A Noncommutative Nullstellensatz for Perfect Two-Answer Quantum Nonlocal Games

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Abstract

This paper introduces a noncommutative version of the Nullstellensatz, motivated by the study of quantum nonlocal games. It has been proved that a two-answer nonlocal game with a perfect quantum strategy also admits a perfect classical strategy. We generalize this result to the infinite-dimensional case, showing that a two-answer game with a perfect commuting operator strategy also admits a perfect classical strategy. This result induces a special case of noncommutative Nullstellensatz.

Keywords

Noncommutative Nullstellensatz, Sum of Squares, GNS construction, Quantum nonlocal games

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1 Introduction

Quantum nonlocal games have been a vibrant area of research across mathematics, physics, and computer science in recent decades. They help understand quantum nonlocality, which was famously verified by the violation of Bell inequalities [2, 11, 21]. In 1969, Clauser et al. first introduced quantum nonlocal games [10]. A nonlocal game typically involves two or more players and a verifier. The verifier sends questions to the players independently, and each player responds without any communication between them. A predefined scoring function determines whether the players win based on the given questions and their answers. The distinction between classical and quantum strategies lies in whether players can share quantum entanglement. For instance, in the CHSH game, the classical strategy limits the winning probability to at most $\frac{3}{4}$. In contrast, quantum strategies using shared entangled states can achieve a success probability of $\cos^2(\frac{\pi}{8}) \approx 0.85$.

The mathematical models of quantum nonlocal games are often described using algebraic structures [3, 4, 12, 17, 19]. Algebraic tools, such as $*$-algebras, (commutative or noncommutative) Gröbner basis (see [5, 18]), semidefinite programming (see [23]), noncommutative Nullstellensatz (see [7–9]) and Positivstellensatz (see [14, 15]) can be used for characterizing the different types of strategies for nonlocal games. Our previous work also gave an algebraic characterization for perfect strategies of mirror games using the universal game algebra, Nullstellensatz, and sums of squares [24].

Nonlocal games with two answers are games in which the set of possible responses consists of only two options [4] (also called binary games in [11]). This paper proposes a noncommutative Nullstellensatz inspired by the perfect commuting operator strategies for two-answer nonlocal games. Specifically, we proved that a two-answer game that admits a perfect commuting operator strategy also has a perfect classical strategy, a generalization of the work [11, Theorem 3]. Combined with the algebraic characterization of perfect commuting operator strategy [4], we get a new form of noncommutative Nullstellensatz. Moreover, based on this result, one can determine whether a two-answer game admits a perfect commuting operator strategy by computing a commutative Gröbner basis [5]. Although our problem is motivated by nonlocal games, our proofs are presented in a purely algebraic form, allowing readers unfamiliar with quantum nonlocal games to engage with the algebraic versions of the theorems directly.

2 Preliminaries

2.1 Motivations

If the readers are familiar with this field, they can skip the content of this subsection.

A quantum nonlocal game $\mathcal{G}$ can be described as a scoring function λ from the finite set $X \times Y \times A \times B$ to $\{0, 1\}$, where the player Alice has a question set X and an answer set A, while the player Bob has a question set Y and an answer set B. In a round of the game, Alice would receive the question $x \in X$ and answer $a \in A$ according to x and her strategy; similarly, Bob would receive the question $y \in Y$ and answer $b \in B$. The players can make arrangements before playing the game, that is to say, Alice knows in advance the conditional probability that Bob will answer b when he receives question y (that is, Bob's strategy). Similarly, Bob also knows Alice's strategy. However, during the game, Alice and Bob cannot communicate with each other. That is, Alice doesn't know which question Bob receives, and similarly for Bob. The players are considered to have won the game when $\lambda(x, y, a, b) = 1$, and they lose in all other cases.

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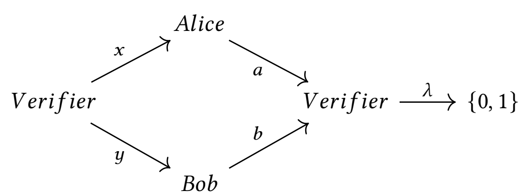

$$\lambda(x,y,a,b) = \begin{cases} 1 & \text{win} \\ 0 & \text{lose} \end{cases}$$

A (deterministic) *classical strategy* involves two mappings

$$u : X \longrightarrow A \text{ and } v : Y \longrightarrow B;$$

when Alice receives a question $x \in X$, she responds with $u(x)$, and similarly, Bob responds with $v(y)$ when he receives $y \in Y$.

If the question pair $(x, y) \in X \times Y$ is chosen randomly according to a distribution $\mu(x, y)$, we can compute the maximal winning expectation over all the deterministic strategies:

$$\omega_c(\mathcal{G}) = \max_{u,v} \sum_{x,y} \mu(x,y)\lambda(x,y,u(x),v(y)),$$

which is called the classical value of $\mathcal{G}$.

Definition 2.1. *We call a deterministic strategy perfect if* $\omega_c(\mathcal{G}) = 1$.

A deterministic strategy $(u(x), v(y))$ is perfect, i.e., the players can always win the game using this strategy, if and only if

$$\lambda(x,y,u(x),v(y)) = 1$$

holds for all $x \in X, y \in Y$. That is, the answers that lead the players to lose the game can not happen, i.e., for all (x, y, a, b) satisfying $\lambda(x, y, a, b) = 0$, we have $a \neq u(x), b \neq v(y)$.

If the players share a quantum state ϕ on a (perhaps infinite-dimensional) Hilbert space $\mathcal{H}$, and for every question pair $(x, y) \in X \times Y$, Alice and Bob perform commuting projection-valued measurements (PVMs)

$$\left\{E_a^x \in \mathcal{B}(\mathcal{H}) : \sum_{a\in A} E_a^x = \mathbf{1}\right\} \text{ and } \left\{F_b^y \in \mathcal{B}(\mathcal{H}) : \sum_{b\in B} F_b^y = \mathbf{1}\right\}$$

respectively to determine their answers, then the game is said to have a *commuting operator strategy.*

$$x \longrightarrow \text{Alice} \xrightarrow{\{E_{a_i}^x,\ a_i\in A\}} \phi \in \mathcal{H} \longrightarrow a$$

$$y \longrightarrow \text{Bob} \xrightarrow{\{F_{b_j}^y,\ b_j\in B\}} \phi \in \mathcal{H} \longrightarrow b$$

The PVMs satisfy the following relations:

$$\begin{aligned}
&E_a^x F_b^y - F_b^y E_a^x = 0,\ \forall (x,y,a,b) \in X \times Y \times A \times B;\\
&(E_a^x)^2 = E_a^x = (E_a^x)^*,\ \forall x \in X, a \in A;\\
&(F_b^y)^2 = F_b^y = (F_b^y)^*,\ \forall y \in Y, b \in B;\\
&E_{a_1}^x E_{a_2}^x = 0,\ \forall x \in X, a_1 \neq a_2 \in A;\\
&F_{b_1}^y F_{b_2}^y = 0,\ \forall y \in Y, b_1 \neq b_2 \in B;\\
&\sum_{a\in A} E_a^x = \mathbf{1},\ \forall x \in X;\\
&\sum_{b\in B} F_b^y = \mathbf{1},\ \forall y \in Y.
\end{aligned}$$

These relations can be abstracted to obtain the universal game algebra for the nonlocal game $\mathcal{G}$ [4, Section 3].

Suppose the distribution on the question set $X \times Y$ is $\mu(x, y)$. Given a commuting operator strategy of $\mathcal{G}$, the conditional probability of the players answering (a, b) when they received (x, y) is

$$p(a, b \mid x, y) = \psi^* E_a^x F_b^y \psi,$$

where ψ^* is the conjugate transpose of ψ. The winning expectation is

$$\sum_{x,y,a,b} \mu(x,y) \cdot \psi^* E_a^x F_b^y \psi \cdot \lambda(x,y,a,b).$$

Then the supremum of winning expectation over all the commuting operator strategies is

$$\omega_{co}(\mathcal{G}) = \sup_{\substack{\mathcal{H},\psi,\\ E_a^x, F_b^y}} \sum_{x,y,a,b} \mu(x,y) \cdot \psi^* E_a^x F_b^y \psi \cdot \lambda(x,y,a,b)$$

which is called the quantum commuting operator value of $\mathcal{G}$. This supremum can be reached (see [13]).

Definition 2.2. *We call a commuting operator strategy perfect if* $\omega_{co}(\mathcal{G}) = 1$.

A commuting operator strategy is perfect if and only if the conditional probability of the players giving answers (a, b) when receiving questions (x, y) is equal to zero, i.e.,

$$\psi^* E_a^x F_b^y \psi = 0,$$

when $\lambda(x, y, a, b) = 0$. That is, the players can certainly win the game $\mathcal{G}$ with this strategy.

Furthermore, if we restrict the quantum state ϕ to be a tensor $\phi_1 \otimes \phi_2$, where ϕ_1 and ϕ_2 are in finite-dimensional Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then we get a (finite-dimensional) *quantum strategy.*

By defining the three types of strategies, we know that the classical strategies are contained in the quantum strategies, which are included in the commuting operator strategies. Therefore, a game that admits a perfect classical strategy also has a perfect commuting operator strategy. However, the converse does not hold. For example, the famous Magic Square game admits a perfect quantum strategy but has no perfect classical strategy [11]. However, in certain exceptional cases, these strategies may be equivalent.

For a two-answer game, that is, one whose answer sets are both $\{0, 1\}$, if it admits a perfect quantum strategy, then Cleve, Hoyer, Toner, and Watrous showed that the two-answer game must have a perfect classical strategy [11, Theorem 3]. We contribute to extending this theorem to the infinite-dimensional case, proving that a two-answer game with a perfect commuting operator strategy also admits a perfect classical strategy. This result, combined with the work of Watts, Helton, and Klep [4, Theorem 4.3], derives a version of the noncommutative Nullstellensatz using a sum of squares (SOS) expression.

2.2 Universal Game Algebra for Two-Answer Games

Let X, Y, A, B be finite sets, where $A = B = \{0, 1\}$, and $\mathbb{C}\langle\{e^x_a, f^y_b\}\rangle$ be the free algebra generated by $\{e^x_a, f^y_b : (x, y, a, b) \in X\times Y\times A\times B\}$.

Define the two-sided ideal

$$\begin{aligned}\mathcal{I} = \langle (e^x_a)^2 - e^x_a,\ (f^y_b)^2 - f^y_b;\\ \sum_{a\in A} e^x_a - 1,\ \sum_{b\in B} f^y_b - 1;\\ e^x_a f^y_b - f^y_b e^x_a \mid x \in X, y \in Y, a \in A, b \in B\rangle\end{aligned}$$

and let

$$\mathcal{A} = \mathbb{C}\langle\{e^x_a, f^y_b\}\rangle/\mathcal{I}. \tag{2.1}$$

Since

$$\begin{aligned}e^x_0 e^x_1 = \frac{1}{2}\Big(\big(e^x_0 + e^x_1 - 1\big)^2 - \big((e^x_0)^2 - e^x_0\big)\\ - \big((e^x_1)^2 - e^x_1\big) + \big(e^x_0 + e^x_1 - 1\big) \Big),\end{aligned}$$

we have

$$e^x_0 e^x_1 \in \mathcal{I},\ \forall x \in X.$$

Similarly, one can show that

$$f^y_0 f^y_1 \in \mathcal{I},\ \forall y \in Y.$$

The elements in $\mathcal{I}$ are the relationships the generators satisfy. We can also equip $\mathcal{A}$ with the natural involution " $*$ " induced by

$$(e^x_a)^* = e^x_a,\ (f^y_b)^* = f^y_b.$$

Then $\mathcal{A}$ is a complex $*-$algebra.

The relations in $\mathcal{A}$ are precisely those satisfied by the PVMs of a two-answer game. Thus, this algebra can characterize the commuting operator strategies of a two-answer game. $\mathcal{A}$ serves as the *universal game algebra* for two-answer games, as discussed in [4, Section 3]. Furthermore, $\mathcal{A}$ is a group algebra.

Let

$$A_x = e^x_0 - e^x_1,\ B_y = f^y_0 - f^y_1 \tag{2.2}$$

for any $x \in X,\ y \in Y$, we have

$$A^2_x = B^2_y = 1, A_x = A^*_x,\ B_y = B^*_y, \tag{2.3}$$

$$e^x_a = \frac{1 + (-1)^a A_x}{2},\ f^y_b = \frac{1 + (-1)^b B_y}{2}. \tag{2.4}$$

Let G be the group generated by elements $A_x,\ x \in X$ and $B_y,\ y \in Y$. Equip the group algebra of G with the natural involution $*$:

$$g^* = g^{-1},\ (g_1g_2)^* = g^*_2 g^*_1,\ \forall g, g_1, g_2 \in G,$$

Then, we observe that

$$\mathcal{A} = \mathbb{C}[G].$$

We denote the set of the sum of Hermitian squares:

$$\mathrm{SOS}_{\mathcal{A}} := \left\{ \sum_{i=1}^{n} \alpha^*_i \alpha_i \mid n \in \mathbb{N},\ \alpha_i \in \mathcal{A} \right\}.$$

It is well known that $\mathrm{SOS}_{\mathcal{A}}$ is Archimedean (see [6, example 3] or [20, Remark 4.1]), that is, for every $\alpha \in \mathcal{A}$, it can be shown that

$$\|a\|^2_1 - \alpha^*\alpha \in \mathrm{SOS}_{\mathcal{A}},$$

where $\|a\|_1 = \sum_{g\in G} |a_g|$.

We also need to introduce the concept of $*-$representation.

Definition 2.3. *A $*$-representation of $\mathcal{A}$ is a unital $*-$homomorphism*

$$\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}),$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on a Hilbert space $\mathcal{H}$ and σ satisfies $\sigma(u^*) = \sigma(u)^*, \forall u \in \mathcal{A}$.

3 Main Results

Let X, Y, A, B be finite sets, where $A = B = \{0, 1\}$, and $\mathbb{C}\langle\{e^x_a, f^y_b\}\rangle$ be the free algebra generated by $\{e^x_a, f^y_b : (x, y, a, b) \in X\times Y\times A\times B\}$. Let $\mathcal{A}$ be the complex $*-$algebra defined in the previous Subsection 2.2.

Definition 3.1. *[4, Definition 3.4] Let $\mathcal{G}$ be a two-answer nonlocal game. Its invalid determining set $\mathcal{N}$ is defined by*

$$\mathcal{N} = \{e^x_a f^y_b \mid \lambda(x, y, a, b) = 0\}. \tag{3.1}$$

Our main result is stated below:

Theorem 3.1. *Let $\mathcal{A}$ denote the universal game algebra for a two-answer game $\mathcal{G}$. Let $\mathcal{N}$ be the invalid determining set of $\mathcal{G}$, and Λ be its index set:*

$$\Lambda = \{(x, y, a, b) \mid \lambda(x, y, a, b) = 0\} \subseteq X \times Y \times A \times B \tag{3.2}$$

Let $\mathcal{L}(\mathcal{N})$ be the left ideal generated by $\mathcal{N}$. Then

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$$

if and only if there exists a $*-$representation

$$\rho : \mathcal{A} \to \mathbb{C}$$

such that

$$\rho(\mathcal{N}) = \{0\}.$$

We prove this theorem by the following propositions.

Proposition 3.2. *([4, Theorem 4.3]) Let $\mathcal{A}$ denote the universal game algebra for two-answer games. If*

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*,$$

there exists a $*-$representation

$$\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

and $0 \neq \psi \in \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space, such that

$$\sigma(\alpha)\psi = 0$$

for all $\alpha \in \mathcal{L}(\mathcal{N})$.

We emphasize that $\mathcal{H}$ is a separable Hilbert space, which will be used in the proof of Proposition 3.3. For completeness, we briefly outline the proof given by Watts, Helton, and Klep in [4, Theorem 4.3]. Furthermore, $\sigma(\alpha)\psi = 0$ holds for all $\alpha \in \mathcal{L}(\mathcal{N})$ if and only if the nonlocal game $\mathcal{G}$ with its invalid determining set $\mathcal{N}$ has a perfect commuting operator strategy.

Proof Sketch. By the Hahn-Banach theorem [1, Theorem III.1.7] and Archimedeanity of $\mathrm{SOS}_{\mathcal{A}}$, there exists a functional $f : \mathcal{A} \to \mathbb{C}$ which strictly separate -1 and $\mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$, i.e

$$f(-1) = -1,\ f(\mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*) \subseteq \mathbb{R}_{\geqslant 0}.$$

We list the properties of f as follows:

- $f(\mathcal{L}(\mathcal{N})) = \{0\}$ and $f(\mathrm{SOS}_{\mathcal{A}}) \subseteq \mathbb{R}_{\geqslant 0}$.
- $f(h^*) = f(h)^*$ for every $h \in \mathcal{A}$.

Now, the GNS construction yields the desired *-representation σ and a cyclic vector ψ. Define the sesquilinear form on $\mathcal{A}$

$$\langle \alpha \mid \beta \rangle = f(\beta^* \alpha),$$

and

$$M = \{\alpha \in \mathcal{A} :\ f(\alpha^* \alpha) = 0\}. \tag{3.3}$$

By Cauchy-Schwarz inequality, M is a left ideal of $\mathcal{A}$. Form the quotient space $\widetilde{\mathcal{H}} := \mathcal{A}/M$, and equip it with the inner product $\langle \cdot \mid \cdot \rangle$. We can complete $\widetilde{\mathcal{H}}$ to the Hilbert space $\mathcal{H}$.

It is worth mentioning that we can assume $\mathcal{H}$ to be a separable Hilbert space, as this assumption holds because $\mathcal{A}$ has only a finite number of generators, allowing us to generate a countable dense subset of $\mathcal{A}$ using these generators with rational coefficients. By applying this to the quotient space, we establish the separability of $\mathcal{H}$.

Define the quotient map

$$\begin{aligned} \phi : \mathcal{A} &\to \mathcal{H} \\ \alpha &\mapsto \alpha + M, \end{aligned}$$

the cyclic vector

$$\psi := \phi(1) = 1 + M,$$

and the left regular representation

$$\begin{aligned} \sigma : \mathcal{A} &\to \mathcal{B}(\mathcal{H}) \\ \alpha &\mapsto (p + M \mapsto \alpha p + M)\,. \end{aligned}$$

By Archimedeanity of $\mathrm{SOS}_{\mathcal{A}}$, it is easy to verify that $\sigma(\alpha)$ is bounded for every $\alpha \in \mathcal{A}$, and thus σ is a $*-$representation. Finally, the result

$$\sigma(\mathcal{L}(\mathcal{N}))\psi = \{0\}$$

follows from

$$\mathcal{L}(\mathcal{N})^* \mathcal{L}(\mathcal{N}) \subseteq \mathcal{L}(\mathcal{N}) \subseteq M.$$

□

Proposition 3.3. *Let $\mathcal{A}$ denote the universal game algebra for two-answer games. Suppose there exists a $*-$representation*

$$\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}),$$

and $0 \neq \psi \in \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space, such that

$$\sigma(\alpha)\psi = 0$$

for all $\alpha \in \mathcal{L}(\mathcal{N})$ ($\mathcal{N}$ is defined in Equation (3.1)). Then there exists a one-dimensional $*-$representation $\rho : \mathcal{A} \to \mathbb{C}$ such that

$$\rho(\mathcal{N}) = \{0\}.$$

Remark 1. *The proof below extends the argument in [11, Theorem 3], which was originally stated for the tensor product of two finite-dimensional Hilbert spaces, to the more general setting of infinite-dimensional Hilbert spaces. In fact, the condition in Proposition 3.3 implies that the quadruple*

$$(\mathcal{H}, \{\sigma(e_a^x)\}, \{\sigma(f_b^y)\}, \psi)$$

defines a perfect commuting operator strategy for the two-answer game with an invalid determining set $\mathcal{N}$. Furthermore, the conclusion of Proposition 3.3 demonstrates that the two mappings induced by ρ:

$$\begin{aligned} u : X &\to A \\ x &\mapsto a \ (\textit{satisfying } \rho(e_a^x) = 1) \end{aligned}$$

and

$$\begin{aligned} v : Y &\to B \\ y &\mapsto b \ (\textit{satisfying } \rho(f_b^y) = 1) \end{aligned}$$

are well-defined, and satisfy: for all $(x, y, u(x), v(y)) \in X \times Y \times A \times B$,

$$\rho(e_{u(x)}^x f_{v(y)}^y) = 1.$$

According to (3.1), it implies $e_{u(x)}^x f_{v(y)}^y \notin \mathcal{N}$. We have

$$\lambda(x, y, u(x), v(y)) = 1,$$

which means that u and v give a perfect deterministic strategy for the game $\mathcal{G}$.

Proof. We construct the one-dimensional representation ρ as follows. Since

$$\sum_{a \in A} \sum_{b \in B} \psi^* \sigma(e_a^x f_b^y)\psi = 1$$

for every fixed pair (x, y), we know that there exist $(x, y, a, b) \in X \times Y \times A \times B$ such that

$$\psi^* \sigma(e_a^x f_b^y)\psi \neq 0.$$

Let

$$\Pi = \{(x, y, a, b) \in X \times Y \times A \times B :\ \psi^* \sigma(e_a^x f_b^y)\psi \neq 0\}, \tag{3.4}$$

we have

$$\Pi \subseteq X \times Y \times A \times B \setminus \Lambda$$

since

$$\sigma(\mathcal{L}(\mathcal{N}))\psi = \{0\},$$

and thus

$$\psi^* \sigma(e_a^x f_b^y)\psi = 0$$

for any $(x, y, a, b) \in \Lambda$, where Λ is the index of the invalid determining set $\mathcal{N}$ (3.1), see Remark 3.1.

Using the generators A_x and B_y defined in (2.2), we can rewrite:

$$\begin{aligned} \psi^* \sigma(e_a^x f_b^y)\psi = &\frac{1}{4} \\ &+ \frac{1}{4}(-1)^a \psi^* \sigma(A_x)\psi \\ &+ \frac{1}{4}(-1)^b \psi^* \sigma(B_y)\psi \\ &+ \frac{1}{4}(-1)^{a+b} \psi^* \sigma(A_x B_y)\psi. \end{aligned} \tag{3.5}$$

Since $\mathcal{H}$ is separable, we can choose an orthogonal basis of $\mathcal{H}$ named

$$\{\psi_1, \psi_2, \ldots\},$$

where $\psi_1 = \psi$. Define

$$\begin{aligned} k &: X \to \mathbb{N} \\ &x \mapsto \min\{j \in \mathbb{N} : \psi_j^* \sigma(A_x)\psi \neq 0\}; \\ l &: Y \to \mathbb{N} \\ &y \mapsto \min\{j \in \mathbb{N} : \psi_j^* \sigma(B_y)\psi \neq 0\}. \end{aligned}$$

Unlike the finite-dimensional case considered in the proof of [11, Theorem 3], here we need to show that for every $x \in X$, $y \in Y$, $k(x)$ and $l(y)$ are well-defined.

As $\psi \neq 0$ and $\sigma(A_x)^2 = 1$, there must exist a $j \in \mathbb{N}$ such that $\psi_j^* \sigma(A_x)\psi \neq 0$ (otherwise, $\sigma(A_x)\psi = 0$, which contradicts the assumption that $\psi \neq 0$ and $\sigma(A_x)^2 = 1$). Similarly, we can also prove that $l(y)$ is well-defined.

Let

$$\begin{aligned} u &: X \to A \\ x &\mapsto \begin{cases} 0, & 0 \leqslant \arg \psi_{k(x)}^* \sigma(A_x)\psi < \pi; \\ 1, & \pi \leqslant \arg \psi_{k(x)}^* \sigma(A_x)\psi < 2\pi. \end{cases} \end{aligned} \tag{3.6}$$

$$\begin{aligned} v &: Y \to B \\ y &\mapsto \begin{cases} 0, & 0 \leqslant \arg \psi_{l(y)}^* \sigma(B_y)\psi < \pi; \\ 1, & \pi \leqslant \arg \psi_{l(y)}^* \sigma(B_y)\psi < 2\pi. \end{cases} \end{aligned} \tag{3.7}$$

□

We have the following claim:

Claim 3.4. *For every $(x, y, u(x), v(y)) \in X \times Y \times A \times B$, we have*

$$(x, y, u(x), v(y)) \in \Pi,$$

where Π is defined in (3.4). In particular, this implies

$$\psi^* \sigma(e_{u(x)}^x f_{v(y)}^y)\psi \neq 0.$$

We will provide the proof of Claim 3.4 after completing the proof of Proposition 3.3.

We construct the one-dimensional $*$−representation ρ as follows. For every $x \in X$, define

$$\rho(e_{u(x)}^x) = 1,\ \rho(e_{1-u(x)}^x) = 0;$$

and for every $y \in Y$, define

$$\rho(f_{v(y)}^y) = 1,\ \rho(f_{1-v(y)}^y) = 0.$$

Then, by linearity and homogeneity, we extend ρ to the entire game algebra $\mathcal{A}$.

Since $\rho(e_a^x)$ and $\rho(f_b^y)$ are either 0 or 1, they are naturally commutative. It is straightforward to check that ρ satisfies:

$$\rho(e_a^x)^2 = \rho(e_a^x),\ \rho(f_b^y)^2 = \rho(f_b^y),$$

and

$$\rho(e_0^x) + \rho(e_1^x) = 1,\ \rho(f_0^y) + \rho(f_1^y) = 1,$$

for all $e_a^x, f_b^y \in \mathcal{A}$, i.e., $\rho(e_a^x)$ and $\rho(f_b^y)$ satisfy the same relations as e_a^x and f_b^y in $\mathcal{A}$. Therefore, ρ is indeed a $*$−representation of $\mathcal{A}$.

Since

$$\rho(e_a^x f_b^y) = 1 \iff (a = u(x)) \wedge (b = v(y)).$$

By Claim 3.4, we have

$$\rho(e_a^x f_b^y) = 1 \Longrightarrow (x, y, a, b) \in \Pi. \tag{3.8}$$

Since the value of $\rho(e_a^x f_b^y)$ can only be 1 or 0, as

$$\Pi \cap \Lambda = \emptyset,$$

the condition (3.8) implies that for every $(x, y, a, b) \in \Lambda$, i.e., for every $e_a^x f_b^y \in \mathcal{N}$,

$$\rho(e_a^x f_b^y) = 0$$

holds, which completes the proof.

Remark 2. *For quantum nonlocal games, the value $\psi^* \sigma(e_a^x f_b^y)\psi$ is the probability that the players provide the answer pair (a, b) for the question pair (x, y). Since we start with a perfect commuting operator strategy, $\psi^* \sigma(e_a^x f_b^y)\psi \neq 0$ implies that the scoring function $\lambda(x, y, a, b) = 1$. In other words, Claim 3.4 indicates that*

$$\lambda(x, y, u(x), v(y)) = 1.$$

That is, the mappings $u : X \to A$ and $v : Y \to B$ defined in equations (3.6) and (3.7) actually define a perfect deterministic classic strategy for the two-answer game.

Now, we provide the proof of Claim 3.4.

Proof of Claim 3.4. We set $a = u(x)$ and $b = v(y)$ in Equation (3.5), and compute

$$\begin{aligned} \psi^* \sigma(e_{u(x)}^x f_{v(y)}^y)\psi = \frac{1}{4} & \\ &+ \frac{1}{4}(-1)^{u(x)} \psi^* \sigma(A_x)\psi \\ &+ \frac{1}{4}(-1)^{v(y)} \psi^* \sigma(B_y)\psi \\ &+ \frac{1}{4}(-1)^{u(x)+v(y)} \psi^* \sigma(A_x B_y)\psi. \end{aligned} \tag{3.9}$$

Notice that $\sigma(A_x)$ and $\sigma(B_y)$ are commutative self-adjoint operators, so $\psi^* \sigma(A_x)\psi$, $\psi^* \sigma(B_y)\psi$ and $\psi^* \sigma(A_x B_y)\psi$ are all real numbers.

If $\psi^* \sigma(A_x)\psi \neq 0$, and since $\psi_1 = \psi$, we conclude that $k(x) = 1$. Moreover, according to (3.6), if $\psi^* \sigma(A_x)\psi > 0$, we have

$$u(x) = 0,\ (-1)^{u(x)} = 1;$$

if $\psi^* \sigma(A_x)\psi < 0$, we have

$$u(x) = 1,\ (-1)^{u(x)} = -1.$$

Therefore, the value below is always positive:

$$(-1)^{u(x)} \psi^* \sigma(A_x)\psi > 0.$$

Similarly, if $\psi^* \sigma(B_y)\psi \neq 0$, we have

$$(-1)^{v(y)} \psi^* \sigma(B_y)\psi > 0.$$

Therefore, if either $\psi^* \sigma(A_x)\psi$ or $\psi^* \sigma(B_y)\psi$ is nonzero, we have

$$\frac{1}{4}(-1)^{u(x)} \psi^* \sigma(A_x)\psi + \frac{1}{4}(-1)^{v(y)} \psi^* \sigma(B_y)\psi > 0.$$

Since $\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)} \psi^* \sigma(A_x B_y)\psi \geqslant 0$, we have

$$\psi^* \sigma(e_{u(x)}^x f_{v(y)}^y)\psi > 0.$$

Hence, we only need to consider the case

$$\psi^*\sigma(A_x)\psi = \psi^*\sigma(B_y)\psi = 0.$$

Since we are working with infinite-dimensional separable Hilbert spaces, we modify the argument in [11, Theorem 3] by incorporating Cauchy-Schwarz inequality and Parseval's identity for proving that

$$\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi > 0.$$

Conversely, suppose

$$(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi = -1 \tag{3.10}$$

holds. By Cauchy-Schwarz's inequality, we know that

$$\begin{aligned}&\left|(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi\right|\\ \leqslant\ & \|(-1)^{u(x)}\sigma(A_x)\psi\| \cdot \|(-1)^{v(y)}\sigma(B_y)\psi\|.\end{aligned}$$

Since ψ is a unit vector and the eigenvalues of $\sigma(A_x), \sigma(B_y)$ can only be ± 1, we know

$$\|(-1)^{u(x)}\sigma(A_x)\psi\| = 1 \text{ and } \|(-1)^{v(y)}\sigma(B_y)\psi\| = 1.$$

Applying the equality condition of the Cauchy-Schwarz inequality, and the assumption (3.10), we obtain

$$(-1)^{u(x)}\sigma(A_x)\psi = -(-1)^{v(y)}\sigma(B_y)\psi. \tag{3.11}$$

By Parseval's identity, we have

$$(-1)^{u(x)}\sigma(A_x)\psi = \sum_{j=1}^{\infty}(-1)^{u(x)}\langle\sigma(A_x)\psi, \psi_j\rangle \cdot \psi_j,$$

and

$$(-1)^{v(y)}\sigma(B_y)\psi = \sum_{j=1}^{\infty}(-1)^{v(y)}\langle\sigma(B_y)\psi, \psi_j\rangle \cdot \psi_j,$$

Then Equation (3.11) gives

$$(-1)^{u(x)}\langle\sigma(A_x)\psi, \psi_j\rangle = -(-1)^{v(y)}\langle\sigma(B_y)\psi, \psi_j\rangle,$$

which implies that

$$(-1)^{u(x)}\psi_j^*\sigma(A_x)\psi = -(-1)^{v(y)}\psi_j^*\sigma(B_y)\psi \tag{3.12}$$

holds for every $j \in \{1, 2, \ldots\ldots\}$. However, Equation (3.12) must fail to hold for $j = \min\{k(x), l(y)\}$. It is clear that Equation (3.12) fails when $k(x) \neq l(y)$. Assume $k(x) = l(y) = j$, we find that the arguments of $\arg\left((-1)^{u(x)}\psi_j^*\sigma(A_x)\psi\right)$ and $\arg\left((-1)^{v(y)}\psi_j^*\sigma(B_y)\psi\right)$ both lie in the range $[0, \pi)$, which contradicts Equation (3.12) once again!

Therefore, when $\psi^*\sigma(A_x)\psi = \psi^*\sigma(B_y)\psi = 0$, we have shown that

$$\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi > 0.$$

That is,

$$\psi^*\sigma(e^x_{u(x)}f^y_{v(y)})\psi > 0,$$

which always holds, thereby proving the claim. □

Finally, we prove Theorem 3.1.

Proof of Theorem 3.1. ($\Longleftarrow$) This direction is straightforward. Suppose, for the sake of contradiction, that this direction does not hold, i.e.,

$$-1 \in \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$$

and there exists a $*-$representation ρ such that

$$\rho(\mathcal{N}) = \{0\},$$

then we have

$$-1 = \rho(-1) \in \rho(\mathrm{SOS}_{\mathcal{A}}) \geqslant 0,$$

which is a contradiction!

($\Longrightarrow$) This follows from Proposition 3.2 and Proposition 3.3. □

We have the following result.

Corollary 3.5. *For any two-answer game $\mathcal{G}$, it admits a perfect commuting operator strategy if and only if it admits a perfect classical strategy, i.e.*

$$\omega_{co}(\mathcal{G}) = 1 \iff \omega_c(\mathcal{G}) = 1.$$

Proof. Any nonlocal game $\mathcal{G}$ with a perfect deterministic strategy also has a perfect commuting operator strategy. On the other hand, according to Proposition 3.2, the condition

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$$

implies that $\mathcal{G}$ has a perfect commuting operator strategy. By Theorem 3.1, in the case of two-answer games, this is further equivalent to $\mathcal{N}$ having a one-dimensional zero point.

Moreover, every deterministic strategy for the game $\mathcal{G}$ can be induced by a one-dimensional complex $*$-representation

$$\pi : \mathcal{A} \to \mathbb{C},$$

i.e.,

$$p(a, b \mid x, y) = \pi(e^x_a f^y_b).$$

Therefore, by Remark 1, the existence of a one-dimensional zero point for $\mathcal{N}$ is equivalent to $\mathcal{G}$ having a perfect deterministic strategy. □

Let $\mathcal{A}$, Λ and $\mathcal{N}$ be the same as in Theorem 3.1. Suppose there exists a $*-$representation

$$\rho : \mathcal{A} \to \mathbb{C}$$

such that

$$\rho(\mathcal{N}) = \{0\}.$$

Since ρ is a one-dimensional representation, all of $\rho(e^x_a)$ and $\rho(f^y_b)$ commute. Thus $\rho(\mathcal{N}) = \{0\}$ if and only if the polynomial system $\mathcal{N} = \{e^x_a f^y_b \mid (x, y, a, b) \in \Lambda\}$ has one-dimensional zeros in the quotient algebra

$$\mathbb{C}[\{e^x_a, f^y_b, \forall x, y, a, b\}]/\mathfrak{K},$$

where $\mathfrak{K}$ is the two-sided ideal generated by the set

$$\begin{aligned}K = \{&(e^x_a)^2 - e^x_a,\ (f^y_b)^2 - f^y_b \mid x \in X, y \in Y, a \in A, b \in B\}\\ &\cup \{e^x_0 + e^x_1 - 1,\ f^y_0 + f^y_1 - 1 \mid x \in X, y \in Y\}.\end{aligned}$$

By Hilbert's Nullstellensatz, this is also equivalent to $1 \notin \mathfrak{I}$, where $\mathfrak{I}$ is a two-sided ideal generated by $K \cup \mathcal{N}$ in the commutative polynomial ring $\mathbb{C}[\{e^x_a, f^y_b, \forall x, y, a, b\}]$.

Combined with Theorem 3.1, we have

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^* \iff 1 \notin \mathfrak{I} \tag{3.13}$$

Determining whether

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$$

is not easy, because $\mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$ is not an ideal. However, the membership problem $1 \notin \mathfrak{I}$ can be determined by computing the Gröbner basis of $\mathfrak{I}$. Hence, we have an algorithm based on Gröbner basis computation to determine whether a two-answer game has a perfect commuting operator strategy.

4 Example and Discussion

Let us look at a graph coloring game in [11, 16]. It is well-known that when n is even, the game has a perfect classical strategy, whereas when n is odd, the color of the last vertex v_n will lead to a contradiction regardless of whether it is chosen as 0 or 1. Thus, there is no perfect classical strategy. An upper bound on the winning probability for commuting operator strategies has been given in [19] to determine that the game has no perfect commuting operator strategy. Below, we show how to use Gröbner basis computation to determine whether there is a perfect commuting operator strategy.

Example 1 (Graph coloring game). *[11, 16] Consider a cycle with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set*

$$E = \{(v_i, v_{i+1}), (v_{i+1}, v_i) \mid i = 1, \ldots, n\},$$

where $v_{n+1} = v_1$. We define the coloring game $\mathcal{G}$ as follows. Let the question set be

$$E \cup \{(v_i, v_i) \mid i = 1, \ldots, n\},$$

and the answer sets of the players are both $\{0, 1\}$. The scoring function is defined by

$$\lambda(v_i, v_j, a, b) = \begin{cases} 0, & \text{if } (v_i = v_j \text{ and } a \neq b) \text{ or } ((v_i, v_j) \in E \text{ and } a = b); \\ 1, & \text{else.} \end{cases}$$

This nonlocal game can be viewed as Alice and Bob attempting to color a cycle graph with two colors so that the vertices of each edge in the graph have different colors.

As a simple example, we compute Gröbner bases of the ideal $\mathfrak{I}$ corresponding to $n = 3$ and $n = 4$ below.

(1) $n = 3$:

$$\begin{aligned} \mathfrak{I} = \langle \{ & (e_0^1)^2 - e_0^1,\ (e_1^1)^2 - e_1^1,\ (e_0^2)^2 - e_0^2,\ (e_1^2)^2 - e_1^2, \\ & (e_0^3)^2 - e_0^3,\ (e_1^3)^2 - e_1^3,\ (f_0^1)^2 - f_0^1,\ (f_1^1)^2 - f_1^1, \\ & (f_0^2)^2 - f_0^2,\ (f_1^2)^2 - f_1^2,\ (f_0^3)^2 - f_0^3,\ (f_1^3)^2 - f_1^3, \\ & e_0^1 + e_1^1 - 1,\ f_0^1 + f_1^1 - 1,\ e_0^2 + e_1^2 - 1,\ f_0^2 + f_1^2 - 1, \\ & e_0^3 + e_1^3 - 1,\ f_0^3 + f_1^3 - 1, \\ & e_0^1 f_1^1,\ e_1^1 f_0^1,\ e_0^2 f_1^2,\ e_1^2 f_0^2,\ e_0^3 f_1^3,\ e_1^3 f_0^3, \\ & e_0^1 f_0^2,\ e_1^1 f_1^2,\ e_0^2 f_0^1,\ e_1^2 f_1^1, \\ & e_0^1 f_0^3,\ e_1^1 f_1^3,\ e_0^3 f_0^1,\ e_1^3 f_1^1, \\ & e_0^2 f_0^3,\ e_1^2 f_1^3,\ e_0^3 f_0^2,\ e_1^3 f_1^2 \} \rangle \end{aligned}$$

Under the graded lexicographic ordering with

$$e_0^1 > e_1^1 > e_0^2 > e_1^2 > \cdots > e_1^3 > f_0^1 > \cdots > f_1^3,$$

the Gröbner basis of $\mathfrak{I}$ is 1. Hence, according to Corollary 3.5 and (3.13), $\mathcal{G}$ has no perfect commuting operator strategy.

(2) $n = 4$:

$$\begin{aligned} \mathfrak{I} = \langle \{ & (e_0^1)^2 - e_0^1,\ (e_1^1)^2 - e_1^1,\ (e_0^2)^2 - e_0^2,\ (e_1^2)^2 - e_1^2, \\ & (e_0^3)^2 - e_0^3,\ (e_1^3)^2 - e_1^3,\ (e_0^4)^2 - e_0^4,\ (e_1^4)^2 - e_1^4, \\ & (f_0^1)^2 - f_0^1,\ (f_1^1)^2 - f_1^1,\ (f_0^2)^2 - f_0^2,\ (f_1^2)^2 - f_1^2, \\ & (f_0^3)^2 - f_0^3,\ (f_1^3)^2 - f_1^3,\ (f_0^4)^2 - f_0^4,\ (f_1^4)^2 - f_1^4 \\ & e_0^1 + e_1^1 - 1,\ f_0^1 + f_1^1 - 1,\ e_0^2 + e_1^2 - 1,\ f_0^2 + f_1^2 - 1, \\ & e_0^3 + e_1^3 - 1,\ f_0^3 + f_1^3 - 1,\ e_0^4 + e_1^4 - 1,\ f_0^4 + f_1^4 - 1 \\ & e_0^1 f_1^1,\ e_1^1 f_0^1,\ e_0^2 f_1^2,\ e_1^2 f_0^2,\ e_0^3 f_1^3,\ e_1^3 f_0^3,\ e_0^4 f_1^4,\ e_1^4 f_0^4 \\ & e_0^1 f_0^2,\ e_1^1 f_1^2,\ e_0^2 f_0^1,\ e_1^2 f_1^1, e_0^2 f_0^3,\ e_1^2 f_1^3,\ e_0^3 f_0^2,\ e_1^3 f_1^2 \\ & e_0^3 f_0^4,\ e_1^3 f_1^4,\ e_0^4 f_0^3,\ e_1^4 f_1^3, e_0^1 f_0^4,\ e_1^1 f_1^4,\ e_0^4 f_0^1,\ e_1^4 f_1^1, \} \rangle \end{aligned}$$

A Gröbner basis of $\mathfrak{I}$ under the graded lexicographic ordering

$$e_0^1 > e_1^1 > e_0^2 > e_1^2 > \cdots > e_1^4 > f_0^1 > \cdots > f_1^4,$$

is

$$\begin{aligned} \{ & f_0^4 + f_1^4 - 1, f_1^3 + f_1^4 - 1, f_0^3 - f_1^4, f_1^2 - f_1^4, \\ & f_0^2 + f_1^4 - 1, f_1^1 + f_1^4 - 1, f_0^1 - f_1^4, e_1^4 - f_1^4, \\ & e_0^4 + f_1^4 - 1, e_1^3 + f_1^4 - 1, e_0^3 - f_1^4, e_1^2 - f_1^4, \\ & e_0^2 + f_1^4 - 1, e_1^1 + f_1^4 - 1, e_0^1 - f_1^4, (f_1^4)^2 - (f_1^4) \}. \end{aligned}$$

By Corollary 3.5 and (3.13), $\mathcal{G}$ has a perfect commuting operator strategy.

Remark 3. *In [4, Section 5], Watts, Helton, and Klep discussed a special type of nonlocal games: torically determined games. A nonlocal game is called a torically determined game if there exists a finite set F with the following form:*

$$F = \{\beta_i g_i - 1 \mid \beta_i \in \mathbb{C}, g_i \in G\},$$

where G is a group and the group algebra $\mathbb{C}[G]$ is isomorphic to the universal game algebra, such that $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \psi \in \mathcal{H})$ defines a perfect commuting operator strategy if and only if $\pi(F)\psi = \mathbf{0}$. They demonstrated that for a torically determined game, the question of whether the game has a perfect commuting operator strategy can be translated into a subgroup membership problem [4, Section 5]. However, this result cannot be used to prove our theorem. The reason is that if we regard $\mathcal{N}$ as the determining set of the game, the elements in $\mathcal{N}$ may not necessarily be expressible in the form $\beta g - 1, \beta \in \mathbb{C}, g \in G$. In other words, a two-answer game is not necessarily a torically determined game.

5 Conclusion

In this paper, we show that for a two-answer nonlocal game,

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$$

if and only if $\mathcal{N}$ has one-dimensional zeros, which implies that a two-answer nonlocal game has a perfect commuting operator strategy if and only if the game has a perfect classical strategy. Moreover, the problem of determining whether $\mathcal{N}$ has one-dimensional zeros is equivalent to an ideal membership problem. Therefore, we can use commutative Gröbner bases to determine whether a two-answer nonlocal game has a perfect commuting operator strategy.

Suppose the answer set A or B contains three or more elements, our main result (Theorem 3.1) fails to hold, as there exists a nonlocal game that has a perfect commuting operator strategy but no perfect classical strategies [11, 22]. Investigating how to generalize our results for more general games is an interesting research topic.

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