in C_{R_3}, which is C_F according to Lemma 2.2.4. It follows that $\phi_1(R_1) = \phi_2(R_2)$ and R_1 and R_2 are isomorphic. □

Definition 2.2.5 *Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ and $\{\partial_i(\mathbf{z}^*) = A_i^*\mathbf{z}^*\}_{1\leq i\leq m}$ be two fully integrable systems of size n over F. They are said to be equivalent if there exists $T \in \mathrm{GL}_n(F)$*

such that

$$A_i^* = T^{-1}A_iT - T^{-1}\delta_i(T), \quad \text{for } i \leq \ell \quad \text{and} \quad A_j^* = \sigma_j(T^{-1})A_jT, \quad \text{for } j > \ell. \tag{2.3}$$

We have

Proposition 2.2.7 *Let* $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ *and* $\{\partial_i(\mathbf{z}^*) = A_i^*\mathbf{z}^*\}_{1\leq i\leq m}$ *be two equivalent fully integrable systems over* F*. Then* E *is a Picard-Vessiot ring of the first system if and only if it is of the second one.*

Proof. Let n be the size of these systems. By Definition 2.2.5, there exists $T \in \mathrm{GL}_n(F)$ such that (2.3) holds. Suppose that E is a Picard-Vessiot ring for the second system, which is generated over F by the entries of the fundamental matrix U and $\det(U)^{-1}$. Set $V = TU$. From (2.3), $\delta_i(V) = (\delta_i(T)+TA_i^*)U = A_iTU = A_iV$ and $\sigma_j(V) = \sigma_j(T)A_j^*U = A_jTU = A_jV$ for $i \leq \ell$ and $j > \ell$, thus V is a fundamental matrix for the first system. Clearly, E is generated by the entries of V and $\det(V)^{-1}$ over F. By definition, E is a Picard-Vessiot ring for the first system. The converse is by symmetry. □

Assume that the ground field F has characteristic 0 with an algebraically closed field of constants. Let R be a Picard-Vessiot ring for a fully integrable system of size n over F and $U \in R^{n\times n}$ be the corresponding fundamental matrix. By Proposition 2.2.3, any solution of the system in R^n is a C_R-linear combination of the n columns of U, which are linearly independent over C_R because U is invertible. Since $C_R = C_F$, all the solutions of the system in R^n form a C_F-vector space of dimension n.

2.2.3 Examples

In this section, we present a few examples for Picard-Vessiot extensions.

Example 2.2.6 *Consider the fully integrable system of size one:*

$$\partial_i(z) = a_iz \quad \text{where } a_i \in F \text{ and } i = 1, \ldots, m. \tag{2.4}$$

Let $E = F[T, T^{-1}]$ *be the orthogonal* Δ*-extension such that* $\delta_i(T) = a_iT$ *and* $\sigma_j(T) = a_jT$ *for* $i \leq \ell$ *and* $j > \ell$*. There are two cases to be considered.*

Case 1. *There does not exist an integer $k > 0$ and a nonzero $r \in F$ such that*

$$\delta_i(r) = ka_i r, \quad \text{for } i \leq \ell \quad \text{and} \quad \sigma_j(r) = a_j^k r, \quad \text{for } j > \ell. \tag{2.5}$$

Since every ideal in E is principal, we define the canonical generator of a nontrivial ideal of E to be the monic polynomial in the intersection of the ideal with $F[T]$ whose degree is minimal. Suppose that E has a nontrivial invariant ideal I whose canonical generator is $f = T^d + \sum_{s=0}^{d-1} f_s T^s$ where $f_s \in F$. We may assume further that $d > 0$ and $f_0 \neq 0$, since T^{-1} is in E. Note that for $i \leq \ell$,

$$\delta_i(f) = dT^{d-1}\delta_i(T) + \sum_{s=0}^{d-1} \left(\delta_i(f_s)\, T^s + s f_s\, \delta_i(T)\, T^{s-1}\right) = d\, a_i\, T^d + \sum_{s=0}^{d-1} (\delta_i(f_s) + s\, a_i\, f_s) T^s,$$

and for $j > \ell$,

$$\sigma_j(f) = \sigma_j(T)^d + \sum_{s=0}^{d-1} \sigma_j(f_s)\sigma_j(T)^s = a_j^d\, T^d + \sum_{s=0}^{d-1} \sigma_j(f_s) a_j^s T^s.$$

Since I is invariant, $\delta_i(f) = da_i f$ for $i \leq \ell$ and $\sigma_j(f) = a_j^d f$ for $j > \ell$. By comparing the coefficients of T^0, we have $\delta_i(f_0) = da_i f_0$ for $i \leq \ell$ and $\sigma_j(f_0) = a_j^d f_0$ for $j > \ell$, i.e., d and f_0 satisfy (2.5), a contradiction to the assumption. So E is simple and therefore a Picard-Vessiot ring of (2.4).

Case 2. *Assume that the integer $k > 0$ is minimal so that the system (2.5) has a nonzero solution $r \in F$. Clearly, the ideal $(T^k + r)$ is neither 0 nor E. In addition,*

$$\delta_i(T^k + r) = ka_i(T^k + r) \quad \text{for } i \leq \ell \quad \text{and} \quad \sigma_j(T^k + r) = a_j^k(T^k + r) \quad \text{for } j > \ell,$$

therefore the ideal $(T^k + r)$ is invariant. We write $F[T, T^{-1}]/(T^k + r)$ as $F[t, t^{-1}]$ where $t = T + (T^k + r)$. Then $t^k = -r + (T^k + r)$, $\delta_i(t) = a_i t$ for $i \leq \ell$ and $\sigma_j(t) = a_j t$ for $j > \ell$.

We claim that $F[t, t^{-1}]$ is a Picard-Vessiot ring of (2.4). It suffices to show that $F[t, t^{-1}]$ is simple. Let I be a nonzero invariant ideal of $F[t, t^{-1}]$. Every element of $F[t, t^{-1}]$ can be written uniquely as $\sum_{i=0}^{k-1} b_i t^i$ with $b_i \in F$. Let $0 \leq d < k$ be minimal such that I contains a nonzero $f = \sum_{i=0}^{d} b_i t^i$. Assume that $d > 0$ and that $b_d = 1$. The minimality of d implies that $b_0 \neq 0$. Since I is invariant, the elements $\delta_i(f) - da_i f$ and $\sigma_j(f) - a_j^d f$ belong to I

for $i \leq \ell$ *and* $j > \ell$, *all with degree less than* d. *Hence all these elements are* 0. *It follows that* $\delta_i(b_0) = da_i b_0$ *for* $i \leq \ell$ *and* $\sigma_j(b_0) = a_j^d b_0$ *for* $j > \ell$, *which contradicts the minimality of* k. *So* $d = 0$, $1 \in I$ *and* $F[t, t^{-1}]$ *is a Picard-Vessiot ring of (2.4).* □

Example 2.2.7 *We now describe a Picard-Vessiot ring of the fully integrable system* $\mathcal{A}$ *in Example 2.2.2. By computing the hyperexponential solutions of the system, as described in [39], we find a change of variable* $\mathbf{z} = T\mathbf{y}$ *where*

$$T = \begin{pmatrix} \frac{x-k}{x} & x^2 \\ (x-k)k & x^2 k \end{pmatrix} \in F^{2\times 2},$$

transforms $\mathcal{A}$ *into* $\mathcal{B} : \{ \delta_x(\mathbf{y}) = B_x\mathbf{y}, \sigma_k(\mathbf{y}) = B_k\mathbf{y} \}$ *where*

$$B_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

From $\mathbf{z} = T\mathbf{y}$ *it follows that* $\delta_x(T) + TB_x = A_xT$ *and* $\sigma_k(T)B_k = A_kT$. *By Definition 2.2.5,* $\mathcal{A}$ *and* $\mathcal{B}$ *are equivalent. Proposition 2.2.7 implies that a Picard-Vessiot ring for* $\mathcal{B}$ *is also a Picard-Vessiot ring for* $\mathcal{A}$. *So it suffices to find a Picard-Vessiot ring of* $\mathcal{B}$.

Let U *be a* 2×2 *matrix with indeterminate entries* u_{11}, u_{12}, u_{21} *and* u_{22} *over* F. *Define* $\delta_x(U) = B_xU$ *and* $\sigma_k(U) = B_kU$ *to turn* $E = F[u_{11}, u_{12}, u_{21}, u_{22}, \det(U)^{-1}]$ *into an orthogonal* Δ*-extension. Clearly,* $I = (u_{12}, u_{21})$ *is an invariant ideal of* E. *Set* $R = E/I$. *Since*

$$\det(U)^{-1} + I = (u_{11}u_{22} - u_{12}u_{21})^{-1} + I = (u_{11} + I)^{-1}(u_{22} + I)^{-1},$$

one has $R = F[\overline{u}_{11}, \overline{u}_{22}, (\overline{u}_{11})^{-1}, (\overline{u}_{22})^{-1}]$ *with* $\overline{u}_{ii} = u_{ii} + I$ *for* $i = 1, 2$. *We now show that* R *is simple. Suppose that* J *is a nontrivial invariant ideal of* R. *Let* f *be a nonzero polynomial in* $J \cap F[\overline{u}_{11}, \overline{u}_{22}]$ *with the smallest number of terms. Then* f *can not be a monomial, for otherwise,* J *would be* R *since* $(\overline{u}_{11})^{-1}$ *and* $(\overline{u}_{22})^{-1}$ *are in* R. *We write*

$$f = \overline{u}_{11}^{d_1}\,\overline{u}_{22}^{d_2} + r\,\overline{u}_{11}^{e_1}\,\overline{u}_{22}^{e_2} + \text{ other terms},$$

with $r \in F$ *nonzero and* $(d_1, d_2) \neq (e_1, e_2)$. *From* $\delta_x(\overline{u}_{11}) = \overline{u}_{11}$ *and* $\delta_x(\overline{u}_{22}) = 0$, *we have*

$$\delta_x(f) = d_1\,\overline{u}_{11}^{d_1}\,\overline{u}_{22}^{d_2} + (\delta_x(r) + e_1 r)\,\overline{u}_{11}^{e_1}\,\overline{u}_{22}^{e_2} + \text{ other terms},$$

in which each monomial has already appeared in f*. Thus* $\delta_x(f) - d_1 f$ *must be zero, because it is in* I *but has fewer terms. It follows that* $\delta_x(r) = (d_1 - e_1)r$*. In the same way, from* $\sigma_k(u_{11}) = u_{11}$ *and* $\sigma_k(u_{22}) = ku_{22}$ *one shows that* $\sigma_k(f) - k^{d_2} f = 0$ *and therefore* $\sigma_k(r) = k^{d_2 - e_2} r$*. But the existence of such a rational element* r *would imply that* $d_1 = e_1$ *and* $d_2 = e_2$*, a contradiction. Thus* R *is simple, and so a Picard-Vessiot ring for* $\mathcal{B}$*, hence also for the system* $\mathcal{A}$*. Since*

$$\delta_x(\overline{u}_{11}) = \overline{u}_{11}, \quad \sigma_k(\overline{u}_{11}) = \overline{u}_{11}, \quad \delta_x(\overline{u}_{22}) = 0, \quad \sigma_k(\overline{u}_{22}) = k\overline{u}_{22},$$

we may understand $\overline{u}_{11}$ *as* e^x *and* $\overline{u}_{22}$ *as* $\Gamma(k)$*, then* $\overline{U} = U + I = \begin{pmatrix} e^x & 0 \\ 0 & \Gamma(k) \end{pmatrix}$ *is a fundamental matrix for* $\mathcal{B}$ *in* $R = F[e^x, e^{-x}, \Gamma(k), \Gamma(k)^{-1}]$ *and*

$$T\overline{U} = \begin{pmatrix} \frac{x-k}{x} e^x & x^2\Gamma(k) \\ k(x-k)e^x & x^2\Gamma(k+1) \end{pmatrix}$$

is for the system $\mathcal{A}$*.* □

Example 2.2.8 *We describe a simple orthogonal* Δ*-extension that contains a solution of the inhomogeneous (compatible) system of size one*

$$\delta_i(z) = a_i \quad \textit{for } i \leq \ell \quad \text{and} \quad \sigma_j(z) = z + a_j \quad \textit{for } j > \ell, \tag{2.6}$$

where the a_i *and* a_j *belong to a simple orthogonal* Δ*-ring* E *of characteristic zero. This is an extension of Example 1.18 in [53].*

Due to the commutativity of the δ_i *and* σ_j*, the* a_i *and* a_j *must satisfy the following integrability conditions:*

$$\begin{cases} \delta_i(a_j) = \delta_j(a_i), & \textit{for } 1 \leq i, j \leq \ell, \\ \delta_i(a_j) = \sigma_j(a_i) - a_i, & \textit{for } i \leq \ell,\ j > \ell, \\ \sigma_i(a_j) - a_j = \sigma_j(a_i) - a_i, & \textit{for } \ell < i, j \leq m. \end{cases} \tag{2.7}$$

If (2.6) has a solution in E*, then there is nothing to do. Otherwise, let* $R = E[\alpha]$ *and extend the* δ_i *and* σ_j *on* R *via the formulas:* $\delta_i(\alpha) = a_i$ *for* $i \leq \ell$ *and* $\sigma_j(\alpha) = \alpha + a_j$ *for* $j > \ell$*.*

The integrability conditions (2.7) imply that R *becomes a well-defined orthogonal* Δ*-ring. Let* I *be a nontrivial invariant ideal of* R *and* $f = f_d\alpha^d + f_{d-1}\alpha^{d-1} + \cdots + f_0$ *be a nonzero element of* I *with minimal degree, where* $f_i \in E$ *and* $f_d \neq 0$*. Let* J *be the set consisting of zero and leading coefficients of elements of* I *with degree* d*. Clearly,* J *is a nonzero ideal of* E *since it contains* f_d*. Moreover, our extensions of* δ_i *and* σ_j *imply that* J *is invariant. Since* E *is simple,* $1 \in J$ *and, therefore, we may assume* $d > 0$ *and* $f_d = 1$*. One verifies that the* $\delta_i(f)$ *and* $\sigma_j(f) - f$ *are elements of* I*, with degrees less than* d*. Since* d *is minimal, they are both* 0*. It follows that* $\delta_i\left(-\frac{f_{d-1}}{d}\right) = a_i$ *for* $i \leq \ell$ *and* $\sigma_j\left(-\frac{f_{d-1}}{d}\right) = -\frac{f_{d-1}}{d} + a_j$ *for* $j > \ell$*, i.e.,* $\frac{-f_{d-1}}{d}$ *is a solution of (2.6) in* E*, a contradiction to our assumption. Thus* R *is simple and contains a solution* α *of (2.6).* □

Example 2.2.9 *This example is a generalization of Example 2.2.8 for the vector case. Consider the inhomogeneous system of size* n

$$\delta_i(\mathbf{z}) = \mathbf{b}_i, \quad \text{for } i \leq \ell \quad \text{and} \quad \sigma_j(\mathbf{z}) = \mathbf{z} + \mathbf{b}_j, \quad \text{for } j > \ell, \tag{2.8}$$

where the $\mathbf{b}_i$ *and* $\mathbf{b}_j$ *are vectors with entries in some simple* Δ*-ring* E *of characteristic* 0*. In addition, these vectors satisfy the integrability conditions (2.7) with the* a_i *and* a_j *replaced by the* $\mathbf{b}_i$ *and* $\mathbf{b}_j$*, respectively.*

Set $\mathbf{b}_i = (b_{i1}, \ldots, b_{in})^\tau$ *for* $1 \leq i \leq m$*. Without loss of generality, we may assume that we have already found a partial solution* $\mathbf{z}_s := (r_1, \ldots, r_s)^\tau \in E^s$ *of (2.8) with* $1 \leq s \leq n$*. In other words, for every* k *with* $1 \leq k \leq s$*, there exists* $r_k \in E$ *satisfying that* $\delta_i(r_k) = b_{ik}$ *and* $\sigma_j(r_k) = r_k + b_{jk}$ *for* $i \leq \ell$ *and* $j > \ell$*. If there is* $r_{s+1} \in E$ *satisfying that* $\delta_i(r_{s+1}) = b_{i,s+1}$ *for* $i \leq \ell$ *and* $\sigma_j(r_{s+1}) = r_{s+1} + b_{j,s+1}$ *for* $j > \ell$*, then* $(r_1, \ldots, r_s, r_{s+1})^\tau$ *is a partial solution of (2.8). Now assume that there does not exist such* r_{s+1} *in* E*. Set* $\overline{E} = E[\alpha]$ *and define* $\delta_i(\alpha) = b_{i,s+1}$ *for* $i \leq \ell$ *and* $\sigma_j(\alpha) = \alpha + b_{j,s+1}$ *for* $j > \ell$*. This turns* $\overline{E}$ *into an orthogonal* Δ*-extension of* F*. One can show that* $\overline{E}$ *is simple by similar argument to that in Example 2.2.8. So,* $\mathbf{z}_{s+1} := (r_1, \ldots, r_s, \alpha)^\tau \in \overline{E}^{s+1}$ *is a partial solution of (2.8) with one more completed entry* r_{s+1}*. Continuing in this way, we will eventually get a complete solution of (2.8).* □

2.3 Completing Partial Solutions

In this section, we use Picard-Vessiot extensions for fully integrable systems to complete solutions of *reducible* systems, *i.e.*, systems that can be put into simultaneous block-triangular form by a change of variable $\mathbf{y} = T\mathbf{z}$ for some invertible matrix T in $F^{n\times n}$. We motivate them by showing that the solutions of a factor can always be extended to solutions of the complete system.

Theorem 2.3.1 *Let $\mathcal{A}$: $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ be a fully integrable system of size n over F, and suppose that there exist a positive integer $d < n$ and matrices B_i in $F^{d\times d}$, C_i in $F^{(n-d)\times d}$ and D_i in $F^{(n-d)\times(n-d)}$ such that*

$$A_i = \begin{pmatrix} B_i & 0 \\ C_i & D_i \end{pmatrix} \quad \text{for } 1 \le i \le m\,. \tag{2.9}$$

Then

(i) *The systems $\mathcal{B}$: $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ and $\mathcal{D}$: $\{\partial_i(\mathbf{y}) = D_i\mathbf{y}\}_{1\le i\le m}$ are both fully integrable.*

(ii) *$(0,\ldots,0,\zeta_{d+1},\ldots,\zeta_n)^\tau$ is a solution of $\mathcal{A}$ whenever $(\zeta_{d+1},\ldots,\zeta_n)^\tau$ is a solution of $\mathcal{D}$.*

(iii) *For any solution $(\eta_1,\ldots,\eta_d)^\tau$ of $\mathcal{B}$ in an orthogonal Δ-extension of F, there exists an orthogonal Δ-extension of F containing $\eta_1,\ldots,\eta_d$ as well as $\eta_{d+1},\ldots,\eta_n$ such that $(\eta_1,\ldots,\eta_n)^\tau$ is a solution of $\mathcal{A}$.*

Proof. (i) Let $\mathbf{z} = (z_1,\ldots,z_n)^\tau$, $\mathbf{x} = (z_1,\ldots,z_d)^\tau$ and $\mathbf{y} = (z_{d+1},\ldots,z_n)^\tau$. The system $\mathcal{A}$ can then be rewritten into a homogeneous system and an inhomogeneous system:

$$\begin{cases} \partial_i(\mathbf{x}) &= B_i\mathbf{x}, \\ \partial_i(\mathbf{y}) &= D_i\mathbf{y} + C_i\mathbf{x}, \end{cases} \quad \text{for } 1 \le i \le m\,. \tag{2.10}$$

Since $\mathcal{A}$ is fully integrable, the matrices A_i satisfy (2.2) and A_j is invertible for $j > \ell$. Hence the B_j and D_j for $j > \ell$ must also be invertible since $\det(A_j) = \det(B_j)\det(D_j)$. In addition, a routine calculation shows that for all $1 \leq i, j \leq m$,

$$\sigma_i(A_j)A_i + \delta_i(A_j) = \begin{pmatrix} \sigma_i(B_j)B_i + \delta_i(B_j) & 0 \\ \sigma_i(C_j)B_i + \sigma_i(D_j)C_i + \delta_i(C_j) & \sigma_i(D_j)D_i + \delta_i(D_j) \end{pmatrix}. \quad (2.11)$$

The compatibility conditions of $\mathcal{A}$ then implies that the B_i and D_i also satisfy the compatibility conditions (2.2). Therefore, $\mathcal{B}$ and $\mathcal{D}$ are both fully integrable.

(ii) It is immediate from (2.10).

(iii) Since $\mathcal{D}$ is fully-integrable, by Theorem 2.2.2 there is a (simple) orthogonal Δ-extension R of F and a fundamental matrix U with entries in R for $\mathcal{D}$. Let $\eta = (\eta_1, \ldots, \eta_d)^\tau$ be a solution of $\mathcal{B}$ in some orthogonal Δ-extension E of F. Viewing R and E as commutative F-algebras, we can extend the δ_i and σ_j to the commutative R-algebra $R \otimes_F E$ via $\delta_i(r \otimes e) = \delta_i(r) \otimes e + r \otimes \delta_i(e)$ for $i \leq \ell$ and $\sigma_j(r \otimes e) = \sigma_j(r) \otimes \sigma_j(e)$ for $j > \ell$. Then

$$\delta_i(1 \otimes \eta_1, \ldots, 1 \otimes \eta_d)^\tau = B_i \left(1 \otimes \delta_i(\eta_1), \ldots, 1 \otimes \delta_i(\eta_d)\right)^\tau, \quad \text{for } i \leq \ell,$$

and

$$\sigma_j(1 \otimes \eta_1, \ldots, 1 \otimes \eta_d)^\tau = B_j \left(1 \otimes \sigma_j(\eta_1), \ldots, 1 \otimes \sigma_j(\eta_d)\right)^\tau, \quad \text{for } j > \ell,$$

and $(1\otimes\eta_1, \ldots, 1\otimes\eta_d)^\tau$ is a solution of $\mathcal{B}$. So, replacing E by $R \otimes_F E$, we can assume without loss of generality that E contains R. Substitute η into (2.10) to get $\partial_i(\mathbf{y}){=}D_i\mathbf{y}{+}C_i\eta$ for each i. Let $\mathbf{v} = (v_1, \ldots, v_{n-d})^\tau$ where the v_k are distinct indeterminates over E, and $G = E[v_1, \ldots, v_{n-d}]$. We extend the δ_i and σ_j to G via $\delta_i(\mathbf{v}) = \mathbf{b}_i$ and $\sigma_j(\mathbf{v}) = \mathbf{v} + \mathbf{b}_j$ where $\mathbf{b}_1, \ldots, \mathbf{b}_m \in E^{n-d}$ are given by $\mathbf{b}_i = U^{-1}C_i\eta$ for $i \leq \ell$ and $\mathbf{b}_j = U^{-1}{D_j}^{-1}C_j\eta$ for $j > \ell$.

To turn G into an orthogonal Δ-extension of E, all the δ_i and σ_j on G should commute, which is equivalent to that the $\mathbf{b}_i$ and $\mathbf{b}_j$ satisfy the integrability conditions (2.7). Although the conditions (2.7) are generally not satisfied for arbitrary $\mathbf{b}_i$'s, we verify that they are satisfied in our case. Since the A_i satisfy the compatibility conditions (2.2), it follows from the bottom-left block in (2.11) that

$$\sigma_i(C_j)B_i + \sigma_i(D_j)C_i + \delta_i(C_j) = \sigma_j(C_i)B_j + \sigma_j(D_i)C_j + \delta_j(C_i), \quad \text{for } 1 \leq i, j \leq m. \quad (2.12)$$

For $1 \leq i,\, j \leq \ell$, we have $\delta_i(\mathbf{b}_j) = \delta_i\left(U^{-1}C_j\eta\right) = -U^{-1}\left(D_iC_j - \delta_i(C_j) - C_jB_i\right)\eta$, which, together with $\sigma_i = \sigma_j = \mathbf{1}$ for $1 \leq i, j \leq \ell$ and (2.12) implies that

$$\delta_i(\mathbf{b}_j) - \delta_j(\mathbf{b}_i) = -U^{-1}\left(D_iC_j - \delta_i(C_j) - C_jB_i - D_jC_i + \delta_j(C_i) + C_iB_j\right)\eta = 0\,.$$

For $i \leq \ell$ and $j > \ell$, we have

$$\begin{aligned}
\delta_i(\mathbf{b}_j) &= U^{-1}\left(-D_iD_j^{-1}C_j - D_j^{-1}\delta_i(D_j)D_j^{-1}C_j + D_j^{-1}\delta_i(C_j) + D_j^{-1}C_jB_i\right)\eta \\
&= U^{-1}\left(-D_j^{-1}\left(D_jD_i + \delta_i(D_j)\right)D_j^{-1}C_j + D_j^{-1}\delta_i(C_j) + D_j^{-1}C_jB_i\right)\eta \\
&= U^{-1}\left(-D_j^{-1}(\sigma_j(D_i)D_j)D_j^{-1}C_j + D_j^{-1}\delta_i(C_j) + D_j^{-1}C_jB_i\right)\eta \\
&= U^{-1}\left(D_j^{-1}(C_jB_i + \delta_i(C_j) - \sigma_j(D_i)C_j)\right)\eta \\
&= U^{-1}\left(D_j^{-1}(\sigma_j(C_i)B_j - D_jC_i)\right)\eta = U^{-1}\left(D_j^{-1}\sigma_j(C_i)B_j - C_i\right)\eta,
\end{aligned}$$

where the last four equalities use (2.12) and the compatibility conditions on the D_i. Hence $\delta_i(\mathbf{b}_j) - (\sigma_j(\mathbf{b}_i) - \mathbf{b}_i) = 0$. Finally, for $\ell < i, j \leq m$, we have

$$\begin{aligned}
\sigma_i(\mathbf{b}_j) - \mathbf{b}_j &= U^{-1}\left(D_i^{-1}\sigma_i(D_j)^{-1}\sigma_i(C_j)B_i - D_j^{-1}C_j\right)\eta \\
&= U^{-1}\left(D_j^{-1}\sigma_j(D_i)^{-1}\sigma_i(C_j)B_i - D_j^{-1}C_j\right)\eta \\
&= U^{-1}\left(D_j^{-1}\sigma_j(D_i)^{-1}(\sigma_i(C_j)B_i - \sigma_j(D_i)C_j)\right)\eta \\
&= U^{-1}\left(D_i^{-1}\sigma_i(D_j)^{-1}(\sigma_j(C_i)B_j - \sigma_i(D_j)C_i)\right)\eta,
\end{aligned}$$

where (2.12) and the compatibility conditions on the D_i are used in the last three equalities. So, $\sigma_i(\mathbf{b}_j) - \mathbf{b}_j = \sigma_j(\mathbf{b}_i) - \mathbf{b}_i$ and the integrability conditions (2.7) are satisfied. Therefore G is an orthogonal Δ-extension of E, hence of F. Let $\zeta = U\mathbf{v} \in G^{n-d}$. Then

$$\partial_i(\zeta) = \delta_i(\zeta) = \delta_i(U)\mathbf{v} + U\delta_i(\mathbf{v}) = D_iU\mathbf{v} + U\mathbf{b}_i = D_i\zeta + C_i\eta, \quad \text{for } i \leq \ell,$$

and $\partial_j(\zeta) = \sigma_j(\zeta) = \sigma_j(U)\sigma_j(\mathbf{v}) = D_jU(\mathbf{v} + \mathbf{b}_j) = D_j\zeta + C_j\eta$ for $j > \ell$. So,

$$\partial_i \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} B_i\eta \\ D_i\zeta + C_i\eta \end{pmatrix} = \begin{pmatrix} B_i & 0 \\ C_i & D_i \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \quad 1 \leq i \leq m\,,$$

and $(\eta^\tau, \zeta^\tau)^\tau$ is a solution of the initial system $\mathcal{A}$. □

We point out here but omit detailed explanation that in the differential case, the quotient systems in [45] yield an alternative approach to completing solutions of factors.

Example 2.3.1 *Let $F = \mathbb{C}(x,k)$. Consider the fully integrable system*

$$\delta_x(\mathbf{z}) = \begin{pmatrix} B_x & 0 \\ C_x & D_x \end{pmatrix}\mathbf{z}, \quad \sigma_k(\mathbf{z}) = \begin{pmatrix} B_k & 0 \\ C_k & D_k \end{pmatrix}\mathbf{z}, \tag{2.13}$$

where $\mathbf{z} = (z_1, z_2, z_3)^\tau$, $B_x = \frac{x+k}{x}$, $B_k = \frac{(k+1)x}{k}$,

$$C_x = \begin{pmatrix} \frac{2x^2-k^2+2x-kx}{x(x-k)} \\ \frac{x^3-x^2k+2x^2-kx+2x-k^2}{(x-k)x} \end{pmatrix}, \quad C_k = \begin{pmatrix} \frac{(k+1)(x^3-2x^2k-3x^2+k^2x+4kx+x-k^2)}{k(x-k-1)^2} \\ \frac{x^2(k+1)}{k} - \frac{(k+1)(x-k)^2}{k(x-k-1)^2} - xk(x-1) \end{pmatrix},$$

and

$$D_x = \begin{pmatrix} \frac{-2-x+k}{x-k} & 0 \\ \frac{-2x-x^2+k^2}{(x-k)x} & \frac{k}{x} \end{pmatrix}, \quad D_k = \begin{pmatrix} \frac{(k+1)(x-k)^2}{k(x-k-1)^2} & 0 \\ \frac{(k+1)(x-k)^2}{k(x-k-1)^2} - kx & xk \end{pmatrix}.$$

We complete the solution $\eta_1 = ke^x x^k$ of the system given by B_x and B_k to a solution of (2.13). First, we compute a fundamental matrix

$$U = \begin{pmatrix} 0 & \frac{ke^{-x}}{(x-k)^2} \\ \Gamma(k)x^k & \frac{ke^{-x}}{(x-k)^2} \end{pmatrix} \in E^{2\times 2}$$

and a Picard-Vessiot ring $E = F[e^x, x^k, \Gamma(k), e^{-x}, x^{-k}, \Gamma(k)^{-1}]$, for the system given by D_x and D_k. Following the proof of Theorem 2.3.1, we let

$$\mathbf{b}_1 = \begin{pmatrix} \frac{kx}{\Gamma(k)}e^x \\ (x-k)(2x^2-k^2+2x-kx)\,x^{k-1}e^{2x} \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} \frac{x+kx+k^2-xk^2-k-1}{\Gamma(k+1)}e^x \\ (x^3-2kx^2-3x^2+k^2x+4kx+x-k^2)x^k e^{2x} \end{pmatrix}.$$

By integration-summation, we find

$$\mathbf{v} = \begin{pmatrix} \frac{\Gamma(k)-ke^x+xke^x}{\Gamma(k)} \\ x^{k+2}e^{2x} - 2x^{k+1}ke^{2x} + x^k k^2 e^{2x} + 1 \end{pmatrix}$$

satisfying $\delta_x(\mathbf{v}) = \mathbf{b}_1$ and $\sigma_k(\mathbf{v}) - \mathbf{v} = \mathbf{b}_2$. Therefore,

$$\begin{pmatrix} \eta_1 \\ U^{-1}\mathbf{v} \end{pmatrix} = \begin{pmatrix} ke^x x^k \\ ke^x x^k + \frac{ke^{-x}}{(x-k)^2} \\ x^{k+1}ke^x + \frac{ke^{-x}}{(x-k)^2} + \Gamma(k)x^k \end{pmatrix}$$

is a solution of (2.13). □

Theorem 2.3.1 also yields fundamental matrices for reducible systems. Consider a fully integrable system $\{\partial_i(\mathbf{z})=A_i\mathbf{z}\}_{1\leq i\leq m}$ where the A_i are as in (2.9). Let $U = (u_{ij})\in E^{d\times d}$ and $V \in R^{(n-d)\times(n-d)}$ be the respective fundamental matrices for $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\leq i\leq m}$ and $\{\partial_i(\mathbf{y}) = D_i\mathbf{y}\}_{1\leq i\leq m}$, where E and R are orthogonal Δ-extensions of F. As in the procedure of completing solutions, we can assume without loss of generality that E contains R. Then a fundamental matrix for the initial system can be constructed as follows: for each $1 \leq i \leq d$, following the procedure of completing solutions, we can find an orthogonal Δ-extension G_i of E and $\xi_i \in G_i^{n-d}$ such that $(u_{1i}, \ldots, u_{di}, \xi_i^\tau)^\tau \in G_i^n$ is a solution of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$. Viewing all the entries of U, V and the ξ_j as elements of $G = G_1 \otimes_F \cdots \otimes_F G_d$, the matrix

$$W = \begin{pmatrix} & U & & 0 \\ \xi_1 & \ldots & \xi_d & V \end{pmatrix} \in G^{n\times n}$$

is easily seen to be a fundamental matrix for $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ (it is invertible because $\det(W) = \det(U)\det(V)$).

2.4 Picard-Vessiot Extensions of Linear Functional Systems

In this section, we generalize the previous notions and results to ∂-finite linear functional systems, which include in particular all (fully) integrable systems.

This section is organized as follows. In Section 2.4.1, we define the notion of cokernels over a matrix over an arbitrary ring R. This notion corresponds to "generic solutions" of a linear functional system over R. The notion of Laurent-Ore algebras $\mathcal{L}$ is introduced in Section 2.4.2. In Section 2.4.3, we define the modules of formal solutions by specializing R to $\mathcal{L}$, and show that the modules of formal solutions capture some general properties of the solutions of a linear functional systems. A connection between ∂-finite linear functional systems and fully integrable systems is given in Section 2.4.4. With the help of this connection, we define the notions of fundamental matrices and Picard-Vessiot extensions for ∂-finite linear functional systems based on the results in Section 2.2.

2.4.1 Cokernels of Matrices over Arbitrary Rings

We study the problem of finding solutions of linear functional systems in a similar way to introduce "generic solutions" for algebraic equations.

Let R be an arbitrary ring. Denote by $Z(R)$ the *center* of R, the set of all elements that commute with every element in R. Then, $Z(R)$ is a subring of R. Consider a $p \times q$ matrix $A = (a_{ij})$ with entries in R. For any R-module N, we can associate to A a $Z(R)$-linear map $\lambda : N^q \rightarrow N^p$ given by

$$\xi := \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_q \end{pmatrix} \mapsto A\xi = \begin{pmatrix} \sum_{j=1}^q a_{1j}\xi_j \\ \vdots \\ \sum_{j=1}^q a_{pj}\xi_j \end{pmatrix}.$$

Note that λ is in general not R-linear. We therefore say that $\xi \in N^q$ is a *solution "in N"* of the system $A(\mathbf{z}) = 0$ if $A(\xi) = 0$, and write $\mathrm{sol}_N(A(\mathbf{z}) = 0)$ for the set of all solutions in N. Clearly, $\mathrm{sol}_N(A(\mathbf{z}) = 0)$ is a $Z(R)$-module.

As in the case of $\mathcal{D}$-modules [47], we can associate to $A \in R^{p\times q}$ an R-module as follows: the matrix A induces the R-linear map $\rho : R^{1\times p} \rightarrow R^{1\times q}$ given by $(r_1, \ldots, r_p) \mapsto (r_1, \ldots, r_p)A$. Let $M = R^{1\times q}/(R^{1\times p}A)$, which is simply the quotient of $R^{1\times q}$ by the image of the map ρ. We call M the *R-cokernel* of A and denote it by $\mathrm{coker}_R(A)$. Clearly, $\mathrm{coker}_R(A)$ is an R-module. Let $\mathbf{e}_{1p}, \ldots, \mathbf{e}_{pp}$ and $\mathbf{e}_{1q}, \ldots, \mathbf{e}_{qq}$ be the canonical bases of $R^{1\times p}$ and $R^{1\times q}$, respectively. Denote by π the canonical map from $R^{1\times q}$ to $\mathrm{coker}_R(A)$, and set $\mathbf{e}_j = \pi(\mathbf{e}_{jq})$ for $1 \leq j \leq q$. Since π is surjective, M is generated by $\mathbf{e}_1, \ldots, \mathbf{e}_q$ over R. Note that $\rho(\mathbf{e}_{ip})$ is the i-th row of A. Hence

$$0 = \pi(\rho(\mathbf{e}_{ip})) = \pi\left(\sum_{j=1}^{q} a_{ij}\mathbf{e}_{jq}\right) = \sum_{j=1}^{q} a_{ij}\pi(\mathbf{e}_{jq}) = \sum_{j=1}^{q} a_{ij}\mathbf{e}_j \quad \text{for } 1 \leq i \leq p,$$

which implies that $A(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = 0$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau$ is a solution of $A(\mathbf{z}) = 0$ in M.

Given two R-modules N_1 and N_2, denote by $\mathrm{Hom}_R(N_1, N_2)$ the set of all R-linear maps from N_1 to N_2. Clearly, $\mathrm{Hom}_R(N_1, N_2)$ is a $Z(R)$-module.

As illustrated by the following theorem, Proposition 1.1 of [47] remains true when $\mathcal{D}$ is replaced by an arbitrary ring R.

Theorem 2.4.1 *Let R be a ring, $A \in R^{p\times q}$ and $M := \operatorname{coker}_R(A) = R^{1\times q}/\left(R^{1\times p}A\right)$. Then $\operatorname{Hom}_R(M,N)$ and $\operatorname{sol}_N(A(\mathbf{z}) = 0)$ are isomorphic as $Z(R)$-modules, for any R-module N.*

Proof. Although the proof of Proposition 1.1 in [47] can be adapted to this theorem in a straightforward way (see [13]), we give here a slightly different but elementary proof. Let $\mathbf{e}_{1q}, \ldots, \mathbf{e}_{qq}$ be the canonical basis of $R^{1\times q}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_q$ be their respective images in M. Consider the map $\pi : \operatorname{Hom}_R(M,N) \to \operatorname{sol}_N(A(\mathbf{z}) = 0)$ given by $\phi \mapsto (\phi(\mathbf{e}_1), \ldots, \phi(\mathbf{e}_q))^\tau$. Since $(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau$ is a solution of $A(\mathbf{z}) = 0$ and ϕ is R-linear, π is well-defined. In addition, π is $Z(R)$-linear, and injective because $\mathbf{e}_1, \ldots, \mathbf{e}_q$ generate M as an R-module. It remains to show that π is surjective. Let $s_1, \ldots, s_q$ be in N such that $(s_1, \ldots, s_q)^\tau$ is a solution of $A(\mathbf{z}) = 0$. Consider the R-linear map $\varphi : R^{1\times q} \to N$ that sends $\mathbf{e}_{iq}$ to s_i for each i. The map $\bar{\varphi} : R^{1\times q}/\ker(\varphi) \to N$ is well-defined and R-linear. Set $A = (a_{ij})$. From $R^{1\times p}A = \sum_{i=1}^p R\mathbf{e}_{ip}A$, any element of $R^{1\times p}A$ has the form $\sum_{i=1}^p r_i(a_{i1}, \ldots, a_{iq})$, then its application to $(s_1, \ldots, s_q)^\tau$ is zero since $A(s_1, \ldots, s_q)^\tau = 0$. Thus $\ker(\varphi)$ contains $R^{1\times p}A$. It follows that the map $\psi : M \to R^{1\times q}/\ker(\varphi)$ given by $a + R^{1\times p}A \mapsto a + \ker(\varphi)$ is well-defined and R-linear. Moreover, $\bar{\varphi}\circ\psi \in \operatorname{Hom}_R(M,N)$ sends $\mathbf{e}_i$ to s_i. Thus π is surjective. □

2.4.2 Laurent-Ore Algebras

Let (F, Φ, ℓ) be an orthogonal Δ-field and $\mathcal{S} = F[\partial_1; \sigma_i, \delta_i] \cdots [\partial_m; \sigma_m, \delta_m]$ be the corresponding Ore algebra over F. Given a linear functional system $A(\mathbf{z}) = 0$ over F. In the differential case, an $\mathcal{S}$-module is classically associated to such a system [47, 53]. However, in the difference case, $\mathcal{S}$-modules may not have appropriate dimensions, as illustrated by the following counterexample.

Example 2.4.1 *Let $\sigma \neq \mathbf{1}$ be an automorphism of F and $\mathcal{S} = F[\partial; \sigma, \mathbf{0}]$ be the corresponding Ore algebra. The equation $\partial(y) = 0$ cannot have a fundamental matrix (u) in any difference ring extension of (F, σ), for otherwise, $0 = \partial(u) = \sigma(u)$, thus $u = 0$. Therefore the equation $\partial(y) = 0$ has only trivial solution in any difference extension. However, the $\mathcal{S}$-module $\mathcal{S}/\mathcal{S}\partial$ has dimension one as an F-vector space.* □

In [52], modules over Laurent algebras are used instead to avoid the above problem. It is therefore natural to introduce in our setting the following extension of $\mathcal{S}$: let $\theta_{\ell+1}, \ldots, \theta_m$ be indeterminates independent of the ∂_i. Since the σ_j^{-1} are automorphisms of F,

$$\overline{\mathcal{S}} = \mathcal{S}[\theta_{\ell+1}; \sigma_{\ell+1}^{-1}, \mathbf{0}] \cdots [\theta_m; \sigma_m^{-1}, \mathbf{0}]$$

is also an Ore algebra. Since

$$(\partial_j \theta_j) a = \partial_j \sigma_j^{-1}(a) \theta_j = \sigma_j(\sigma_j^{-1}(a)) \partial_j \theta_j = a \partial_j \theta_j$$

for all $a \in F$ and $j > \ell$, $\partial_j \theta_j$ is in the center of $\bar{\mathcal{S}}$. Therefore the left ideal

$$T = \sum_{j=\ell+1}^{m} \bar{\mathcal{S}}(\partial_j \theta_j - 1)$$

is a two-sided ideal of $\bar{\mathcal{S}}$ and $\mathcal{L} := \bar{\mathcal{S}}/T = F[\overline{\partial}_1, \ldots, \overline{\partial}_m, \overline{\theta}_{\ell+1}, \ldots, \overline{\theta}_m]$ is a factor ring where the $\overline{\partial}_i$ and $\overline{\theta}_j$ denote the canonical images of ∂_i and θ_j, respectively. It follows that $\overline{\partial}_i a = \overline{\partial_i a} = \sigma_i(a)\overline{\partial}_i + \delta_i(a)$, $\overline{\theta}_j a = \overline{\theta_j a} = \sigma_i^{-1}(a)\overline{\theta}_j$ and $\overline{\partial}_j \overline{\theta}_j = \overline{\theta}_j \overline{\partial}_j = 1$ for all $a \in F$ and i, j with $1 \le i \le m$, $\ell + 1 \le j \le m$. Identifying $\overline{\partial}_i$ with ∂_i and writing ∂_j^{-1} for $\overline{\theta}_j$, we can write $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ and view it as an extension of $\mathcal{S}$.

Definition 2.4.2 *Let (F, Φ, ℓ) be an orthogonal Δ-field, $\bar{\mathcal{S}}$ and T be as above. We call the factor ring $\bar{\mathcal{S}}/T$ the* Laurent-Ore algebra *over F, and write it by convention*

$$\mathcal{L} = F[\partial_1; \mathbf{1}, \delta_1] \cdots [\partial_\ell; \mathbf{1}, \delta_\ell][\partial_{\ell+1}, \partial_{\ell+1}^{-1}; \sigma_{\ell+1}, \mathbf{0}] \cdots [\partial_m, \partial_m^{-1}; \sigma_m, \mathbf{0}],$$

where $[\partial_j, \partial_j^{-1}; \sigma_j, \mathbf{0}]$ means that the ∂_j and ∂_j^{-1} are pseudo-linear operators ([14]) associated to $(\sigma_j, \mathbf{0})$ and $(\sigma_j^{-1}, \mathbf{0})$, respectively. Simply, we write $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$.

For linear ordinary difference equations, $\mathcal{L} = F[\sigma, \sigma^{-1}]$ is the algebra used in [52]. For linear partial difference equations with constant coefficients, $\mathcal{L}$ is the Laurent polynomial ring used in [49, 63]. We remark that, except for the purely differential case where ℓ=0, Laurent-Ore algebras $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ are not Ore algebras, because the $\partial_j \partial_j^{-1} = \partial_j^{-1} \partial_j = 1$.

Remark 2.4.3 *Every element of $\mathcal{L}$ can be written as an element of $\mathcal{S}$ multiplied by some monomial in $\partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}$, either from the left-hand side or from the right-hand side.*

Let us revisit the equation in Example 2.4.1 with the newly defined Laurent-Ore algebras.

Example 2.4.4 *Let $\mathcal{L} = F[\partial, \partial^{-1}]$ be the corresponding Laurent-Ore algebra. Since ∂ is invertible, the left ideal generated by ∂ in $\mathcal{L}$ is $\mathcal{L}$, therefore the $\mathcal{L}$-module $\mathcal{L}/(\mathcal{L}\partial) = 0$ and has zero dimension over F.* □

2.4.3 Modules of Formal Solutions

The introduction of Laurent-Ore algebras allows us to construct fundamental matrices and Picard-Vessiot extensions for linear functional systems.

Let (F, Φ, ℓ) be an orthogonal Δ-field, $\mathcal{S}$ and $\mathcal{L}$ be the corresponding Ore and Laurent-Ore algebras, respectively. Given a linear functional system $A(\mathbf{z}) = 0$ where $A \in \mathcal{S}^{p\times q} \subseteq \mathcal{L}^{p\times q}$, we can construct both its $\mathcal{S}$-cokernel $\text{coker}_{\mathcal{S}}(A)$ and $\mathcal{L}$-cokernel $\text{coker}_{\mathcal{L}}(A)$. However, as illustrated by Examples 2.4.1 and 2.4.4, when difference operators are involved, $\text{coker}_{\mathcal{S}}(A)$ may not have the right dimension while $\text{coker}_{\mathcal{L}}(A)$ always has the right one. This motivates us to define

Definition 2.4.5 *Let $A \in \mathcal{S}^{p\times q} \subseteq \mathcal{L}^{p\times q}$. The $\mathcal{L}$-module $M = \mathcal{L}^{1\times q}/(\mathcal{L}^{1\times p}A)$ is called the* module of formal solutions *of the system $A(\mathbf{z}) = 0$. The dimension of M as an F-vector space is called the* linear dimension *of the system. The system is said to be* of finite linear dimension*, or simply,* ∂-finite*, if* $0 < \dim_F M < +\infty$.

Remark 2.4.6

(i) *The notion of modules of formal solutions for systems has already appeared in [2].*

(ii) *We choose to exclude in our definition the systems $A(\mathbf{z}) = 0$ with $\dim_F M = 0$, since $\dim_F M = 0$ implies that $\mathcal{L}^{1\times q} = \mathcal{L}^{1\times p}A$, consequently the system has only trivial solution in any $\mathcal{L}$-module, which includes all orthogonal Δ-extensions of F.*

(iii) *Let* $\mathbf{e}_1, \ldots, \mathbf{e}_q$ *be the images of the* $\mathbf{e}_{1q}, \ldots, \mathbf{e}_{qq}$ *in* M*. Then* $A(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = 0$*. Therefore* M *always contains a set of* $\mathcal{L}$*-generators that solve* $A(\mathbf{z}) = 0$*.*

As we saw earlier, although a linear functional system may be represented by a form other than $A(\mathbf{z}) = 0$, it can always be rewritten equivalently into the latter form. We therefore understand the module of formal solutions for $A(\mathbf{z}) = 0$ to be the module of formal solutions for the original system.

In terms of modules of formal solutions for linear functional systems, Theorem 2.4.1 can be rephrased as

Theorem 2.4.2 *Let* $A \in \mathcal{S}^{p\times q}$ *and* M *be the module of formal solutions for the system* $A(\mathbf{z}) = 0$*. Then* $\mathrm{Hom}_{\mathcal{L}}(M, N)$ *and* $\mathrm{sol}_N(A(\mathbf{z}) = 0)$ *are isomorphic as* C_F*-vector spaces for any* $\mathcal{L}$*-module* N*. In particular,* $\mathrm{Hom}_{\mathcal{L}}(M, E)$ *and* $\mathrm{sol}_E(A(\mathbf{z}) = 0)$ *are isomorphic as* C_F*-vector spaces for any orthogonal* Δ*-extension* E *of* F*.*

Remark 2.4.7 *Since every orthogonal* Δ*-extension* E *of* F *is turned into an* $\mathcal{L}$*-module via the actions* $\partial_i(e) = \delta_i(e)$ *for* $i \leq \ell$*,* $\partial_j(e) = \sigma_j(e)$ *and* $\partial_j^{-1}(e) = \sigma_j^{-1}(e)$ *for* $j>\ell$*,* $\mathrm{sol}_E(A(\mathbf{z}){=}0)$ *in Theorem 2.4.2 is well-defined.*

Proposition 2.4.3 *Let* $A \in \mathcal{S}^{p\times q}$ *and* $B \in \mathcal{S}^{n\times d}$*, and* M_A *and* M_B *be the modules of formal solutions of* $A(\mathbf{z}) = 0$ *and* $B(\mathbf{x}) = 0$*, respectively. If* M_A *is isomorphic to* M_B *as* $\mathcal{L}$*-modules, then there exists* $Q \in \mathcal{L}^{q\times d}$ *such that for any orthogonal* Δ*-extension* E *of* F*, the correspondence* $\xi \mapsto Q\xi$ *is an isomorphism of* C_E*-modules between* $\mathrm{sol}_E(B(\mathbf{x}) = 0)$ *and* $\mathrm{sol}_E(A(\mathbf{z}) = 0)$*.*

Proof. To simplify notation, we denote $\mathrm{sol}_E(A(\mathbf{z}){=}0)$ and $\mathrm{sol}_E(B(\mathbf{x}){=}0)$ by W_A and W_B, respectively. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_d\}$ be the sets of $\mathcal{L}$-generators of M_A and M_B such that $A(\mathbf{e}) = 0$ and $B(\mathbf{f}) = 0$, respectively, where $\mathbf{e}{=}(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau$ and $\mathbf{f}{=}(\mathbf{f}_1, \ldots, \mathbf{f}_d)^\tau$. Let $\pi : M_A \to M_B$ be the $\mathcal{L}$-module isomorphism. Then there exists $Q \in \mathcal{L}^{q\times d}$ such that $\pi(\mathbf{e}) = Q\,\mathbf{f}$. Since π is $\mathcal{L}$-linear, $A(Q\,\mathbf{f}) = A(\pi(\mathbf{e})) = \pi(A(\mathbf{e})) = 0$. Let $\xi \in W_B$. By

Theorem 2.4.2, there exists $\phi \in \mathrm{Hom}_{\mathcal{L}}(M_B, E)$ such that $\xi = \phi(\mathbf{f})$. Hence

$$A(Q\,\xi) = A(Q\,\phi(\mathbf{f})) = \phi(A(Q\,\mathbf{f})) = 0,$$

so $Q\xi$ belongs to W_A. Thus the correspondence $\xi \mapsto Q\,\xi$ is a homomorphism of C_E-modules from W_B to W_A.

For every $\eta \in W_A$, there exists $\psi \in \mathrm{Hom}_{\mathcal{L}}(M_A, E)$ such that

$$\eta = \psi(\mathbf{e}) = \psi(\pi^{-1}(Q\,\mathbf{f})) = Q\,\psi(\pi^{-1}(\mathbf{f})).$$

Clearly, $B(\psi(\pi^{-1}(\mathbf{f}))) = \psi \circ \pi^{-1}(B(\mathbf{f})) = 0$, thus $\psi(\pi^{-1}(\mathbf{f}))$ is in W_B and the correspondence $\xi \mapsto Q\,\xi$ is surjective. If $\xi \in W_B$ and $Q\,\xi = 0$, there is $\varphi \in \mathrm{Hom}_{\mathcal{L}}(M_B, E)$ such that $\xi = \varphi(\mathbf{f})$. Hence $0 = Q\,\xi = \varphi(Q\,\mathbf{f}) = \varphi(\pi(\mathbf{e}))$. This means that $\varphi \circ \pi$ maps everything to 0 as M_A is generated by $\mathbf{e}_1, \ldots, \mathbf{e}_q$ over $\mathcal{L}$. Hence $\varphi = (\varphi \circ \pi) \circ \pi^{-1} = 0$. Thus $\xi = 0$ and the correspondence is bijective. □

Proposition 2.4.3 reveals that, for two linear functional systems, if their modules of formal solutions are isomorphic then there is a one-to-one correspondence between their solution spaces in an orthogonal Δ-extension of F.

We now establish a connection between submodules of $\mathcal{S}^{1\times q}$ and those of $\mathcal{L}^{1\times q}$.

Recall that a (left) ideal I of $\mathcal{S}$ is said to be *of finite rank* ([39]) if $\mathcal{S}/I$ is a finite-dimensional F-vector space.

Lemma 2.4.4 *Let J be a finite-rank left ideal of $\mathcal{S}$ and I be the left ideal generated by J in $\mathcal{L}$. Then every element of $\mathcal{L}$ is congruent to some element of $\mathcal{S}$ modulo I.*

Proof. If J contains some monomial in $\partial_{\ell+1}, \ldots, \partial_m$, then $I = \mathcal{L}$ and every $f \in \mathcal{L}$ is congruent to 0 modulo I. Assume that J does not contain any monomial in $\partial_{\ell+1}, \ldots, \partial_m$. Set $\bar{J}$ to be the contraction $I \cap \mathcal{S}$ and let H be the set of all monomials in $\partial_{\ell+1}, \ldots, \partial_m$. Since every element of H is invertible in $\mathcal{L}$, one can verify that

$$\bar{J} = \{a \in \mathcal{S} \mid ha \in J \text{ for some } h \in H\}. \tag{2.14}$$

Since $J \subseteq \bar{J}$, $\dim_F \mathcal{S}/\bar{J}$ is finite and $\bar{J}$ is of finite rank. For $j > \ell$, let f_j be a nonzero polynomial in $F[\partial_j] \cap \bar{J}$ with minimal degree. Then each f_j is of positive degree with a nonzero coefficient of ∂_j^0, for otherwise, $\bar{J}$ would contain 1, and, hence by (2.14), J would have a nonempty intersection with H, a contradiction to the assumption. Since $\partial_j^{-1} f_j \in I$, ∂_j^{-1} is congruent to an element of $F[\partial_j]$ modulo I. It follows that every element of $\mathcal{L}$ is congruent to an element of $\mathcal{S}$ modulo I. □

The following lemma is a generalization of Lemma 2.4.4 to submodules of free $\mathcal{S}$-modules.

Lemma 2.4.5 *Let N be a left submodule of $\mathcal{S}^{1\times n}$ where $n \geq 1$ such that $\mathcal{S}^{1\times n}/N$ is finite-dimensional over F, and M be the submodule generated by N in $\mathcal{L}^{1\times n}$. Then every element of $\mathcal{L}^{1\times n}$ is congruent to some element of $\mathcal{S}^{1\times n}$ modulo M.*

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of $\mathcal{S}^{1\times n}$ and $J_i = \{f \in \mathcal{S} \mid f(\mathbf{e}_i) \in N\}$ for $1 \leq i \leq n$. Then J_i is a left ideal of $\mathcal{S}$ for each i. Since $\mathcal{S}^{1\times n}/N$ is finite-dimensional over F, there exists $f_{ij} \in F[\partial_j]$ such that $f_{ij}(\mathbf{e}_i)$ belongs to N for all i, j with $1 \leq i \leq n$ and $1 \leq j \leq m$. This implies that a rectangular system([19]) $\{f_{i1}, \ldots, f_{im}\}$ is contained in J_i for each i, hence J_i is of finite rank by Proposition 2.1 of [19] or by Lemma 2.1 in [43]. Since every element $\mathbf{w}$ of M has the form $\sum_{i=1}^n g_i \mathbf{u}_i$ where $g_i \in \mathcal{L}$ and $\mathbf{u}_i \in N$, we have $\mathbf{w} = p \sum_{i=1}^n g_i' \mathbf{u}_i$ where $g_i' \in \mathcal{S}$ and $p \in P := \{\partial_{\ell+1}^{k_{\ell+1}} \cdots \partial_m^{k_m} \mid k_j \leq 0\}$. So $M = \{p \cdot \mathbf{u} \mid p \in P, \mathbf{u} \in N\}$. Let I_i be the ideal generated by J_i in $\mathcal{L}$ and $H_i = \{f \in \mathcal{L} \mid f\mathbf{e}_i \in M\}$ for $i = 1, \ldots, n$. If $f \in H_i$ then $f\mathbf{e}_i \in M$ and $f\mathbf{e}_i = p\mathbf{u}$ with $p \in P$ and $\mathbf{u} \in N$. Since $\mathbf{e}_i$ is the unit vector where 1 is the ith entry and 0 elsewhere, we must have $f\mathbf{e}_i = pa\mathbf{e}_i$ for some $a \in \mathcal{S}$ and hence $\mathbf{u} = a\mathbf{e}_i$. It follows that $a \in J_i$ and $f = pa \in I_i$. Conversely, if $h \in I_i$ then $h = a_1 f_1 + \cdots + a_t f_t$ with $a_j \in \mathcal{L}$ and $f_j \in J_i$. Since $f_j \mathbf{e}_i \in N$ for each j, $h\mathbf{e}_i \in M$. So $h \in H_i$ and $I_i = H_i$.

For any $\mathbf{w} \in \mathcal{L}^{1\times q}$, we have $\mathbf{w} = \sum_{i=1}^n f_i \mathbf{e}_i$ where $f_i \in \mathcal{L}$. From Lemma 2.4.4 and $I_i = H_i$, each f_i is congruent to some $g_i \in \mathcal{S}$ modulo H_i, thus $f_i = g_i + h_i$ for some $h_i \in H_i$. So $\mathbf{w} = (\sum_{i=1}^n g_i \mathbf{e}_i) + (\sum_{i=1}^n h_i \mathbf{e}_i) \in \mathcal{S}^{1\times n} + M$ and the lemma follows. □

Proposition 2.4.6 *Let N be a left submodule of $\mathcal{S}^{1\times n}$ such that $\mathcal{S}^{1\times n}/N$ has finite dimension over F, M the submodule generated by N in $\mathcal{L}^{1\times n}$ and $\overline{N} = M \cap \mathcal{S}^{1\times n}$. Then $\mathcal{S}^{1\times n}/\overline{N}$ and $\mathcal{L}^{1\times n}/M$ are isomorphic as F-vector spaces.*

Proof. Let ϕ be the map from $\mathcal{S}^{1\times n}/\overline{N}$ to $\mathcal{L}^{1\times n}/M$ given by $\mathbf{u}+\overline{N}\mapsto \mathbf{u}+M$. The map ϕ is well-defined because $\overline{N}\subseteq M$. Suppose that $\phi(\mathbf{u}+\overline{N})=\mathbf{u}+M=M$. Then $\mathbf{u}\in M\cap\mathcal{S}^{1\times n}=\overline{N}$ and ϕ is injective. From Lemma 2.4.5, every $\mathbf{w}\in\mathcal{L}^{1\times n}$ is congruent to some $\mathbf{u}\in\mathcal{S}^{1\times n}$ modulo M. Thus $\mathbf{w}+M=\mathbf{u}+M=\phi(\mathbf{u}+\overline{N})$ and ϕ is surjective. □

Remark 2.4.8 *One can check easily that*

$$\overline{N}=\left\{\mathbf{w}\in\mathcal{S}^n \mid \partial_{\ell+1}^{k_{\ell+1}}\cdots\partial_m^{k_m}(\mathbf{w})\in N,\ \textit{for some } k_{\ell+1},\ldots,k_m\in\mathbb{N}\right\}.$$

When $n=1$, Proposition 2.4.6 specializes to Lemma 2 in [13]. For later references, we recite this specialization as follows.

Corollary 2.4.7 *Let J be a finite-rank left ideal of $\mathcal{S}$, I be the left ideal generated by J in $\mathcal{L}$ and $\overline{I}=I\cap\mathcal{S}$. Then $\mathcal{S}/\overline{I}$ and $\mathcal{L}/I$ are isomorphic as F-vector spaces.*

Example 2.4.9 *Let $A_1,\ldots,A_m$ be arbitrary matrices in $F^{n\times n}$ and A be the stacking of the blocks of the form $(\partial_i\cdot\mathbf{1}_n-A_i)$:*

$$A=\begin{pmatrix}\partial_1\cdot\mathbf{1}_n-A_1\\ \vdots\\ \partial_m\cdot\mathbf{1}_n-A_m\end{pmatrix}\in\mathcal{S}^{mn\times n}\subseteq\mathcal{L}^{mn\times n}.$$

The system $A(\mathbf{z})=0$ then corresponds to the system $\{\partial_i(\mathbf{z})=A_i\mathbf{z}\}_{1\le i\le m}$. Let M be the module of formal solutions of $A(\mathbf{z})=0$. Let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be the images of $\mathbf{e}_{1n},\ldots,\mathbf{e}_{nn}$ in M, respectively. For $\mathbf{e}=(\mathbf{e}_1,\ldots,\mathbf{e}_n)^\tau\in M^n$, we have $A(\mathbf{e})=0$ or $\partial_i(\mathbf{e})=A_i\mathbf{e}$ for each i. Since the entries of A_i are in F, $\partial_i(\mathbf{e}_j)\in\sum_{s=1}^n F\mathbf{e}_s$ for all i,j, thus $\mathcal{L}\mathbf{e}_j\subseteq\sum_{s=1}^n F\mathbf{e}_s$ for all j. So $M=\sum_{s=1}^n\mathcal{L}\mathbf{e}_s=\sum_{s=1}^n F\mathbf{e}_s$, i.e., $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ is also a set of F-generators of M. In particular, $\dim_F M\le n$. □

Example 2.4.10 *Consider a $p\times 1$ matrix $A=(L_1,\ldots,L_p)^\tau$ with $L_i\in\mathcal{S}$. Let J and I be the ideals generated by $L_1,\ldots,L_p$ in $\mathcal{S}$ and $\mathcal{L}$, respectively. The module of formal solutions for $A(\mathbf{z})=0$ is $M=\mathcal{L}/I$, which by Corollary 2.4.7 is finite-dimensional over F if $\dim_F\mathcal{S}/J$ is finite.* □

Proposition 2.4.6 indicates that the linear dimension of $A(\mathbf{z}) = 0$ can be computed without using Laurent-Ore algebras if $\operatorname{coker}_{\mathcal{S}}(A)$ is finite-dimensional over F. However, the following example shows that there are systems whose $\mathcal{S}$-cokernels are infinite-dimensional over F. □

Example 2.4.11 *Let $A = (L_1, L_2)^\tau$ with $L_1 = \partial_1\partial_2(\partial_1 + 1)$ and $L_2 = \partial_1\partial_2(\partial_2 + 1)$, J the ideal in $\mathcal{S}$ generated by L_1 and L_2, and $M = \operatorname{coker}_{\mathcal{L}}(A)$ (i.e., the module of formal solutions of $A(\mathbf{z}) = 0$). Since ∂_1 and ∂_2 are invertible in $\mathcal{L}$,*

$$M = \mathcal{L}/(\mathcal{L}L_1 + \mathcal{L}L_2) = \mathcal{L}/(\mathcal{L}(\partial_1 + 1) + \mathcal{L}(\partial_2 + 1)),$$

thus $\dim_F M = 1$. However, $\mathcal{S}/J$ is infinite-dimensional over F. □

Example 2.4.12 *Consider the case $\ell = 0$ and $m = 2$. Let $A = \partial_1 + 1$. Then both its $\mathcal{S}$-cokernel $\mathcal{S}/\mathcal{S}A$ and its $\mathcal{L}$-cokernel $\mathcal{L}/\mathcal{L}A$ are infinite-dimensional over F. Systems of this kind are out of scope of the thesis.* □

2.4.4 Integrable Connections

In this section, we study systems of the form $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$. We are interested in systems of this kind, because the problem of finding solutions of a general linear functional system can be finally reduced to that of finding solutions of such a system.

We first look at some properties of $\mathcal{L}$-modules of finite dimension.

Proposition 2.4.8 *Let M be an $\mathcal{L}$-module with a finite basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ over F. Suppose that $\partial_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = B_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ where $B_i \in F^{n\times n}$ for $1 \le i \le m$. Then the system $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ is fully integrable.*

Proof. Set $\mathbf{b}=(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$. Since ∂_i and ∂_j commute for any i and j, $\partial_i(\partial_j(\mathbf{b}))=\partial_j(\partial_i(\mathbf{b}))$. From $\partial_i(\mathbf{b}) = B_i\mathbf{b}$ and the linear independence of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ over F, it follows that

$$\sigma_i(B_j)B_i + \delta_i(B_j) = \sigma_j(B_i)B_j + \delta_j(B_i), \quad \text{for } 1 \le i, j \le m,$$

i.e., $B_1, \dots, B_m$ satisfy the compatibility conditions (2.2). Therefore $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ is integrable.

Suppose that B_t is singular for some $t > \ell$. Then there is a nonzero $v \in F^{1\times n}$ such that $vB_t = 0$, thus $v\partial_t(\mathbf{b}){=}vB_t\mathbf{b} = 0$. Since M is an $\mathcal{L}$-module on which ∂_t^{-1} acts, $0{=}\partial_t^{-1}(v\partial_t(\mathbf{b})){=}\sigma_t^{-1}(v)\partial_t^{-1}(\partial_t(\mathbf{b})){=}\sigma_t^{-1}(v)\mathbf{b}$, which implies that $\mathbf{b}_1, \dots, \mathbf{b}_n$ are linearly dependent over F, a contradiction. So the B_j are invertible for $\ell + 1 \le j \le m$ and the system $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ is fully integrable. □

As we saw in Example 2.4.9, for a system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$, the dimension of its module of formal solutions over F is not greater than the size of the system. Furthermore,

Proposition 2.4.9 *A system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ over F is fully integrable if and only if the dimension of its module M of formal solutions over F is equal to the size of the system.*

Proof. Let n be the size of the system and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a set of $\mathcal{L}$-generators of M such that $\partial_i(\mathbf{e}_1, \dots, \mathbf{e}_n)^\tau = A_i(\mathbf{e}_1, \dots, \mathbf{e}_n)^\tau$. If $\dim_F M = n$, then $\mathbf{e}_1, \dots, \mathbf{e}_n$ form an F-basis of M, therefore $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ is fully integrable by Proposition 2.4.8. Conversely, if $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ is fully integrable, then by Theorem 2.2.2 it has a fundamental matrix $U = (u_1, \dots, u_n) \in E^{n\times n}$ where the u_j are columns of U and E is a (simple) orthogonal Δ-extension of F. Since u_j is a solution of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$, by Theorem 2.4.2 there exists $\varphi_j \in \mathrm{Hom}_{\mathcal{L}}(M, E)$ such that $u_j = \varphi_j(\mathbf{e})$ for $1 \le j \le n$. Let $\sum_{i=1}^n \lambda_i\mathbf{e}_i = 0$ where $\lambda_i \in F$. It follows that $(\lambda_1, \dots, \lambda_n)u_j = (\lambda_1, \dots, \lambda_n)\varphi_j(\mathbf{e}) = 0$ for $1 \le j \le n$, *i.e.*, $(\lambda_1, \dots, \lambda_n)U = 0$. Since U is invertible, $\lambda_i = 0$ for each i and $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent over F. □

Recall the notion of a tensor product of modules over noncommutative rings given in [54]. Let R be a noncommutative ring, M a right R-module and N a left R-module. We can define the abelian group $M \otimes_R N$, called the *tensor product* of M and N over R, as the free abelian group generated by the $\mathbb{Z}$-linear combinations of all the pairs $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u} \in M$ and $\mathbf{v} \in N$, modulo the subgroup generated by the expressions of the form

$$(\mathbf{u}+\mathbf{u}', \mathbf{v}) - (\mathbf{u}, \mathbf{v}) - (\mathbf{u}', \mathbf{v}), \quad (\mathbf{u}, \mathbf{v}+\mathbf{v}') - (\mathbf{u}, \mathbf{v}) - (\mathbf{u}, \mathbf{v}'), \quad (\mathbf{u}\,r, \mathbf{v}) - (\mathbf{u}, r\,\mathbf{v}),$$

for all $\mathbf{u}, \mathbf{u}' \in M$, $\mathbf{v}, \mathbf{v}' \in N$ and $r \in R$. The equivalence class of $(\mathbf{u}, \mathbf{v}) \in M \times N$ is denoted $\mathbf{u} \otimes \mathbf{v}$ in $M \otimes_R N$.

For a matrix $A \in \mathcal{S}^{p\times q} \subseteq \mathcal{L}^{p\times q}$, we can construct its $\mathcal{S}$-cokernel $\mathrm{coker}_{\mathcal{S}}(A)$ and $\mathcal{L}$-cokernel $\mathrm{coker}_{\mathcal{L}}(A)$. Since $\mathcal{L}$ is a right $\mathcal{S}$-module and $\mathrm{coker}_{\mathcal{S}}(A)$ is a left $\mathcal{S}$-module, the tensor product $\mathcal{L} \otimes_{\mathcal{S}} \mathrm{coker}_{\mathcal{S}}(A)$ is well-defined by the above argument and we have the following relations in $\mathcal{L} \otimes_{\mathcal{S}} \mathrm{coker}_{\mathcal{S}}(A)$:

$$(l + l') \otimes \mathbf{v} = l \otimes \mathbf{v} + l' \otimes \mathbf{v}, \quad l \otimes (\mathbf{v} + \mathbf{v}') = l \otimes \mathbf{v} + l \otimes \mathbf{v}', \quad l\, s \otimes \mathbf{v} = l \otimes s\, \mathbf{v},$$

for $l, l' \in \mathcal{L}$, $\mathbf{v}, \mathbf{v}' \in \mathrm{coker}_{\mathcal{S}}(A)$ and $s \in \mathcal{S}$. Further, with $\mathcal{L}$ viewed as a left $\mathcal{L}$-module, the tensor product $\mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A)$ becomes a left $\mathcal{L}$-module endowed with the action

$$l'(l \otimes \mathbf{v}) = (l'l) \otimes \mathbf{v}, \quad \text{for all } l, l' \in \mathcal{L},\ \mathbf{v} \in \mathrm{coker}_{\mathcal{S}}(A).$$

Lemma 2.4.10 *Let $A \in \mathcal{S}^{p\times q}$. Then $\mathrm{coker}_{\mathcal{L}}(A)$ and $\mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A)$ are isomorphic as $\mathcal{L}$-modules.*

Proof. Let $\mathbf{e}_1^*, \ldots, \mathbf{e}_q^*$ and $\mathbf{e}_1, \ldots, \mathbf{e}_q$ be the respective images of $\mathbf{e}_{1q}, \ldots, \mathbf{e}_{qq}$ in $\mathrm{coker}_{\mathcal{S}}(A)$ and $\mathrm{coker}_{\mathcal{L}}(A)$. Let π be the map $\mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A) \to \mathrm{coker}_{\mathcal{L}}(A)$ given by

$$\sum_{i=1}^{q} l_i \otimes \mathbf{e}_i^* \mapsto \sum_{i=1}^{q} l_i \mathbf{e}_i, \ \text{ where } \ l_i \in \mathcal{L},$$

and λ the map $\mathrm{coker}_{\mathcal{L}}(A) \to \mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A)$ given by $\sum_{i=1}^{q} l_i \mathbf{e}_i \mapsto \sum_{i=1}^{q} l_i \otimes \mathbf{e}_i^*$ where $l_i \in \mathcal{L}$. We need to show that λ is well-defined since any element in $\mathrm{coker}_{\mathcal{L}}(A)$ may be written in various combinations of $\mathbf{e}_1, \ldots, \mathbf{e}_q$. Suppose that $\sum_{i=1}^{q} l_i \mathbf{e}_i = \sum_{i=1}^{q} k_i \mathbf{e}_i$ with $l_i, k_i \in \mathcal{L}$. Let us set $A = (a_{st})_{1\le s\le p,\, 1\le t\le q}$ where $a_{st} \in \mathcal{S}$. It follows that $\sum_{i=1}^{q} (l_i - k_i)\mathbf{e}_{iq} \in \mathcal{L}^{1\times p} A$ and there exist $h_1, \ldots, h_p \in \mathcal{L}$ such that

$$\sum_{i=1}^{q} (l_i - k_i)\mathbf{e}_{iq} = h_1(a_{11}, \ldots, a_{1q}) + \cdots + h_p(a_{p1}, \ldots, a_{pq}),$$

which implies that $l_i - k_i = h_1 a_{1i} + \cdots + h_p a_{pi}$ for $i = 1, \ldots, q$. Hence

$$\begin{aligned} & \lambda\left(\sum_{i=1}^{q} l_i \mathbf{e}_i - \sum_{i=1}^{q} k_i \mathbf{e}_i\right) = \sum_{i=1}^{q}(l_i - k_i) \otimes \mathbf{e}_i^* = \sum_{i=1}^{q} (h_1 a_{1i} + \cdots + h_p a_{pi}) \otimes \mathbf{e}_i^* \\ = \;& (h_1 \otimes (a_{11}\mathbf{e}_1^*) + \cdots + h_p \otimes (a_{p1}\mathbf{e}_1^*)) + \cdots + \big(h_1 \otimes (a_{1q}\mathbf{e}_q^*) + \cdots + h_p \otimes (a_{pq}\mathbf{e}_q^*)\big) \\ = \;& h_1 \otimes (a_{11}\mathbf{e}_1^* + \cdots + a_{1q}\mathbf{e}_q^*) + \cdots + h_p \otimes (a_{p1}\mathbf{e}_1^* + \cdots + a_{pq}\mathbf{e}_q^*) \\ = \;& h_1 \otimes ((a_{11}, \ldots, a_{1q}) + \mathcal{S}^{1\times p} A) + \cdots + h_p \otimes ((a_{p1}, \ldots, a_{pq}) + \mathcal{S}^{1\times p} A) = 0. \end{aligned}$$

Thus λ is well-defined. In addition, one can verify that $\pi \circ \lambda$ and $\lambda \circ \pi$ are identity maps on $\mathrm{coker}_{\mathcal{L}}(A)$ and $\mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A)$, respectively. So $\mathrm{coker}_{\mathcal{L}}(A) \cong_{\mathcal{L}} \mathcal{L} \otimes \mathrm{coker}_{\mathcal{S}}(A)$. □

From Lemma 2.4.10, for any matrix A over $\mathcal{S}$, its linear dimension, $\dim_F \mathrm{coker}_{\mathcal{L}}(A)$, is not greater than $\dim_F \mathrm{coker}_{\mathcal{S}}(A)$.

Let $A(\mathbf{z})$=0 with $A \in \mathcal{S}^{p\times q}$ be a system of finite linear dimension n and M=$\mathrm{coker}_{\mathcal{L}}(A)$ be its module of formal solutions with an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ satisfying

$$\partial_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = B_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau, \quad \text{where } B_i \in F^{n\times n}, \quad 1 \le i \le m.$$

According to Proposition 2.4.8, $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ is a fully integrable system, which is called an *integrable connection* of $A(\mathbf{z}) = 0$ with respect to a basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of M. Observe that the integrable connections of $A(\mathbf{z}) = 0$ with respect to different F-bases of M are equivalent to each other. Indeed, let $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ and $\{\partial_i(\widetilde{\mathbf{x}}) = \widetilde{B}_i\widetilde{\mathbf{x}}\}_{1\le i\le m}$ be the integrable connections of $A(\mathbf{z}) = 0$ with respect to two F-bases $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and $\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_n$ of M, respectively. Write $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ and $\widetilde{\mathbf{b}} = (\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_n)^\tau$. Let $T \in \mathrm{GL}_n(F)$ be given by $\widetilde{\mathbf{b}} = T\mathbf{b}$. From $\partial_i(\mathbf{b}) = B_i\mathbf{b}$ and $\partial_i(\widetilde{\mathbf{b}}) = \widetilde{B}_i\widetilde{\mathbf{b}}$, we get that

$$\widetilde{B}_i T = T B_i + \delta_i(T), \quad \text{for } i \le \ell \quad \text{and} \quad \sigma_j(T) B_j = \widetilde{B}_j T, \quad \text{for } j > \ell.$$

By Definition 2.2.5, $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\le i\le m}$ and $\{\partial_i(\widetilde{\mathbf{x}}) = \widetilde{B}_i\widetilde{\mathbf{x}}\}_{1\le i\le m}$ are equivalent. So in the sequel, we sometimes omit mentioning the basis when referring to an integrable connection.

Proposition 2.4.11 *Let $A, \mathbf{b}_1, \ldots, \mathbf{b}_n, B_1, \ldots, B_m$ be as above, and B be the stacking of the blocks $(\partial_i \cdot \mathbf{1}_n - B_i)$. We have the following two statements:*

(i) $\text{coker}_{\mathcal{L}}(A) \cong_{\mathcal{L}} \text{coker}_{\mathcal{L}}(B)$.

(ii) *Suppose that* $\text{coker}_{\mathcal{S}}(A)$ *has a finite F-basis* $\mathbf{f}_1, \ldots, \mathbf{f}_d$ *and* $\partial_i(\mathbf{f}_1, \ldots, \mathbf{f}_d)^\tau = D_i(\mathbf{f}_1, \ldots, \mathbf{f}_d)^\tau$ *where* $D_i \in F^{d \times d}$ *for* $1 \leq i \leq m$. *Let D be the stacking of the blocks* $(\partial_i \cdot \mathbf{1}_d - D_i)$. *Then*

$$\text{coker}_{\mathcal{S}}(A) \cong_{\mathcal{S}} \text{coker}_{\mathcal{S}}(D) \quad \textit{and} \quad \text{coker}_{\mathcal{L}}(A) \cong_{\mathcal{L}} \text{coker}_{\mathcal{L}}(D).$$

Proof. (i) Let $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$ denote the images of $\mathbf{e}_{1n}, \ldots, \mathbf{e}_{nn}$ in $\text{coker}_{\mathcal{L}}(B)$, respectively. Set $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ and $\bar{\mathbf{e}} = (\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n)^\tau$. As seen in Example 2.4.9, $\{\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n\}$ is a set of F-generators of $\text{coker}_{\mathcal{L}}(B)$. Since $B(\mathbf{b}) = 0$, there exists an $\mathcal{L}$-module homomorphism φ from $\text{coker}_{\mathcal{L}}(B)$ to $\text{coker}_{\mathcal{L}}(A)$ such that $\varphi(\bar{\mathbf{e}}) = \mathbf{b}$ by Theorem 2.4.1 or by Theorem 2.4.2. The F-linear independence of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ then implies that of $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$. So $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$ form an F-basis of $\text{coker}_{\mathcal{L}}(B)$ and φ is bijective.

(ii) $\text{coker}_{\mathcal{S}}(A) \cong_{\mathcal{S}} \text{coker}_{\mathcal{S}}(D)$ follows from a similar argument to the proof of (i), by replacing $\mathcal{L}$ with $\mathcal{S}$, the $\mathbf{b}_i$ with the $\mathbf{f}_i$, and B with D. From Lemma 2.4.10, we have

$$\text{coker}_{\mathcal{L}}(A) \cong_{\mathcal{L}} \mathcal{L} \otimes \text{coker}_{\mathcal{S}}(A) \quad \text{and} \quad \text{coker}_{\mathcal{L}}(D) \cong_{\mathcal{L}} \mathcal{L} \otimes \text{coker}_{\mathcal{S}}(D).$$

It then follows that $\text{coker}_{\mathcal{L}}(A) \cong_{\mathcal{L}} \text{coker}_{\mathcal{L}}(D)$. □

Proposition 2.4.11 (i) reveals that, for a ∂-finite system $A(\mathbf{z}) = 0$, its module of formal solutions is isomorphic to that of its integrable connection $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1 \leq i \leq m}$. Consequently, Proposition 2.4.3 implies that there is a one-to-one correspondence between all solutions of $A(\mathbf{z}) = 0$ and those of $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1 \leq i \leq m}$. In fact, this correspondence is given by a matrix over F, as shown in the following

Proposition 2.4.12 *Let* $A(\mathbf{z})$=0 *with* $A \in \mathcal{S}^{p \times q}$ *be a system of finite linear dimension* n, M_A *be its module of formal solutions,* $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ *be a set of $\mathcal{L}$-generators for* M_A *and* $\mathbf{b}_1, \ldots, \mathbf{b}_n$ *be an F-basis of* M_A *such that* $A(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = 0$ *and* $\partial_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = B_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ *where* $B_i \in F^{n \times n}$ *for each i. Then there exists* $P \in F^{q \times n}$ *such that for any orthogonal Δ-extension E of F, the correspondence* $\xi \mapsto P\xi$ *is an isomorphism of C_E-modules between* $\text{sol}_E(\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1 \leq i \leq m})$ *and* $\text{sol}_E(A(\mathbf{z}) = 0)$.

Proof. Let B be the stacking of the blocks of the form $(\partial_i \cdot \mathbf{1}_n - B_i)$, M_B be the module of formal solutions for $B(\mathbf{x}) = 0$ and $\{\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n\}$ be a set of $\mathcal{L}$-generators of M_B such that $\partial_i(\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n)^\tau = B_i(\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n)^\tau$ for each i. As seen in Example 2.4.9, $\{\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n\}$ is a set of F-generators of M_B. From Proposition 2.4.11 (i), $M_A \cong_{\mathcal{L}} M_B$, thus $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$ form an F-basis of M_B. With the $\mathcal{L}$-generators $\mathbf{f}_1, \ldots, \mathbf{f}_d$ of M_B in the proof of Proposition 2.4.3 replaced by the F-basis $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$, one sees that the matrix $Q \in \mathcal{L}^{q\times d}$ in the proof of Proposition 2.4.3 can be chosen to be a matrix in $F^{q\times d}$, which gives the desirable correspondence. □

Let $A \in \mathcal{S}^{p\times q}$. Suppose that $\operatorname{coker}_{\mathcal{S}}(A)$ has finite dimension d over F and has an F-basis $\mathbf{f}_1, \ldots, \mathbf{f}_d$ such that $\partial_i(\mathbf{f}_1, \ldots, \mathbf{f}_d)^\tau = D_i(\mathbf{f}_1, \ldots, \mathbf{f}_d)^\tau$ where $D_i \in F^{d\times d}$ for $1 \le i \le m$. One can show that $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ is an integrable system by a similar argument to the first part of the proof for Proposition 2.4.8. From Proposition 2.4.11 (ii), $\operatorname{coker}_{\mathcal{L}}(A)$ is isomorphic to the $\mathcal{L}$-cokernel of $\{\partial_i(\mathbf{y}) = D_i\mathbf{y}\}_{1\le i\le m}$. Hence, for a system $A(\mathbf{z}) = 0$ such that $\operatorname{coker}_{\mathcal{S}}(A)$ is finite-dimensional, to compute its linear dimension it suffices to compute the linear dimension of the integrable system $\{\partial_i(\mathbf{y}) = D_i\mathbf{y}\}_{1\le i\le m}$. This task is discussed later in Section 2.5.

2.4.5 Fundamental Matrices and Picard-Vessiot Extensions

Based on the results in previous sections, we now generalize the notions and results of fundamental matrices and Picard-Vessiot extensions for ∂-finite linear functional systems.

Definition 2.4.13 *Let $A(\mathbf{z}) = 0$ with $A \in \mathcal{S}^{p\times q}$ be a ∂-finite system, M be its module of formal solutions, $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ be a set of $\mathcal{L}$-generators of M and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be an F-basis of M such that $A(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = 0$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = P(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ where $P \in F^{q\times n}$.*

A $q \times n$ matrix V with entries in an orthogonal Δ-extension E of F is called a fundamental matrix *for $A(\mathbf{z}) = 0$ if $V = PU$ where $U \in E^{n\times n}$ is a fundamental matrix of the integrable connection of $A(\mathbf{z}) = 0$ with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$.*

A Picard-Vessiot ring for the integrable connection of $A(\mathbf{z}) = 0$ is called a Picard-Vessiot ring *for $A(\mathbf{z}) = 0$.*

Although this is not stated in the definition, it follows from Proposition 2.4.12 that the columns of a fundamental matrix form a C_E-basis of the C_E-module $\operatorname{sol}_E(A(\mathbf{z})=0)$: denote $\text{sol}_E(A(\mathbf{z})=0)$ and $\text{sol}_E(\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\leq i\leq m})$ by W_A and W_B respectively. Then the columns of $V=PU$ are in W_A by Proposition 2.4.12. Let $\mathbf{c}\in C_E^{n\times 1}$ be such that $0=\mathbf{V}\mathbf{c}=PU\mathbf{c}$. Since $U\mathbf{c} \in W_B$, we have $U\mathbf{c} = 0$ by Proposition 2.4.12, hence $\mathbf{c} = 0$ since U is invertible. Thus the columns of V are linearly independent over C_E. For any $\eta \in W_A$ there exists $\xi\in W_B$ such that $\eta = P\xi$. By Proposition 2.2.3 there exists $\mathbf{c} \in C_E^{n\times 1}$ such that $\xi=U\mathbf{c}$. Hence $\eta=PU\mathbf{c} = V\mathbf{c}$.

As remarked earlier, integrable connections with respect to different F-bases of M are equivalent to each other. Therefore the last part of Definition 2.4.13 is justified by Proposition 2.2.7.

As a final consequence of Theorems 2.2.2 and 2.2.5, we have

Theorem 2.4.13 *Every ∂-finite system $A(\mathbf{z}) = 0$ has a fundamental matrix in some simple orthogonal Δ-extension of F and has a Picard-Vessiot ring E. If F has characteristic 0 and C_F is algebraically closed, then $C_E = C_F$.*

Proof. Let $n > 0$ be the linear dimension of $A(\mathbf{z}) = 0$, M be its module of formal solutions, $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ be a set of $\mathcal{L}$-generators of M and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be an F-basis of M such that $A(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = 0$ and $\partial_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau = B_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$ for each i. Let $P \in F^{q\times n}$ be given by $(\mathbf{e}_1, \ldots, \mathbf{e}_q)^\tau = P(\mathbf{b}_1, \ldots, \mathbf{b}_n)^\tau$. Since $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\leq i\leq m}$ is a fully integrable system, there exists, by Theorem 2.2.2, a fundamental matrix $U \in E^{n\times n}$ for that system where E is a simple orthogonal Δ-extension of F. Then $V := PU \in E^{q\times n}$ is a fundamental matrix for $A(\mathbf{z}) = 0$. The existence of the Picard-Vessiot ring and the second statement follow directly from Theorem 2.2.5. □

Assume that F has characteristic 0 with an algebraically closed field of constants. Let E be a Picard-Vessiot ring for the system $A(\mathbf{z}) = 0$. As mentioned after Theorem 2.2.5, $\text{sol}_E(\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\leq i\leq m})$ is of dimension n over C_F. But that space is isomorphic to $\text{sol}_E(A(\mathbf{z}) = 0)$ by Proposition 2.4.12. Therefore, the dimension of $\text{sol}_E(A(\mathbf{z}) = 0)$ as a C_F-vector space equals the linear dimension of $A(\mathbf{z}) = 0$, whenever the latter is finite.

Example 2.4.14 *Let* $F=\mathbb{C}(x,k)$ *and* $\mathcal{A}$ *be the system* $\{\delta_x(\mathbf{z})=A_x\mathbf{z}, \sigma_k(\mathbf{z})=A_k\mathbf{z}\}$ *where*

$$A_x = \begin{pmatrix} \frac{x+1}{x} & \frac{k(x+1-k)}{x^2(k-1)} & -\frac{k(x+1-k)}{x^2(k-1)} \\ x+1 & \frac{xk-k^2+2x^2+kx^2+k-1}{x(k-1)} & -\frac{xk-k^2+2x^2+kx^2}{x(k-1)} \\ x+1 & \frac{xk+2x^2+kx^2-2k^2+k}{x(k-1)} & -\frac{xk+2x^2+kx^2-2k^2+1}{x(k-1)} \end{pmatrix},$$

$$A_k = \begin{pmatrix} \frac{k+1}{k} & \frac{k+1-xk-x}{x(k-1)} & \frac{xk+x-k-1}{x(k-1)} \\ \frac{x(k+1)}{k} & \frac{1-2x+k-xk+x^3}{k-1} & \frac{2x+xk-x^3-k-1}{k-1} \\ \frac{x(k+1)}{k} & \frac{1-2xk-2x+k+x^3}{k-1} & \frac{2xk+2x-k-x^3-1}{k-1} \end{pmatrix}.$$

Note that A_x *and* A_k *satisfy the compatibility conditions (2.2) but* A_k *is singular, so the system is not fully integrable. We will show in Example 2.5.2 that all solutions* $\mathbf{z}$ *of* $\mathcal{A}$ *can be found by a change of variable* $\mathbf{z} = P\mathbf{y}$ *where*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{x(k-1)}{x^2-1} & \frac{x^2-k}{x^2-1} \end{pmatrix}$$

and $\mathbf{y}$ *is a solution of the fully integrable system* $\mathcal{B} : \{\delta_x(\mathbf{y}) = B_x\mathbf{y},\, \sigma_k(\mathbf{y}) = B_k\mathbf{y}\}$ *with*

$$B_x = \begin{pmatrix} \frac{-x+x^3-1+x^2-xk-k+k^2}{x(x^2-1)} & \frac{k(x+1-k)}{x^2(x^2-1)} \\ \frac{-x-xk+x^3-1-x^2+k^2-kx^2}{x^2-1} & \frac{-k^2+xk+kx^2+3x^2-1}{x(x^2-1)} \end{pmatrix},$$

$$B_k = \begin{pmatrix} \frac{xk+x+k^2+2k+1}{k(x+1)} & -\frac{k+1}{x(x+1)} \\ -\frac{(kx^2-x-k^2-2k-1)x}{k(x+1)} & \frac{x^2+x-1-k}{x+1} \end{pmatrix}.$$

So it suffices to compute a Picard-Vessiot extension of $\mathcal{B}$*. The same method to construct a fundamental matrix for the system in Example 2.2.2 yields a fundamental matrix for* $\mathcal{B}$*:*

$$U = \begin{pmatrix} xke^x & -kx^k \\ kx^2e^x & (x^2-k-1)x^{k+1} \end{pmatrix},$$

hence PU *is for* $\mathcal{A}$*. In addition, a Picard-Vessiot ring* $E = \mathbb{C}(x,k)[e^x, e^{-x}, x^k, x^{-k}]$ *for* $\mathcal{B}$ *is a Picard-Vessiot ring for* $\mathcal{A}$*.* □

2.5 Computing Linear Dimension of Integrable Systems

In this section, we present an algorithm for computing linear dimensions of integrable systems by linear algebra. For general linear functional systems, their linear dimensions will be computed by Gröbner basis techniques described in Chapter 3.

2.5.1 Notation

As in previous sections, let $\mathcal{S}$ and $\mathcal{L}$ be the Ore algebra and the Laurent-Ore algebra over F. Given a linear functional system $A(\mathbf{z}) = 0$ where $A \in \mathcal{S}^{p\times q}$, by the *submodule* N_A *defined by* the system we mean the submodule generated by the row vectors of A over $\mathcal{L}$. Clearly, the quotient module $\mathcal{L}^{1\times q}/N_A$ is the module of formal solutions of the system.

Let $\mathbf{z} = (z_1, \ldots, z_n)^\tau$ be a vector of unknowns. Two subvectors $\mathbf{u} = (z_{i_1}, \ldots, z_{i_d})^\tau$ and $\mathbf{v} = (z_{j_1}, \ldots, z_{j_{n-d}})^\tau$ form a *partition* of $\mathbf{z}$ if $\{z_{i_1}, \ldots, z_{i_d}\}$ and $\{z_{j_1}, \ldots, z_{j_{n-d}}\}$ form a partition of $\{z_1, \ldots, z_n\}$. Let $\mathbf{u} = (z_{i_1}, \ldots, z_{i_d})^\tau$ be a subvector of $\mathbf{z}$ and B be a square matrix of size n over $\mathcal{L}$. The submatrix of B consisting of its i_1th, $\ldots$, i_dth rows is denoted $B^{\mathbf{u}}$. Assume that $\{\mathbf{u}, \mathbf{v}\}$ is a partition of $\mathbf{z}$. Then the submodule defined by the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ where $A_i \in F^{n\times n}$ equals that defined by

$$\partial_i(\mathbf{u}) = A_i^{\mathbf{u}}\mathbf{z}, \quad \partial_i(\mathbf{v}) = A_i^{\mathbf{v}}\mathbf{z}, \quad i = 1, \ldots, m.$$

2.5.2 Linear Reduction

Lemma 2.5.1 *Let $\mathcal{A}$ be the linear functional system consisting of*

$$\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m} \quad \textit{and} \quad P\mathbf{z} = 0,$$

where P is a matrix over F with n columns. Set $d = n - \operatorname{rank}(P)$. Then one can either assert that $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ has only trivial solution or find a partition $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbf{z}$ such that the system $\mathcal{B}$ consisting of

$$\mathbf{v} = Q\mathbf{u}, \quad \partial_i\mathbf{u} = B_i\mathbf{u}, \quad R_i\mathbf{u} = 0, \quad i = 1, \ldots, m, \qquad (2.15)$$

defines the same submodule as $\mathcal{A}$, where $\mathbf{u}$ is a column vector of size d, Q and R_i belong to $F^{(n-d)\times d}$ and the matrices B_i belong to $F^{d\times d}$.

Proof. If all the entries of P are zero, then we set $\mathbf{u}$ to be $\mathbf{z}$ and B_i to be A_i. If $d = 0$ then all the z_i are zero. Thus we may assume that $0 < d < n$.

Solve the linear algebraic system $P\mathbf{z} = 0$ to get a partition $\{\mathbf{u}, \mathbf{v}\}$ of $\mathbf{z}$ such that $\mathbf{v}{=}Q\mathbf{u}$, where $\mathbf{u}$ has size d and $\mathbf{v}$ has size $(n-d)$. Using the linear relations $\mathbf{v}{=}Q\mathbf{u}$ to eliminate all the unknowns in $\mathbf{v}$ that appear in the system $\partial_i(\mathbf{u}){=}A_i^{\mathbf{u}}\mathbf{z}$ yields $\partial_i(\mathbf{u}){=}B_i\mathbf{u}$, for $i = 1, \ldots, m$. Using $\mathbf{v}{=}Q\mathbf{u}$ and $\partial_i(\mathbf{u}) = B_i\mathbf{u}$ to eliminate the $\partial_i(\mathbf{v})$ and $\mathbf{v}$ in $\partial_i(\mathbf{v}) = A_i^{\mathbf{v}}\mathbf{z}$ yields $R_i\mathbf{u}{=}0$, for $i = 1, \ldots, m$.

Note that any matrix representation of the system $\mathcal{B}$ has n columns since $\mathbf{z}$ is also the vector of unknowns for $\mathcal{B}$.

Let $N_{\mathcal{A}}$ and $N_{\mathcal{B}}$ be the submodules defined by the systems $\mathcal{A}$ and $\mathcal{B}$, respectively. The submodule $N_{\mathcal{B}}$ is contained in $N_{\mathcal{A}}$, because all the row vectors in the matrix representation (2.15) of $\mathcal{B}$ belong to $N_{\mathcal{A}}$. Conversely, the row vectors in the matrix representation of $P\mathbf{u} = 0$ are in the submodule defined by $\mathbf{v} = Q\mathbf{u}$; the row vectors in the matrix representation of $\partial_i(\mathbf{u}) = A_i^{\mathbf{u}}\mathbf{z}$ are in the submodule defined by $\mathbf{v} = Q\mathbf{u}$ and $\partial_i(\mathbf{u}) = B_i\mathbf{u}$; and the row vectors in the matrix representation of $\partial_i(\mathbf{v}) = A_i^{\mathbf{v}}\mathbf{z}$ are in the submodule defined by $\mathbf{v} = Q\mathbf{u}$, $\partial_i(\mathbf{u}) = B_i\mathbf{u}$ and $R_i\mathbf{u} = 0$. Thus $N_{\mathcal{A}} = N_{\mathcal{B}}$. □

The proof of Lemma 2.5.1 contains an algorithm for separating unknowns of the system $\mathcal{A}$ by linear algebra.

We present formulas for the matrices B_i and R_i in (2.15). These formulas have an interesting consequence to be described in Lemma 2.5.2. Without loss of generality, assume that $\mathbf{u}$ and $\mathbf{v}$ in (2.15) are $(z_1, \ldots, z_d)^\tau$ and $(z_{d+1}, \ldots, z_n)^\tau$, respectively, where $1 \leq d < n$. Partition the matrices

$$A_i = \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{pmatrix}, \quad i = 1, \ldots, m,$$

where $A_{i1} \in F^{d\times d}$, $A_{i2} \in F^{d\times(n-d)}$, $A_{i3} \in F^{(n-d)\times d}$ and $A_{i4} \in F^{(n-d)\times(n-d)}$. From $\mathbf{v} = Q\mathbf{u}$

and $\partial_i(\mathbf{u}) = (A_{i1}, A_{i2})(\mathbf{u}, \mathbf{v})^\tau = (A_{i1}, A_{i2})(\mathbf{u}, Q\mathbf{u})^\tau$, it follows that

$$B_i = A_{i1} + A_{i2}Q, \quad i = 1, \ldots, m.$$

From $\mathbf{v} = Q\mathbf{u}$ and $\partial_i(\mathbf{v}) = (A_{i3}, A_{i4})(\mathbf{u}, \mathbf{v})^\tau = (A_{i3}, A_{i4})(\mathbf{u}, Q\mathbf{u})^\tau$, we obtain

$$\sigma_i(Q)\partial_i(\mathbf{u}) + \delta_i(Q)\mathbf{u} = (A_{i3} + A_{i4}Q)\mathbf{u},$$

hence $(\sigma_i(Q)B_i + \delta_i(Q))\mathbf{u} = (A_{i3} + A_{i4}Q)\mathbf{u}$ because $\partial_i(\mathbf{u}) = B_i\mathbf{u}$. Consequently,

$$R_i = (\sigma_i(Q)B_i + \delta_i(Q)) - (A_{i3} + A_{i4}Q), \quad i = 1, \ldots, m. \tag{2.16}$$

The next lemma plays a key role in our algorithm for computing linear dimension of an integrable system.

Lemma 2.5.2 *If $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ in the system $\mathcal{A}$ given in Lemma 2.5.1 is integrable and the R_i in* (2.16) *are all zero, then the system $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1\le i\le m}$ in (2.15) is integrable.*

Proof. To simplify notation, we assume that $\mathbf{u} = (z_1, \ldots, z_d)^\tau$ and $\mathbf{v} = (z_{d+1}, \ldots, z_n)^\tau$. From (2.16) and $R_i = 0$, we have

$$A_{i3} = \sigma_i(Q)B_i + \delta_i(Q) - A_{i4}Q, \quad i = 1, \ldots, m. \tag{2.17}$$

For all i, j with $1 \le i, j \le m$, we have

$$\begin{aligned} & \sigma_i(A_j)A_i + \delta_i(A_j) \\ = & \begin{pmatrix} \sigma_i(A_{j1})A_{i1} + \sigma_i(A_{j2})A_{i3} + \delta_i(A_{j1}) & \sigma_i(A_{j1})A_{i2} + \sigma_i(A_{j2})A_{i4} + \delta_i(A_{j2}) \\ \sigma_i(A_{j3})A_{i1} + \sigma_i(A_{j4})A_{i3} + \delta_i(A_{j3}) & \sigma_i(A_{j3})A_{i2} + \sigma_i(A_{j4})A_{i4} + \delta_i(A_{j4}) \end{pmatrix}. \end{aligned} \tag{2.18}$$

Substitute (2.17) into the top-left block of (2.18) to yield

$$\begin{aligned} & \sigma_i(A_{j1})A_{i1} + \sigma_i(A_{j2})A_{i3} + \delta_i(A_{j1}) \\ = \; & \sigma_i(A_{j1})A_{i1} + \sigma_i(A_{j2})\left(\sigma_i(Q)B_i + \delta_i(Q) - A_{i4}Q\right) + \delta_i(A_{j1}) \\ = \; & \alpha_{ij} - \sigma_i(A_{j2})A_{i4}Q \end{aligned}$$

where $\alpha_{ij} = \sigma_i(A_{j1})A_{i1} + \sigma_i(A_{j2})\sigma_i(Q)\,(A_{i1} + A_{i2}Q) + \sigma_i(A_{j2})\delta_i(Q) + \delta_i(A_{j1})$. From the compatibility conditions (2.2) satisfied by the A_i, it follows that

$$\alpha_{ij} - \alpha_{ji} = (\sigma_i(A_{j2})A_{i4} - \sigma_j(A_{i2})A_{j4})\,Q \tag{2.19}$$

and

$$(\sigma_i(A_{j1})A_{i2} + \delta_i(A_{j2})) - (\sigma_j(A_{i1})A_{j2} + \delta_j(A_{i2})) = \sigma_j(A_{i2})A_{j4} - \sigma_i(A_{j2})A_{i4}. \tag{2.20}$$

A routine calculation shows that

$$\begin{aligned} \sigma_i(B_j)B_i + \delta_i(B_j) &= \sigma_i(A_{j1} + A_{j2}Q)(A_{i1} + A_{i2}Q) + \delta_i(A_{j1} + A_{j2}Q) \\ &= \alpha_{ij} + \sigma_i(A_{j1})A_{i2}Q + \delta_i(A_{j2})Q. \end{aligned}$$

From (2.19) and (2.20), we get

$$\begin{aligned} & (\sigma_i(B_j)B_i + \delta_i(B_j)) - (\sigma_j(B_i)B_j + \delta_j(B_i)) \\ = \; & (\alpha_{ij} + \sigma_i(A_{j1})A_{i2}Q + \delta_i(A_{j2})Q) - (\alpha_{ji} + \sigma_j(A_{i1})A_{j2}Q + \delta_i(A_{j2})Q) \\ = \; & \alpha_{ij} - \alpha_{ji} + (\sigma_i(A_{j1})A_{i2} + \delta_i(A_{j2}))\,Q - (\sigma_j(A_{i1})A_{j2} + \delta_j(A_{i2}))\,Q \\ = \; & \alpha_{ij} - \alpha_{ji} + (\sigma_j(A_{i2})A_{j4} - \sigma_i(A_{j2})A_{i4})\,Q = 0, \end{aligned}$$

for all i, j with $1 \leq i, j \leq m$, *i.e.*, the B_i satisfy the compatibility conditions (2.2). So $\{\partial_i(\mathbf{x}) = B_i\mathbf{x}\}_{1\leq i\leq m}$ is integrable. □

2.5.3 An algorithm

Given a system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ where $A_1, \ldots, A_m \in F^{n\times n}$. From all those singular matrices among $A_{\ell+1}, \ldots, A_m$, we can derive linear relations among the coordinates of $\mathbf{z}$. To see this, assume that A_j is a singular matrix for some $\ell + 1 \leq j \leq m$ and the rank of A_j equals r_j. From $\partial_j(\mathbf{z}) = A_j\mathbf{z}$, we derive $(n - r_j)$ linear relations among the coordinates of $\partial_j(\mathbf{z})$. Applying ∂_j^{-1} to these relations yields $(n - r_j)$ linear relations among the coordinates of $\mathbf{z}$.

We present an algorithm based on Lemmas 2.5.1 and 2.5.2.

Algorithm LinearReduction: *Given a system* $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ *and certain linear relations* $P\mathbf{z} = 0$*, where the* $A_i \in F^{n\times n}$ *and* P *is a matrix over* F *with* n *columns, either conclude that the system has only trivial solution, or find a partition* $\{\mathbf{u}, \mathbf{v}\}$ *of* $\mathbf{z}$*, and matrices* $Q, B_1, \ldots, B_m$ *such that*

(a) The system consisting of $\mathbf{v} = Q\mathbf{u}$ *and* $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1\le i\le m}$ *defines the same submodule as the given system;*

(b) The matrices B_j *are invertible for* $j = \ell + 1, \ldots, m$*.*

1. [Initialize] Set $\bar{P}$ to be P and $\bar{A}_i$ to be A_i for $i = 1, \ldots, m$. Set $\bar{n} = n$ to be n and $\bar{\mathbf{z}}$ to be $\mathbf{z}$. Set $\mathbf{v}$ to be the null vector and Q to be the null matrix.

2. [Collect new linear relations] For $i = \ell + 1, \ldots, m$, compute $r_i = \text{rank}(\bar{A}_i)$. If $r_i < \bar{n}$, then update $\bar{P}$ by adding $(\bar{n} - r_i)$ linear relations derived from $\bar{A}_i$ among the coordinates in $\bar{\mathbf{z}}$, as described above. If $\bar{P}$ is the null matrix, then set $\mathbf{u}$ to be $\mathbf{z}$, $\mathbf{v}$ to be the null vector, Q to be the null matrix, and return the partition $\{\mathbf{u}, \mathbf{v}\}$, Q and A_i, $i = 1, \ldots, m$.

3. [Solve linear algebraic equations] Solve $\bar{P}\bar{\mathbf{z}} = 0$. If $\bar{\mathbf{z}} = 0$, then exit [the input system has only trivial solution.] Otherwise, find a partition $\{\bar{\mathbf{u}}, \bar{\mathbf{v}}\}$ of $\bar{\mathbf{z}}$ such that $\bar{\mathbf{v}} = \bar{Q}\bar{\mathbf{u}}$ for some matrix $\bar{Q}$ over F.

4. [Update the transformation for unknowns] Update $\mathbf{v}$ by combining the value of $\mathbf{v}$ and $\bar{\mathbf{v}}$, and update Q by combining the value of Q and $\bar{Q}$. We then have a new partition $\{\mathbf{u}, \mathbf{v}\}$ of $\mathbf{z}$ and a new transformation $\mathbf{v} = Q\mathbf{u}$, in which the size of the current value of $\mathbf{v}$ is greater than that of the previous value of $\mathbf{v}$.

5. [Reduce] For $i = 1, \ldots, m$, reduce $\partial_i(\mathbf{u}) = \bar{A}_i^{\mathbf{u}}\mathbf{z}$ by $\mathbf{v} = Q\mathbf{u}$ to get $\partial_i(\mathbf{u}) = B_i\mathbf{u}$, and reduce $\partial_i(\mathbf{v}) = \bar{A}_i^{\mathbf{v}}\mathbf{z}$ by $\mathbf{v} = Q\mathbf{u}$ and $\partial_i(\mathbf{u}) = B_i\mathbf{u}$ to get $R_i\mathbf{u} = 0$.

6. [Case distinction]

(i) [new linear relations found by reduction] If $R_i \neq 0$ for some $i \in \{1, \dots, m\}$, then set $\bar{P}$ be the matrix consisting all the nonzero row vectors of the R_i. Set $\bar{A}_i$ to be B_i. Set $\bar{n}$ to be the size of $\mathbf{u}$ and $\bar{\mathbf{z}}$ to be $\mathbf{u}$. Go to step 2.

(ii) [new linear relations found by rank computation] If B_j is not of full rank for some $j \in \{\ell+1, \dots, m\}$, then set $\bar{P}$ to be the matrix given by the linear relations produced by all the singular matrices among $B_{\ell+1}, \dots, B_m$. Set $\bar{A}_i$ to be B_i. Set $\bar{n}$ to be the size of $\mathbf{u}$ and $\bar{\mathbf{z}}$ to be $\mathbf{u}$. Go to step 3.

(iii) [Done] Return the partition $\{\mathbf{u}, \mathbf{v}\}$, Q, and B_i, $i = 1, \dots, m$.

The termination of the algorithm **LinearReduction** is guaranteed by step 4 because the size of $\mathbf{v}$ is bounded by n. As shown in the proof of Lemma 2.5.1, the submodule N defined by the input system contains all the row vectors in the matrix representation of the system consisting of $\mathbf{v} = Q\mathbf{u}$, $\partial_i(\mathbf{u}) = B_i\mathbf{u}$, and $R_i\mathbf{u} = 0$ for $i = 1, \dots, m$, as the algorithm proceeds step by step. The new linear relations found by some singular B_j or $\bar{A}_j$ for $j \in \{\ell + 1, \dots, m\}$, are contained in N, because N is an $\mathcal{L}$-module, which is closed under the action of ∂_j^{-1}. Therefore, the output and input systems define the same module since the R_i are all zero matrices when the algorithm stops, and $P\mathbf{z}$ can be reduced to zero by $\mathbf{v} = Q\mathbf{u}$ in every iteration. The algorithm **LinearReduction** is correct.

If $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$ in the input system is integrable, then $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1 \leq i \leq m}$ is fully integrable by Lemma 2.5.2. Note that the module of formal solutions of the input system equals that of the output one, which is isomorphic to that of $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1 \leq i \leq m}$. Hence, by Proposition 2.4.9, the linear dimension of the input system is the size of $\mathbf{u}$. The algorithm **LinearReduction** has another application in Section 4.2.

Example 2.5.1 *Let $F = \mathbb{C}(k)$ and σ_k be the shift operator $k \mapsto k+1$ on F. Consider the system $\sigma_k(\mathbf{z}) = A_k\mathbf{z}$ where $\mathbf{z} = (z_1, z_2)^\tau$ and $A_k = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. Note that A_k is singular. Solve the linear system $(v_1, v_2)A_k = 0$ to yield a basis $\{(1,1)\}$ of all its solutions. Thus $\sigma_k(z_1)+\sigma_k(z_2) = 0$. By an application of σ_k^{-1}, we get $z_2 = -z_1$. This relation together* *with* A_k *yields* $\sigma_k(z_1) = -z_1$*, which is obviously a fully integrable system. So the original system has linear dimension one.* □

Example 2.5.2 *Let* $\mathcal{A}$ *be the system* $\{\delta_x(\mathbf{z}) = A_x\mathbf{z}, \sigma_k(\mathbf{z}) = A_k\mathbf{z}\}$ *where* A_x *and* A_k *are given in Example 2.4.14. Note that* A_k *is singular. Solve the linear system* $(v_1, v_2, v_3)A_k = 0$ *to get* $\sigma_k(z_3) = \frac{xk}{x^2-1}\sigma_k(z_1) + \frac{x^2-k-1}{x^2-1}\sigma_k(z_2)$*. An application of* σ_k^{-1} *then yields*

$$z_3 = \frac{x(k-1)}{x^2-1}z_1 + \frac{x^2-k}{x^2-1}z_2.$$

Substitute the above relation into the system $\mathcal{A}$*, we get that* $\delta_x(z_1, z_2)^\tau = B_x(z_1, z_2)^\tau$ *and* $\sigma_k(z_1, z_2)^\tau = B_k(z_1, z_2)^\tau$ *where*

$$B_x = \begin{pmatrix} \frac{-x+x^3-1+x^2-xk-k+k^2}{x(x^2-1)} & \frac{k(x+1-k)}{x^2(x^2-1)} \\ \frac{-x-xk+x^3-1-x^2+k^2-kx^2}{x^2-1} & \frac{-k^2+xk+kx^2+3x^2-1}{x(x^2-1)} \end{pmatrix} \text{ and } B_k = \begin{pmatrix} \frac{xk+x+k^2+2k+1}{k(x+1)} & -\frac{k+1}{x(x+1)} \\ -\frac{(kx^2-x-k^2-2k-1)x}{k(x+1)} & \frac{x^2+x-1-k}{x+1} \end{pmatrix}.$$

A straightforward but tedious calculation shows that both R_1 *and* R_2 *in* (2.16) *are zero. Hence the system* $\mathcal{B}$ *given by* B_x *and* B_k *is fully integrable since* B_k *is invertible. So* $\mathcal{A}$ *has linear dimension two.* □

To conclude, we explain how to determine whether a linear functional system is ∂-finite. In practice, it suffices to compute the dimension of its module M of formal solutions as an F-vector space. As seen in previous sections, when the system is given as an integrable system, we have a set of generators of M over F, so computing $\dim_F M$ can be done by linear algebra (as explained in the algorithm **LinearReduction**). Note that $\dim_F M = n$ if and only if this integrable system is fully integrable. When the system is given by an ideal in $\mathcal{S}$, Corollary 2.4.7 shows that either $M = 0$ (if the ideal contains a monomial in $\partial_{\ell+1}, \ldots, \partial_m$) or an F-basis of M can be computed via Gröbner bases of ideals in $\mathcal{S}$. There are algorithms and implementations for this task [18, 19]. For more general matrices $A \in \mathcal{S}^{p\times q}$, one can use the Gröbner basis technique to be developed in Chapter 3 for computing F-bases of $\mathcal{L}$-modules. However, if $\operatorname{coker}_{\mathcal{S}}(A)$ has finite dimension over F, it suffices to compute the linear dimension of an integrable system according to Proposition 2.4.12. The algorithm **LinearReduction** in this section supplies a tool for this task. Therefore, Gröbner basis techniques in $\mathcal{L}$ are only necessary when $\operatorname{coker}_{\mathcal{S}}(A)$ is infinite-dimensional over F.

Chapter 3

Gröbner Basis Computation in Laurent-Ore Algebras

Recall some terminologies introduced in Section 2.4.2. We denote by $\mathcal{S}$ the Ore algebra $F[\partial_1, \ldots, \partial_m]$, by $\bar{\mathcal{S}}$ the Ore algebra $F[\partial_1, \ldots, \partial_m, \theta_{\ell+1}, \ldots, \theta_m]$ and by $\mathcal{L}$ the Laurent-Ore algebra $F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ over F.

As discussed in Chapter 2, we are interested in finding a linear basis of modules of formal solutions for a linear functional system. This problem can be formulated as: *Let M be an $\mathcal{L}$-submodule of $\mathcal{L}^n$. Decide whether $\mathcal{L}^n/M$ is a finite-dimensional vector space over F. Moreover, find a basis of $\mathcal{L}^n/M$ over F if the dimension is finite.*

In this chapter, $\mathbf{e}_i$ denotes the unit vector in $\mathcal{S}^n$ with 1 in the ith position and 0 elsewhere. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form the canonical basis of $\mathcal{S}^n$, $\bar{\mathcal{S}}^n$, as well as $\mathcal{L}^n$.

Let I be the ideal of $\bar{\mathcal{S}}$ generated by $\partial_{\ell+1}\theta_{\ell+1} - 1, \ldots, \partial_m\theta_m - 1$. By definition, $\mathcal{L} = \bar{\mathcal{S}}/I$. Then $\mathcal{L}^n = \oplus_{i=1}^n \mathcal{L}\mathbf{e}_i \cong_{\mathcal{L}} \oplus_{i=1}^n (\bar{\mathcal{S}}/I)\,\mathbf{e}_i$. On the other hand, let I_n be the submodule of $\bar{\mathcal{S}}^n$ generated by $\partial_j\theta_j\mathbf{e}_i - \mathbf{e}_i$ for $1 \leq i \leq n$ and $\ell+1 \leq j \leq m$. Then, the map

$$\bar{\mathcal{S}}^n/I_n \to (\bar{\mathcal{S}}/I)\,\mathbf{e}_1 \oplus \cdots \oplus (\bar{\mathcal{S}}/I)\,\mathbf{e}_n, \quad (s_1, \ldots, s_n) + I_n \mapsto (s_1 + I, \ldots, s_n + I),$$

with $s_i \in \bar{\mathcal{S}}$, is an isomorphism of $\bar{\mathcal{S}}$-modules. Since $(\bar{\mathcal{S}}/I)\,\mathbf{e}_1 \oplus \cdots \oplus (\bar{\mathcal{S}}/I)\,\mathbf{e}_n$ is naturally an $\mathcal{L}$-module, so is $\bar{\mathcal{S}}^n/I_n$. It follows that $\bar{\mathcal{S}}^n/I_n \cong_{\mathcal{L}} \mathcal{L}^n$. This observation enables us to introduce Gröbner bases for submodules of $\mathcal{L}^n$ by Zampieri's approach [63].

This chapter has two sections. In Section 3.1, we describe the notion of Gröbner bases in finitely generated free modules over Ore algebras. The results of this section are rather straightforward. In Section 3.2, we extend Zampieri's approach to computing Gröbner bases in finitely generated free modules over Laurent-Ore algebras.

3.1 Gröbner Bases for Modules over Ore Algebras

In this section, R denotes a general Ore algebra $F[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$ over a field F where σ_i is a monomorphism on F and δ_i is a σ_i-derivation on F. Both $\mathcal{S}$ and $\bar{\mathcal{S}}$ are special instances of R. We follow the approach in [24, Ch.5] to extend the notion of Gröbner bases for ideals of R to R^n.

Let P be the set of all power products $\partial_1^{k_1} \cdots \partial_m^{k_m}$ in R for all $k_1, \ldots, k_m \in \mathbb{N}$. Note that P is a commutative monoid as in the usual commutative case.

We recall some terminologies given in [23]. A *monomial* in R^n is understood as an element of the form $p\,\mathbf{e}_i$ for some $p \in P$ and some $i \in \{1, \ldots, n\}$. The set of all monomials in R^n is denoted P_n, which forms a basis for R^n over F. For two monomials $p\,\mathbf{e}_i$ and $q\,\mathbf{e}_j$, we say that $p\,\mathbf{e}_i$ *divides* $q\,\mathbf{e}_j$ if i equals j and p divides q in P. We note that Dickson's lemma holds for P_n, that is, for every subset S of P_n, there exists a finite subset $D \subset S$ such that every element of S is divisible by some element of D.

Remark that we do not use the notion of terms, which are usually referred as elements of the form $f\mathbf{u}$ with $f \in F$ and $\mathbf{u} \in P_n$, because the product of a power product in P and a term is not necessarily a term due to the presence of derivation operators.

A basic theory of Gröbner bases involves three things: orderings on monomials, a division algorithm and the Buchberger algorithm. Let us consider them one by one.

An ordering relation $\succ$ on P_n is called a *monomial order* if the following conditions are satisfied:

1. $\succ$ is a total order,

2. for every pair $\mathbf{u}, \mathbf{v} \in P_n$ with $\mathbf{u} \succ \mathbf{v}$, we have $p\,\mathbf{u} \succ p\,\mathbf{v}$ for every $p \in P$,

3. $p\,\mathbf{u} \succ \mathbf{u}$ for all $\mathbf{u} \in P_n$ and $p \in P$ with $p \neq 1$.

Dickson's lemma implies that a monomial order is a well-ordering, *i.e.*, every infinite descending sequence of monomials stabilizes.

From now on we fix a monomial order $\succ$ on P_n.

For an element $u \in R^n$, we use $\mathrm{lm}(u)$ and $\mathrm{lc}(u)$ to denote the leading monomial and leading coefficient of u, respectively. For a subset G of R^n, denote $\mathrm{lm}(G) = \{\mathrm{lm}(\mathbf{g}) \mid \mathbf{g} \in G\}$.

Let $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_k\}$ be a finite subset of nonzero elements in R^n. Every $\mathbf{g} \in R^n$ can be written as

$$\mathbf{g} = q_1\,\mathbf{g}_1 + \ldots + q_k\,\mathbf{g}_k + r,$$

where $q_1, \ldots, q_k \in R$, $\mathrm{lm}(q_i\mathbf{g}_i) \preceq \mathrm{lm}(\mathbf{g})$, and either $r = 0$ or r is an F-linear combination of monomials, none of which is divisible by any of $\mathrm{lm}(\mathbf{g}_1), \ldots, \mathrm{lm}(\mathbf{g}_k)$. One can devise a division algorithm for finding $q_1, \ldots, q_k$ and r in the same way as in the commutative case ([24]). We call r the *remainder* of $\mathbf{g}$ on division by G.

For a submodule M of R^n, we say that a finite set $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_k\} \subset M$ is a *Gröbner basis* if the leading monomial of any element of M is divisible by some $\mathrm{lm}(\mathbf{g}_i)$. As in the commutative case, if G is a Gröbner basis of M, then, for every $\mathbf{g} \in R^n$, $\mathbf{g}$ is in M if and only if its remainder on division by G is zero.

A characterization of commutative Gröbner bases with respect to division extends immediately to R^n in the following

Proposition 3.1.1 *Let M be a submodule of R^n generated by a finite set G. Let V be the F-vector space generated by all the monomials not divisible by any element of* $\mathrm{lm}(G)$. *Then G is a Gröbner basis of M if and only if R^n is the direct sum of M and V as F-vector spaces.*

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero elements of R^n with $\mathrm{lm}(\mathbf{u}) = p\,\mathbf{e}_i$ and $\mathrm{lm}(\mathbf{v}) = q\mathbf{e}_j$ where p and q are in P. If $i = j$, we define the *S-polynomial* of $\mathbf{u}$ and $\mathbf{v}$ to be

$$S(\mathbf{u}, \mathbf{v}) = \left(\frac{t}{p}\right) \mathrm{lc}(\mathbf{u})^{-1}\mathbf{u} - \left(\frac{t}{q}\right) \mathrm{lc}(\mathbf{v})^{-1}\mathbf{v},$$

where t is the least common multiple of p and q. If $i \neq j$, the S-polynomial of $\mathbf{u}$ and $\mathbf{v}$ is defined to be zero. One can verify that, if $i = j$, the leading monomial of the S-polynomial of $\mathbf{u}$ and $\mathbf{v}$ is lower than $\operatorname{lcm}(p, q)\,\mathbf{e}_i$.

Now we prepare for proving Buchberger's criterion.

By the same telescoping trick in §6 of [23, Ch.2], one has

Lemma 3.1.2 *Let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be elements of R^n with the same leading monomial $p\,\mathbf{e}_i$. If the leading monomial of the sum $\mathbf{u} = a_1\mathbf{u}_1 + \ldots + a_r\mathbf{u}_r$, where $a_1, \ldots, a_r \in F$, is lower than $p\,\mathbf{e}_i$, then $\mathbf{u}$ is an F-linear combination of the S-polynomials $S(\mathbf{u}_j, \mathbf{u}_k)$ where $1 \leq j < k \leq r$. In particular, the leading monomial of $S(\mathbf{u}_j, \mathbf{u}_k)$ is lower than $p\,\mathbf{e}_i$.*

Observe that, for $p = \partial_1^{k_1} \cdots \partial_m^{k_m} \in P$ and $a \in F$, the product $p\,a$ is not necessarily a term R. In fact, the multiplication rules in R imply

$$p\,a = \sigma_1^{k_1} \circ \cdots \circ \sigma_m^{k_m}(a)p + \sum_i a_i p_i, \tag{3.1}$$

where $a_i \in F$ and the p_i are proper divisors of p. This observation motivates us to prove the following lemma, which is trivial in the usual commutative case (see the equality between equations (4) and (5) in §6 of [23, Ch.2]).

Lemma 3.1.3 *Let $\mathbf{u}$ and $\mathbf{v}$ be in R^n such that $\operatorname{lm}(\mathbf{u}) = s\,\mathbf{e}_i$ and $\operatorname{lm}(\mathbf{v}) = t\,\mathbf{e}_i$ where $s, t \in P$. Let $p, q \in P$ be such that $p\,s = q\,t$, which is denoted w. Then*

$$S(p\,\mathbf{u},\, q\,\mathbf{v}) - \left(\frac{w}{\operatorname{lcm}(s,t)}\right) S(\mathbf{u},\, \mathbf{v}) = g\,\mathbf{u} + h\,\mathbf{v}, \tag{3.2}$$

where $g, h \in R$ such that $\operatorname{lm}(g\mathbf{u})$ and $\operatorname{lm}(h\mathbf{v})$ are both lower than $w\,\mathbf{e}_i$, which is equal to $\operatorname{lm}(p\,\mathbf{u})$ and $\operatorname{lm}(q\,\mathbf{v})$.

Proof. Suppose that $\operatorname{lcm}(s,t) = f\,s = g\,t$ with $f, g \in P$. Since w is a common multiple of s and t, $w = r\operatorname{lcm}(s,t)$ for some $r \in P$. It follows that $p = rf$ and $q = rg$. We have

$$\begin{aligned}
rS(\mathbf{u}, \mathbf{v}) &= r\,f\operatorname{lc}(\mathbf{u})^{-1}\mathbf{u} - r\,g\operatorname{lc}(\mathbf{v})^{-1}\,\mathbf{v} \\
&= p\operatorname{lc}(\mathbf{u})^{-1}\,\mathbf{u} - q\operatorname{lc}(\mathbf{v})^{-1}\,\mathbf{v} \\
&= \operatorname{lc}(p\,\mathbf{u})^{-1}(p + g_0)\mathbf{u} - \operatorname{lc}(q\,\mathbf{v})^{-1}(q + h_0)\mathbf{v} \quad \text{(by (3.1))} \\
&= S(p\,\mathbf{u},\, q\,\mathbf{v}) + \operatorname{lc}(p\mathbf{u})^{-1}g_0\mathbf{u} - \operatorname{lc}(q\mathbf{v})^{-1}h_0\mathbf{v},
\end{aligned}$$

where g_0 and h_0 are in R such that $\mathrm{lm}(g_0)$ is lower than p, and $\mathrm{lm}(h_0)$ is lower than q. Then $g = -\mathrm{lc}(p\,\mathbf{u})^{-1} g_0$ and $h = \mathrm{lc}(q\,\mathbf{v})^{-1} h_0$ are as desired. □

With the aid of Lemmas 3.1.2 and 3.1.3, we get the Buchberger criterion in the same way as in the usual commutative case.

Proposition 3.1.4 *Let M be a submodule generated by $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_k\}$ in R^n. Then G is a Gröbner basis of M if and only if the remainder of $S(\mathbf{g}_i, \mathbf{g}_j)$ on division by G is zero for all $i < j$.*

Proof. The proof is the same as that of Theorem 6 in §6 of [23, Ch.2] except that we need to replace the equality between equations (4) and (5) on page 83 of [23] by (3.2). □

The Buchberger algorithm is again the same as its commutative counterpart. Its correctness is guaranteed by Proposition 3.1.4 and its termination is by Dickson's lemma.

Similar to the usual commutative case, a Gröbner basis G is said to be *reduced* if, for every $\mathbf{g} \in G$, $\mathrm{lc}(\mathbf{g}) = 1$ and $\mathbf{g}$ is reduced with respect to $G \setminus \{\mathbf{g}\}$. By the same proof to Proposition 6 in [23, Ch.2, §7], one can show

Proposition 3.1.5 *For a given monomial order, every submodule M of R^n has a unique reduced Gröbner basis G.*

Example 3.1.1 *Let $R = \mathcal{S}$ and $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ be an integrable system of size n over F. Consider the corresponding matrices $(\partial_i \cdot \mathbf{1}_n - A_i)$ for $i = 1, \ldots, m$. Let M be the submodule generated by the row vectors of those matrices over $\mathcal{S}$. These row vectors form a Gröbner basis of M with respect to a monomial order in which $\partial_i\mathbf{e}_j \succ \mathbf{e}_k$ for all $1 \le i \le m$ and $1 \le j, k \le n$. This is because compatibility conditions* (2.2) *imply that all the remainders of S-polynomials on division by those vectors are zero. But such a system may have only trivial solution in any orthogonal Δ-extension (see Example 2.4.1).* □

Example 3.1.2 *Let $R = \bar{\mathcal{S}}$ and $T_n = \{\partial_j\theta_j\mathbf{e}_i - \mathbf{e}_i \mid 1 \le i \le n,\ \ell+1 \le j \le m\} \subset \bar{\mathcal{S}}^n$. Recall that I_n is the submodule generated by T_n over $\bar{\mathcal{S}}$. A straightforward calculation verifies that T_n is a (reduced) Gröbner basis of I_n with respect to every monomial order. Denote* *by* P_n^T *the set of all the monomials in* P_n *that are not divisible by any element of* $\mathrm{lm}(T_n)$ *with respect to* T_n. *Clearly,* P_n^T *consists of all the monomials of the form* $p\,\mathbf{e}_i$ *such that* ∂_j *and* θ_j *do not appear simultaneously in* p *for all* i, j *with* $1 \leq i \leq n$ *and* $\ell + 1 \leq j \leq m$. □

3.2 Gröbner Bases for Modules over Laurent-Ore Algebras

Set the Ore algebra R in Section 3.1 to be $\bar{\mathcal{S}}$. Based on the observation that $\mathcal{L}^n \cong_{\mathcal{L}} \bar{\mathcal{S}}^n / I_n$, the previous results can be applied to develop a Gröbner basis technique in $\mathcal{L}^n$.

Let $P^* \subset \mathcal{L}$ be the set of all power products $\partial_1^{k_1} \cdots \partial_\ell^{k_\ell} \partial_{\ell+1}^{k_{\ell+1}} \cdots \partial_m^{k_m}$ for all $k_1, \ldots, k_\ell \in \mathbb{N}$ and $k_{\ell+1}, \ldots, k_m \in \mathbb{Z}$. Since all monomials of the form $\partial_{\ell+1}^{k_{\ell+1}} \cdots \partial_m^{k_m}$ constitute a commutative group, the usual divisibility on monomials is not useful for our purpose.

Definition 3.2.1 *Let* p *and* q *be two elements of* P^*. *We say that* p divides q in the sense of Laurent*, if the following conditions are satisfied:*

1. $\deg_{\partial_k} p \leq \deg_{\partial_k} q$ *for all* k *with* $1 \leq k \leq \ell$,

2. *either* $0 \leq \deg_{\partial_k} p \leq \deg_{\partial_k} q$ *or* $\deg_{\partial_k} q \leq \deg_{\partial_k} p \leq 0$, *for* $\ell + 1 \leq k \leq m$.

Remark 3.2.2 *Unlike in the usual sense,* ∂_k^{-s} *does not divide* ∂_k^t *in the sense of Laurent for* $s, t \in \mathbb{Z}^+$ *and any* k *with* $\ell + 1 \leq k \leq m$, *by the condition 2 of Definition 3.2.1.*

A *monomial* of $\mathcal{L}^n$ is an element of the form $p\,\mathbf{e}_i$ with $p \in P^*$ and $i \in \{1, \ldots, n\}$. The set of monomials in $\mathcal{L}^n$ is denoted P_n^*. We say that a monomial $p\,\mathbf{e}_i$ *divides* $q\,\mathbf{e}_j$ in the sense of Laurent if i equals j and p divides q in the same sense.

Let P_n be the set of all monomials in $\bar{\mathcal{S}}^n$, and P_n^T be the same as in Example 3.1.2. Define ρ to be the map $P_n^T \to P_n^*$ that sends ∂_i to ∂_i for $1 \leq i \leq m$, θ_j to ∂_j^{-1} for $\ell+1 \leq j \leq m$, and $\mathbf{e}_k$ to $\mathbf{e}_k$ for $1 \leq k \leq n$. Clearly, ρ is bijective. Moreover, for any pair $p\,\mathbf{e}_i$, $q\,\mathbf{e}_j$ in P_n^*, $p\,\mathbf{e}_i$ divides $q\,\mathbf{e}_j$ in the sense of Laurent if and only if $\rho^{-1}(p\,\mathbf{e}_i)$ divides $\rho^{-1}(q\,\mathbf{e}_j)$. Consequently, Dickson's lemma holds for the divisibility in Definition 3.2.1. Remark that ρ extends to an isomorphism from the F-vector space generated by P_n^T to $\mathcal{L}^n$.

Let $\succ$ be a monomial order on P_n. Then $\succ$ is a total order on P_n^T. For two monomials $p\,\mathbf{e}_i$ and $q\,\mathbf{e}_j$ in P_n^*, we define $p\,\mathbf{e}_i \succ q\,\mathbf{e}_j$ if $\rho^{-1}(p\,\mathbf{e}_i) \succ \rho^{-1}(q\,\mathbf{e}_j)$ in P_n^T. Such an ordering is called an *induced* order on P_n^*. Leading monomials and leading coefficients for elements of $\mathcal{L}^n$ are then defined likewise. As an induced order on P_n^* is a well-ordering, a division algorithm can be devised in the same way.

Then the following definition is quite natural.

Definition 3.2.3 *Let M be a submodule of $\mathcal{L}^n$. A finite subset $G \subset M$ is called a* Gröbner basis *with respect to an induced order on P_n^*, if the leading monomial of every element of M is divisible in the sense of Laurent by the leading monomial of some element of G.*

One can easily show that, if G is a Gröbner basis of a submodule M of $\mathcal{L}^n$, then an element $u \in \mathcal{L}^n$ is in M if and only if the remainder on division of u is zero. Moreover,

Proposition 3.2.1 *Let G be a Gröbner basis of a submodule M in $\mathcal{L}^n$ and V be the F-vector space generated by the elements of P_n^* that are not divisible in the sense of Laurent by any element of $\operatorname{lm}(G)$. Then $\mathcal{L}^n = M \oplus V$ where $\oplus$ denotes a direct sum of F-vector spaces.*

Example 3.2.4 *Let $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ be a fully integrable system over F. We then have $\partial_j^{-1}(\mathbf{z}) = \sigma_j^{-1}\left(A_j^{-1}\right)\mathbf{z}$ for all j with $\ell+1 \le j \le m$. Consider the matrices*

$$\partial_i \cdot \mathbf{1}_n - A_i, \quad i = 1,\ldots,m, \quad \text{and} \quad \partial_j \cdot \mathbf{1}_n - \sigma_j^{-1}\left(A_j^{-1}\right), \quad j = \ell+1,\ldots,m.$$

Let M be the submodule generated by the row vectors of these matrices over $\mathcal{L}$. We show that these row vectors form a Gröbner basis of M with respect to an induced order in which $\partial_i\mathbf{e}_k \succ \mathbf{e}_l$ and $\partial_j^{-1}\mathbf{e}_k \succ \mathbf{e}_l$ for all i, j and $1 \le k, l \le n$. Indeed, if $p\,\mathbf{e}_k \in \operatorname{lm}(M)$ is not divisible in the sense of Laurent by any of $\partial_i\mathbf{e}_k$ and $\partial_j^{-1}\mathbf{e}_k$, then p must be one. It follows that there exists a nontrivial F-linear combination among $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in M. On the other hand, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are all the possible monomials that are not divisible in the sense of Laurent by any element of $\operatorname{lm}(M)$. Thus $\mathcal{L}^n/M$ has dimension less than n over F, a contradiction to Proposition 2.4.9. $\square$

As described in Section 2.5.1, there is a surjective $\bar{\mathcal{S}}$-module homomorphism ϕ from $\bar{\mathcal{S}}^n$ to $\mathcal{L}^n$ that replaces $\theta_{\ell+1}, \ldots, \theta_m$ appearing in an element of $\bar{\mathcal{S}}^n$ by $\partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}$, respectively. Clearly, the kernel of ϕ is I_n, the submodule generated by T_n. Moreover, ρ is the restriction of ϕ on the F-vector space generated by P_n^T in $\bar{\mathcal{S}}^n$.

Lemma 3.2.2 *Let $p\,\mathbf{e}_i \in P_n^T$ and $q\,\mathbf{e}_j \in P_n$. If $p\,\mathbf{e}_i \succ q\,\mathbf{e}_j$, then $\phi(p\,\mathbf{e}_i) \succ \phi(q\,\mathbf{e}_j)$.*

Proof. Write $q = q_1 q_2$ where $q_1 \in P_n^T$ and q_2 is a power product of $\partial_{\ell+1}\theta_{\ell+1}, \ldots, \partial_m\theta_m$. Then $q\,\mathbf{e}_j \succeq q_1\,\mathbf{e}_j$, and, consequently, $p\,\mathbf{e}_i \succ q_1\,\mathbf{e}_j$. The lemma then follows from the fact that $\phi(q\,\mathbf{e}_j) = \phi(q_1\,\mathbf{e}_j)$. □

The next proposition yields an algorithm for computing Gröbner bases in $\mathcal{L}^n$.

Proposition 3.2.3 *Let M be a submodule of $\mathcal{L}^n$ and ϕ be defined as above. If G is a Gröbner basis of $\phi^{-1}(M)$ with respect to a monomial order $\succ$, then $\phi(G)$ is a Gröbner basis of M with respect to the induced order on P_n^*.*

Proof. Let $\mathbf{u} = \sum_i a_i \mathbf{u}_i$ be in M with $a_i \in F$ and $\mathbf{u}_i \in P_n^*$. Then $\tilde{\mathbf{u}} = \sum a_i \rho^{-1}(\mathbf{u}_i)$ is in $\phi^{-1}(M)$. Suppose that $\operatorname{lm}(\mathbf{u}) = \mathbf{u}_1$. Then $\operatorname{lm}(\tilde{\mathbf{u}}) = \rho^{-1}(\mathbf{u}_1)$. Since G is a Gröbner basis of $\phi^{-1}(M)$, there exists $\mathbf{g} \in G$ such that $\operatorname{lm}(\mathbf{g})$ divides $\rho^{-1}(\mathbf{u}_1)$. Consequently, $\operatorname{lm}(\mathbf{g})$ is in P_n^T. By Lemma 3.2.2, $\operatorname{lm}(\phi(\mathbf{g}))$ is equal to $\rho(\operatorname{lm}(\mathbf{g}))$, which divides $\mathbf{u}_1$ in the sense of Laurent. Hence, $\phi(G)$ is a Gröbner basis of M. □

By Proposition 3.2.3 we compute a Gröbner basis of a submodule M of $\mathcal{L}^n$ as follows. Let $\mathbf{s}_1, \ldots, \mathbf{s}_m$ be a set of generators of M over $\mathcal{L}$. Then $\rho^{-1}(\mathbf{s}_1), \ldots, \rho^{-1}(\mathbf{s}_m)$ and the elements of T_n form a set of generators of $\phi^{-1}(M)$ over $\bar{\mathcal{S}}$. Apply the Buchberger algorithm in Section 3.1 for $R = \bar{\mathcal{S}}$, we compute a Gröbner basis G of $\phi^{-1}(M)$. Then $\phi(G)$ is a Gröbner basis of M.

Example 3.2.5 *Let $\mathcal{S} = F[\partial_1, \partial_2]$, $L_1 = \partial_1\partial_2(\partial_1 + 1)$ and $L_2 = \partial_1\partial_2(\partial_2 + 1)$ be as given in Example 2.4.11. Then $\bar{\mathcal{S}} = F[\partial_1, \partial_2, \theta_1, \theta_2]$ and $\mathcal{L} = F[\partial_1, \partial_2, \partial_1^{-1}, \partial_2^{-1}]$. Let us compute a Gröbner basis of the ideal I generated by L_1 and L_2 in $\mathcal{L}$. View L_1, L_2 as elements in $\bar{\mathcal{S}}$ and let $g_1 = \partial_1\theta_1 - 1$ and $g_2 = \partial_2\theta_2 - 1$. We compute a Gröbner basis of the ideal generated* *by $H = \{L_1, L_2, g_1, g_2\}$ in $\bar{\mathcal{S}}$ with respect to a total degree order. The remainders on division by H of $S(L_1, g_1), S(L_1, g_2), S(L_2, g_1)$ and $S(L_2, g_2)$ are*

$$r_{11} = \partial_1\partial_2 + \partial_2, \quad r_{12} = \partial_1^2 + \partial_1, \quad r_{21} = \partial_2^2 + \partial_2 \quad \text{and} \quad r_{22} = \partial_1\partial_2 + \partial_1,$$

respectively. Set $H_1 = H \cup \{r_{11}, r_{12}, r_{21}, r_{22}\}$. Both the remainder of $S(r_{11}, g_2)$ and that of $S(r_{12}, g_1)$ on division by H_1 are $h_1 = \partial_1 + 1$, while both the remainder of $S(r_{21}, g_2)$ and that of $S(r_{22}, g_1)$ on division by H_1 are $h_2 = \partial_2 + 1$. Note that all the elements of $H_1 \setminus \{g_1, g_2\}$ can be reduced to zero by $\{h_1, h_2\}$ and that the remainders of g_1 and g_2 on division by $\{h_1, h_2\}$ are $h_3 = \theta_1 + 1$ and $h_4 = \theta_2 + 1$, respectively. Let $G = \{h_1, h_2, h_3, h_4\}$. One can easily verify that G is a Gröbner basis of the ideal J generated by H in $\bar{\mathcal{S}}$. Proposition 3.2.3 then implies that $\{\partial_1 + 1,\ \partial_2 + 1,\ \partial_1^{-1} + 1,\ \partial_2^{-1} + 1\}$ is a Gröbner basis of I. Hence the linear dimension of the system in Example 2.4.11 equals one. $\square$

Chapter 4

Factorization of Modules over Laurent-Ore Algebras

The work of this chapter is motivated by the algorithm **FactorWithSpecifiedLeaders** in [44, 45], where the idea of associated equations is fully generalized to factor linear PDE's with finite-dimensional solution spaces. In terms of modules over an Ore algebra

$$\mathcal{S} = F[\partial_1; \mathbf{1}, \delta_1] \cdots [\partial_\ell; \mathbf{1}, \delta_\ell][\partial_{\ell+1}; \sigma_{\ell+1}, \mathbf{0}] \cdots [\partial_m; \sigma_m, \mathbf{0}],$$

where $\ell = m$, the problem solved by their algorithm can be formulated as follows: given a submodule N of $\mathcal{S}^n$ such that $M = \mathcal{S}^n/N$ is finite-dimensional over the field F, find all submodules of $\mathcal{S}^n$ that contain N. Such a submodule is called a factor of N since all its solutions are solutions of N. In their algorithm a factor is represented by a Gröbner basis with respect to a pre-chosen monomial order. Observe that, for a (right) factor of a given order, there is only one possibility for its leading derivative in the ordinary case, whereas, there are many possibilities in the partial case. Due to this complication, the algorithm has to check every possibility to compute all the factors of a given order. This is an ideal-theoretic approach because the quotient module M does not come into play.

In the module-theoretic approach to be described in this chapter, we compute all submodules of the above quotient module M, and then recover the factors of N in the sense of [44, 45] via the canonical map from $\mathcal{S}^n$ to M. As all submodules of M are represented by linear bases over F, the problem of guessing leading derivatives goes away. The same idea carries over to $\mathcal{L}$-modules of finite dimension and results in a factorization algorithm for ∂-finite linear functional systems. Recall that $\mathcal{L}$ denotes the Laurent-Ore algebra $F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$.

This chapter is organized as follows. We present some constructions on $\mathcal{L}$-modules in Section 4.1, and propose an algorithm for computing hyperexponential solutions of fully integrable systems in Section 4.2. Two building blocks of the factorization algorithm are described in Section 4.3, and a factorization algorithm together with some examples is given in Section 4.4. Finally in Section 4.5, we discuss the eigenring method for factoring $\mathcal{L}$-modules of finite dimension.

4.1 Constructions on Modules over Laurent-Ore Algebras

Let R be a ring. The notions of reducibility of R-modules are defined in [55] as follows.

An R-module M is *reducible* if M has a submodule other than 0 and M. Otherwise, M is *irreducible* or *simple*.

An R-module M is *completely reducible* or *semisimple* if for every submodule N_1 there exists a submodule N_2 such that $M = N_1 \oplus N_2$. Note that an irreducible module is completely reducible as well.

An R-module M is *decomposable* if M can be written as $N_1 \oplus N_2$ where N_1 and N_2 are nontrivial submodules of M. Otherwise, M is *indecomposable*.

Clearly, an R-module M is reducible if it is decomposable, and M is irreducible when it is both indecomposable and completely reducible.

By *factoring* an R-module, we mean finding its R-submodules.

Let F be an orthogonal Δ-field with C the field of constants, $\mathcal{S} = F[\partial_1, \ldots, \partial_m]$ and $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ be the corresponding Ore algebra and Laurent-Ore algebra, respectively. Throughout this chapter, assume that F has characteristic 0 and C is algebraically closed.

Ordinary and partial differential modules in [53] are special cases of $\mathcal{L}$-modules. The constructions in [53, §2.2] can be carried on $\mathcal{L}$-modules in a similar way.

Let N be a submodule of M. The F-vector space M/N endowed with the induced actions: $\partial_i(\mathbf{w}+N)=\partial_i(\mathbf{w})+N$ for $1 \leq i \leq m$ and $\partial_j^{-1}(\mathbf{w}+N)=\partial_j^{-1}(\mathbf{w})+N$ for $\ell+1 \leq j \leq m$, becomes an $\mathcal{L}$-module, which is called the *quotient module*.

Let M_1 and M_2 be two $\mathcal{L}$-modules. The *direct sum* of M_1 and M_2 is $M_1 \oplus M_2$ equipped with the actions: $\partial_i(\mathbf{w}_1 + \mathbf{w}_2) = \partial_i(\mathbf{w}_1) + \partial_i(\mathbf{w}_2)$ and $\partial_j^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \partial_j^{-1}(\mathbf{w}_1) + \partial_j^{-1}(\mathbf{w}_2)$ for $\mathbf{w}_1 \in M_1$, $\mathbf{w}_2 \in M_2$, $1 \leq i \leq m$ and $\ell + 1 \leq j \leq m$.

Recall that the F-vector space $M_1 \otimes_F M_2$ is formed by the free abelian group generated by the $\mathbb{Z}$-linear combination of all the pairs $(\mathbf{w}_1, \mathbf{w}_2) \in M_1 \times M_2$, modulo the subgroup G generated by expressions of the form:

$$(\mathbf{w}_1 + \mathbf{w}_1^*, \mathbf{w}_2) - (\mathbf{w}_1, \mathbf{w}_2) - (\mathbf{w}_1^*, \mathbf{w}_2), \quad (\mathbf{w}_1, \mathbf{w}_2 + \mathbf{w}_2^*) - (\mathbf{w}_1, \mathbf{w}_2) - (\mathbf{w}_1, \mathbf{w}_2^*),$$
$$(a\mathbf{w}_1, \mathbf{w}_2) - a(\mathbf{w}_1, \mathbf{w}_2), \quad (\mathbf{w}_1, a\mathbf{w}_2) - a(\mathbf{w}_1, \mathbf{w}_2),$$

for all $\mathbf{w}_1, \mathbf{w}_1^* \in M_1$, $\mathbf{w}_2, \mathbf{w}_2^* \in M_2$ and $a \in F$. Define

$$\partial_i(\mathbf{w}_1, \mathbf{w}_2) = (\partial_i(\mathbf{w}_1), \mathbf{w}_2) + (\mathbf{w}_1, \partial_i(\mathbf{w}_2)) \quad \text{and} \quad \partial_j^s(\mathbf{w}_1, \mathbf{w}_2) = (\partial_j^s(\mathbf{w}_1), \partial_j^s(\mathbf{w}_2)),$$

for $i \leq \ell$, $j > \ell$ and $s \in \{-1, 1\}$. One can verify that G is closed under the actions of all the ∂_i and ∂_j^{-1}. Thus, endowed with the induced actions:

$$\partial_i(\mathbf{w}_1 \otimes \mathbf{w}_2) = \partial_i(\mathbf{w}_1) \otimes \mathbf{w}_2 + \mathbf{w}_1 \otimes \partial_i(\mathbf{w}_2) \quad \text{and} \quad \partial_j^s(\mathbf{w}_1 \otimes \mathbf{w}_2) = \partial_j^s(\mathbf{w}_1) \otimes \partial_j^s(\mathbf{w}_2)$$

for $\mathbf{w}_1 \in M_1$, $\mathbf{w}_2 \in M_2$, $i \leq \ell$, $j > \ell$ and $s \in \{-1, 1\}$, $M_1 \otimes_F M_2$ becomes an $\mathcal{L}$-module, which is called the *tensor product* of M_1 and M_2, and denoted by $M_1 \otimes M_2$. The tensor product of several $\mathcal{L}$-modules can be defined similarly.

Consider the tensor product $M \otimes \cdots \otimes M$ of d copies of an $\mathcal{L}$-module M. Denote by W the subspace of this tensor product generated by the expressions of the form $\mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_d$ where there are (at least) indices $i \neq j$ with $\mathbf{w}_i = \mathbf{w}_j$. The exterior power $\wedge_F^d M$ is defined to be the quotient space of $M \otimes \cdots \otimes M$ and W. One can verifies that W is closed under the actions of all ∂_i and ∂_j^{-1}, so $\wedge^d M$ becomes an $\mathcal{L}$-module with the induced actions and is called the d-th *exterior power* of M.

Let M be an $\mathcal{L}$-module. The *internal Hom* of two $\mathcal{L}$-modules M and N is the F-vector space $\mathrm{Hom}_F(M,N)$ of all the F-linear maps from M to N endowed with the ∂_i given by

$$(\partial_i\varphi)(\mathbf{w})=\partial_i(\varphi(\mathbf{w}))-\varphi(\partial_i(\mathbf{w}))\quad\text{for } i\le\ell \quad\text{and}\quad (\partial_j\varphi)(\mathbf{w})=\partial_j(\varphi(\partial_j^{-1}(\mathbf{w})))\quad\text{for } j>\ell,$$

for $\varphi\in\mathrm{Hom}_F(M,N)$ and $\mathbf{w}\in M$. A straightforward calculation shows that

$$(\partial_i(\partial_j\varphi))(\mathbf{w})=\partial_i\partial_j(\varphi(\mathbf{w}))-\partial_i(\varphi(\partial_j\mathbf{w}))-\partial_j(\varphi(\partial_i\mathbf{w}))+\varphi(\partial_i\partial_j\mathbf{w}),\quad 1\le i<j\le\ell,$$

$$(\partial_i(\partial_j\varphi))(\mathbf{w})=\partial_i\partial_j(\varphi(\partial_j^{-1}\partial_i^{-1}\mathbf{w})),\quad \ell+1\le i<j\le m,$$

and

$$(\partial_i(\partial_j\varphi))(\mathbf{w})=\partial_i\partial_j(\varphi(\partial_j^{-1}(\mathbf{w})))-\partial_j(\varphi(\partial_j^{-1}\partial_i\mathbf{w}))=(\partial_j(\partial_i\varphi))(\mathbf{w}),$$

for $1\le i\le\ell$ and $\ell+1\le j\le n$. Hence $\mathrm{Hom}_F(M,N)$ is a well-defined $\mathcal{L}$-module. It follows that

$$\mathrm{Hom}_{\mathcal{L}}(M,N)=\{\varphi\in\mathrm{Hom}_F(M,N)\mid \partial_i\varphi=0 \text{ for } i\le\ell \text{ and } \partial_j\varphi=\varphi \text{ for } j>\ell\}.$$

A special case of internal Hom is the *dual module* $M^*:=\mathrm{Hom}_F(M,F)$ of an $\mathcal{L}$-module M.

Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$. Suppose that

$$\partial_i(\mathbf{e}_1,\ldots,\mathbf{e}_n)^\tau=B_i(\mathbf{e}_1,\ldots,\mathbf{e}_n)^\tau,\quad i=1,\ldots,m,$$

where $B_i\in F^{n\times n}$. Then the B_j are invertible for $j>\ell$. Let M^* be the dual module of M and $\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*$ be its dual basis such that $\mathbf{e}_i^*(\mathbf{e}_j)$ is 1 if $i=j$ and is 0 otherwise. Write $B_i=(b_{ikl})$ for $i\le\ell$ and $B_i^{-1}=(a_{ikl})$ for $i>\ell$. For $i\le\ell$, we have

$$\partial_i\left(\mathbf{e}_j^*\right)(\mathbf{e}_k)=\delta_i\left(\mathbf{e}_j^*(\mathbf{e}_k)\right)-\mathbf{e}_j^*(\partial_i(\mathbf{e}_k))=-\mathbf{e}_j^*\left(\sum_{l=1}^n b_{ikl}\mathbf{e}_l\right)=-b_{ikj},$$

for $k=1,\ldots,n$, thus $\partial_i(\mathbf{e}_j^*)=-\sum_{s=1}^n b_{isj}\mathbf{e}_s^*$, and for $i>\ell$,

$$(\partial_i\mathbf{e}_j^*)(\mathbf{e}_k)=\partial_i(\mathbf{e}_j^*(\partial_i^{-1}(\mathbf{e}_k)))=\partial_i\left(\mathbf{e}_j^*\left(\sum_{l=1}^n\sigma_i^{-1}(a_{ikl})\mathbf{e}_l\right)\right)=\partial_i(\sigma_i^{-1}(a_{ikj}))=a_{ikj},$$

for $k=1,\ldots,n$, thus $\partial_i(\mathbf{e}_j^*)=\sum_{s=1}^n a_{isj}\mathbf{e}_s^*$. The above argument leads to

$$\partial_i(\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*)^\tau=-B_i^\tau(\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*)^\tau \text{ for } i\le\ell \text{ and } \partial_i(\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*)^\tau=(B_i^{-1})^\tau(\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*)^\tau \text{ for } i>\ell.$$

We call the sequence of matrices

$$-B_1^\tau,\ \ldots,\ -B_\ell^\tau,\ (B_{\ell+1}^{-1})^\tau,\ \ldots,\ (B_m^{-1})^\tau, \tag{4.1}$$

the *dual sequence* of $B_1, \ldots, B_m$.

As a consequence of Proposition 2.4.8, we have

Lemma 4.1.1 *Let $B_1, \ldots, B_m \in F^{n\times n}$ where the B_j are invertible for $j > \ell$. Then they satisfy the compatibility conditions (2.2) if and only if its dual sequence (4.1) satisfies (2.2).*

Remark 4.1.1 *Lemma 4.1.1 can be proved alternatively by a straightforward but tedious verification of the conditions (2.2) for these matrices.*

4.2 Hyperexponential Solutions of Fully Integrable Systems

As a preparation for our factorization algorithm, we present an algorithm for computing hyperexponential solutions of fully integrable systems.

We first recall some definitions introduced in [39].

Let E be an orthogonal Δ-extension of F. A nonzero element $h \in E$ is said to be *hyperexponential over F with respect to ∂_i* if $\partial_i(h) = r_i h$ for some $r_i \in F$. The element h is *hyperexponential over F* if it is hyperexponential over F with respect to all ∂_i. In the sequel, we abbreviate "hyperexponential" as "hyperexp". Two hyperexp elements $h_1, h_2 \in E$ are said to be *similar* over F, denoted by $h_1 \sim h_2$, if there exist $c_1, c_2 \in C_E$ and $r_1, r_2 \in F$ such that $c_1 r_2 h_1 + c_2 r_2 h_2 = 0$.

The above two notions can be extended to vectors. We say that a nonzero vector $\mathbf{h} \in E^n$ is *hyperexp* over F *with respect to ∂_i* if $\mathbf{h}$ can be written as $h\mathbf{v}$ where $\mathbf{v} \in F^n$ and $h \in E$ is hyperexp over F with respect to ∂_i. The vector $\mathbf{h} \in E^n$ is *hyperexp over F* if $\mathbf{h}$ is hyperexp over F with respect to all ∂_i. Observe that $\mathbf{h} \in E^n$ is hyperexp over F if and only if $\mathbf{h}$ can be written as $h\mathbf{v}$ where $\mathbf{v} \in F^n$ and $h \in E$ is hyperexp over F. Indeed, if $\mathbf{h} \in E^n$ is hyperexp over F then $\mathbf{h} = h_i \mathbf{v}_i$ where $\mathbf{v}_i \in F^n$ and $\partial_i(h_i) = r_i h_i$ with $r_i \in F$ for $i = 1, \ldots, m$. Fix an index j. Since $h_i\mathbf{v}_i = h_j\mathbf{v}_j$, there exists $q_i \in F$ such that $h_i = q_i h_j$ for each i. Applying the δ_i, for $i \leq \ell$, and the σ_i, for $i > \ell$, to the above relation yields

$$\delta_i(h_j) = \left(r_i - q_i^{-1}\delta_i(q_i)\right) h_j \quad \text{for } i \leq \ell \quad \text{and} \quad \sigma_i(h_j) = r_i q_i \sigma_i(q_i)^{-1} h_j \quad \text{for } i > \ell.$$

Hence h_j is hyperexp over F and $\mathbf{h} = h_j\mathbf{v}_j$ is of the desired form.

Unlike in the purely differential case, a hyperexp element of an orthogonal Δ-extension of F is not necessarily invertible. However, the following lemma shows that a hyperexp element in a simple orthogonal Δ-ring extension is always invertible.

Lemma 4.2.1 *Let E be a simple orthogonal Δ-extension of F. If $h \in E$ is hyperexp over F then it is invertible in E.*

Proof. By definition, $\partial_i(h)$ and h are linearly dependent over F for $i = 1, \ldots, m$. If follows that the algebraic ideal (h) generated by h in E is invariant. Since E is simple, $(h) = E$ and so h is invertible. □

Remark 4.2.1 *From Theorem 2.2.5, every fully integrable system has a Picard-Vessiot ring E, which is a simple ring containing "all" solutions of the system. So we can assume that, for every hyperexp solution $h\mathbf{v}$ of a fully integrable system, h is invertible. In addition, we have $C_E = C$, as F has characteristic zero and C is algebraically closed. Hence two hyperexp elements over F are similar if and only if their ratio is an element of F.*

We now describe two algorithms for computing hyperexp solutions of fully integrable systems.

The first algorithm is a natural generalization of the "cyclic vectors" method used in purely differential ([53]) and purely difference ([31]) cases.

Let M be an $\mathcal{L}$-module of finite dimension. An element $\mathbf{w}$ of M is called a *cyclic vector* if there is $k \in \{1, \ldots, m\}$ such that $\mathbf{w}$, $\partial_k(\mathbf{w})$, $\ldots$, $\partial_k^{n-1}(\mathbf{w})$, with $n \geq 1$, form an F-basis of M.

Given a fully integrable system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ of size n over F, let A be the stacking of blocks $(\partial_i \cdot \mathbf{1}_n - A_i)$, M be the module of formal solutions of $A(\mathbf{z}) = 0$ and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$

be a set of $\mathcal{L}$-generators of M such that $A(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = 0$, *i.e.*,

$$\partial_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = A_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau, \quad i = 1, \ldots, m. \tag{4.2}$$

Hence $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a set of F-generators of M by Example 2.4.9 and moreover, by Proposition 2.4.8, it is an F-basis of M.

Assume that F contains a nonconstant a. Then either $\delta_k(a) \neq 0$ for some $k \leq \ell$ or $\sigma_k(a) \neq a$ for some $k > \ell$. In the former case, M as an $F[\partial_k]$-module contains a cyclic vector by Proposition 2.9 in [53], and in the latter case, a is not periodic, therefore M as an $F[\partial_k, \partial_k^{-1}]$-module has a cyclic vector by Theorem 7.2 in [31]. So, in both cases, M contains a cyclic vector $\mathbf{w}$ such that $\mathbf{w}, \partial_k(\mathbf{w}), \ldots, \partial_k^{n-1}(\mathbf{w})$ form an F-basis of M. Then there exists $P \in \mathrm{GL}_n(F)$ such that $(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = P(\mathbf{w}, \partial_k(\mathbf{w}), \ldots, \partial_k^{n-1}(\mathbf{w}))^\tau$. Suppose that

$$\partial_i(\mathbf{w}, \partial_k(\mathbf{w}), \ldots, \partial_k^{n-1}(\mathbf{w}))^\tau = B_i(\mathbf{w}, \partial_k(\mathbf{w}), \ldots, \partial_k^{n-1}(\mathbf{w}))^\tau, \quad i = 1, \ldots, m, \tag{4.3}$$

where $B_i \in F^{n \times n}$. By Proposition 2.4.8, $\{\partial_i(\mathbf{y}) = B_i \mathbf{y}\}_{1 \leq i \leq m}$ with $\mathbf{y} = (y_1, \ldots, y_n)^\tau$ is a fully integrable system, for which M is the module of formal solutions. Theorem 2.4.1, together with (4.2) and (4.3), implies that the map $\mathbf{y} \to P\mathbf{y}$ from $\mathrm{sol}_E(\{\partial_i(\mathbf{y}) = B_i \mathbf{y}\}_{1 \leq i \leq m})$ to $\mathrm{sol}_E(\{\partial_i(\mathbf{z}) = A_i \mathbf{z}\}_{1 \leq i \leq m})$ is bijective, for any orthogonal Δ-extension E of F. By linear algebra, we find a rectangular system $L : \{L_1(y) = 0, \ldots, L_m(y) = 0\}$ annihilating y_1, where each $L_i \in F[\partial_i]$ is of minimal order. Clearly, there is a one-to-one correspondence between $\mathrm{sol}_E(L) \to \mathrm{sol}_E(\{\partial_i(\mathbf{y}) = B_i \mathbf{y}\}_{1 \leq i \leq m})$ given by $y \mapsto (y, \partial_k(y), \ldots, \partial_k^{n-1}(y))^\tau$, for any orthogonal Δ-extension E of F. Hence every hyperexp solution of $\{\partial_i(\mathbf{z}) = A_i \mathbf{z}\}_{1 \leq i \leq m}$ has the form $P\left(h, \partial_k(h), \ldots, \partial_k^{n-1}(h)\right)^\tau$ where h is a hyperexp solution of L in E. Therefore it suffices to find all hyperexp solutions of the system L.

By Proposition 1 in [46], every rectangular system has only a finite number of dissimilar hyperexp solutions. Apply the main algorithm in [39, 40] to L to compute all its hyperexp solutions. Suppose that we find

$$h_1 r_{11}, \quad \ldots, \quad h_1 r_{1,t_1}, \quad \ldots, \quad h_s r_{s1}, \quad \ldots, \quad h_s r_{s,t_s},$$

where $h_1, \ldots, h_s$ are dissimilar hyperexp elements over F and $r_{i1}, \ldots, r_{i,t_i} \in F$ are linearly independent over C. Every hyperexp solution h of L has the form $h_i(c_1 r_{i1} + \cdots + c_{t_i} r_{i,t_i})$ with some $i \in \{1, \ldots, s\}$ and $c_j \in C$, not all zero.

For $i = 1, \ldots, s$ and $j = 1, \ldots, t_i$, we get that

$$P\left(h_i r_{ij}, \, \partial_k(h_i r_{ij}), \, \ldots, \, \partial_k^{n-1}(h_i r_{ij})\right)^\tau = h_i \mathbf{v}_{ij},$$

where $\mathbf{v}_{ij} \in F^n$. Since $r_{i1}, \ldots, r_{i,t_i}$ are linearly independent over C, so are $\mathbf{v}_{i1}, \ldots, \mathbf{v}_{i,t_i}$. Thus, every hyperexp solution of $\{\partial_i(\mathbf{z}) = A_i \mathbf{z}\}_{1 \le i \le m}$ has the form

$$h_i(c_1 \mathbf{v}_{i1} + \cdots + c_{t_i} \mathbf{v}_{i,t_i}),$$

with $i \in \{1, \ldots, s\}$ and $c_1, \ldots, c_{t_i} \in C$, not all zero.

Although in theory, by finding a cyclic vector, we can reduce the problem of finding hyperexp solutions of $\{\partial_i(\mathbf{z}) = A_i \mathbf{z}\}_{1 \le i \le m}$ to finding hyperexp solutions of the associated rectangular system L, finding such a system and computing its hyperexp solutions can be of high complexity.

We now present an alternative approach to computing hyperexp solutions of fully integrable systems. This algorithm is based on the assumption that *we are able to find all rational solutions of a fully integrable system*. Indeed, for the ordinary differential case, an algorithm to find rational solutions of the system $\mathbf{y}' = A\mathbf{y}$ without using cyclic vectors has been given in [1, 3], and the method in [39, 40] for finding rational solutions of rectangular systems can be adapted easily to finding rational solutions of fully integrable systems.

Given a fully integrable system $\{\partial_i(\mathbf{z}) = A_i \mathbf{z}\}_{1 \le i \le m}$ where $\mathbf{z} = (z_1, \ldots, z_n)^\tau$, find by linear algebra a rectangular system $L : \{L_1(z) = 0, \, \ldots, \, L_m(z) = 0\}$ annihilating z_1 where each $L_i \in F[\partial_i]$ is of minimal order. We then proceed as follows.

Step. 1. Apply the algorithm in [39, 40] to compute all hyperexp solutions of L. If L has no hyperexp solutions then go to Step 2. Otherwise, suppose that we find $\{h_1 r_1, \ldots, h_s r_s\}$ where $h_1, \ldots, h_s$ are pairwise dissimilar hyperexp elements over F and the $r_i \in F$ may contain some unspecified constants. For each $k \in \{1, \ldots, s\}$, let $y_1, \ldots, y_n$ be new unknowns and substitute $z_1 = h_k y_1, \ldots, z_n = h_k y_n$ into $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ to get

$$\partial_i(y_1, \ldots, y_n)^\tau = B_i^{(k)}(y_1, \ldots, y_n)^\tau, \quad i = 1, \ldots, m, \tag{4.4}$$

where $B_i^{(k)} = A_i - \frac{\delta_i(h_k)}{h_k}\mathbf{1}_n$ for $i \le \ell$ and $B_i^{(k)} = \frac{h_k}{\sigma_i(h_k)}A_i$ for $i > \ell$. A straightforward verification shows that (4.4) is a fully integrable system, of which we can find all rational solutions of (4.4) by the assumption. Suppose that $\mathbf{v}_{k1}, \ldots, \mathbf{v}_{k,t_k}$ form a basis of all rational solutions of (4.4), for $k = 1, \ldots, s$. Hence,

$$h_1\mathbf{v}_{11}, \ \ldots, \ h_1\mathbf{v}_{1,t_1}, \quad \ldots, \quad h_s\mathbf{v}_{s,1}, \ \ldots, \ h_s\mathbf{v}_{s,t_s}$$

are hyperexp solutions of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ whose first coordinates are nonzero. Moreover, all hyperexp solutions of this system, whose first coordinates are nonzero, are C-linear combinations of $h_k\mathbf{v}_{k1}, \ \ldots, \ h_k\mathbf{v}_{k,t_k}$ for some $k \in \{1, \ldots, s\}$. Indeed, if $h\mathbf{v}$ is a hyperexp solution, where $\mathbf{v} = (v_1, \ldots, v_n)^\tau \in F^n$ and $v_1 \ne 0$, of this system, then hv_1 is a hyperexp solution of L. Therefore h is similar to some h_k with $k \in \{1, \ldots, s\}$. There is $r \in F$ such that $h = rh_k$, thus $rh_k\mathbf{v}$ is a hyperexp solution of the original system. It follows that $r\mathbf{v}$ is a rational solution of (4.4). Hence

$$h\mathbf{v} = rh_k\mathbf{v} = h_k\,(c_1\mathbf{v}_{k,1} + \cdots + c_{t_k}\mathbf{v}_{k,t_k}), \quad \text{where } c_1, \ldots, c_{t_k} \in C, \text{ not all zero,}$$

are hyperexp solutions of the original system.

Step. 2. Substitute $z_1 = 0$ into all the first rows in the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ to yield a linear system $P\mathbf{z} = 0$ where P is a matrix over F with n columns. Apply the algorithm **LinearReduction** to $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ and $P\mathbf{z} = 0$ finally yields a partition $\{\mathbf{u}, \mathbf{v}\}$ of $\mathbf{z}$ such that $\mathbf{v}=Q\mathbf{u}$ where Q is a matrix over F, and a fully integrable system $\{\partial_i(\mathbf{u})=B_i\mathbf{u}\}_{1\le i\le m}$ over F which has less unknowns than the original one.

The above process can be repeated recursively until we find all hyperexp solutions of $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1\le i\le m}$. We therefore get hyperexp solutions of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ via the formula $\mathbf{v} = Q\mathbf{u}$. Combine all the hyperexp solutions obtained in these two steps, we obtain all hyperexp solutions of the original system.

Algorithm HyperexpSolutions (Find all hyperexp solutions of a fully integrable system)

Input: A fully integrable system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ over F where $\mathbf{z} = (z_1, \ldots, z_n)^\tau$.

Output: All its hyperexp solutions.

1. [Initialize] Set H to be the null set.

2. [Construct a rectangular system] Find by linear algebra a rectangular system

$$L: \quad \{\, L_1(z) = 0, \ \ldots, \ L_m(z) = 0 \,\}$$

that annihilates z_1.

3. [Compute hyperexp solutions whose first coordinates are nonzero] Apply the algorithm in [39, 40] to compute all hyperexp solutions of L. If L has no hyperexp solutions, then go to Step 4. Otherwise, suppose that we find $\{h_1r_1, \ldots, h_sr_s\}$ where $h_1, \ldots, h_s$ are pairwise dissimilar hyperexp elements over F and the $r_i \in F$ may contain some unspecified constants. For $g = 1, \ldots, s$, construct the fully integrable system $\mathcal{B}^{(g)} : \{\partial_i(\mathbf{y}) = B_i^{(g)}\mathbf{y}\}_{1\le i\le m}$ where $B_i^{(g)} = A_i - \frac{\delta_i(h_g)}{h_g}\mathbf{1}_n$ for $i \le \ell$ and $B_i^{(g)} = \frac{h_g}{\sigma_i(h_g)}A_i$ for $i > \ell$. If $\mathcal{B}^{(g)}$ has only trivial solution for $g = 1, \ldots, s$, then go to Step 4. Otherwise, suppose that $\mathcal{B}^{(j_1)}, \ldots, \mathcal{B}^{(j_q)}$ have nonzero rational solutions where $1 \le j_1 < \cdots < j_q \le s$. Let $\mathbf{v}_{j_p,1}, \ldots, \mathbf{v}_{j_p,t_{j_p}}$ be a basis of rational solutions of $\mathcal{B}^{(j_p)}$ for $p = 1, \ldots, q$. Set

$$\mathbf{h}_{j_1} = h_{j_1}\left(c_{j_1,1}\mathbf{v}_{j_1,1} + \cdots + c_{j_1,t_{j_1}}\mathbf{v}_{j_1,t_{j_1}}\right), \ \ldots, \ \mathbf{h}_{j_q} = h_{j_q}\left(c_{j_q,1}\mathbf{v}_{j_q,1} + \cdots + c_{j_q,t_{j_q}}\mathbf{v}_{j_q,t_{j_q}}\right),$$

where $c_{j_p,1}, \ldots, c_{j_p,t_{j_p}} \in C$, not all zero, for $p = 1, \ldots, q$. Set H to be $\{\mathbf{h}_{j_1}, \ldots, \mathbf{h}_{j_q}\}$.

4. [Compute hyperexp solutions whose first coordinates are zero] Substitute $z_1 = 0$ into the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ to yield $P\mathbf{z} = 0$ where P is a matrix over F with n columns. Apply the algorithm **LinearReduction** to $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$ and $P\mathbf{z}=0$ to finally produce a partition $\{\mathbf{u}, \mathbf{v}\}$ of $\mathbf{z}$ such that $\mathbf{v} = Q\mathbf{u}$ with Q a matrix over F, and a fully integrable system $\{\partial_i(\mathbf{u})=B_i\mathbf{u}\}_{1\le i\le m}$ over F. Apply the algorithm **HyperexpSolutions** recursively to compute hyperexp solutions of the system $\{\partial_i(\mathbf{u}) = B_i\mathbf{u}\}_{1\le i\le m}$. If this system has no hyperexp solutions, then return H. Otherwise, suppose that we find $\mathbf{g}_1, \ldots, \mathbf{g}_l$ where $\mathbf{g}_i$ may contain some unspecified constants. Use the formula $Q\mathbf{g}_i$ for $i = 1, \ldots, l$, to retrieve other components. We thus obtain some hyperexp solutions $\{\widetilde{\mathbf{h}}_1, \ldots, \widetilde{\mathbf{h}}_l\}$ of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\le i\le m}$. Update H by combining the values of H and $\{\widetilde{\mathbf{h}}_1, \ldots, \widetilde{\mathbf{h}}_l\}$, and return H.

Remark 4.2.2 *The algorithm* **HyperexpSolutions** *is applicable when the coefficient field is the field of rational functions in $x_1, \ldots, x_n$ over $\overline{\mathbb{Q}}$, and ∂_i only acts nontrivially on x_i for $1 \leq i \leq n$. This is because the algorithm is based on the work in [39, 40].*

Example 4.2.3 *Let $F = \mathbb{C}(x, k)$ and $\mathcal{S} = F[\partial_x; 1, \delta_x][\partial_k; \sigma_k, 0]$. We now compute hyperexponential solutions of the fully integrable system $\mathcal{A}^{(2)} : \{\delta_x(\mathbf{z}) = A_{2x}\mathbf{z},\, \sigma_k(\mathbf{z}) = A_{2k}\mathbf{z}\}$ where $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6)^\tau$,*

$$A_{2x} = \begin{pmatrix} -\frac{2(x^2-x-k^2)}{(x-k)x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & \frac{x^3+x^2k-2x^2-xk-k^2x-k^2-k^3}{x^2(x-k)} & 0 & 0 \\ 0 & -1 & -\frac{2(x^2-x-k^2-2k-1)}{(x-k-1)x} & 0 & a_{35} & 0 \\ 0 & -1 & 0 & -\frac{2(x^2-x-k^2)}{(x-k)x} & a_{45} & 0 \\ 0 & 0 & -1 & -1 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2(x^2-x-k^2-2k-1)}{(x-k-1)x} \end{pmatrix},$$

in which

$$a_{23} = \frac{x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3}{x^2(x-k-1)}, \quad a_{35} = \frac{x^3+x^2k-2x^2-xk-k^2x-k^2-k^3}{x^2(x-k)},$$
$$a_{45} = \frac{x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3}{x^2(x-k-1)}, \quad a_{55} = -\frac{2(2x^3-2x^2k-3x^2-2k^2x+2k^3+3k^2+k}{(x-k)x(x-k-1)},$$

and

$$A_{2k} = \begin{pmatrix} \frac{4(x-k-1)^2}{(x-k)^2x^2} & 0 & \frac{2(x-k-1)}{(x-k)x} & -\frac{2(x-k-1)}{(x-k)x} & -\frac{2(x^2-2xk-2x+k^2+k)}{x^2(x-k)^2} & 1 \\ \frac{2(x^2-2xk-4x+k^2+3k+2)}{(x-k)^2x^4} & \frac{(x-k-2)}{(x-k)x^2} & 0 & -\frac{2(x^2-2xk-3x+k^2+2k)}{x^3(x-k)^2} & 0 & 0 \\ -\frac{2(x-k-2)(x-k-1)}{(x-k)^2x^3} & 0 & 0 & \frac{x-k-2}{(x-k)x^2} & 0 & 0 \\ \frac{2(x-k-2)(x-k-1)}{(x-k)^2x^3} & 0 & \frac{x-k-2}{(x-k)x^2} & 0 & -\frac{2(x^2-2xk-3x+k^2+2k)}{x^3(x-k)^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{x-k-2}{(x-k)x^2} & 0 \\ \frac{(x-k-2)^2}{(x-k)^2x^4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By linear algebra, we obtain two annihilators

$$L_x = \partial_x + \frac{2(x^2-x-k^2)}{(x-k)x} \quad \textit{and} \quad L_k = \partial_k^3 - \frac{3(x-k-3)^2}{(x-k-2)^2x^2}\partial_k^2 + \frac{3(x-k-3)^2}{(x-k-1)^2x^4}\partial_k - \frac{(x-k-3)^2}{x^6(x-k)^2},$$

of z_1 in ∂_x and ∂_k, respectively, both of minimal order. Applying the algorithm in [39] to the rectangular system $L : \{L_x(z_1) = 0,\, L_k(z_1) = 0\}$, we find the common associate of all hyperexp solutions of L is $e^{-2x}x^{-2k}$. Substituting $z_i = e^{-2x}x^{-2k}y_i$ for $i = 1, \ldots, 6$, into $\mathcal{A}^{(2)}$ with the new unknowns y_i yields the following system $\mathcal{B} : \{\delta_x(\mathbf{y}) = B_x\mathbf{y},\, \sigma_k(\mathbf{y}) = B_k\mathbf{y}\}$

where $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6)^\tau$,

$$B_x=\begin{pmatrix} \frac{2}{x-k} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2(x+k)}{x} & b_{23} & \frac{x^3+x^2k-2x^2-xk-k^2x-k^2-k^3}{x^2(x-k)} & 0 & 0 \\ 0 & -1 & \frac{2(k+1)}{(x-k-1)x} & 0 & \frac{x^3+x^2k-2x^2-xk-k^2x-k^2-k^3}{x^2(x-k)} & 0 \\ 0 & -1 & 0 & \frac{2}{x-k} & \frac{x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3}{x^2(x-k-1)} & 0 \\ 0 & 0 & -1 & -1 & -\frac{2(x^3-x^2k-2x^2-k^2x+k^3+2k^2+k)}{(x-k)x(x-k-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(k+1)}{(x-k-1)x} \end{pmatrix},$$

in which $b_{23} = \frac{x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3}{x^2(x-k-1)}$, *and*

$$B_k=\begin{pmatrix} \frac{4(x-k-1)^2}{(x-k)^2} & 0 & \frac{2x(x-k-1)}{x-k} & -\frac{2x(x-k-1)}{x-k} & -\frac{2(x^2-2xk-2x+k^2+k)}{(x-k)^2} & x^2 \\ \frac{2(x^2-2xk-4x+k^2+3k+2)}{x^2(x-k)^2} & \frac{x-k-2}{x-k} & 0 & -\frac{2(x^2-2xk-3x+k^2+2k)}{x(x-k)^2} & 0 & 0 \\ -\frac{2(x-k-2)(x-k-1)}{x(x-k)^2} & 0 & 0 & \frac{x-k-2}{x-k} & 0 & 0 \\ \frac{2(x-k-2)(x-k-1)}{x(x-k)^2} & 0 & \frac{x-k-2}{x-k} & 0 & -\frac{2(x^2-2xk-3x+k^2+2k)}{x(x-k)^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{x-k-2}{x-k} & 0 \\ \frac{(x-k-2)^2}{x^2(x-k)^2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis of all rational solutions of $\mathcal{B}$ *is* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ *where*

$$\mathbf{v}_1 = \begin{pmatrix} -x^2+2xk-k^2 \\ -\frac{2k^3+k^2-2x^2-x^3+x+k^4-2x^2k^2+3xk+3k^2x-4x^2k+x^4}{x^3} \\ \frac{x+2x^3+k^3+2xk+k^2-3x^2k-3x^2}{x^2} \\ \frac{-2xk+k+k^3+x^2k+2k^2-x-2k^2x}{x^2} \\ -\frac{k+k^2+x^2-x-2xk}{x} \\ -\frac{x^2-2xk-2x+k^2+2k+1}{x^2} \end{pmatrix}, \qquad \mathbf{v}_2=\begin{pmatrix} -x^2+2xk-k^2 \\ -\frac{k^2-2xk+k-2x+x^2}{x^2} \\ \frac{k+k^2+x^2-x-2xk}{x} \\ -\frac{k+k^2+x^2-x-2xk}{x} \\ 0 \\ -\frac{x^2-2xk-2x+k^2+2k+1}{x^2} \end{pmatrix},$$

$$\mathbf{v}_3=\begin{pmatrix} k^3+5/2x^2+x^2k+5/2k^2-2k^2x-5xk \\ -\frac{2x^4+2k^4+2k^2+10xk-k-4x^2k^2+4k^2x+6x-7x^2k-5x^2-2x^3+5k^3}{x^2} \\ \frac{-5x^2+5xk+k^3+3x+2x^3-k-3x^2k}{x} \\ \frac{-4k^2x+3k^3-x^2k+5k-5x-9xk+8k^2+2x^3+x^2}{x} \\ -2k-2k^2+4xk+2x-2x^2 \\ \frac{-10xk+3+8k-6x+2x^2k-4k^2x+3x^2+7k^2+2k^3}{2x^2} \end{pmatrix},$$

$$\mathbf{v}_4 = \begin{pmatrix} -2k^2x + x^2 + k^2 + x^2k - 2xk + k^3 \\ -\frac{2x-x^3+k^2+5xk+x^4+3k^2x-4x^2k+k^4-2x^2-2x^2k^2+2k^3}{x^2} \\ x^2 - 2xk - 2x + k^2 + 2k + 1 \\ \frac{x^3-3k^2x+2k-2x+2k^3-4xk+4k^2}{x} \\ -x^2 - k^2 - k + x + 2xk \\ \frac{k(x^2-2xk-2x+k^2+2k+1)}{x^2} \end{pmatrix},$$

$$\mathbf{v}_5 = \begin{pmatrix} -2k^3 - 2k^2 - 2x^2k - 2x^2 + 4k^2x + 4xk \\ -\frac{-2k^4-k^2x-4x+x^5+k^4x+4x^2-2x^4+4x^2k^2-4x^3k-2x^3k^2+8x^2k-2k^2+4xk^3-8xk-4k^3}{x^2} \\ x^3 + 4x - 3x^2 + 3xk - x^2k - 3k - 2 + k^3 - k^2x \\ \frac{-2x^2-3x^2k+4x-8k^2-2x^3-x^2k^2+9k^2x-4k^3-4k+x^4+10xk-x^3k+xk^3}{x} \\ 2k^2 - 2x - k^2x - 5xk - x^3 + 3x^2 + 2x^2k + 2k \\ -\frac{2k(x^2-2xk-2x+k^2+2k+1)}{x^2} \end{pmatrix},$$

$$\mathbf{v}_6 = \begin{pmatrix} -2xk^3 + 5k^3 + 5x^2k + k^4 + 6x^2 - 10k^2x + 6k^2 + x^2k^2 - 12xk \\ \frac{x^3-4x^3k^2-5k^2+k-4k^4-5k^2x-21xk+19x^2k+11x^2+2k^4x-12x-5x^4+2x^5+11x^2k^2-10k^3-8x^3k+6xk^3}{x^2} \\ -\frac{-7x^3+k+7x^2k-x^2k^2+2x^4+11x^2-9xk+k^4-6x-2x^3k}{x} \\ -\frac{-k^4-5x^2k-2x^3k+11x+25xk-20k^2+20k^2x-5x^3-5x^2-3x^2k^2-11k-10k^3+2x^4+4xk^3}{x} \\ -4x^2k - 5k + 2x^3 - 5k^2 + 12xk + 5x + 2k^2x - 7x^2 \\ \frac{-4x+3x^2k+x^2k^2+k^4+2x^2+9k^2+5k^3-2xk^3-10xk+2+7k-8k^2x}{x^2} \end{pmatrix}.$$

So all hyperexp solutions of the original system $\mathcal{A}$ are of the form $e^{-2x}x^{-2k}\left(\sum_{i=1}^{6} c_i\mathbf{v}_i\right)$ where the c_i are in $\mathbb{C}$, not all zero. □

4.3 A Module-Theoretic Approach to Factorization

We describe an idea on factoring $\mathcal{L}$-modules, which generalizes the module-theoretic method for factoring differential modules [53, 62].

4.3.1 Reduction from M to $\wedge^d M$

Recall that a *decomposable* ([48]) element $\mathbf{w} \in \wedge^d M$ is an exterior product of d elements in M, *i.e.*, $\mathbf{w} = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d$.

The following theorem is a generalization of Lemma 10 in [22] or the corresponding statement in [53, §4.2.1]:

Theorem 4.3.1 *An $\mathcal{L}$-module M has a d-dimensional submodule if and only if $\wedge^d M$ has a one-dimensional submodule generated by a decomposable element.*

Proof. Let N be a d-dimensional submodule of M and $\mathbf{w}_1, \ldots, \mathbf{w}_d$ be an F-basis of N. Suppose that $\partial_i(\mathbf{w}_1, \ldots, \mathbf{w}_d)^\tau = A_i(\mathbf{w}_1, \ldots, \mathbf{w}_d)^\tau$ where $A_i = (a_{ist})_{1\le s,\, t\le d} \in F^{d\times d}$ for each i. Then $\wedge_F^d N$ is an F-subspace of $\wedge_F^d M$ generated by $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d$. Moreover,

$$\begin{aligned} &\partial_i(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d) = \textstyle\sum_{s=1}^d \mathbf{w}_1 \wedge \cdots \wedge \partial_i(\mathbf{w}_s) \wedge \cdots \wedge \mathbf{w}_d \\ = \;\; & \textstyle\sum_{s=1}^d \left(\mathbf{w}_1 \wedge \cdots \wedge \left(\sum_{t=1}^d a_{ist}\, \mathbf{w}_t\right) \wedge \cdots \wedge \mathbf{w}_d\right) = \mathrm{tr}(A_i)\, (\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d) \in \wedge^d N, \end{aligned}$$

for $i \le \ell$ and $\partial_j(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d) = \partial_j(\mathbf{w}_1) \wedge \cdots \wedge \partial_j(\mathbf{w}_d) = \det(A_j)\, (\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d) \in \wedge^d N$ for $j > \ell$, where "tr" and "det" denote respectively the trace and the determinant of a matrix. So, $\wedge^d N$ is a one-dimensional submodule generated by $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d$, which is decomposable.

Conversely, let $\mathbf{u} = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d$ be a decomposable element of $\wedge^d M$ that generates a one-dimensional submodule. Suppose that $\partial_i(\mathbf{u}) = a_i \mathbf{u}$ with $a_i \in F$ for $i = 1, \ldots, m$. Clearly, $a_j \neq 0$ for $j > \ell$ since M is an $\mathcal{L}$-module. Since $\mathbf{u} \neq 0$, $\mathbf{w}_1, \ldots, \mathbf{w}_d$ are linearly independent over F. Therefore there is a basis B of M containing $\mathbf{w}_1, \ldots, \mathbf{w}_d$. Pick arbitrarily a finite number of distinct $\mathbf{b}_1, \ldots, \mathbf{b}_s$ in $B \setminus \{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$. The F-linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_d, \mathbf{b}_1, \ldots, \mathbf{b}_s$ then implies that of $\mathbf{b}_1 \wedge \mathbf{u}, \ldots, \mathbf{b}_s \wedge \mathbf{u}$. In particular, $\mathbf{b} \wedge \mathbf{u} \neq 0$ for any $\mathbf{b} \in B \setminus \{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$.

Consider a map $\phi_{\mathbf{u}} : M \to \wedge^{d+1} M$ given by $\mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{u}$. Clearly, $\ker(\phi_{\mathbf{u}})$ is a vector space over F. Let $\mathbf{v} \in \ker(\phi_{\mathbf{u}})$ then $\mathbf{v} \wedge \mathbf{u} = 0$. For $i \le \ell$, we have

$$0 = \partial_i(\mathbf{v} \wedge \mathbf{u}) = \partial_i(\mathbf{v}) \wedge \mathbf{u} + \mathbf{v} \wedge (a_i \mathbf{u}) = \partial_i(\mathbf{v}) \wedge \mathbf{u} + a_i(\mathbf{v} \wedge \mathbf{u}),$$

thus $\partial_i(\mathbf{v}) \wedge \mathbf{u} = 0$ and $\partial_i(\mathbf{v}) \in \ker(\phi_{\mathbf{u}})$. For $j > \ell$, we have

$$0 = \partial_j(\mathbf{v} \wedge \mathbf{u}) = \partial_j(\mathbf{v}) \wedge (a_j \mathbf{u}) = a_j(\partial_j(\mathbf{v}) \wedge \mathbf{u}),$$

which implies that $\partial_j(\mathbf{v}) \wedge \mathbf{u} = 0$ and thus $\partial_j(\mathbf{v}) \in \ker(\phi_{\mathbf{u}})$. Likewise, $\partial_j^{-1}(\mathbf{v}) \in \ker(\phi_{\mathbf{u}})$ for $j > \ell$. So, $\ker(\phi_{\mathbf{u}})$ is an $\mathcal{L}$-module.

Clearly, $\oplus_{i=1}^d F\mathbf{w}_i$ is contained in $\ker(\phi_{\mathbf{u}})$. Let $\mathbf{w}$ be in $\ker(\phi_{\mathbf{u}}) \subset M$. Then there exist $\mathbf{b}_1, \ldots, \mathbf{b}_s$ in $B \backslash \{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$ such that $\mathbf{w} = \sum_{i=1}^d \lambda_i \mathbf{w}_i + \sum_{j=1}^s \mu_j \mathbf{b}_j$ where $\lambda_i, \mu_j \in F$. Therefore $0 = \mathbf{w} \wedge \mathbf{u} = \sum_{j=1}^s \mu_j \left(\mathbf{b}_j \wedge \mathbf{u}\right)$. The F-linear independence of $\mathbf{b}_1 \wedge \mathbf{u}, \ldots, \mathbf{b}_s \wedge \mathbf{u}$ therefore implies that $\mu_j = 0$ for each j. So $\ker(\phi_{\mathbf{u}}) = \oplus_{i=1}^d F\mathbf{w}_i$ and is a d-dimensional submodule of M. □

Theorem 4.3.1 converts the problem of finding d-dimensional submodules of M into that of finding one-dimensional submodules of $\wedge^d M$ whose generator are decomposable, and thus reduces our factorization problem to two "subproblems": finding one-dimensional submodules and determining the decomposability of their generators.

4.3.2 One-Dimensional Submodules

As we saw, a building block for factoring is computing one-dimensional submodules. In the ordinary differential case, an efficient algorithm for finding one-dimensional submodules is described in [7] and implemented in the ISOLDE package.

In this section, we first set up a correspondence between one-dimensional submodules of a finite-dimensional $\mathcal{L}$-module and hyperexponential solutions of its associated fully integrable system. Such a correspondence results naturally in an algorithm for finding one-dimensional submodules.

Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and let $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau$. Suppose that $\partial_i(\mathbf{e}) = B_i\mathbf{e}$ where $B_i \in F^{n\times n}$ for $i = 1, \ldots, m$. By Proposition 2.4.8, $B_1, \ldots, B_m$ satisfy the compatibility conditions (2.2) and the B_j are invertible for $j > \ell$. By applying ∂_j^{-1} to both sides of $\partial_j(\mathbf{e}) = B_j\mathbf{e}$, we get $\partial_j^{-1}(\mathbf{e}) = \sigma_j^{-1}(B_j^{-1})\mathbf{e}$ for $j > \ell$, which means the $\mathcal{L}$-module structure of M is uniquely determined by the actions of $\partial_1, \ldots, \partial_m$. Let $\mathbf{w} = \sum_{i=1}^n a_i\mathbf{e}_i \in M$ with $a_i \in F$. Then

$$\partial_i(\mathbf{w}) = \partial_i((a_1, \ldots, a_n)\mathbf{e}) = (\delta_i(a_1, \ldots, a_n) + (a_1, \ldots, a_n)B_i)\,\mathbf{e}$$

for $i \leq \ell$ and $\partial_j(\mathbf{w}) = \partial_j((a_1, \ldots, a_n)\mathbf{e}) = \sigma_j(a_1, \ldots, a_n)B_j\mathbf{e}$ for $j > \ell$. The condition

$$\partial_i(\mathbf{w}) = 0, \quad i \leq \ell \quad \text{and} \quad \partial_j(\mathbf{w}) = \mathbf{w}, \quad j > \ell$$

has therefore a translation that the vector $(a_1, \ldots, a_n)^\tau$ of coefficients of $\mathbf{w}$ is a solution of the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$ where $A_i = -B_i^\tau$ for $i \leq \ell$ and $A_i = (B_i^{-1})^\tau$ for $i > \ell$. From Lemma 4.1.1, $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$ is a fully integrable system, which is called the *fully integrable system associated to* M with respect to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be another F-basis of M with $(\mathbf{f}_1, \ldots, \mathbf{f}_n) = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\,T$ for some $T \in \mathrm{GL}_n(F)$. Substitute $\mathbf{z} = T\mathbf{z}^*$ into $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$, we obtain the fully integrable system $\{\partial_i(\mathbf{z}^*) = A_i^*\mathbf{z}^*\}_{1 \leq i \leq m}$ for the new basis, where

$$A_i^* = T^{-1}A_iT - T^{-1}\delta_i(T), \quad \text{for } i \leq \ell \quad \text{and} \quad A_j^* = \sigma_j(T^{-1})A_jT, \quad \text{for } j > \ell.$$

By Definition 2.2.5, $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$ and $\{\partial_i(\mathbf{z}^*) = A_i^*\mathbf{z}^*\}_{1 \leq i \leq m}$ are equivalent. Indeed, the fully integrable systems associated to a finite-dimensional $\mathcal{L}$-module with respect to its different bases are equivalent.

Conversely, any fully integrable system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1 \leq i \leq m}$ comes from an $\mathcal{L}$-module $M := F^n$ with the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and the ∂_i given by

$$\partial_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = -A_i^\tau(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau \quad \text{and} \quad \partial_j(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = (A_j^{-1})^\tau(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau,$$

for $i \leq \ell$ and $j > \ell$. So we have set up a correspondence between $\mathcal{L}$-modules of finite dimension and fully integrable systems of equal dimension.

[Convention] Any element of F^n is considered as a column vector.

For convenience of later discussion, we give the following proposition, which is an analogue to Proposition 5.1 in [45]. It describes a correspondence between one-dimensional submodules of a finite-dimensional $\mathcal{L}$-module and hyperexp solutions of its associated fully integrable systems. Although this proposition is obvious in the ordinary (differential and difference) cases, a detailed proof seems necessary because the compatibility conditions should be taken into account.

Proposition 4.3.2 *Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathcal{A}$ be the fully integrable system associated to M with respect to $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then*

(i) *If $h\mathbf{v}$ is a hyperexp solution of $\mathcal{A}$ where $\mathbf{v} \in F^n$ and h is hyperexp over F, then the element $(\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v}$ generates a one-dimensional submodule of M.*

(ii) *If the element $(\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v}$ with $\mathbf{v} \in F^n$ generates a one-dimensional submodule of M, then there exists a hyperexp element h over F such that $h\mathbf{v}$ is a solution of $\mathcal{A}$.*

(iii) *For $k = 1, 2$, let $h_k\mathbf{v}_k$, where $\mathbf{v}_k \in F^n$ and h_k is hyperexp over F, be a hyperexp solution of $\mathcal{A}$, and N_k be the one-dimensional submodule generated by $(\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v}_k$. Then N_1 is isomorphic to N_2 if and only if $h_1 \sim h_2$. In particular, $N_1 = N_2$ if and only if $h_1 \sim h_2$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent over F.*

Proof. (i) Let $\mathbf{u} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)h\mathbf{v}$. Since $h\mathbf{v}$ is a solution of $\mathcal{A}$, we have $\partial_i(\mathbf{u}) = 0$ for $i \leq \ell$ and $\partial_j(\mathbf{u}) = \mathbf{u}$ for $j > \ell$. Set $\mathbf{w} = \frac{\mathbf{u}}{h} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v} \in M$. Then $\partial_i(\mathbf{w}) = -\frac{\delta_i(h)}{h}\mathbf{w}$ for $i \leq \ell$ and $\partial_j(\mathbf{w}) = \frac{h}{\sigma_j(h)}\mathbf{w}$ for $j > \ell$, so $F\mathbf{w}$ is a one-dimensional submodule of M.

(ii) Let $\mathbf{w} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v} \in M$. Suppose that $\partial_i(\mathbf{w}) = a_i\mathbf{w}$ with $a_i \in F$ for $1 \leq i \leq m$. Proposition 2.4.8 together with Lemma 4.1.1 implies that the system

$$\{\, \partial_i(y) = -a_i y, \quad 1 \leq i \leq \ell, \quad \partial_j(y) = a_j^{-1} y, \quad \ell + 1 \leq j \leq m \,\},$$

is fully integrable. Thus, either by Theorem 2.2.2 or by Example 2.2.6, this system has a solution h in a simple orthogonal Δ-extension of F. One sees that $\partial_i(h\mathbf{w}) = 0$ for $i \leq \ell$ and $\partial_j(h\mathbf{w}) = h\mathbf{w}$ for $j > \ell$, thus $h\mathbf{v}$ is a hyperexp solution of $\mathcal{A}$.

(iii) Set $\mathbf{w}_k = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v}_k$ for $k = 1, 2$. By the same argument in (i), we have

$$\partial_i(\mathbf{w}_k) = -\frac{\delta_i(h_k)}{h_k}\mathbf{w}_k, \quad \text{for } i \leq \ell \quad \text{and} \quad \partial_j(\mathbf{w}_k) = \frac{h_k}{\sigma_j(h_k)}\mathbf{w}_k, \quad \text{for } j > \ell. \tag{4.5}$$

If $h_1 \sim h_2$ then $\frac{h_2}{h_1} = r$ for some $r \in F$ by Remark 4.2.1. It follows that the map from N_1 to N_2 given by $\mathbf{w}_1 \mapsto r\mathbf{w}_2$ is an isomorphism of $\mathcal{L}$-modules. Conversely, let ϕ be an isomorphism of $\mathcal{L}$-modules from N_1 to N_2 sending $\mathbf{w}_1$ to $r\mathbf{w}_2$ where $r \in F$. The $\mathcal{L}$-linearity of ϕ, together with the relations (4.5), implies that

$$\frac{\delta_i(h_2)}{h_2} = \frac{\delta_i(h_1)}{h_1} + \frac{\delta_i(r)}{r}, \quad \text{for } i \leq \ell \quad \text{and} \quad \frac{\sigma_j(h_2)}{h_2} = \frac{\sigma_j(h_1)}{h_1}\frac{\sigma_j(r)}{r}, \quad \text{for } j > \ell,$$

which implies $h_1 \sim h_2$. The second statement in (iii) is then obvious. □

At the end of this section, we describe an algorithm for finding one-dimensional submodules of $\mathcal{L}$-modules of finite dimension.

Let M be an $\mathcal{L}$-module of dimension n and $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$ be the fully integrable system associated to M with respect to a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of M. Suppose that, by applying the algorithm **HyperexpSolutions**, we find a finite collection of hyperexp solutions of $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$:

$$h_1\mathbf{v}_{11}, \ \ldots, \ h_1\mathbf{v}_{1,t_1}, \quad \ldots, \quad h_s\mathbf{v}_{s1}, \ \ldots, \ h_s\mathbf{v}_{s,t_s},$$

where $h_1, \ldots, h_s$ are pairwise dissimilar hyperexp elements over F and $\mathbf{v}_{k,1}, \ldots, \mathbf{v}_{k,t_k}$ are linearly independent over C for $k = 1, \ldots, s$. Let $h\mathbf{v}$ be a hyperexp solution of the system $\{\partial_i(\mathbf{z}) = A_i\mathbf{z}\}_{1\leq i\leq m}$. Then h is similar to some h_k with $1 \leq k \leq s$ and $\mathbf{v}$ is a C-linear combination of $\mathbf{v}_{k1}, \ldots, \mathbf{v}_{k,t_k}$. Set $\mathbf{w}_{kl} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{v}_{kl}$ for $l = 1, \ldots, t_k$ and

$$I_k = \left\{F(c_1\mathbf{w}_{k1} + \cdots + c_{t_k}\mathbf{w}_{k,t_k}) \mid c_1, \ldots, c_{t_k} \in C, \text{ not all zero}\right\},$$

for $k = 1, \ldots, s$. From Proposition 4.3.2, it follows that $I_1, \ldots, I_s$ constitute a partition of all one-dimensional submodules of M by the equivalence relation "$\cong_{\mathcal{L}}$", an isomorphism between $\mathcal{L}$-modules.

Example 4.3.1 [Legendre's system] *Let $F = \mathbb{C}(x,k)$ and $\mathcal{L} = F[\partial_x, \partial_k, \partial_k^{-1}]$ be the Laurent-Ore algebra. A Gröbner basis of the ideal generated by the Legendre's system* (1.1) *is as follows:*

$$g_1 = xk + x + (x^2 - 1)\partial_x - (k+1)\partial_k, \quad g_2 = k + 1 + (k+2)\partial_k^2 - (2xk + 3x)\partial_k.$$

Let $A = (g_1, g_2)^\tau \in \mathcal{L}^{2\times 1}$, $M = \mathcal{L}/(\mathcal{L}g_1 + \mathcal{L}g_2)$ and $\mathbf{e}_1, \mathbf{e}_2$ be the images of 1 *and ∂_k in M, respectively. Then $\mathbf{e}_1, \mathbf{e}_2$ form a basis of M over F and, in addition,*

$$\partial_x \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \frac{-xk-x}{x^2-1} & \frac{k+1}{x^2-1} \\ \frac{-k-1}{x^2-1} & \frac{xk+x}{x^2-1} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \partial_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k-1}{k+2} & \frac{2xk+3x}{k+2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Apply the algorithm **HyperexpSolutions** *to the fully integrable system associated to M with respect to $\mathbf{e}_1, \mathbf{e}_2$. We find that this fully integrable system has no hyperexp solutions. So M has no one-dimensional submodules, or, M is irreducible.* □

4.3.3 Decomposability of Elements of $\wedge^d M$

We now return to the problem of finding d-dimensional submodules of a finite-dimensional $\mathcal{L}$-module M.

Apply the algorithm described in Section 4.3.2 to find all one-dimensional submodules of $\wedge^d M$. Suppose that we obtain a finite collection $\{F\mathbf{w}_1, \ldots, F\mathbf{w}_s\}$ of one-dimensional submodules of $\wedge^d M$, where each $\mathbf{w}_k$ may contain some unspecified constants. By Theorem 4.3.1, $F\mathbf{w}_k$ corresponds to a d-dimensional submodule of M if and only if $\mathbf{w}_k$ is decomposable. It remains to determine the decomposability of each $\mathbf{w}_k$, or, in other words, to find constraints on the unspecified constants, for which $\mathbf{w}_k$ is decomposable.

For each $\mathbf{w}$ in $\{\mathbf{w}_1, \ldots, \mathbf{w}_s\}$, consider the map $\phi_{\mathbf{w}} : M \to \wedge^{d+1} M$ given by $\mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{w}$. From Theorem 1.1 in [48, Ch.4] and the proof of Theorem 4.3.1, $\mathbf{w}$ is decomposable if and only if $\ker(\phi_{\mathbf{w}})$ is of dimension d. The latter is equivalent to the condition that the matrix P of $\phi_{\mathbf{w}}$ has rank $(n-d)$. Hence determining the decomposability of $\mathbf{w}$ amounts to a rank computation of P, i.e., identifying the unspecified constants $c_1, \ldots, c_t$ in $\mathbf{w}$ such that all $(n-d+1) \times (n-d+1)$ minors of P are zero and P has a nonzero $(n-d) \times (n-d)$ minor. This further amounts to solving a nonlinear system in $c_1, \ldots, c_t$. We observe that this is the Plücker relations described in [53, 61] (for more details, please see [29, 30, 32]). If this nonlinear system has no solutions in C, then $\mathbf{w}$ is not decomposable and does not lead to any d-dimensional submodule of M. Otherwise, substitute the values of $c_1, \ldots, c_t$ into P and compute a basis $\mathbf{r}_1, \ldots, \mathbf{r}_d$ of the rational kernel of P where $\mathbf{r}_j \in F^n$. Set $\mathbf{u}_j = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{r}_j$ for $j = 1, \ldots, d$. Then $\oplus_{j=1}^d F\mathbf{u}_j$ is a d-dimensional submodule of M.

Remark 4.3.2 *There are alternative ways to compute ranks of parameterized matrices, for example, the Gaussian method with branching, a Gröbner basis method using the linear structure [23] or the algorithm described in [60] for computing the rank of a parameterized* *linear system. These methods may be more efficient than computing minors.*

Example 4.3.3 *Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$. Suppose that*

$$\partial_x(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau = A_x(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau \quad \text{and} \quad \partial_k(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau = A_k(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau,$$

where

$$A_x = \begin{pmatrix} 0 & -\frac{k(k+1)}{x^2-1} & 0 & 0 \\ -1 & \frac{2x}{x^2-1} & 0 & 0 \\ 0 & 0 & 0 & -\frac{(k+1)(k+2)}{x^2-1} \\ 0 & 0 & -1 & \frac{2x}{x^2-1} \end{pmatrix} \text{ and } A_k = \begin{pmatrix} \frac{(2k+3)x}{k+1} & \frac{2k+3}{k+1} & 1 & 0 \\ 0 & \frac{(2k+3)x}{k+1} & 0 & 1 \\ -\frac{k+2}{k+1} & 0 & 0 & 0 \\ 0 & -\frac{k+2}{k+1} & 0 & 0 \end{pmatrix}.$$

Then $\{\delta_x(\mathbf{z}) = -A_x^\tau \mathbf{z}, \sigma_k(\mathbf{z}) = (A_k^{-1})^\tau \mathbf{z}\}$ is the fully integrable system associated to M with respect to the given basis. Apply the algorithm **HyperexpSolutions***, we find that the above system has no hyperexp solutions. So M has no one-dimensional submodules.*

In a similar way, we find that the fully integrable system associated to $\wedge^3 M$ with respect to the basis $\{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4\}$ has no hyperexp solutions. Thus $\wedge^3 M$ has no one-dimensional submodules and consequently M has no three-dimensional submodules.

Let us compute all two-dimensional submodules of M. Clearly,

$$\mathbf{f}_1 = \mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{f}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{f}_3 = \mathbf{e}_1 \wedge \mathbf{e}_4, \quad \mathbf{f}_4 = \mathbf{e}_2 \wedge \mathbf{e}_3, \quad \mathbf{f}_5 = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad \mathbf{f}_6 = \mathbf{e}_3 \wedge \mathbf{e}_4,$$

form a basis of $\wedge^2 M$ over F and, moreover,

$$\partial_x(\mathbf{f}_1, \ldots, \mathbf{f}_6)^\tau = B_x(\mathbf{f}_1, \ldots, \mathbf{f}_6)^\tau \quad \text{and} \quad \partial_k(\mathbf{f}_1, \ldots, \mathbf{f}_6)^\tau = B_k(\mathbf{f}_1, \ldots, \mathbf{f}_6)^\tau,$$

where

$$B_x = \begin{pmatrix} \frac{2x}{x^2-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{(k+1)(k+2)}{x^2-1} & -\frac{k(k+1)}{x^2-1} & 0 & 0 \\ 0 & -1 & \frac{2x}{x^2-1} & 0 & -\frac{k(k+1)}{x^2-1} & 0 \\ 0 & -1 & 0 & \frac{2x}{x^2-1} & -\frac{(k+1)(k+2)}{x^2-1} & 0 \\ 0 & 0 & -1 & -1 & \frac{4x}{x^2-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2x}{x^2-1} \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} \frac{(2k+3)^2x^2}{(k+1)^2} & 0 & \frac{(2k+3)x}{k+1} & -\frac{(2k+3)x}{k+1} & \frac{2k+3}{k+1} & 1 \\ \frac{(k+2)(2k+3)}{(k+1)^2} & \frac{k+2}{k+1} & 0 & 0 & 0 & 0 \\ -\frac{(k+2)(2k+3)x}{(k+1)^2} & 0 & 0 & \frac{k+2}{k+1} & 0 & 0 \\ \frac{(k+2)(2k+3)x}{(k+1)^2} & 0 & \frac{k+2}{k+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{k+2}{k+1} & 0 \\ \frac{(k+2)^2}{(k+1)^2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\{\delta_x(\mathbf{z}) = -B_x^\tau, \sigma_k(\mathbf{z}) = (B_k^{-1})^\tau \mathbf{z}\}$ *is the fully integrable system associated to* $\wedge^2 M$. *Apply the algorithm* **HyperexpSolutions** *to find all its hyperexp solutions:*

$$\mathbf{h} = c\left(-\tfrac{1}{x^2-1}, -\tfrac{1}{k+1}, -\tfrac{x}{x^2-1}, \tfrac{x}{x^2-1}, \tfrac{k+1}{x^2-1}, -\tfrac{1}{x^2-1}\right)^\tau, \quad \textit{for any } c \in \mathbb{C}^*.$$

Hence the generator of any one-dimensional submodule of $\wedge^2 M$ *has the form*

$$\mathbf{w} = -\tfrac{1}{x^2-1}\mathbf{f}_1 - \tfrac{1}{k+1}\mathbf{f}_2 - \tfrac{x}{x^2-1}\mathbf{f}_3 + \tfrac{x}{x^2-1}\mathbf{f}_4 + \tfrac{k+1}{x^2-1}\mathbf{f}_5 - \tfrac{1}{x^2-1}\mathbf{f}_6.$$

Thus $\wedge^2 M$ *has only one-dimensional submodule* $F\mathbf{w}$. *Consider the map* $\phi_{\mathbf{w}} : M \to \wedge^3 M$ *given by* $\mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{w}$. *The matrix* P *of* $\phi_{\mathbf{w}}$ *is*

$$\begin{pmatrix} \frac{x}{x^2-1} & \frac{1}{k+1} & -\frac{1}{x^2-1} & 0 \\ \frac{k+1}{x^2-1} & \frac{x}{x^2-1} & 0 & -\frac{1}{x^2-1} \\ -\frac{1}{x^2-1} & 0 & \frac{x}{x^2-1} & -\frac{1}{k+1} \\ 0 & -\frac{1}{x^2-1} & -\frac{k+1}{x^2-1} & \frac{x}{x^2-1} \end{pmatrix},$$

and has exactly rank 2. *This means that* M *has two-dimensional submodules. To retrieve two-dimensional submodules of* M*, we compute the rational kernel of* P *and find its* F*-basis*

$$\mathbf{r}_1 = \left(\frac{xk+x}{k+1}, -k-1, 1, 0\right)^\tau \in F^4, \quad \mathbf{r}_2 = \left(-\frac{x^2-1}{k+1}, x, 0, 1\right)^\tau \in F^4.$$

Set

$$\mathbf{w}_1 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)\mathbf{r}_1 = \frac{xk+x}{k+1}\mathbf{e}_1 - (k+1)\mathbf{e}_2 + \mathbf{e}_3$$

and

$$\mathbf{w}_2 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)\mathbf{r}_2 = -\frac{x^2-1}{k+1}\mathbf{e}_1 + x\mathbf{e}_2 + \mathbf{e}_4.$$

Then $F\mathbf{w}_1 \oplus F\mathbf{w}_2$ is the only two-dimensional submodule of M. In addition,

$$\partial_x \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{(k+1)(k+2)}{x^2-1} \\ -1 & \frac{2x}{x^2-1} \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} \quad \text{and} \quad \partial_k \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} = \begin{pmatrix} x & -k-2 \\ -\frac{x^2-1}{k+2} & x \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix}.$$

We remark that the $\mathcal{L}$-module M in this example is reducible but not completely reducible, since it has only one two-dimensional submodule. □

4.4 An Factorization Algorithm for $\mathcal{L}$-Modules of Finite Dimension

We now describe a factorization algorithm for $\mathcal{L}$-modules of finite dimension.

Algorithm FactorModule (Factor $\mathcal{L}$-modules of finite dimension)

Input: An $\mathcal{L}$-module M with an F-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and the actions of ∂_i on this basis:

$$\partial_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = B_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau, \quad \text{for } i = 1, \ldots, m, \tag{4.6}$$

where $B_1, \ldots, B_m \in F^{n\times n}$ and the B_j are invertible for $j > \ell$.

Output: For $0 < d < n$, all d-dimensional submodules of M given by their F-bases and the actions of ∂_i on the bases.

1. [Construct the exterior power] From (4.6), construct an F-basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ of $\wedge^d M$ with $g = \binom{n}{d}$, and the fully integrable system $\{\partial_i(\mathbf{z}) = \widetilde{B}_i \mathbf{z}\}_{1\le i\le m}$, where $\widetilde{B}_i \in F^{g\times g}$, associated with $\wedge^d M$ with respect to $\mathbf{f}_1, \ldots, \mathbf{f}_g$.

2. [Compute one-dimensional submodules] Apply the algorithm **HyperexpSolutions** to find all hyperexp solutions of the system $\{\partial_i(\mathbf{z}) = \widetilde{B}_i \mathbf{z}\}_{1\le i\le m}$. If the output is NULL, then exit [M has no d-dimensional submodules]. Otherwise, construct a finite collection $\{F\mathbf{w}_1, \ldots, F\mathbf{w}_s\}$ of one-dimensional submodules of $\wedge^d M$ where each $\mathbf{w}_q$ may contain unspecified constants $c_1, \ldots, c_{t_q}$.

3. [Determine the decomposability] For $q = 1, \ldots, s$, consider the map $\phi_q : M \to \wedge^{d+1} M$ given by $\mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{w}_q$. Construct the matrix P_q of ϕ_q, which is an $\binom{n}{d+1} \times n$ matrix with

entries in $F(c_1, \ldots, c_{t_q})$. The condition that all $(n-d+1)\times(n-d+1)$ minors of P_q are zero and that P_q has a nonzero $(n-d)$ minor then yields a nonlinear system in $c_1, \ldots, c_{t_q}$, which we denote by T_q. If none of the systems $T_1, \ldots, T_s$ has solutions in C, then exit [M has no d-dimensional submodules]. Otherwise, suppose that, for any $q \in \{l_1, \ldots, l_e\} \subseteq \{1, \ldots, s\}$ with $e \le s$, T_q has a solution in C. For every such q, substitute the values of $c_1, \ldots, c_{t_q}$ into P_q and compute an F-basis $\mathbf{r}_{q1}, \ldots, \mathbf{r}_{qd}$ for the rational kernel of P_q.

4. [Retrieve d-dimensional submodules] For each $q \in \{l_1, \ldots, l_e\}$, set $\mathbf{v}_{qj} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)\mathbf{r}_{qj}$ for j=$1, \ldots, d$. Then $\{\oplus_{j=1}^{d} F\mathbf{v}_{l_1,j}, \ \ldots, \ \oplus_{j=1}^{d} F\mathbf{v}_{l_e,j}\}$ are all d-dimensional submodules of M.

Example 4.4.1 *Let $F = \mathbb{C}(x,k)$, $\mathcal{L} = F[\partial_x; 1, \delta_x][\partial_k, \partial_k^{-1}; \sigma_k, 0]$. Let M be an $\mathcal{L}$-module with an F-basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ satisfying that $\partial_x(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau = A_x(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau$ and $\partial_k(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau = A_k(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^\tau$ where*

$$A_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-x^3-x^2k+2x^2+xk+k^2x+k^2+k^3}{x^2(-x+k)} & \frac{2(x^2-x-k^2)}{(x-k)x} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-x^3-x^2k+x^2+3xk+2x+k^2x+4k^2+5k+2+k^3}{x^2(-x+k+1)} & -\frac{2(-x^2+x+k^2+2k+1)}{(-x+k+1)x} \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(x-k)x^2}{x-k-2} & 0 & \frac{2x(x-k-1)}{x-k-2} & 0 \\ -\frac{2x(x^2-2xk-3x+k^2+2k)}{(x-k-2)^2} & -\frac{(x-k)x^2}{x-k-2} & \frac{2(x^2-2xk-4x+k^2+3k+2)}{(x-k-2)^2} & \frac{2x(x-k-1)}{x-k-2} \end{pmatrix}.$$

Let us compute all two-dimensional submodules of M. Clearly,

$$\mathbf{f}_1 = \mathbf{e}_1 \wedge \mathbf{e}_2, \ \mathbf{f}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3, \ \mathbf{f}_3 = \mathbf{e}_1 \wedge \mathbf{e}_4, \ \mathbf{f}_4 = \mathbf{e}_2 \wedge \mathbf{e}_3, \ \mathbf{f}_5 = \mathbf{e}_2 \wedge \mathbf{e}_4, \ \mathbf{f}_6 = \mathbf{e}_3 \wedge \mathbf{e}_4$$

form a basis of $\wedge^2 M$ over F. The fully integrable system associated to $\wedge^2 M$ is the $\mathcal{A}^{(2)}$ in Example 4.2.3, whose hyperexp solutions are of the form $H = e^{-2x}x^{-2k}\left(\sum_{i=1}^{6} c_i \mathbf{v}_i\right)$ where the $\mathbf{v}_i$ are as in Example 4.2.3 and the c_i are in $\mathbb{C}$, not all zero. Hence every one-dimensional submodule of $\wedge^2 M$ has a generator of the form $\mathbf{w} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6)\left(\sum_{i=1}^{6} c_i \mathbf{v}_i\right)$. It remains to determine the decomposability of $\mathbf{w}$. Consider the map $M \to \wedge^3 M$ given by $\mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{w}$, whose matrix is some $P \in F^{4\times 4}$. The matrix P has rank 2 if and only *if all its 3×3 minors are zero and there exists a nonzero 2×2 minor. This yields four sets of solutions for the c_i:*

$$\{c_4 = c_4, \quad c_1 = c_1, \quad c_2 = c_2, \quad c_5 = 0, \quad c_3 = 0, \quad c_6 = 0\},$$

$$\left\{c_3 = c_3, \quad c_1 = c_1, \quad c_2 = c_2, \quad c_5 = 0, \quad c_6 = 0, \quad c_4 = -\tfrac{3c_3}{2}\right\},$$

$$\left\{c_3 = c_3, \quad c_5 = c_5, \quad c_4 = c_4, \quad c_2 = c_2, \quad c_6 = 0, \quad c_1 = \tfrac{c_3(2c_4+3c_3-4c_5)}{4c_5}\right\},$$

$$\left\{c_3{=}c_3, c_5{=}c_5, c_4{=}c_4, c_1{=}c_1, c_6{=}c_6, c_2{=}-\tfrac{8c_6c_5-2c_3c_4-3c_3^2-4c_6c_4-20c_3c_6-4c_1c_6+4c_1c_5+4c_3c_5-24c_6^2}{4c_6}\right\}.$$

Therefore M has two-dimensional submodules if and only if the c_i in $\mathbf{w}$ satisfy one of these four relations.

Substitute these four relations into P respectively and compute the corresponding F-bases for the rational kernel of P. Finally, we get all two-dimensional submodules given below:

$$N_i = \{a_1\mathbf{u}_{i,1} + a_2\mathbf{u}_{i,2} \mid a_1, a_2 \in F\}, \quad i = 1, 2, 3, 4.$$

For N_1,

$$\begin{aligned}
\mathbf{u}_{1,1} =& (2c_1x^2k + c_1k^2x + c_4x^5 + c_1x^2 + 2c_1x^4 + c_2x^2k - 2x^3c_2k - 3x^3c_1k - 2x^4c_4k \\
&+x^2c_2k^2 + xk^3c_1 + x^4c_2 + c_4x^3k^2 - c_2x^3 - 2c_4x^4 - 3c_1x^3 + c_4x^3 + 2c_4x^3k)/(x(2c_1x - c_1x^2 + 2c_1xk \\
&-2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))e_1 \\
&-(c_4x^3 + c_1x^2 - kc_4x^2 - c_4x^2 - c_1xk - c_1x)(x - k)/(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 \\
&+2k^2c_4 - 2c_1k - c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2)e_2 + e_3,
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_{1,2} =& (c_1x^4 - 2c_1x^2 + c_4x^5 + c_1x - 2c_4x^3 - c_1x^3 - c_4x^4 + c_2x^3 - 2c_2x^2 + 2c_4x^2 + c_1k^2 + 2c_1k^3 + c_1k^4 - 2c_2x^2k \\
&+3c_1k^2x - 4c_1x^2k + 5kc_4x^2 + k^2c_4x + c_2xk + 3c_1xk - 4c_4x^3k - 2c_4x^3k^2 - 2c_1x^2k^2 + 2c_4xk^3 + 3c_4x^2k^2 + c_4xk^4 \\
&+c_2xk^2)/(x(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 + 2c_2xk - 2kc_4x \\
&-2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))e_1 - (c_4x^3 - c_2x^2 + kc_4x^2 + c_2xk - 2k^2c_4x + c_2x - 2xc_4 + c_1xk - 4kc_4x - c_1 \\
&-c_1k^2 - 2c_1k)(x - k)/((2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 \\
&+2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2))e_2 + e_4,
\end{aligned}$$

and its $\mathcal{L}$-module structures are given by

$$\partial_x \begin{pmatrix} \mathbf{u}_{1,1} \\ \mathbf{u}_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3)}{x^2(x-k-1)} & \frac{2(x^2-x-k^2-2k-1)}{(x-k-1)x} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{1,1} \\ \mathbf{u}_{1,2} \end{pmatrix},$$

and

$$\partial_k \begin{pmatrix} \mathbf{u}_{1,1} \\ \mathbf{u}_{1,2} \end{pmatrix} = \begin{pmatrix} d_{11}^{(1)} & d_{12}^{(1)} \\ d_{21}^{(1)} & d_{22}^{(1)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{1,1} \\ \mathbf{u}_{1,2} \end{pmatrix},$$

in which

$$d_{11}^{(1)} = \frac{c_4x^3-c_2x^2+kc_4x^2+c_2xk-2k^2c_4x+c_2x-2xc_4+c_1xk-4kc_4x-c_1-c_1k^2-2c_1k}{(x-k-2)(-c_1-c_2+kc_4+c_4)},$$

$$d_{12}^{(1)} = -\frac{(x-k-1)(xc_4+c_1)x}{(-c_1x-c_2x+kc_4x+xc_4+c_2k+2c_2+c_1k+2c_1-k^2c_4-3kc_4-2c_4)},$$

$$\begin{aligned} d_{21}^{(1)} = \Big(& 4c_1 - c_2xk^2 + c_4xk^4 + c_1x^4 - c_2x^3 - c_4x^4 - 3c_1x^3 - 6c_4x^3 + 3c_1x - 2c_2x + 4c_2x^2 + 12c_1k + 13c_1k^2 + 2c_4x^2 + 6c_1k^3 \\ &+c_1k^4 + 8xc_4 - kc_4x^2 - 4c_1x^2k + c_1k^2x + 2c_2x^2k - 2c_4x^3k^2 - 6c_4x^3k + 3c_1xk - 3c_2xk + 22kc_4x \\ &+21k^2c_4x - c_4x^2k^2 + 8c_4xk^3 - 2c_1x^2k^2 + c_4x^5\Big) /((x-k-2)^2(-c_1-c_2+kc_4+c_4)x), \end{aligned}$$

$$\begin{aligned} d_{22}^{(1)} = -&\Big((c_4x^3 + c_2x^2 + 2c_1x^2 - c_4x^2 - kc_4x^2 - c_1xk - 4xc_4 - 3c_1x - 2kc_4x - c_2xk - 2c_2x - 4c_1 - 4c_1k - c_1k^2)(x-k-1)\Big) / \\ &\Big((x-k-2)(-c_1x - c_2x + kc_4x + xc_4 + c_2k + 2c_2 + c_1k + 2c_1 - k^2c_4 - 3kc_4 - 2c_4)\Big). \end{aligned}$$

For N_2,

$$\begin{aligned} \mathbf{u}_{2,1} = &-(4c_1x^2k + 2c_1k^2x + 2c_1x^2 + 4c_1x^4 + 2c_2x^2k + 4c_3x^3k + c_3x^5 - 4x^3c_2k - 6x^3c_1k \\ &+2x^2c_2k^2 + 2xk^3c_1 + 2x^4c_2 + 2x^2c_3k^3 - 4c_3x^4 - 2c_2x^3 - 3c_3x^3k^2 - 6c_1x^3 + 3c_3x^3 - 2c_3x^2k)/(x(-4c_1x \\ &+2c_1x^2 + c_3k^3 - c_3k^2 - 3c_3x^2 + 6c_3x - 5c_3k - 4c_1xk + 4c_2k + 2c_2k^2 - 4c_2x + 2c_2x^2 + 4c_1k + 2c_1k^2 + c_3x^2k + 4c_3xk \\ &-4c_2xk + 2c_1 + 2c_2 - 2c_3k^2x - 3c_3))e_1 + (c_3x^3 - c_3x^2 - c_3x^2k + 2c_1x^2 - 2c_1xk - 2c_1x)(x-k)/(-4c_1x + 2c_1x^2 \\ &+c_3k^3 - c_3k^2 - 3c_3x^2 + 6c_3x - 5c_3k - 4c_1xk + 4c_2k + 2c_2k^2 - 4c_2x + 2c_2x^2 + 4c_1k + 2c_1k^2 + c_3x^2k + 4c_3xk \\ &-4c_2xk + 2c_1 + 2c_2 - 2c_3k^2x - 3c_3)e_2 + e_3, \end{aligned}$$

$$\begin{aligned} \mathbf{u}_{2,2} = &-(2c_1x^4 - 4c_1x^2 + 2c_1x - 2c_1x^3 + 2c_2x^3 - 4c_2x^2 + 2c_1k^2 + 4c_1k^3 + 2c_1k^4 - c_3x^4 \\ &+c_3x^5 + 6c_3x^2 - 4c_3x^3 - 4c_2x^2k + 6c_1k^2x - 8c_1x^2k + 2c_2xk + 6c_1xk - 4c_1x^2k^2 + 2c_2xk^2 + 5c_3x^2k - 2c_3x^3k^2 \\ &-2c_3x^3k + 4c_3xk^3 + c_3k^2x - 2c_3xk + c_3xk^4 - c_3x^2k^2)/(x(-4c_1x + 2c_1x^2 + c_3k^3 - c_3k^2 - 3c_3x^2 + 6c_3x - 5c_3k - 4c_1xk \\ &+4c_2k + 2c_2k^2 - 4c_2x + 2c_2x^2 + 4c_1k + 2c_1k^2 + c_3x^2k + 4c_3xk - 4c_2xk + 2c_1 + 2c_2 - 2c_3k^2x - 3c_3))e_1 + (c_3x^3 \\ &+2c_3x^2 - 2c_2x^2 - c_3x^2k + 2c_1xk + 2c_2x - 4c_3x - 4c_3xk + 2c_2xk - 4c_1k - 2c_1 - 2c_1k^2)(x-k)/(-4c_1x + 2c_1x^2 + c_3k^3 \\ &-c_3k^2 - 3c_3x^2 + 6c_3x - 5c_3k - 4c_1xk + 4c_2k + 2c_2k^2 - 4c_2x + 2c_2x^2 + 4c_1k + 2c_1k^2 + c_3x^2k + 4c_3xk - 4c_2xk \\ &+2c_1 + 2c_2 - 2c_3k^2x - 3c_3)e_2 + e_4, \end{aligned}$$

and its $\mathcal{L}$-module structures are given by

$$\partial_x \begin{pmatrix} \mathbf{u}_{2,1} \\ \mathbf{u}_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3)}{x^2(x-k-1)} & \frac{2(x^2-x-k^2-2k-1)}{(x-k-1)x} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{2,1} \\ \mathbf{u}_{2,2} \end{pmatrix}$$

and

$$\partial_k \begin{pmatrix} \mathbf{u}_{2,1} \\ \mathbf{u}_{2,2} \end{pmatrix} = \begin{pmatrix} d_{11}^{(2)} & d_{12}^{(2)} \\ d_{21}^{(2)} & d_{22}^{(2)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{2,1} \\ \mathbf{u}_{2,2} \end{pmatrix},$$

in which

$$d_{11}^{(2)} = -\frac{(c_3x^3+2c_3x^2-2c_2x^2-c_3x^2k+2c_1xk+2c_2x-4c_3x-4c_3xk+2c_2xk-4c_1k-2c_1-2c_1k^2)}{(x-k-2)(2c_1-2c_3+2c_2+c_3k)},$$

$$d_{12}^{(2)} = \frac{(x-k-1)(c_3x+2c_1)x}{2c_1x-2c_3x+2c_2x+c_3xk-4c_1-4c_2+4c_3-c_3k^2-2c_2k-2c_1k},$$

$$\begin{aligned} d_{21}^{(2)} = -\Big(& 8c_1 - 2c_2xk^2 + 2c_1x^4 - 2c_2x^3 - 6c_1x^3 + 6c_1x - 4c_2x + 8c_2x^2 + 24c_1k + 26c_1k^2 + 12c_1k^3 + 2c_1k^4 \\ &-8c_3x^3k - 2c_3x^3k^2 + 3c_3x^2k + 24c_3xk + 17c_3k^2x + 3c_3x^2k^2 + c_3xk^4 + 6c_3xk^3 + c_3x^5 - c_3x^4 - 4c_3x^3 \\ &-6c_3x^2 + 12c_3x - 8c_1x^2k + 2c_1k^2x + 4c_2x^2k + 6c_1xk - 6c_2xk - 4c_1x^2k^2\Big) / \Big((x-k-2)^2(2c_1 - 2c_3 + 2c_2 + c_3k)x\Big), \end{aligned}$$

$$\begin{aligned} d_{22}^{(2)} = &\Big((c_3x^3 - 3c_3x^2 + 4c_1x^2 + 2c_2x^2 + c_3x^2k - 2c_1xk - 2c_2xk - 2c_3k^2x - 4c_2x - 4c_3xk - 6c_1x - 8c_1 - 8c_1k - 2c_1k^2)(x-k-1)\Big) \\ &/ \Big(2c_1x - 2c_3x + 2c_2x + c_3xk - 4c_1 - 4c_2 + 4c_3 - c_3k^2 - 2c_2k - 2c_1k)(x-k-2)\Big). \end{aligned}$$

For N_3,

$$
\begin{aligned}
\mathbf{u}_{3,1} = {} & (-9c_3^2x^3 + 16c_5^2x^4 - 12c_5^2x^5 - 8c_5^2x^3 + 4c_5^2x^6 + 3c_3^2x^2 + 6c_3^2x^4 \\
& -28c_3x^4c_5 + 12c_5^2x^4k + 4c_3x^4c_4 + 6c_3^2x^2k - 6c_3x^3c_4 - 4c_2x^3c_5 + 4c_4x^3c_5 + 3c_3^2xk^2 + 2c_3x^2c_4 \\
& -8c_5x^4c_4 + 4c_5x^5c_4 + 8c_3x^5c_5 - 4c_3x^2c_5 - 4c_5^2x^4k^2 - 4c_5^2x^5k + 4c_5^2x^3k^3 + 4c_5x^4c_2 + 3c_3^2xk^3 \\
& -12c_5^2x^3k + 24c_3x^3c_5 - 9c_3^2x^3k - 12c_3x^2kc_5 + 4c_3x^2kc_4 - 4c_3xk^2c_5 - 4c_3xk^3c_5 + 4c_2x^2k^2c_5 \\
& -8c_5x^4kc_4 - 12c_5x^4c_3k + 2c_4xk^3c_3 + 8c_5x^3kc_4 + 4c_4x^3k^2c_5 + 32c_3x^3kc_5 + 4c_2x^2kc_5 + 2c_4xk^2c_3 \\
& -6c_3x^3kc_4 + 4c_3x^2k^3c_5 - 8c_2x^3kc_5)/(x(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k \\
& -4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 \\
& -8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 \\
& +8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 \\
& +4c_3xkc_4 - 3c_3^2))e_1 - (-12c_5^2x^3 + 4c_5^2x^4 - 3c_3^2x + 8c_5^2x^2 + 3c_3^2x^2 + 8c_5^2x^2k - 4c_4x^2c_5 + 8c_3x^3c_5 \\
& -4c_5^2x^3k - 12c_3x^2c_5 - 4c_5x^2kc_4 + 2c_3x^2c_4 + 4c_4x^3c_5 - 8c_3x^2kc_5 - 2c_3xc_4 + 4c_3xkc_5 + 4c_3xc_5 \\
& -2c_3xkc_4 - 3c_3^2xk)(x-k)/(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k - 4c_5c_2 - 6c_3^2k - 8c_5^2k^3 \\
& -3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 - 8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 \\
& +6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 \\
& -4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2)e_2 + e_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u}_{3,2} = {} & (3c_3^2x - 3c_3^2x^3 - 8c_5^2x^5 + 16c_5^2x^3 + 4c_5^2x^6 - 6c_3^2x^2 + 3c_3^2x^4 + 3c_3^2k^4 \\
& +6c_3^2k^3 - 16c_5^2x^2 + 3c_3^2k^2 - 12c_3x^4c_5 - 16c_5^2x^4k + 2c_3x^4c_4 - 12c_3^2x^2k - 2c_3x^3c_4 + 4c_2x^3c_5 \\
& -8c_4x^3c_5 + 9c_3^2xk^2 - 4c_3x^2c_4 - 4c_5x^4c_4 + 4c_5x^5c_4 + 8c_3x^5c_5 + 32c_3x^2c_5 - 8c_5^2x^4k^2 + 32c_5^2x^3k \\
& -16c_3x^3c_5 - 4c_3xc_5 + 2c_3xc_4 + 9c_3^2xk - 32c_5^2x^2k - 8c_2x^2c_5 - 4c_3k^2c_5 + 2c_3k^2c_4 - 8c_5^2xk^2 - 8c_3k^3c_5 \\
& +8c_4x^2c_5 - 4c_3k^4c_5 + 2c_4k^4c_3 + 4c_4k^3c_3 - 4k^2c_5^2x^2 - 16c_5^2xk^3 - 8c_5^2xk^4 - 6c_3^2x^2k^2 + 16c_5^2x^3k^2 \\
& +16c_5^2x^2k^3 + 4c_5^2x^2k^4 + 56c_3x^2kc_5 - 8c_3x^2kc_4 - 4c_3xk^2c_5 + 20c_3xk^3c_5 - 16c_5x^3kc_4 - 8c_4x^3k^2c_5 \\
& -28c_3x^3kc_5 - 8c_2x^2kc_5 + 6c_4xk^2c_3 + 4c_2xkc_5 + 20c_5x^2kc_4 + 4c_4xk^2c_5 - 16c_3xkc_5 + 6c_3xkc_4 + 8c_3xk^4c_5 \\
& +4c_2xk^2c_5 + 4c_4xk^4c_5 + 8c_4xk^3c_5 - 4c_3x^2k^2c_4 + 24c_3x^2k^2c_5 - 16c_3x^3k^2c_5 + 12k^2c_5x^2c_4)/(x(6c_3^2x - 2c_3c_4 \\
& +10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k - 4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x \\
& +4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 - 8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 \\
& -2c_3k^2c_4 + 16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 \\
& +8c_2xkc_5 - 8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2))e_1 - (-2c_3c_4 + 4c_3c_5 \\
& +4c_5^2x^4 - 8c_5^2x^3 - 8c_5^2x^2 - 6c_3^2k - 3c_3^2k^2 + 16xc_5^2 + 4c_4x^3c_5 + 4c_3x^2c_5 + 8c_3x^3c_5 + 32kc_5^2x \\
& +4c_2xc_5 - 20c_3xc_5 + 3c_3^2xk - 20c_5^2x^2k - 4c_2x^2c_5 + 4c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 + 8c_3kc_5 - 4c_3kc_4 \\
& -4k^2c_5^2x^2 - 8xc_4c_5 + 4c_3x^2kc_5 - 12c_3xk^2c_5 + 4c_2xkc_5 - 16kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 36c_3xkc_5 \\
& +2c_3xkc_4 - 3c_3^2)(x-k)/(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k - 4c_5c_2 - 6c_3^2k - 8c_5^2k^3 \\
& -3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 - 8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 \\
& +6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 \\
& -4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2)e_2 + e_4,
\end{aligned}
$$

and its $\mathcal{L}$-module structures are given by

$$
\partial_x \begin{pmatrix} \mathbf{u}_{3,1} \\ \mathbf{u}_{3,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(x^3+x^2k-x^2-3xk-2x-k^2x-4k^2-5k-2-k^3)}{x^2(x-k-1)} & \frac{2(x^2-x-k^2-2k-1)}{(x-k-1)x} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{3,1} \\ \mathbf{u}_{3,2} \end{pmatrix},
$$

$$
\partial_k \begin{pmatrix} \mathbf{u}_{3,1} \\ \mathbf{u}_{3,2} \end{pmatrix} = \begin{pmatrix} d_{11}^{(3)} & d_{12}^{(3)} \\ d_{21}^{(3)} & d_{22}^{(3)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{3,1} \\ \mathbf{u}_{3,2} \end{pmatrix},
$$

in which

$$\begin{aligned}
d_{11}^{(3)} = &\Big(4c_2xc_5 + 32kc_5^2x - 2c_3k^2c_4 + 8c_3kc_5 - 20c_3xc_5 + 3c_3^2xk - 4c_3kc_4 - 4c_2x^2c_5 + 4c_3k^2c_5 + 16c_5^2xk^2 \\
&-20c_5^2x^2k + 16xc_5^2 - 4k^2c_5^2x^2 - 8xc_4c_5 + 8c_3x^3c_5 + 4c_4x^3c_5 - 3c_3^2 - 8c_5^2x^3 + 4c_5^2x^4 \\
&-6c_3^2k - 2c_3c_4 + 4c_3c_5 - 3c_3^2k^2 - 8c_5^2x^2 + 4c_3x^2c_5 - 12c_3xk^2c_5 + 4c_3x^2kc_5 + 2c_3xkc_4 \\
&-36c_3xkc_5 - 8c_4xk^2c_5 + 4c_5x^2kc_4 - 16kc_4xc_5 + 4c_2xkc_5\Big) \,/ \\
&\Big((x-k-2)(-4c_5c_2 - 3c_3^2 + 4c_3kc_5 - 8c_5^2k + 4c_5kc_4 - 2c_3c_4 + 14c_3c_5 + 4c_4c_5 - 8c_5^2)\Big)\,,
\end{aligned}$$

$$\begin{aligned}
d_{12}^{(3)} = &-\Big((x-k-1)(-8xc_5^2 + 4c_5^2x^2 + 4xc_4c_5 + 8c_3xc_5 - 4c_3c_5 + 2c_3c_4 + 3c_3^2)x\Big) \,/\, (-4c_2xc_5 \\
&-8c_4c_5 - 8kc_5^2x + 4c_2kc_5 - 22c_3kc_5 - 4k^2c_4c_5 + 14c_3xc_5 + 2c_3kc_4 - 4c_3k^2c_5 - 2c_3xc_4 - 8xc_5^2 \\
&+4xc_4c_5 - 12c_5kc_4 + 16c_5^2 + 6c_3^2 + 3c_3^2k + 8c_5c_2 + 4c_3c_4 - 28c_3c_5 + 8k^2c_5^2 + 24c_5^2k - 3c_3^2x \\
&+4c_3xkc_5 + 4kc_4xc_5)\,,
\end{aligned}$$

$$\begin{aligned}
d_{21}^{(3)} = &\Big(-8c_2xc_5 - 4c_2x^3c_5 + 8c_3x^5c_5 + 2c_3x^4c_4 - 12c_3^2x^2k - 8c_5^2x^4k^2 + 4c_5x^5c_4 - 176kc_5^2x + 26c_3k^2c_4 - 48c_3kc_5 \\
&+60c_3xc_5 + 9c_3^2xk + 24c_3kc_4 + 16c_2x^2c_5 - 52c_3k^2c_5 - 168c_5^2xk^2 - 24c_3k^3c_5 + 120c_5^2x^2k + 6c_3xc_4 + 8c_4x^2c_5 \\
&-4c_5x^4c_4 - 64xc_5^2 + 16c_5^2x^3k^2 + 100k^2c_5^2x^2 + 32c_5^2x^2k^3 + 4c_5^2x^2k^4 + 2c_4k^4c_3 + 12c_4k^3c_3 \\
&-6c_3^2x^2k^2 - 4c_3k^4c_5 - 64c_5^2xk^3 - 8c_5^2xk^4 + 32xc_4c_5 - 4k^2c_5x^2c_4 - 16c_3x^3k^2c_5 + 32c_4xk^3c_5 \\
&+4c_4xk^4c_5 - 4c_2xk^2c_5 + 8c_3xk^4c_5 - 4c_3x^2k^2c_4 + 8c_3x^2k^2c_5 - 32c_5^2x^4k + 48c_5^2x^3k - 32c_3x^3c_5 \\
&-6c_3x^3c_4 + 3c_3^2xk^2 - 24c_4x^3c_5 - 12c_3x^4c_5 + 12c_3^2 + 3c_3^2x^4 + 48c_5^2x^3 - 9c_3^2x^3 - 8c_5^2x^5 + 4c_5^2x^6 \\
&-24c_5^2x^4 + 36c_3^2k + 8c_3c_4 - 16c_3c_5 + 39c_3^2k^2 + 9c_3^2x + 32c_5^2x^2 + 3c_3^2k^4 + 18c_3^2k^3 + 8c_2x^2kc_5 \\
&-52c_3x^3kc_5 - 8c_4x^3k^2c_5 - 24c_5x^3kc_4 + 60c_3xk^3c_5 + 156c_3xk^2c_5 - 8c_3x^2kc_4 + 16c_3x^2kc_5 + 2c_4xk^2c_3 \\
&+6c_3xkc_4 + 168c_3xkc_5 + 84c_4xk^2c_5 - 4c_5x^2kc_4 + 88kc_4xc_5 - 12c_2xkc_5\Big) \,/\, \Big((x-k-2)^2(-4c_5c_2 - 3c_3^2 \\
&+4c_3kc_5 - 8c_5^2k + 4c_5kc_4 - 2c_3c_4 + 14c_3c_5 + 4c_4c_5 - 8c_5^2)x\Big)\,,
\end{aligned}$$

$$\begin{aligned}
d_{22}^{(3)} = &-\Big((-8c_2xc_5 + 4c_3x^2c_4 + 16kc_5^2x - 2c_3k^2c_4 + 16c_3kc_5 - 12c_3xc_5 - 3c_3^2xk - 8c_3kc_4 + 4c_2x^2c_5 + 4c_3k^2c_5 \\
&-12c_5^2x^2k - 6c_3xc_4 - 4c_4x^2c_5 + 32xc_5^2 - 4k^2c_5^2x^2 - 16xc_4c_5 + 8c_3x^3c_5 + 4c_4x^3c_5 - 12c_3^2 - 8c_5^2x^3 \\
&+4c_5^2x^4 + 6c_3^2x^2 - 12c_3^2k - 8c_3c_4 + 16c_3c_5 - 3c_3^2k^2 - 9c_3^2x - 16c_5^2x^2 - 20c_3x^2c_5 - 4c_3xk^2c_5 \\
&-4c_3x^2kc_5 - 2c_3xkc_4 - 16c_3xkc_5 - 4c_5x^2kc_4 - 8kc_4xc_5 - 4c_2xkc_5)(x-k-1)\Big) \,/\, \Big((x-k-2)(-4c_2xc_5 - 8c_4c_5 - 8kc_5^2x \\
&+4c_2kc_5 - 22c_3kc_5 - 4k^2c_4c_5 + 14c_3xc_5 + 2c_3kc_4 - 4c_3k^2c_5 - 2c_3xc_4 - 8xc_5^2 + 4xc_4c_5 - 12c_5kc_4 + 16c_5^2 + 6c_3^2 \\
&+3c_3^2k + 8c_5c_2 + 4c_3c_4 - 28c_3c_5 + 8k^2c_5^2 + 24c_5^2k - 3c_3^2x + 4c_3xkc_5 + 4kc_4xc_5\Big)\,.
\end{aligned}$$

For N_4,

$$\begin{aligned}
\mathbf{u}_{4,1} = &(-9c_3^2x^3 + 16c_5^2x^4 - 12c_5^2x^5 - 8c_5^2x^3 + 4c_5^2x^6 + 3c_3^2x^2 + 6c_3^2x^4 \\
&-28c_3x^4c_5 + 12c_5^2x^4k + 4c_3x^4c_4 + 6c_3^2x^2k - 6c_3x^3c_4 - 4c_2x^3c_5 + 4c_4x^3c_5 + 3c_3^2xk^2 + 2c_3x^2c_4 \\
&-8c_5x^4c_4 + 4c_5x^5c_4 + 8c_3x^5c_5 - 4c_3x^2c_5 - 4c_5^2x^4k^2 - 4c_5^2x^5k + 4c_5^2x^3k^3 + 4c_5x^4c_2 + 3c_3^2xk^3 \\
&-12c_5^2x^3k + 24c_3x^3c_5 - 9c_3^2x^3k - 12c_3x^2kc_5 + 4c_3x^2kc_4 - 4c_3xk^2c_5 - 4c_3xk^3c_5 + 4c_2x^2k^2c_5 \\
&-8c_5x^4kc_4 - 12c_5x^4c_3k + 2c_4xk^3c_3 + 8c_5x^3kc_4 + 4c_4x^3k^2c_5 + 32c_3x^3kc_5 + 4c_2x^2kc_5 + 2c_4xk^2c_3 \\
&-6c_3x^3kc_4 + 4c_3x^2k^3c_5 - 8c_2x^3kc_5)/(x(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k \\
&-4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 \\
&-8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 \\
&+4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 \\
&+4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2))e_1 - (-12c_5^2x^3 + 4c_5^2x^4 - 3c_3^2x + 8c_5^2x^2 \\
&+3c_3^2x^2 + 8c_5^2x^2k - 4c_4x^2c_5 + 8c_3x^3c_5 - 4c_5^2x^3k - 12c_3x^2c_5 - 4c_5x^2kc_4 + 2c_3x^2c_4 + 4c_4x^3c_5 \\
&-8c_3x^2kc_5 - 2c_3xc_4 + 4c_3xkc_5 + 4c_3xc_5 - 2c_3xkc_4 - 3c_3^2xk)(x-k)/(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 \\
&-16k^2c_5^2 - 8c_5^2k - 4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 \\
&-4c_2k^2c_5 + 8c_2xc_5 - 8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 \\
&+16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 \\
&-8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2)e_2 + e_3,
\end{aligned}$$

$$
\begin{aligned}
\mathbf{u}_{4,2} = {} & (3c_3^2x - 3c_3^2x^3 - 8c_5^2x^5 + 16c_5^2x^3 + 4c_5^2x^6 - 6c_3^2x^2 + 3c_3^2x^4 + 3c_3^2k^4 \\
& +6c_3^2k^3 - 16c_5^2x^2 + 3c_3^2k^2 - 12c_3x^4c_5 - 16c_5^2x^4k + 2c_3x^4c_4 - 12c_3^2x^2k - 2c_3x^3c_4 + 4c_2x^3c_5 \\
& -8c_4x^3c_5 + 9c_3^2xk^2 - 4c_3x^2c_4 - 4c_5x^4c_4 + 4c_5x^5c_4 + 8c_3x^5c_5 + 32c_3x^2c_5 - 8c_5^2x^4k^2 + 32c_5^2x^3k \\
& -16c_3x^3c_5 - 4c_3xc_5 + 2c_3xc_4 + 9c_3^2xk - 32c_5^2x^2k - 8c_2x^2c_5 - 4c_3k^2c_5 + 2c_3k^2c_4 - 8c_5^2xk^2 - 8c_3k^3c_5 \\
& +8c_4x^2c_5 - 4c_3k^4c_5 + 2c_4k^4c_3 + 4c_4k^3c_3 - 4k^2c_5^2x^2 - 16c_5^2xk^3 - 8c_5^2xk^4 - 6c_3^2x^2k^2 + 16c_5^2x^3k^2 \\
& +16c_5^2x^2k^3 + 4c_5^2x^2k^4 + 56c_3x^2kc_5 - 8c_3x^2kc_4 - 4c_3xk^2c_5 + 20c_3xk^3c_5 - 16c_5x^3kc_4 - 8c_4x^3k^2c_5 \\
& -28c_3x^3kc_5 - 8c_2x^2kc_5 + 6c_4xk^2c_3 + 4c_2xkc_5 + 20c_5x^2kc_4 + 4c_4xk^2c_5 - 16c_3xkc_5 + 6c_3xkc_4 + 8c_3xk^4c_5 \\
& +4c_2xk^2c_5 + 4c_4xk^4c_5 + 8c_4xk^3c_5 - 4c_3x^2k^2c_4 + 24c_3x^2k^2c_5 - 16c_3x^3k^2c_5 + 12k^2c_5x^2c_4)/(x(6c_3^2x \\
& -2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k - 4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 \\
& +16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 - 8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 \\
& +18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 + 4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 \\
& -8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 + 4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2))e_1 - (-2c_3c_4 \\
& +4c_3c_5 + 4c_5^2x^4 - 8c_5^2x^3 - 8c_5^2x^2 - 6c_3^2k - 3c_3^2k^2 + 16xc_5^2 + 4c_4x^3c_5 + 4c_3x^2c_5 + 8c_3x^3c_5 \\
& +32kc_5^2x + 4c_2xc_5 - 20c_3xc_5 + 3c_3^2xk - 20c_5^2x^2k - 4c_2x^2c_5 + 4c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 \\
& +8c_3kc_5 - 4c_3kc_4 - 4k^2c_5^2x^2 - 8xc_4c_5 + 4c_3x^2kc_5 - 12c_3xk^2c_5 + 4c_2xkc_5 - 16kc_4xc_5 + 4c_5x^2kc_4 \\
& -8c_4xk^2c_5 - 36c_3xkc_5 + 2c_3xkc_4 - 3c_3^2)(x-k)/(6c_3^2x - 2c_3c_4 + 10c_3c_5 - 3c_3^2x^2 - 16k^2c_5^2 - 8c_5^2k \\
& -4c_5c_2 - 6c_3^2k - 8c_5^2k^3 - 3c_3^2k^2 - 2c_3x^2c_4 + 10c_3x^2c_5 + 16kc_5^2x + 4c_4k^3c_5 - 4c_2k^2c_5 + 8c_2xc_5 \\
& -8c_2kc_5 - 20c_3xc_5 + 4c_3xc_4 + 6c_3^2xk - 8c_5^2x^2k - 4c_2x^2c_5 + 18c_3k^2c_5 - 2c_3k^2c_4 + 16c_5^2xk^2 \\
& +4c_3k^3c_5 + 4c_5kc_4 + 8k^2c_4c_5 + 24c_3kc_5 - 4c_3kc_4 + 4c_3x^2kc_5 - 8c_3xk^2c_5 + 8c_2xkc_5 - 8kc_4xc_5 \\
& +4c_5x^2kc_4 - 8c_4xk^2c_5 - 28c_3xkc_5 + 4c_3xkc_4 - 3c_3^2)e_2 + e_4,
\end{aligned}
$$

and its $\mathcal{L}$-module structures are given by

$$
\partial_x \begin{pmatrix} \mathbf{u}_{4,1} \\ \mathbf{u}_{4,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(-x^3-x^2k+x^2+3xk+2x+k^2x+4k^2+5k+2+k^3)}{x^2(-x+k+1)} & \frac{2(-x^2+x+k^2+2k+1)}{(-x+k+1)x} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{4,1} \\ \mathbf{u}_{4,2} \end{pmatrix}
$$

and

$$
\partial_k \begin{pmatrix} \mathbf{u}_{4,1} \\ \mathbf{u}_{4,2} \end{pmatrix} = \begin{pmatrix} d_{11}^{(4)} & d_{12}^{(4)} \\ d_{21}^{(4)} & d_{22}^{(4)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{4,1} \\ \mathbf{u}_{4,2} \end{pmatrix},
$$

in which

$$
\begin{aligned}
d_{11}^{(4)} = {} & \Big(4c_2xc_5 + 32kc_5^2x - 2c_3k^2c_4 + 8c_3kc_5 - 20c_3xc_5 + 3c_3^2xk - 4c_3kc_4 - 4c_2x^2c_5 + 4c_3k^2c_5 + 16c_5^2xk^2 \\
& -20c_5^2x^2k + 16xc_5^2 - 4k^2c_5^2x^2 - 8xc_4c_5 + 8c_3x^3c_5 + 4c_4x^3c_5 - 3c_3^2 - 8c_5^2x^3 + 4c_5^2x^4 - 6c_3^2k - 2c_3c_4 \\
& +4c_3c_5 - 3c_3^2k^2 - 8c_5^2x^2 + 4c_3x^2c_5 - 12c_3xk^2c_5 + 4c_3x^2kc_5 + 2c_3xkc_4 - 36c_3xkc_5 - 8c_4xk^2c_5 + 4c_5x^2kc_4 \\
& -16kc_4xc_5 + 4c_2xkc_5\Big) \Big/ \Big((-x+k+2)(8c_5^2k - 4c_5kc_4 - 4c_3kc_5 + 3c_3^2 + 8c_5^2 + 4c_5c_2 + 2c_3c_4 - 14c_3c_5 - 4c_4c_5)\Big),
\end{aligned}
$$

$$
\begin{aligned}
d_{12}^{(4)} = {} & \Big((-x+k+1)(-8xc_5^2 + 4c_5^2x^2 + 4xc_4c_5 + 8c_3xc_5 - 4c_3c_5 + 2c_3c_4 + 3c_3^2)x\Big) \Big/ \Big(-4c_2xc_5 - 8c_4c_5 - 8kc_5^2x + 4c_2kc_5 \\
& -22c_3kc_5 - 4k^2c_4c_5 + 14c_3xc_5 + 2c_3kc_4 - 4c_3k^2c_5 - 2c_3xc_4 - 8xc_5^2 + 4xc_4c_5 - 12c_5kc_4 + 16c_5^2 + 6c_3^2 \\
& +3c_3^2k + 8c_5c_2 + 4c_3c_4 - 28c_3c_5 + 8k^2c_5^2 + 24c_5^2k - 3c_3^2x + 4c_3xkc_5 + 4kc_4xc_5\Big),
\end{aligned}
$$

$$
\begin{aligned}
d_{21}^{(4)} = {} & \Big(8c_2xc_5 + 4c_2x^3c_5 - 8c_3x^5c_5 - 2c_3x^4c_4 + 12c_3^2x^2k + 8c_5^2x^4k^2 - 4c_5x^5c_4 + 176kc_5^2x - 26c_3k^2c_4 + 48c_3kc_5 \\
& -60c_3xc_5 - 9c_3^2xk - 24c_3kc_4 - 16c_2x^2c_5 + 52c_3k^2c_5 + 168c_5^2xk^2 + 24c_3k^3c_5 - 120c_5^2x^2k - 6c_3xc_4 \\
& -8c_4x^2c_5 + 4c_5x^4c_4 + 64xc_5^2 - 16c_5^2x^3k^2 - 100k^2c_5^2x^2 - 32c_5^2x^2k^3 - 4c_5^2x^2k^4 - 2c_4k^4c_3 \\
& -12c_4k^3c_3 + 6c_3^2x^2k^2 + 4c_3k^4c_5 + 64c_5^2xk^3 + 8c_5^2xk^4 - 32xc_4c_5 + 4k^2c_5x^2c_4 + 16c_3x^3k^2c_5 \\
& -32c_4xk^3c_5 - 4c_4xk^4c_5 + 4c_2xk^2c_5 - 8c_3xk^4c_5 + 4c_3x^2k^2c_4 - 8c_3x^2k^2c_5 + 32c_5^2x^4k - 48c_5^2x^3k \\
& +32c_3x^3c_5 + 6c_3x^3c_4 - 3c_3^2xk^2 + 24c_4x^3c_5 + 12c_3x^4c_5 - 12c_3^2 - 3c_3^2x^4 - 48c_5^2x^3 + 9c_3^2x^3 \\
& +8c_5^2x^5 - 4c_5^2x^6 + 24c_5^2x^4 - 36c_3^2k - 8c_3c_4 + 16c_3c_5 - 39c_3^2k^2 - 9c_3^2x - 32c_5^2x^2 - 3c_3^2k^4 \\
& -18c_3^2k^3 - 8c_2x^2kc_5 + 52c_3x^3kc_5 + 8c_4x^3k^2c_5 + 24c_5x^3kc_4 - 60c_3xk^3c_5 - 156c_3xk^2c_5 + 8c_3x^2kc_4 \\
& -16c_3x^2kc_5 - 2c_4xk^2c_3 - 6c_3xkc_4 - 168c_3xkc_5 - 84c_4xk^2c_5 + 4c_5x^2kc_4 - 88kc_4xc_5 + 12c_2xkc_5\Big) \Big/ \\
& \Big((-x+k+2)^2(8c_5^2k - 4c_5kc_4 - 4c_3kc_5 + 3c_3^2 + 8c_5^2 + 4c_5c_2 + 2c_3c_4 - 14c_3c_5 - 4c_4c_5)x\Big),
\end{aligned}
$$

$$
\begin{aligned}
d_{22}^{(4)} = \Big(& (8c_2xc_5 - 4c_3x^2c_4 - 16kc_5^2x + 2c_3k^2c_4 - 16c_3kc_5 + 12c_3xc_5 + 3c_3^2xk + 8c_3kc_4 - 4c_2x^2c_5 - 4c_3k^2c_5 \\
& +12c_5^2x^2k + 6c_3xc_4 + 4c_4x^2c_5 - 32xc_5^2 + 4k^2c_5^2x^2 + 16xc_4c_5 - 8c_3x^3c_5 - 4c_4x^3c_5 + 12c_3^2 \\
& +8c_5^2x^3 - 4c_5^2x^4 - 6c_3^2x^2 + 12c_3^2k + 8c_3c_4 - 16c_3c_5 + 3c_3^2k^2 + 9c_3^2x + 16c_5^2x^2 + 20c_3x^2c_5 \\
& +4c_3xk^2c_5 + 4c_3x^2kc_5 + 2c_3xkc_4 + 16c_3xkc_5 + 4c_5x^2kc_4 + 8kc_4xc_5 + 4c_2xkc_5)(-x+k+1)\Big) \Big/ \\
& \Big((-x+k+2)(-4c_2xc_5 - 8c_4c_5 - 8kc_5^2x + 4c_2kc_5 - 22c_3kc_5 - 4k^2c_4c_5 + 14c_3xc_5 + 2c_3kc_4 - 4c_3k^2c_5 \\
& -2c_3xc_4 - 8xc_5^2 + 4xc_4c_5 - 12c_5kc_4 + 16c_5^2 + 6c_3^2 + 3c_3^2k + 8c_5c_2 + 4c_3c_4 - 28c_3c_5 + 8k^2c_5^2 \\
& +24c_5^2k - 3c_3^2x + 4c_3xkc_5 + 4kc_4xc_5)\Big) .
\end{aligned}
$$

□

4.5 Eigenrings and Factorization

We discuss another approach to factoring $\mathcal{L}$-modules, which is not based on the associated equations method. This method is first introduced in [59] to factor linear ordinary differential operators using eigenrings of the operators. Three algorithms are presented there for computing eigenrings. Significant improvements on these algorithms are described in [8, 33]. Although the eigenring method does not always factor reducible operators, it often yields factors quickly. This method has been generalized in [4, 11] for systems of linear difference equations, and in [6] recently for systems of linear PDEs in positive characteristic. We will generalize in this section this method for the factorization of $\mathcal{L}$-modules of finite dimension.

Let R be an arbitrary ring and let M be an R-module. Recall that $\mathrm{End}_R(M)$ is the set of all R-linear maps on M. Clearly, $\mathrm{End}_R(M)$ becomes a ring with the usual addition and the composition of maps adopted as the multiplication.

Definition 4.5.1 *Let M be an R-module. A set of elements $\{\pi_1, \ldots, \pi_s\}$ of the ring $\mathrm{End}_R(M)$ is called* a set of orthogonal idempotents *if they satisfy*

$$\sum_{i=1}^{s} \pi_i = \mathbf{1} \quad \text{and} \quad \pi_i\pi_j = \mathbf{0} \quad \text{whenever } i \neq j, \tag{4.7}$$

where $\mathbf{1}$ and $\mathbf{0}$ are the identity map and the zero map on M, respectively.

Remark 4.5.2 *Although it is not stated in Definition 4.5.1, the π_i are all idempotent. Indeed, the condition (4.7) implies that $\pi_i^2 = \sum_{j=1}^{s} \pi_i\pi_j = \pi_i \left(\sum_{j=1}^{s} \pi_j\right) = \pi_i$ for each i.*

It is stated in Exercise 7 in §1 of [41, Ch.1] that

Proposition 4.5.1 *Let M be an R-module. If* $\mathrm{End}_R(M)$ *has a set of orthogonal idempotents* $\pi_1, \ldots, \pi_s$ *then* $M = \oplus_{i=1}^s \pi_i(M)$. *Conversely, if M can be written as a direct sum of submodules* $M = N_1 \oplus \cdots \oplus N_s$ *then* $\{\pi_1, \ldots, \pi_s\}$ *is a set of orthogonal idempotents of* $\mathrm{End}_R(M)$ *where π_i is the projection from M to N_i.*

For any R-module M, $\mathrm{End}_R(M)$ always has a set of orthogonal idempotents $\{\mathbf{0}, \mathbf{1}\}$, which is called the *trivial orthogonal idempotents* of $\mathrm{End}_R(M)$. From Proposition 4.5.1, one sees that if $\mathbf{1}$ is contained in a set π of orthogonal idempotents, then $\pi = \{\mathbf{0}, \mathbf{1}\}$ and π is trivial.

As a direct consequence of Proposition 4.5.1, an R-module M is decomposable if and only if $\mathrm{End}_R(M)$ has a nontrivial set of orthogonal idempotents.

Let F be an orthogonal Δ-field, C its field of constants and $\mathcal{L}=F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ the Laurent-Ore algebra over F.

Definition 4.5.3 *For an $\mathcal{L}$-module M, the endomorphism ring* $\mathrm{End}_{\mathcal{L}}(M)$ *is called the* eigenring *of M, denoted by $\mathcal{E}(M)$.*

By Definition 4.5.3, a map $\phi \in \mathrm{End}_F(M)$ belongs to $\mathcal{E}(M)$ if and only if ϕ commutes with the ∂_i and ∂_j^{-1} for all i, j with $1 \leq i \leq m$ and $\ell + 1 \leq j \leq m$. However, since M is an $\mathcal{L}$-module on which the ∂_j^{-1} act, the commutativity of ϕ with the ∂_j for $\ell + 1 \leq j \leq m$ implies that $\partial_j \circ \phi \circ \partial_j^{-1}(\mathbf{w}) = \phi(\mathbf{w})$ and further $\phi \circ \partial_j^{-1}(\mathbf{w}) = \partial_j^{-1} \circ \phi(\mathbf{w})$ for $\mathbf{w} \in M$. Hence, $\phi \in \mathcal{E}(M)$ if and only if ϕ commutes with all the ∂_i for $1 \leq i \leq m$.

Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Suppose that

$$\partial_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau = B_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau, \quad i = 1, \ldots, m,$$

where $B_i \in F^{n \times n}$ for $1 \leq i \leq m$ and the B_j are invertible for $j > \ell$. For the actual calculation of $\mathcal{E}(M)$ we now interpret elements of $\mathcal{E}(M)$ in terms of the B_i. Let ϕ be in $\mathrm{End}_F(M)$ and $P \in F^{n \times n}$ be its transformation matrix given by

$$(\phi(\mathbf{e}_1), \ldots, \phi(\mathbf{e}_n))^\tau = P(\mathbf{e}_1, \ldots, \mathbf{e}_n)^\tau.$$

For any $\mathbf{w} = \sum_{i=1}^{n} a_i\mathbf{e}_i \in M$ where $a_i \in F$, we write

$$\phi(\mathbf{w}) = \sum_{i=1}^{n} a_i\phi(\mathbf{e}_i) = (a_1, \ldots, a_n)(\phi(\mathbf{e}_1), \ldots, \phi(\mathbf{e}_n))^{\tau} = (a_1, \ldots, a_n)P(\mathbf{e}_1, \ldots, \mathbf{e}_n)^{\tau}.$$

One can verify that the conditions $\partial_i(\phi(\mathbf{w})) = \phi(\partial_i(\mathbf{w}))$, for all $\mathbf{w} \in M$ and $1 \leq i \leq m$, hold if and only if P satisfies that $\delta_i(P) = B_iP - PB_i$ for $i \leq \ell$ and $\sigma_j(P) = B_jPB_j^{-1}$ for $j > \ell$. Hence the eigenring $\mathcal{E}(M)$ can be defined equivalently to be

$$\mathcal{E}(M) = \{P \in F^{n\times n} \mid \delta_i(P) = B_iP - PB_i \text{ for } i \leq \ell \text{ and } \sigma_j(P) = B_jPB_j^{-1} \text{ for } j > \ell\}. \quad (4.8)$$

Clearly, $\mathbf{1}_n \in \mathcal{E}(M)$ and $\mathcal{E}(M)$ is a C-subalgebra of $F^{n\times n}$ of dimension $\leq n^2$. Denote by $C \cdot \mathbf{1}_n$ the set of all matrices of the form $c \cdot \mathbf{1}_n$ where $c \in C$. Then $C \cdot \mathbf{1}_n \subseteq \mathcal{E}(M)$.

As a natural generalization of the results in [5], [53, Ch.2,4] or [59] for the case of linear ODEs, we have

Theorem 4.5.2 *Let M be an $\mathcal{L}$-module of dimension n. Then*

(i) *If $\mathcal{E}(M) \neq C \cdot \mathbf{1}_n$ then M is reducible.*

(ii) *If M is decomposable then $\mathcal{E}(M) \neq C \cdot \mathbf{1}_n$.*

(iii) *If M is completely reducible, then M is irreducible if and only if $\mathcal{E}(M) = C \cdot \mathbf{1}_n$.*

Proof. (i) Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form an F-basis of M. Suppose that P is a nonzero matrix in $\mathcal{E}(M)$ such that $P \notin C \cdot \mathbf{1}_n$. Since $\mathcal{E}(M)$ is of dimension $\leq n^2$ over C, $\mathbf{1}_n, P, \ldots, P^{n^2}$ are linearly dependent over C. Then there exists a monic polynomial $f(t) \in C[t]$ of minimal degree such that $f(P) = 0$. It follows that the characteristic polynomial $\det(P - t\mathbf{1}_n)$ of P belongs to $C[t]$ and the roots of $\det(P - t\mathbf{1}_n)$ are roots of $f(t)$, which all belong to C since C is algebraically closed. Let λ be a root of $\det(P - t\mathbf{1}_n)$ and $N = \{\mathbf{w} \in M \mid P\mathbf{w} = \lambda\mathbf{w}\}$. Obviously, N is closed under addition. Since $\lambda \in C$,

$$P(\partial_i(\mathbf{w})) = (\phi \circ \partial_i)(\mathbf{w}) = (\partial_i \circ \phi)(\mathbf{w}) = \partial_i(P\mathbf{w}) = \partial_i(\lambda\mathbf{w}) = \lambda\partial_i(\mathbf{w}),$$

for $i = 1, \ldots, m$, where ϕ is the transformation on M induced by the matrix P. Thus N is an $\mathcal{L}$-submodule of M. Suppose that $N = M$. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ belong to N and therefore

$$P - \lambda\mathbf{1}_n = (P - \lambda\mathbf{1}_n)(\mathbf{e}_1, \ldots, \mathbf{e}_n) = (\,(P - \lambda\mathbf{1}_n)\,\mathbf{e}_1,\, \ldots,\, (P - \lambda\mathbf{1}_n)\,\mathbf{e}_n\,) = 0,$$

hence $P = \lambda \mathbf{1}_n \in C \cdot \mathbf{1}_n$, a contradiction to our assumption. Suppose that $N = 0$. It follows that $(a_1, \ldots, a_n)(P - \lambda \mathbf{1}_n) = 0$ with $a_i \in F$ implies that $(a_1, \ldots, a_n) = 0$, *i.e.*, the linear system $(P^\tau - \lambda \mathbf{1}_n)\mathbf{x} = 0$ has only trivial solution. Hence $\det(P^\tau - \lambda \mathbf{1}_n) \neq 0$, a contradiction to the assumption. So N is a nontrivial submodule of M and M is reducible.

(ii) Now suppose that M is decomposable. Then $M = N_1 \oplus N_2$ where N_1 and N_2 are nontrivial submodules of M. Let $d = \dim_F N_1$. Then $0 < d < n$. One can verify easily that $P = \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & 0 \end{pmatrix}$ satisfies the conditions (4.8). So $P \in \mathcal{E}(M)$ but $P \notin C \cdot \mathbf{1}_n$.

(iii) It is immediately from (i) and (ii). □

Remark 4.5.4 *Observe that the condition that "$\mathcal{E}(M)$ is not a division ring", is used in [5, 6] instead of the condition "$\mathcal{E}(M) \neq C \cdot \mathbf{1}_n$". This is because the field C of constants is not assumed to be algebraically closed there. To check if "$\mathcal{E}(M)$ is not a division ring", one needs to find a C-basis $\{P_1, \ldots, P_s\}$ of $\mathcal{E}(M)$ and then decide if there exist $c_1, \ldots, c_s \in C$, not all zero, such that the determinant of $\sum_{i=1}^s c_i P_i$ is zero.*

However, if C is algebraically closed, the above two conditions are equivalent. Indeed, if $\mathcal{E}(M) \neq C \cdot \mathbf{1}_n$ then there is $P \in \mathcal{E}(M)$ but $P \notin C \cdot \mathbf{1}_n$. By previous discussion, the characteristic polynomial $\det(P - t\mathbf{1}_n)$ *of P belongs to $C[t]$ where t is an indeterminate over C. Let λ be a root of* $\det(P - t\mathbf{1}_n) = 0$. *Clearly, $P - \lambda \mathbf{1}_n$ is not invertible. Since C is algebraically closed, $\lambda \in C$. Moreover $P - \lambda \mathbf{1}_n$ is nonzero, for otherwise, P and $\mathbf{1}_n$ would be linearly dependent over C. Therefore $\mathcal{E}(M)$ is not a division ring.*

Given an $\mathcal{L}$-module M of finite dimension over F, the representation (4.8) of eigenrings allows us to compute $\mathcal{E}(M)$. Let $P \in \mathcal{E}(M)$ be a matrix of n^2 indeterminates z_{ij}. From (4.8), construct the system

$$\partial_i(z_{11}, \ldots, z_{1n}, \ldots, z_{n1}, \ldots, z_{nn})^\tau = A_i(z_{11}, \ldots, z_{1n}, \ldots, z_{n1}, \ldots, z_{nn})^\tau, \quad i = 1, \ldots, m,$$

where $A_i \in F^{n^2 \times n^2}$. A C-basis of all rational solutions of the above system yields a C-basis $\{P_1, \ldots, P_r\}$ of all rational solutions of (4.8). Without loss of generality, we assume that $P_1 = \mathbf{1}_n$. Therefore $\mathcal{E}(M) = \oplus_{i=1}^r C \cdot P_i$. If r=1, then $\mathcal{E}(M)$ is trivial and M is indecomposable by Theorem 4.5.2 (ii). Otherwise, each eigenvalue λ of a nontrivial $P \in \mathcal{E}(M)$

will produce a submodule $\{\mathbf{w} \in M \mid P\mathbf{w} = \lambda\mathbf{w}\}$ of M. If $\mathcal{E}(M)$ has a set of nontrivial idempotents $\pi_1, \ldots, \pi_s$, then a decomposition of M is derived:

$$M = \pi_1(M) \oplus \cdots \oplus \pi_s(M). \tag{4.9}$$

If M is furthermore completely reducible, a maximal decomposition of M can be obtained by applying the eigenring method recursively on the submodules in the decomposition (4.9).

Example 4.5.5 *Let $F = \mathbb{C}(x,k)$, M be an $\mathcal{L}$-module of dimension two and $\mathbf{e}_1, \mathbf{e}_2$ be a basis of M given by $\partial_x(\mathbf{e}_1, \mathbf{e}_2)^\tau = B_x(\mathbf{e}_1, \mathbf{e}_2)^\tau$ and $\partial_k(\mathbf{e}_1, \mathbf{e}_2)^\tau = B_k(\mathbf{e}_1, \mathbf{e}_2)^\tau$ where $B_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.*

We now compute the eigenring of M. Let $P \in \mathcal{E}(M)$ be a 2×2 matrix with indeterminate entries z_{11}, z_{12}, z_{21} and z_{22}. The conditions $\delta_x(P) = B_xP - PB_x$ and $\sigma_k(P) = B_kPB_k^{-1}$ induce the system $\mathcal{A} : \{\delta_x(\mathbf{z}) = A_x\mathbf{z},\ \sigma_k(\mathbf{z}) = A_k\mathbf{z}\}$ where $\mathbf{z} = (z_{11}, z_{12}, z_{21}, z_{22})^\tau$,

$$A_x = \begin{pmatrix} 0 & \frac{k(-x+x^2-kx+2k)}{(x-k)(x-1)} & \frac{x^2-kx+3k-2x}{x(x-k)k(x-1)} & 0 \\ -\frac{x^2-kx+3k-2x}{x(x-k)k(x-1)} & -\frac{x^3-kx^2-2x+3k+kx}{x(x-k)(x-1)} & 0 & \frac{x^2-kx+3k-2x}{x(x-k)k(x-1)} \\ -\frac{k(-x+x^2-kx+2k)}{(x-k)(x-1)} & 0 & \frac{x^3-kx^2-2x+3k+kx}{x(x-k)(x-1)} & \frac{k(-x+x^2-kx+2k)}{(x-k)(x-1)} \\ 0 & -\frac{k(-x+x^2-kx+2k)}{(x-k)(x-1)} & -\frac{x^2-kx+3k-2x}{x(x-k)k(x-1)} & 0 \end{pmatrix}$$

and

$$A_k = \frac{1}{\gamma}\begin{pmatrix} \frac{x^2-2kx-x+k^2}{k}\alpha & -x\alpha\beta & -\frac{x^2-2kx-x+k^2}{k^2}\beta & \frac{x}{k}\beta^2 \\ \frac{1}{k(k+1)}\alpha\beta & \frac{1}{k+1}\alpha^2 & -\frac{1}{k^2(k+1)}\beta^2 & -\frac{1}{k(k+1)}\alpha\beta \\ \frac{x(k+1)(x^2-2kx-x+k^2)}{k}\beta & -(k+1)x^2\beta^2 & \frac{(k+1)(x^2-2kx-x+k^2)^2}{k^2} & \frac{x(k+1)(2kx+x-x^2-k^2)}{k}\beta \\ \frac{x}{k}\beta^2 & x\alpha\beta & \frac{x^2-2kx-x+k^2}{k^2}\beta & \frac{x^2-2kx-x+k^2}{k}\alpha, \end{pmatrix},$$

with $\alpha = k+1+kx^2-k^2x-x$, $\beta = k+1+kx-k^2-x$ and $\gamma = (x-k)(x-k-1)(x-1)^2$. All rational solutions of $\mathcal{A}$ are of the form

$$c_1\left(\frac{1}{x-1}, -\frac{1}{k(x-1)}, \frac{xk}{x-1}, -\frac{x}{x-1}\right) + c_2\left(-\frac{x}{x-1}, \frac{1}{k(x-1)}, -\frac{xk}{x-1}, \frac{1}{x-1}\right),$$

for $c_1, c_2 \in \mathbb{C}$*. So*

$$\begin{aligned} \mathcal{E}(M) &= \left\{ \begin{pmatrix} \frac{c_1-c_2x}{x-1} & \frac{c_2-c_1}{k(x-1)} \\ \frac{(c_1-c_2)xk}{x-1} & \frac{c_2-c_1x}{x-1} \end{pmatrix}, \quad \textit{for any } c_1, c_2 \in \mathbb{C} \right\} \\ &= \mathbb{C} \begin{pmatrix} \frac{1}{x-1} & -\frac{1}{k(x-1)} \\ \frac{kx}{x-1} & -\frac{x}{x-1} \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} -\frac{x}{x-1} & \frac{1}{k(x-1)} \\ -\frac{kx}{x-1} & \frac{1}{x-1} \end{pmatrix}. \end{aligned}$$

Recall that the necessary condition for $\{P_1, \ldots, P_s\} \subset \mathcal{E}(M)$ *being a set of orthogonal idempotents is that* $P_i^2 = P_i$ *for each* i*. Substitute*

$$P = \begin{pmatrix} \frac{c_1-c_2x}{x-1} & \frac{c_2-c_1}{k(x-1)} \\ \frac{(c_1-c_2)xk}{x-1} & \frac{c_2-c_1x}{x-1} \end{pmatrix}$$

into the relation $P^2 = P$*, we obtain three solutions:*

$$P_0 = \mathbf{1}_2, \qquad P_1 = \begin{pmatrix} -\frac{1}{x-1} & \frac{1}{k(x-1)} \\ -\frac{kx}{x-1} & \frac{x}{x-1} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{x}{x-1} & -\frac{1}{k(x-1)} \\ \frac{kx}{x-1} & -\frac{1}{x-1} \end{pmatrix}.$$

Among which, we find $P_1 P_2 = 0$ *and* $P_1 + P_2 = \mathbf{1}_2$*. So* $\{P_1, P_2\}$ *is a set of nontrivial orthogonal idempotents of* $\mathcal{E}(M)$*. We have*

$$P_1(M) = \{P_1(\mathbf{w}) \mid \mathbf{w} \in M\} = \{(a_1, a_2)P_1(\mathbf{e}_1, \mathbf{e}_2)^\tau \mid a_1, a_2 \in F\} = F \cdot \left(\mathbf{e}_1 - \frac{1}{k}\mathbf{e}_2\right)$$

and

$$P_2(M) = \{P_2(\mathbf{w}) \mid \mathbf{w} \in M\} = \{(a_1, a_2)P_2(\mathbf{e}_1, \mathbf{e}_2)^\tau \mid a_1, a_2 \in F\} = F \cdot \left(\mathbf{e}_1 - \frac{1}{kx}\mathbf{e}_2\right).$$

Therefore, $P_1(M) \oplus P_2(M)$ *is a decomposition of* M *into two nontrivial submodules.* □

The eigenring method, however, may fail to find any factor of an $\mathcal{L}$-module of finite dimension even if this module is reducible. This happens when the eigenring of that module is trivial. The four-dimensional $\mathcal{L}$-module M in Example 4.3.3 is such an example. This M has a two-dimensional submodule, so is reducible. But M can not be completely reducible, as it has dimension four but has only one two-dimensional submodule.

More work needs to be done on the eigenring method. For example, how does one find a set of orthogonal idempotents efficiently and how does one compute the complement of a known submodule?

Chapter 5

Concluding Remarks

In the thesis we define the modules of formal solutions and Picard-Vessiot extensions for ∂-finite linear functional systems. These two notions enable us to describe solutions of ∂-finite systems in an algebraic setting. We anticipate that these two notions can be generalized to linear functional systems where difference operators are not necessarily automorphisms. Such a generalization would rely on the notion of reflexive closure of submodules of $\mathcal{S}^{1\times n}$, the $\overline{N}$ in the proof of Proposition 2.4.6 and Remark 2.4.8.

There is however a challenging problem. Our methods for computing linear dimensions make essential use of the assumption that the maps $\sigma_{\ell+1}, \ldots, \sigma_m$ are automorphisms. Can we still compute linear dimensions of systems without this assumption?

Another direction is to investigate the possibility to extend the Galois theory of linear ordinary differential and difference equations to ∂-finite linear functional systems.

Although the thesis contains a complete algorithm for factoring $\mathcal{L}$-modules with finite dimension, the algorithm has not yet been fully implemented. We plan to complete a prototype for the algorithm in MAPLE soon, and try to improve the practical efficiency. In particular, we would like to improve the algorithm for computing one-dimensional modules, and to find a more efficient way to decide the decomposability of an element in an exterior power of a finite-dimensional $\mathcal{L}$-module. We will also consider how to factor ∂-finite linear functional systems with more general coefficients.

Complexity analysis is an important issue in computer algebra. We will try to analyze the algorithms presented in this thesis. In this direction, the monograph [25] would be a valuable reference.

All the examples in the thesis contain two operators, each of which acts on only one variable nontrivially. It would be interesting to see whether we are able to factor a module for modelling differential-delay equations. In this case, both differential and shift operators act on one variable. Along this direction, we need to consider how to compute rational and hyperexponential solutions of a linear differential-delay equation.

It would be interesting to see applications of the factorization algorithm in the handling of holonomic objects [19, 50], which are usually represented by ∂-finite systems with finitely many initial values. Holonomic ideals ([56]) are left ideals in Weyl Algebras, in which linear functional operators have polynomial coefficients. A factorization algorithm for holonomic ideals is not known and is thus worth investigating.

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