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A Formal Proof of the Irrationality of $\zeta(3)$ in Lean 4

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Abstract We formalize a proof of the irrationality of $\zeta(3)$ in Lean 4, using Beukers' method. To support this, we extend the Lean mathematical library (`Mathlib`) by formalizing shifted Legendre polynomials and important results in analytic number theory that were previously missing. As part of the Lean 4 `PrimeNumberTheoremAnd` project, we also formalize the asymptotic behavior of the prime counting function, giving the first formal proof in Lean 4 of a version of the Prime Number Theorem with an error term which is stronger than what had previously been formalized. This result is a crucial ingredient in proving the irrationality of $\zeta(3)$. Our complete Lean 4 formalization is publicly available on GitHub. *

Keywords formal proof, irrationality, Riemann zeta function, shifted Legendre polynomial, Prime Number Theorem, number theory.

1 Introduction

The Riemann zeta function is a crucial concept in mathematics. For real values of s with $s > 1$, the Riemann zeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In 1978, Apéry proved that $\zeta(3)$ is irrational [21]. This result was the first dent in the problem of the irrationality of the values of the Riemann zeta function at odd positive integers; see [20] for an informal report on Apéry's proof by Van der Poorten. In 1979, this proof was shortened by Beukers [22], who used an integral method to connect $\zeta(3)$ with a specific double improper integral over the unit square $(0,1)^2$.

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*See `https://github.com/ahhwuhu/zeta_3_irrational`.

Formal proof assistants can rigorously verify the correctness of mathematical proofs, thereby avoiding potential gaps in human reasoning. At present, a large number of mathematical results have already been formalized and verified in interactive theorem provers such as Coq, Isabelle, and Lean [28]. A computer algebra based formal proof of the irrationality of $\zeta(3)$ using the Coq proof assistant was given by Salvy [14], Chyzak et al. [12] and Mahboubi and Sibut-Pinote [†][17]. It follows Apéry's original proof [21] and uses the method of creative telescoping [18] for dealing with recurrence relations that appeared in Apéry's proof.

Eberl formalized Beukers' proof using the Isabelle proof assistant [26] based on the lecture notes of Filaseta [16]. The asymptotic upper bound on $\text{lcm}\{1, \dots, n\} \leq O(c^n)$ for any $c > e$ (Euler's number) used by both Apéry and Beukers is available in Isabelle [15].

Our work contributes to the mathematical library `Mathlib` [24] for the Lean 4 theorem proof assistant [25], a system based on dependent type theory augmented with quotient types and classical reasoning. `Mathlib` is a decentralized and continuously evolving library, with contributions from over 340 authors. While Lean's library excels in many areas, it has lagged behind other theorem provers, such as Isabelle, in certain analytical domains. Our project aims to bridge this gap by formalizing important theorems in areas like calculus and analytic number theory, thereby enhancing Lean's analytical content and further enriching `Mathlib`'s diverse body of work.

Although Lean trails behind Isabelle in formalizing some foundational theorems, such as the Prime Number Theorem, significant progress is being made. Terence Tao, Alex Kontorovich, and others are actively working on the `PrimeNumberTheoremAnd` project [‡] in Lean 4. As part of this effort, we have formalized related results — Theorem 25 and Corollary 9 [§]—which are crucial for proving the irrationality of $\zeta(3)$. Additionally, during this process, we identified and corrected a typo in their formal theorem statements.

Our formalization of the proof of the irrationality of $\zeta(3)$ in Lean 4 follows mainly Beukers' method [22]. A key idea in Beukers' proof for showing that a real number x is irrational is to construct a non-zero sequence $\{a_n + b_n x\}$, where $a_n, b_n \in \mathbb{Z}$, that tends to zero as $n \to \infty$. If x were rational, say $x = \frac{p}{q}, q > 0$, the sequence $\{|a_n + b_n x|\}$ would have a lower bound of $\frac{1}{q}$, independent from n, leading to a contradiction. Our main steps are outlined by referring to Beukers' proof. In all the following multiple integrals, we denote by μ a measure on the region of integration, and use $d\mu$ to indicate differential element.

- Consider the integral

$$\int_{(x,y)\in(0,1)^2} -P_n(x)P_n(y)\frac{\log(xy)}{1-xy}d\mu \tag{1}$$

  where $P_n(x)$ is the shifted Legendre polynomial

$$P_n(x) := \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n]. \tag{2}$$

[†]`https://github.com/coq-community/apery`

[‡]`https://github.com/AlexKontorovich/PrimeNumberTheoremAnd`

[§]`https://alexkontorovich.github.io/PrimeNumberTheoremAnd/web/sect0004.html`

According to Lemma 1 in [22], the integral (1) equals to $\frac{a_n+b_n\zeta(3)}{d_n^3}$, where

$$d_n := \operatorname{lcm}\{1, 2, \dots, n\}, \tag{3}$$

the least common multiple of $1, 2, \dots, n$ and $a_n, b_n \in \mathbb{Z}$.

- According to the Prime Number Theorem [9–11], for sufficiently large values of n, we have

$$d_n^3 \leq \left(e^3\right)^n.$$

Besides, the integral (1) is positive and bounded above by $2\left(\frac{1}{24}\right)^n \zeta(3)$. Hence, for sufficiently large n, one has

$$0 < |a_n + b_n\zeta(3)| < 2\left(\frac{1}{24}\right)^n \zeta(3)d_n^3 < \left(\frac{21}{24}\right)^n 2\zeta(3),$$

which implies the irrationality of $\zeta(3)$.

The figure below illustrates the dependencies between all the definitions, lemmas, and theorems which are used in the process of formally proving the irrationality of $\zeta(3)$.

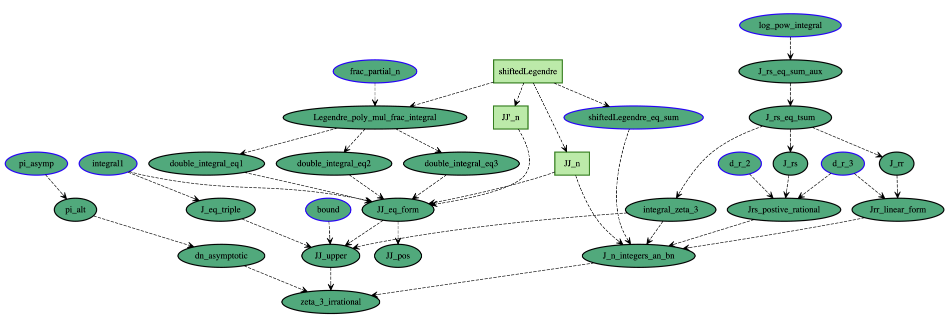

Figure 1 The Dependency Graph of the Formalization Project

In this paper, we make the following main contributions:

- We introduce and rigorously define shifted Legendre polynomials, formalizing key properties within Lean 4, thus advancing the formalization of special functions in the Lean ecosystem.
- We provide the first formal proof in Lean 4 of a strengthened version of the Prime Number Theorem, establishing that the prime counting function $\pi(x)$ is asymptotic to $\frac{x}{\log x}$. This key result (Corollary 9 in the `PrimeNumberTheoremAnd` project) enhances Lean's capabilities in analytic number theory.
- We present a complete formal proof of the irrationality of $\zeta(3)$ in Lean 4, following Beukers' method, contributing to the formal verification of an important result in analytic number theory.

To enhance readability, we provide both formal and informal statements of the definitions and theorems. The proofs of the theorems are primarily presented informally, with the corresponding formal proofs available in Lean 4, which can be accessed and downloaded on GitHub¶.

The paper is organized as follows. In Section 2, we introduce the concept of Lebesgue integrability in Lean 4, focusing on the lower Lebesgue integral `lintegral`, which is more convenient for dealing with multiple integrals. In Section 3, we formally define the shifted Legendre polynomials and establish their fundamental properties in Lean 4. Section 4 presents two key results related to the Prime Number Theorem. Finally, in Section 5, we formally show the irrationality of $\zeta(3)$ in Lean 4.

2 Lean Preliminaries

Lean is an open-source theorem prover with a small trusted kernel based on dependent type theory [23]. One of its most exciting applications is in training large language models (LLMs) for theorem proving, leveraging Lean 4's formal framework to enable AI systems to assist in automated reasoning and proof generation [7, 8]. Google DeepMind has translated one million problems written in natural language into Lean, without including human-written solutions, for training AlphaProof to solve International Mathematical Olympiad problems at a silver medalist level. By formalizing the proof of irrationality of $\zeta(3)$ in Lean 4, we aim to add some knowledge in the fields of analysis, combinatorics, and number theory to Lean's mathematical library.

As the formal proof of the irrationality of $\zeta(3)$ is closely tied to demonstrating that certain improper integrals converge, we begin by introducing basic definitions related to the integrability of functions in `Mathlib`.

In `Mathlib`, the integrability of a function f is defined using the concept of the measurability of the function and its Lebesgue integrability over a given domain. The formal definition of integrability of f in Lean 4 is:

```
def MeasureTheory.Integrable {α} {_ : MeasurableSpace α} (f : α → β)
   (μ : Measure α := by volume_tac) : Prop :=
   AEStronglyMeasurable f μ ∧ HasFiniteIntegral f μ
```

The above definition involves two key parts: measurability and integrability. The formal definition of measurability in Lean 4 is:

```
def MeasureTheory.AEStronglyMeasurable {_ : MeasurableSpace α} (f : α → β)
   (μ : Measure α := by volume_tac) : Prop := ∃ g, StronglyMeasurable g ∧ f =ᵐ[μ] g
```

A function is `AEStronglyMeasurable` if it is almost everywhere equal to the limit of a sequence of simple functions. A simple function is a measurable function whose image consists of only a finite set of real numbers, and any simple function can be expressed as a linear combination of a finite number of characteristic functions [27].

¶https://github.com/ahhwuhu/zeta_3_irrational

The functions considered in this paper are elementary functions, which are functions generated by a finite number of basic operations such as addition, multiplication, inversion, and composition involving basic functions like polynomial functions, rational functions, exponential functions, logarithmic functions, and trigonometric functions. Elementary functions are always measurable, and since measurable functions can be approximated by simple functions, elementary functions are always `AEStronglyMeasurable`. Hence, the functions we are considering will be `AEStronglyMeasurable` as well.

A function f is Lebesgue integrable if its Lebesgue lower integral over the domain is finite. The formal definition of a function having a finite integral in Lean 4 is:

```
def MeasureTheory.HasFiniteIntegral {_ : MeasurableSpace α} (f : α → β)
  (μ : Measure α := by volume_tac) : Prop := (∫⁻ a, ‖f a‖₊ ∂μ) < ∞
```

The "$\int^-$" symbol in the definition is a notation for `lintegral`, mathematically the lower Lebesgue integral of a $[0, \infty]$ valued function. The lower Lebesgue integral of a function is obtained by approximating the integral from below by simple functions and take the infimum of the integrals of those approximating functions.

The lower Lebesgue integral `lintegral` is the extended real-valued version of the integral. In Lean 4, the set of extended non-negative real numbers $[0, \infty]$ is defined as `ENNReal`. According to dependent type theory, the objects on both sides of an equation should have the same type; otherwise, the equation would be ill-typed. Therefore, in all equations appearing later in this article, if one side of the mathematical expression is a `lintegral`, which means the "$\int^-$" symbol appears on one side, and the other side is implicitly assumed to be of the type `ENNReal`. There are two functions `ENNReal.ofReal` and `ENNReal.toReal` which can be used to convert types between non-negative real numbers and `ENNReal`.

To prove the integral of a function is finite, we compute the lower Lebesgue integral `lintegral`, and check whether it is finite.

Compared to the standard Lebesgue integral, `integral`, the benefit of using `lintegral` is that issues of integrability or summability do not arise at all. We only need to calculate the specific `lintegral` value.

We can connect the lower Lebesgue integral and integral by the following theorem:

```
theorem MeasureTheory.integral_eq_lintegral_of_nonneg_ae {f : α → ℝ}
    (hf : 0 ≤ᵐ[μ] f) (hfm : AEStronglyMeasurable f μ) :
    ∫ a, f a ∂μ = ENNReal.toReal (∫⁻ a, ENNReal.ofReal (f a) ∂μ)
```

Listing 1: the lower Lebesgue integral and integral

If f is an elementary function, then f must be `AESstronglyMeasurable`. The condition $0 \leq^{\mathsf{m}}[\mu]\ f$ means that f is non-negative almost everywhere under the measure μ, i.e., the set of points where $f(x) < 0$ has measure zero. We can prove it through the following theorem in `Mathlib`:

```
theorem MeasureTheory.ae_nonneg_restrict_of_forall_setIntegral_nonneg_inter
```

```
    {f : α → ℝ} {t : Set α} (hf : IntegrableOn f t μ) (hf_zero :
    ∀ s, MeasurableSet s → μ (s ∩ t) < ⊤ → 0 ≤ ∫ x in s ∩ t, f x ∂μ) :
    0 ≤ᵐ[μ.restrict t] f
```

Listing 2: function nonnegative almost everywhere

To apply theorem listing 1, we need to check two conditions: `hf_zero` and `IntegrableOn`. The condition `hf_zero` can be checked easily due to the non-negativity of the function at every point in $(0,1)^2$. Below, we focus on checking the `IntegrableOn` condition.

The condition `IntegrableOn` is satisfied if a function is integrable with respect to the restricted measure on a set `s`, essentially, it is to prove that the value of `lintegral` is finite.

```
theorem MeasureTheory.hasFiniteIntegral_iff_norm (f : α → β) :
    HasFiniteIntegral f μ ↔ (∫⁻ a, ENNReal.ofReal ‖f a‖ ∂μ) < ∞
```

The following theorem shows that `ENNReal.ofReal` and `ENNReal.toReal` can be cancelled out:

```
theorem ENNReal.toReal_ofReal_eq_iff {a : ℝ} : (ENNReal.ofReal a).toReal = a ↔ 0 ≤ a
```

In Lean 4, we can relate an integral in the `ENNReal` space to a `lintegral` using the following theorem, which allows us to move `ENNReal.ofReal` from outside to inside the `lintegral`. Formally, the theorem is stated as:

```
theorem MeasureTheory.ofReal_integral_eq_lintegral_ofReal {f : α → ℝ}
    (hfi : Integrable f μ) (f_nn : 0 ≤ᵐ[μ] f) :
    ENNReal.ofReal (∫ x, f x ∂μ) = ∫⁻ x, ENNReal.ofReal (f x) ∂μ
```

Listing 3: `ENNReal` of integral equal to `lintegral`

Additionally, in the proof of Lemma 5.10, we will also make use of the substitution formula for integrals:

```
theorem intervalIntegral.integral_comp_mul_deriv {f f' g : ℝ → ℝ}
    (h : ∀ x ∈ uIcc a b, HasDerivAt f (f' x) x)
    (h' : ContinuousOn f' (uIcc a b)) (hg : Continuous g) :
    (∫ x in a..b, (g ∘ f) x * f' x) = ∫ x in f a..f b, g x
```

Listing 4: change of variables

To prove that a multiple integral is equal to a repeated integral, we use the following theorem in `Mathlib`:

```
theorem MeasureTheory.integral_prod (f : α × β → E) (hf : Integrable f (μ.prod ν)) :
    ∫ z, f z ∂μ.prod ν = ∫ x, ∫ y, f (x, y) ∂ν ∂μ
```

Listing 5: multiple integral equal to repeated integral

The following theorem in `Mathlib` states that for a non-negative integral function f, the integral of f is positive if and only if the measure of the support of f is positive.

```
theorem MeasureTheory.integral_pos_iff_support_of_nonneg_ae {f : α → ℝ}
    (hf : 0 ≤ᵐ[μ] f) (hfi : Integrable f μ) :
    (0 < ∫ x, f x ∂μ) ↔ 0 < μ (Function.support f)
```

Listing 6: integral positive iff support nonnegetive almost everywhere

3 Shifted Legendre Polynomial

The shifted Legendre polynomials

$$P_n(x) := \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n]$$

have been used in Beukers' proof for constructing a convergent sequence $\{a_n + b_n\zeta(3)\}$.

In this section, we formally define the shifted Legendre polynomial and outline its fundamental properties in Lean 4. These definitions and properties have been added to `Mathlib`.

In Lean 4, the shifted Legendre polynomial is defined as

```
noncomputable def shiftedLegendre (n : ℕ) : ℝ[X] :=
  C (n ! : ℝ)⁻¹ * derivative^[n] (X ^ n * (1 - X) ^ n)
```

where `C` is the embedding of $\mathbb{R}$ into its polynomial ring and `X` represents the variable.

By expanding the polynomial $(x-x^2)^n$, and combining it with the linearity of the derivative operator, the shifted Legendre polynomials $P_n(x)$ can be written as polynomials with integer coefficients:

$$P_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n}x^k, \tag{4}$$

which is formalized as the following theorem in Lean 4:

```
theorem shiftedLegendre_eq_sum (n : ℕ) : shiftedLegendre n =
    Σ k in Finset.range (n + 1), C ((- 1) ^ k : ℝ) *
    (Nat.choose n k : ℝ[X]) * (Nat.choose (n + k) n : ℝ[X]) * X ^ k
```

One can prove a more abstract version of the above theorem, which generalizes the result as follows:

```
lemma shiftedLegendre_eq_int_poly (n : ℕ) : ∃ a : ℕ → ℤ, shiftedLegendre n =
    Σ k in Finset.range (n + 1), (a k : ℝ[X]) * X ^ k
```

The shifted Legendre polynomials have good properties for performing integration by parts. For all $n \in \mathbb{N}$ and a $\mathcal{C}^n$ functions $f : [0,1] \to \mathbb{R}$, we have:

$$\int_0^1 P_n(x)f(x)\,dx = \frac{(-1)^n}{n!}\int_0^1 x^n(1-x)^n\frac{d^n f}{dx^n}\,dx \tag{5}$$

We present the formalization of equation (5) for

$$f(y) := \frac{1}{1-(1-xy)z},$$

which will be used in Lemma 5.9 and Lemma 5.11, with $0 < x, z < 1$.

Lemma 3.1 *For $0 < x, z < 1$, we have*

$$\frac{d^n}{dy^n}\left(\frac{1}{1-(1-xy)z}\right) = (-1)^n n! \frac{(xz)^n}{(1-(1-xy)z)^{n+1}}. \tag{6}$$

The formal statement in Lean 4 is:

```
lemma n_derivative' {x z : ℝ} (n : ℕ) (hx : x ∈ Set.Ioo 0 1) (hz : z ∈ Set.Ioo 0 1) :
  (deriv^[n] fun y ↦ 1 / (1 - (1 - x * y) * z)) =
  (fun y ↦ (-1) ^ n * n ! * (x * z) ^ n / (1 - (1 - x * y) * z) ^ (n + 1))
```

Proof When we formalize equality (6) in Lean 4, we need to discuss whether $1-(1-xy)z$ is equal to 0 for $x, z \in (0, 1)$.

- For any y, when $1-(1-xy)z \neq 0$, $\frac{1}{1-(1-xy)z}$ is differentiable to the n-th order by induction. It is straightforward to check the equality (6).
- If $1-(1-xy)z = 0$, then, because functions with a zero denominator are defined as 0 in Lean 4 (since all functions are total), the values on the right-hand side of equality (6) are zero. In Lean 4, the derivative at points where a function is not differentiable is defined as 0. Hence, we prove the left side of equality (6) is 0 by demonstrating, using an $\varepsilon - \delta$ argument, that $\frac{1}{1-(1-xy)z}$ is not differentiable at the point y where $1-(1-xy)z = 0$. Consequently, the equality (6) holds with a value of 0.

□

Lemma 3.2 *For $0 < x, z < 1$, one has*

$$\int_0^1 P_n(y)\frac{1}{1-(1-xy)z}\,dy = \int_0^1 \frac{(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}}\,dy. \tag{7}$$

The formal statement in Lean 4 is:

```
lemma legendre_integral_special {x z : ℝ} (n : ℕ) (hx : x ∈ Set.Ioo 0 1)
    (hz : z ∈ Set.Ioo 0 1) :
  ∫ (y : ℝ) in (0)..1,
    eval y (shiftedLegendre n) * (1 / (1 - (1 - x * y) * z)) =
  ∫ (y : ℝ) in (0)..1,
    (x * y * z) ^ n * (1 - y) ^ n / (1 - (1 - x * y) * z) ^ (n + 1)
```

Proof It can be proven by induction and integration by parts that

$$\int_0^1 P_n(y)\frac{1}{1-(1-xy)z}\,dy = \frac{(-1)^n}{n!}\int_0^1 y^n(1-y)^n\frac{d^n}{dy^n}\left(\frac{1}{1-(1-xy)z}\right)dy$$

By substituting equality (6) into the right-hand side of the above equation, we obtain equality (7). □

4 Prime Number Theorem

Suppose the prime factorization of $d_n = \mathrm{lcm}\{1, ..., n\}$ is

$$d_n = \prod_{p \le n} p^m, \text{ where } m = \max_{k \in \mathbb{N}} \{p^k \le n\}.$$

Since log is strictly monotonic, we have

$$m = \left\lfloor \frac{\log n}{\log p} \right\rfloor.$$

We have the following estimate for d_n:

$$d_n = \prod_{p \le n} p^{\lfloor \frac{\log n}{\log p} \rfloor} \le \prod_{p \le n} p^{\frac{\log n}{\log p}} = \prod_{p \le n} p^{\log_p n} = \prod_{p \le n} n = n^{\pi(n)},$$

where $\pi(n)$ is the number of primes less than or equal to n.

The Prime Number Theorem states that:

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \log x} = 1.$$

There are several equivalent ways to express the Prime Number Theorem. One of the most common is:

$$\lim_{x \to \infty} \frac{\pi(x)}{\int_2^x \frac{dt}{\log t}} = 1.$$

The Prime Number Theorem was first proved in 1896 by Jacques Hadamard [10] and by Charles de la Vallée Poussin [11] independently. A modern proof was given by Atle Selberg and Paul Erdős, 1948, independently [9]. In [6], Avigad et al. presented the first formalization of Selberg's elementary proof [1] of the Prime Number Theorem using the Isabelle/HOL prover [4], which was later reproved in Metamath by Carneiro [3]. Subsequently, Harrison provided a formal proof of the Prime Number Theorem based on Newman's presentation [2] using the HOL-Light prover [5]. This work was later extended by Eberl and Paulson in Isabelle [15]. Later, Song and Yao present a formalized version of the Prime Number Theorem with an explicit error term in Isabelle [29].

Currently, no formal proof of the Prime Number Theorem exists in Lean. Terence Tao, Alex Kontorovich, and others are actively working on the `PrimeNumberTheoremAnd` project in Lean 4. The goal of this project is to formalize the Prime Number Theorem in Lean, including a classical error term, along with several related results in analytic number theory. A long-term objective is the formalization of the Chebotarev Density Theorem—a fundamental theorem in algebraic number theory that generalizes the Prime Number Theorem to Galois extensions of number fields.

We prove Theorem 4.1 and Theorem 4.2 corresponding to Theorem 25 and Corollary 9 in the `PrimeNumberTheoremAnd` project. ‖ These constitute the first formal proof in Lean of

‖ `https://alexkontorovich.github.io/PrimeNumberTheoremAnd/web/sect0004.html` a strengthened version of the Prime Number Theorem incorporating an error term which is stronger than what had previously been formalized. Furthermore, Theorem 4.2 plays a crucial role in the formal proof of the irrationality of $\zeta(3)$. The formalization of Theorem 4.1 and Theorem 4.2 in Lean 4 has been contributed to the `PrimeNumberTheoremAnd` project and is publicly available.**

Theorem 4.1 *The prime counting function admits the following asymptotic estimate as* $x \to \infty$

$$\pi(x) = (1 + o(1)) \int_2^x \frac{1}{\log t} dt.$$

The formal statement in Lean 4 is:

```
theorem pi_asymp :
    ∃ c : ℝ → ℝ, c =o[atTop] (fun _ ↦ (1 : ℝ)) ∧
    ∀ᶠ (x : ℝ) in atTop, Nat.primeCounting ⌊x⌋₊ = (1 + c x) * ∫ t in Set.Icc 2 x, 1 /
    (log t) ∂ volume
```

A precise description of the auxiliary constants involved in $o(1)$ and the concept of "sufficiently large" is essential for formalization. Therefore, we present the proof in great detail, ensuring that each step can be easily transcribed into Lean 4.

Proof We aim to show that $\frac{\pi(x)}{\int_2^x \frac{1}{\log t} dt} - 1$ is $o(1)$, that is, for every ε, there exists $M_\varepsilon \in \mathbb{R}$ such that for all $x > M_\varepsilon$, we have

$$\left| \frac{\pi(x)}{\int_2^x \frac{1}{\log t} dt} - 1 \right| \leq \varepsilon.$$

For all $x \geq 2$, it has been formalized in Lean 4 that:

$$\pi(x) = \frac{1}{\log x} \sum_{p \leq \lfloor x \rfloor} \log p + \int_2^x \frac{\sum_{p \leq \lfloor t \rfloor} \log p}{t \log^2 t} dt.$$

We also know that for every $\varepsilon > 0$, there exists a function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $f_\varepsilon = o(\varepsilon)$ and f_ε is integrable on $(2, x)$ for all $x \geq 2$. Furthermore, for x sufficiently large, say $x > N_\varepsilon \geq 2$, we have

$$\sum_{p \leq \lfloor x \rfloor} \log p = x + x f_\varepsilon(x).$$

Hence, for every $\varepsilon > 0$ and for sufficiently large x, such a function f_ε satisfies

$$\pi(x) = \frac{x + x f_\varepsilon(x)}{\log x} + \int_2^{N_\varepsilon} \frac{\sum_{p \leq \lfloor x \rfloor} \log p}{t \log^2 t} dt + \int_{N_\varepsilon}^x \frac{t + t f_\varepsilon(t)}{t \log^2 t} dt,$$

which can be simplified to

$$\pi(x) = \left(\frac{x}{\log x} + \int_{N_\varepsilon}^x \frac{1}{\log^2 t} dt \right) + \left(\frac{x f_\varepsilon(x)}{\log x} + \int_{N_\varepsilon}^x \frac{f_\varepsilon(t)}{\log^2 t} dt \right) + \int_2^{N_\varepsilon} \frac{\sum_{p \leq \lfloor x \rfloor} \log p}{t \log^2 t} dt.$$

**https://github.com/AlexKontorovich/PrimeNumberTheoremAnd/pull/211

Using integration by parts, we obtain

$$\frac{x}{\log x} + \int_{N_\varepsilon}^{x} \frac{1}{\log^2 t} dt = \int_{N_\varepsilon}^{x} \frac{1}{\log t} dt + \frac{N_\varepsilon}{\log N_\varepsilon} = \int_{2}^{x} \frac{1}{\log t} dt + \left(\frac{N_\varepsilon}{\log N_\varepsilon} - \int_{2}^{N_\varepsilon} \frac{1}{\log t} dt \right).$$

Hence

$$\pi(x) = \int_2^x \frac{1}{\log t} dt + \left(\frac{x f_\varepsilon(x)}{\log x} + \int_{N_\varepsilon}^{x} \frac{f_\varepsilon(t)}{\log^2 t} dt \right) + C_\varepsilon,$$

for some constant $C_\varepsilon \in \mathbb{R}$. Therefore, we have

$$\frac{\pi(x)}{\int_2^x \frac{1}{\log t} dt} - 1 = \left(\frac{x f_\varepsilon(x)}{\log x} + \int_{N_\varepsilon}^{x} \frac{f_\varepsilon(t)}{\log^2 t} dt \right) \Big/ \int_2^x \frac{1}{\log t} dt + \frac{C_\varepsilon}{\int_2^x \frac{1}{\log t} dt}.$$

Recall that $f_\varepsilon = o(\varepsilon)$, so for all $c > 0$, there exists $M_{c,\varepsilon}$ such that for all $x > M_{c,\varepsilon}$, we have

$$|f_\varepsilon(x)| \le c\varepsilon.$$

Therefore, for $x > M_{c,\varepsilon} > 2$, we have

$$\begin{aligned}
\frac{x f_\varepsilon(x)}{\log x} &\le \frac{c\varepsilon \cdot x}{\log x} \\
\left| \int_{N_\varepsilon}^{x} \frac{f_\varepsilon(t)}{\log^2 t} dt \right| &\le \int_{N_\varepsilon}^{M_{c,\varepsilon}} \left| \frac{f_\varepsilon(t)}{\log^2 t} \right| dt + \int_{M_{c,\varepsilon}}^{x} \left| \frac{f_\varepsilon(t)}{\log^2 t} \right| dt \\
&\le \int_{N_\varepsilon}^{M_{c,\varepsilon}} \frac{|f_\varepsilon(t)|}{\log^2 t} dt + c\varepsilon \int_{M_{c,\varepsilon}}^{x} \frac{1}{\log^2 t} dt \\
&= \int_{N_\varepsilon}^{M_{c,\varepsilon}} \frac{|f_\varepsilon(t)|}{\log^2 t} dt + c\varepsilon \left(\int_{M_{c,\varepsilon}}^{x} \frac{1}{\log t} dt + \frac{M_{c,\varepsilon}}{\log M_{c,\varepsilon}} - \frac{x}{\log x} \right).
\end{aligned}$$

Hence, for $x > M_{c,\varepsilon} > 2$, we have

$$\begin{aligned}
\left| \frac{x f_\varepsilon(x)}{\log x} + \int_{N_\varepsilon}^{x} \frac{f_\varepsilon(t)}{\log^2 t} dt \right| &\le \int_{N_\varepsilon}^{M_{c,\varepsilon}} \frac{|f_\varepsilon(t)|}{\log^2 t} dt + c\varepsilon \left(\int_{M_{c,\varepsilon}}^{x} \frac{1}{\log t} dt + \frac{M_{c,\varepsilon}}{\log M_{c,\varepsilon}} \right) \\
&= \int_{N_\varepsilon}^{M_{c,\varepsilon}} \frac{|f_\varepsilon(t)|}{\log^2 t} dt + c\varepsilon \left(\int_{2}^{x} \frac{1}{\log t} dt + \frac{M_{c,\varepsilon}}{\log M_{c,\varepsilon}} - \int_{M_{c,\varepsilon}}^{2} \frac{1}{\log t} dt \right).
\end{aligned}$$

Let $D_{c,\varepsilon}$ denote the value of

$$\int_{N_\varepsilon}^{M_{c,\varepsilon}} \frac{|f_\varepsilon(t)|}{\log^2 t} dt + c\varepsilon \frac{M_{c,\varepsilon}}{\log M_{c,\varepsilon}} - c\varepsilon \int_{M_{c,\varepsilon}}^{2} \frac{1}{\log t} dt,$$

we observe that

$$\begin{aligned}
\left| \frac{\pi(x)}{\int_2^x \frac{1}{\log t} dt} - 1 \right| &\le \left(c\varepsilon \int_2^x \frac{1}{\log t} dt + D_{c,\varepsilon} \right) \Big/ \int_2^x \frac{1}{\log t} dt + \frac{C_\varepsilon}{\int_2^x \frac{1}{\log t} dt} \\
&= c\varepsilon + \frac{D_{c,\varepsilon}}{\int_2^x \frac{1}{\log t} dt} + \frac{C_\varepsilon}{\int_2^x \frac{1}{\log t} dt}.
\end{aligned}$$

In particular, for $c = \frac{1}{2}$, there exists a constant D, such that for all $x > \max\left(M_{\frac{1}{2},\varepsilon}, N_\varepsilon\right)$, we have

$$\left|\frac{\pi(x)}{\int_2^x \frac{1}{\log t}dt} - 1\right| \leq \frac{\varepsilon}{2} + \frac{D}{\int_2^x \frac{1}{\log t}dt}.$$

Note that

$$\int_2^x \frac{1}{\log t}dt \geq \frac{(x-2)}{\log x},$$

for $x > e^s$, $s > 1$, we have the inequality:

$$\int_2^x \frac{1}{\log t}dt \geq \frac{e^s - 2}{s}.$$

Consequently, for sufficiently large $s > A_\varepsilon > 1$, we have

$$\frac{D}{\int_2^x \frac{1}{\log t}dt} \leq \frac{sD}{e^s - 2} \leq \frac{sD}{e^s} \leq \frac{\varepsilon}{2}.$$

Thus, for all $x > \max\left(M_{\frac{1}{2},\varepsilon}, N_\varepsilon, e^{A_\varepsilon}\right)$, we obtain

$$\left|\frac{\pi(x)}{\int_2^x \frac{1}{\log t}dt} - 1\right| \leq \varepsilon.$$

This completes the proof that $\frac{\pi(x)}{\int_2^x \frac{1}{\log t}dt} - 1$ is $o(1)$ for sufficiently large x. □

Theorem 4.1 allows us to express the asymptotic distribution law of prime numbers as follows:

Theorem 4.2 *The prime counting function $\pi(x)$ satisfies the asymptotic estimate*

$$\pi(x) = (1 + o(1))\frac{x}{\log x},$$

as $x \to \infty$.

The formal statement in Lean 4 is:

```
theorem pi_alt : ∃ c : ℝ → ℝ, c =o[atTop] (fun _ ↦ (1:ℝ)) ∧
    ∀ x : ℝ, Nat.primeCounting ⌊x⌋₊ = (1 + c x) * x / log x
```

Proof There exists a constant c_1 such that for sufficiently large x. we have

$$\int_2^{\sqrt{x}} \frac{1}{(\log t)^2}\,dt \leq \frac{1}{(\log 2)^2}(\sqrt{x} - 2) \leq c_1\sqrt{x}.$$

Similarly, there exists a constant c_2 such that for sufficiently large x, we have

$$\int_{\sqrt{x}}^x \frac{1}{(\log t)^2}\,dt \leq \frac{1}{(\log\sqrt{x})^2}(x - \sqrt{x}) \leq \frac{1}{4(\log x)^2}x \leq c_2\frac{x}{(\log x)^2}.$$

Since for sufficiently large x, $(\log x)^2 \leq \sqrt{x}$, $\sqrt{x} \leq \frac{x}{(\log x)^2}$, there exists a constant c such that

$$\int_2^x \frac{1}{(\log t)^2}\,dt \leq c_1\frac{x}{(\log x)^2} + c_2\frac{x}{(\log x)^2} \leq c\frac{x}{(\log x)^2}.$$

By integrating by parts, we obtain:

$$\int_2^x \frac{1}{\log t}\,dt = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2}\,dt. \tag{8}$$

Let

$$g(x) = \left(\int_2^x \frac{1}{(\log t)^2}\,dt - \frac{2}{\log 2}\right)\frac{\log x}{x}.$$

For sufficiently large x, we have

$$\begin{aligned}|g| &= \left|\int_2^x \frac{1}{(\log t)^2}\,dt - \frac{2}{\log 2}\right|\frac{\log x}{x}\\ &\le \left|\int_2^x \frac{1}{(\log t)^2}\,dt\right|\frac{\log x}{x} + \left|\frac{2}{\log 2}\right|\frac{\log x}{x}\\ &\le c\frac{x}{(\log x)^2}\frac{\log x}{x} + \left|\frac{2}{\log 2}\right|\frac{\log x}{x}\\ &= \frac{c}{\log x} + \left|\frac{2}{\log 2}\right|\frac{\log x}{x}.\end{aligned}$$

When $x \to \infty$, we have $\frac{1}{\log x} \to 0$, and $\frac{\log x}{x} \to 0$ as x increases much faster than $\log x$. Hence, we have

$$g(x) = o(1).$$

By equation (8), we obtain

$$\begin{aligned}\int_2^x \frac{1}{\log t}\,dt &= \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2}\,dt\\ &= \frac{x}{\log x} + \left(\left(\int_2^x \frac{1}{(\log t)^2}\,dt - \frac{2}{\log 2}\right)\frac{\log x}{x}\right)\frac{x}{\log x}\\ &= (1+g(x))\,\frac{x}{\log x}.\end{aligned}$$

By Theorem 4.1, there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f = o\,(1)$, and for sufficiently large x,

$$\pi(x) = (1+f(x))\int_2^x \frac{1}{\log t}dt.$$

For sufficiently large x, we can complete the proof by replacing $\int_2^x \frac{1}{\log t}\,dt$ by $(1+g(x))\,\frac{x}{\log x}$:

$$\begin{aligned}\pi(x) &= (1+f(x))\int_2^x \frac{1}{\log t}dt\\ &= (1+f(x))\,(1+g(x))\,\frac{x}{\log x}\\ &= (1+o(1))\frac{x}{\log x},\end{aligned}$$

as $f(x) + g(x) + f(x)g(x) = o(1)$. $\square$

By Theorem 4.2, for sufficiently large n, we have

$$d_n \leq n^{\pi(n)} \sim n^{\frac{n}{\log n}} = \left(e^{\log n}\right)^{\frac{n}{\log n}} = e^n.$$

Consequently, for sufficiently large n, we have

$$d_n^3 \leq (e^n)^3 = \left(e^3\right)^n \leq 21^n. \tag{9}$$

The upper bound (9) will be used to bound the sequence of $\{|a_n + b_n\zeta(3)|\}$.

5 Formal Proof of Irrationality of $\zeta(3)$ in Lean 4

We first consider an essential class of double integrals given in [22, Lemma 1 (b)]. Let r and s be natural numbers. We define

$$J_{rs} := \int_{(x,y)\in(0,1)^2} -\frac{\log(xy)}{1-xy} x^r y^s d\mu. \tag{10}$$

The formal definition of the function J_{rs} in Lean 4 is:

```
noncomputable abbrev J (r s : ℕ) : ℝ := ∫ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
  -(x.1 * x.2).log / (1 - x.1 * x.2) * x.1 ^ r * x.2 ^ s
```

Since $-\frac{\log(xy)}{1-xy}x^r y^s$ is not negative for any $x, y \in (0,1)$, we can define the lower Lebesgue integral `J_ENN_rs` in Lean 4:

```
noncomputable abbrev J_ENN (r s : ℕ) : ENNReal :=
   ∫⁻ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
   ENNReal.ofReal (- (x.1 * x.2).log / (1 - x.1 * x.2) * x.1 ^ r * x.2 ^ s)
```

For all $x, y \in (0,1)$, expanding $\frac{1}{1-xy}$ using the geometric series, we have

$$-\frac{\log(xy)}{1-xy} x^r y^s = \sum_{n\in\mathbb{N}} -\log(xy) x^{r+n} y^{s+n}. \tag{11}$$

Furthermore, we aim to express J_{rs} as $\sum_{n\in\mathbb{N}} \int_{(x,y)\in(0,1)^2} -\log(xy)x^{n+r}y^{n+s}$. In other words, in Lean 4, we seek to prove the following equality by interchanging the order of the lower Lebesgue integral "$\int^-$", summation sign $\sum_{n\in\mathbb{N}}$ and the type conversion `ENNReal.ofReal`:

```
J_ENN r s = Σ' (n : ℕ), ∫⁻ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
   ENNReal.ofReal (- (x.1 * x.2).log * x.1 ^ (n + r) * x.2 ^ (n + s))
```

This is another example of the advantages of using `lintegral` over `integral`: the lower Lebesgue integral commutes with infinite sums without needing to check integrability or summability conditions. To interchange the order of summation and type conversion, we must verify convergence using the following theorem:

```
theorem ENNReal.ofReal_tsum_of_nonneg {f : α → ℝ}
  (hf_nonneg : ∀ n, 0 ≤ f n) (hf : Summable f) :
  ENNReal.ofReal (Σ' n, f n) = Σ' n, ENNReal.ofReal (f n)
```

5.1 Linear Form of J_{rs}

We dedicate this section to demonstrating the following theorem.

Theorem 5.1 *The integral J_{rs} (10) can be expressed by the formula,*

$$J_{rs} = a_{rs}\zeta(3) + \frac{b_{rs}}{d^3_{\max\{r,s\}}}, \tag{12}$$

where a_{rs} and b_{rs} are integers, and $d_{\max\{r,s\}} = \mathrm{lcm}\{1, 2, \ldots, \max\{r, s\}\}$.

The formal statement of the theorem in Lean 4 is:

```
lemma linear_int_aux : ∃ a b : ℕ → ℕ → ℤ, ∀ r s : ℕ, J r s =
    b r s * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3 + a r s / (d (Finset.Icc 1 (Nat.max r
    s))) ^ 3
```

The connection between J_{rs} and $\zeta(3)$ has been recorded in the following lemmas given in [22, Lemma 1].

Lemma 5.2 *The integral J_{rr} can be written as:*

$$J_{rr} = 2\zeta(3) - 2\sum_{m=1}^{r} \frac{1}{m^3}. \tag{13}$$

we let $\sum_{m=1}^{r} \frac{1}{m^3} = 0$ for $r = 0$. In particular, we have $J_{00} = 2\zeta(3)$.

The formal statement in Lean 4 is:

```
theorem J_rr (r : ℕ) : J r r = 2 * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3 -
    2 * Σ m in Finset.Icc 1 r, 1 / (m : ℝ) ^ 3
```

Lemma 5.3 *For $r, s \in \mathbb{N}$, assume $r \neq s$, we have*

$$J_{rs} = \frac{\sum_{m=1}^{r} \frac{1}{m^2} - \sum_{m=1}^{s} \frac{1}{m^2}}{r - s}. \tag{14}$$

The formal statement in Lean 4 is:

```
theorem J_rs {r s : ℕ} (h : r ≠ s) : J r s =
    (Σ m in Icc 1 r, 1 / (m : ℝ) ^ 2 - Σ m in Icc 1 s, 1 / (m : ℝ) ^ 2) / (r - s)
```

We will prove Lemma 5.2 and Lemma 5.3 later. For now, assume they are true, according to equality (13) and equality (14), one can show that for all distinct $r, s \in \mathbb{N}$, there exist integers z_r and z_{rs} such that

$$J_{rr} = 2\zeta(3) - \frac{z_r}{d_r^3} \quad \text{and} \quad J_{rs} = \frac{z_{rs}}{d_r^3}. \tag{15}$$

where $d_r := \mathrm{lcm}\{1, 2, \ldots, r\}$.

By equalities (15), we immediately derive a unified form of J_{rs} (as given in equalities (12)), thereby concluding the proof of Theorem 5.1.

To establish the equalities (15), we begin by calculating a special family of lower Lebesgue integrals which:

$$\int^{-}_{(x,y)\in(0,1)^2} -\log(xy)x^{k+r}y^{k+s}d\mu,$$

for natural numbers k, r, s. That is, we formalize the following lemma:

Lemma 5.4

$$\int^{-}_{(x,y)\in(0,1)^2} -\log(xy)x^{k+r}y^{k+s}d\mu = \frac{1}{(k+r+1)^2(k+s+1)} + \frac{1}{(k+r+1)(k+s+1)^2}.$$

The formal statement in Lean 4 is:

```
lemma J_ENN_rs_eq_tsum_aux_intergal (r s k : ℕ) :
    ∫⁻ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, ENNReal.ofReal (- (x.1 * x.2).log *
    x.1 ^ (k + r) * x.2 ^ (k + s)) = ENNReal.ofReal (1 / ((k + r + 1) ^ 2 * (k + s +
    1)) + 1 / ((k + r + 1) * (k + s + 1) ^ 2))
```

Proof For lower Lebesgue integrals, the double integral is equivalent to the repeated integral without any assumption on integrability.

We regard x as a parameter and integrate y to get

$$\begin{aligned}
&\int^{-}_{(x,y)\in(0,1)^2} -\log(xy)x^{k+r}y^{k+s}d\mu \\
=&\int^{-}_{x\in(0,1)}\int^{-}_{y\in(0,1)} -\log(xy)x^{k+r}y^{k+s} \\
=&\int^{-}_{x\in(0,1)}\int^{-}_{y\in(0,1)} \left(-\log(x)x^{k+r}y^{k+s} + \left(-\log(y)x^{k+r}y^{k+s}\right)\right) \\
=&\int^{-}_{x\in(0,1)}\int^{-}_{y\in(0,1)} -\log(x)x^{k+r}y^{k+s} + \int^{-}_{x\in(0,1)}\int^{-}_{y\in(0,1)} -\log(y)x^{k+r}y^{k+s}.
\end{aligned}$$

We consider the following two special integrals, which can be directly calculated :

$$\int^{-}_{x\in(0,1)} -\log(x)x^n = \frac{1}{(n+1)^2}, \text{ and } \int^{-}_{x\in(0,1)} x^n = \frac{1}{n+1}.$$

The formal statements in Lean 4 are:

```
lemma ENN_log_pow_integral (n : ℕ) : ∫⁻ (x : ℝ) in Set.Ioo 0 1,
    ENNReal.ofReal (-x.log * x ^ n) = ENNReal.ofReal (1 / (n + 1) ^ 2)
```

and

```
lemma ENN_pow_integral (n : ℕ) : ∫⁻ (x : ℝ) in Set.Ioo 0 1, ENNReal.ofReal (x ^ n) =
    ENNReal.ofReal (1 / (n + 1))
```

Using the above two lemmas twice, one can get

$$\begin{aligned}&\int^{-}_{(x,y)\in(0,1)^2} -\log(xy)x^{k+r}y^{k+s}d\mu\\ =&\int^{-}_{x\in(0,1)} -\log(x)x^{k+r}\frac{1}{k+s+1}+\int^{-}_{x\in(0,1)} x^{k+r}\frac{1}{(k+s+1)^2}\\ =&\frac{1}{(k+r+1)^2(k+s+1)}+\frac{1}{(k+r+1)(k+s+1)^2}.\end{aligned}$$

This proves Lemma 5.4. □

Let us now proceed with the proofs of Lemma 5.2 and Lemma 5.3. By equation (11) and Lemma 5.4, we have

$$\begin{aligned}&\int^{-}_{(x,y)\in(0,1)^2} -\frac{\log(xy)}{1-xy}x^r y^s d\mu\\ =&\sum_{k=0}^{\infty}\left(\frac{1}{(k+r+1)^2(k+s+1)}+\frac{1}{(k+r+1)(k+s+1)^2}\right).\end{aligned} \tag{16}$$

The formal statement in Lean 4 is:

```
lemma J_ENN_rs_eq_tsum (r s : ℕ) : J_ENN r s = Σ' (k : ℕ), ENNReal.ofReal
    (1 / ((k + r + 1) ^ 2 * (k + s + 1)) + 1 / ((k + r + 1) * (k + s + 1) ^ 2))
```

In order to show Lemma 5.2 and Lemma 5.3, we discuss the convergence of the series

$$\sum_{k=0}^{\infty}\left(\frac{1}{(k+r+1)^2(k+s+1)}+\frac{1}{(k+r+1)(k+s+1)^2}\right),$$

in two cases: $r=s$ and $r\neq s$ corresponding to Lemma 5.2 and Lemma 5.3.

Proof [Proof of Lemma 5.2]

For $r=s$, the right side of equality (16) is equal to

$$2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}-2\sum_{m=1}^{r}\frac{1}{m^3}=2\zeta(3)-2\sum_{m=1}^{r}\frac{1}{m^3},$$

which implies

$$\int^{-}_{(x,y)\in(0,1)^2} -\frac{\log(xy)}{1-xy}x^r y^s d\mu = 2\zeta(3)-2\sum_{m=1}^{r}\frac{1}{m^3}. \tag{17}$$

The formal statement in Lean 4 is:

```
lemma J_ENN_rr (r : ℕ) : J_ENN r r = ENNReal.ofReal
    (2 * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3 - 2 * Σ m in Finset.Icc 1 r, 1 / (m : ℝ) ^
    3)
```

Listing 7: lintegral form of J_{rr}

As $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent, the above series is convergent. Therefore, we have proved Lemma 5.2. □

Proof [Proof of Lemma 5.3] For $r \neq s$, as the right side of equality (16) is symmetric with respect to r and s, without loss of generality, we can assume $r > s$. By rewriting the right side of equality (16), we have

$$\begin{aligned} &\sum_{k\in\mathbb{N}} \left(\frac{1}{(k+r+1)^2(k+s+1)} + \frac{1}{(k+r+1)(k+s+1)^2} \right) \\ =&\sum_{k\in\mathbb{N}} \frac{1}{r-s} \left(\frac{1}{(k+s+1)^2} - \frac{1}{(k+r+1)^2} \right) \\ =&\frac{1}{r-s} \sum_{k=s+1}^{r} \frac{1}{k^2}. \end{aligned}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we know that the above series converges. We have then proved Lemma 5.3. □

Now we proceed to prove Theorem 5.1.

Proof The expressions of $2\zeta(3)-2\sum_{k=1}^{r} \frac{1}{k^3}$ and $\frac{1}{r-s}(\sum_{k=1}^{r} \frac{1}{k^2} - \sum_{k=1}^{s} \frac{1}{k^2})$ are non-negative as they result from simplifying the non-negative series in (16).

Let d_r denote the least common multiple from 1 to r. As every number in $\{1,\dots,r\}$ is divisible by d_r, $\sum_{k=1}^{r} \frac{d_r^3}{k^3}, \sum_{k=1}^{s} \frac{d_r^2}{k^2}, \sum_{k=1}^{r} \frac{d_r^2}{k^2}$ and $\frac{d_r}{r-s}$ are integers. Hence, we have

$$J_{rr} d_r^3 = \left(2\zeta(3) - 2\sum_{k=1}^{r} \frac{1}{k^3} \right) d_r^3 = 2\zeta(3) d_r^3 - 2\sum_{k=1}^{r} \frac{d_r^3}{k^3} \in 2\zeta(3) d_r^3 - \mathbb{Z}$$

and for $r > s$,

$$J_{rs} d_r^3 = \left(\frac{1}{r-s} \left(\sum_{k=1}^{r} \frac{1}{k^2} - \sum_{k=1}^{s} \frac{1}{k^2} \right) \right) d_r^3 = \frac{d_r}{r-s} \left(\sum_{k=1}^{r} \frac{d_r^2}{k^2} - \sum_{k=1}^{s} \frac{d_r^2}{k^2} \right) \in \mathbb{Z}.$$

For the case of $r < s$, we can prove that $J_{rs} = J_{sr}$ by multiplying both the numerator and denominator of J_{rs} in equality (14) by -1.

Therefore, we immediately obtain equalities (15), and its formal form in Lean 4:

```
lemma J_rr_linear (r : ℕ) : ∃ a : ℤ, J r r =
    2 * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3 - a / (d (Finset.Icc 1 r)) ^ 3

lemma J_rs_linear {r s : ℕ} (h : r > s) : ∃ a : ℤ, J r s = a / d (Finset.Icc 1 r) ^ 3
```

By equalities (15), we immediately complete the proof of Theorem 5.1. □

5.2 Integer Sequence

In order to construct an integer sequence of the form $\{a_n + b_n\zeta(3)\}$, we introduce the following integral

$$\mathfrak{J}_n := \int_{(x,y)\in(0,1)^2} -P_n(x)P_n(y)\frac{\log(xy)}{1-xy} d\mu.$$

The formal statement in Lean 4 is:

```
noncomputable abbrev JJ (n : ℕ) : ℝ :=
    ∫ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
    (-(x.1 * x.2).log / (1 - x.1 * x.2) * (shiftedLegendre n).eval x.1 *
    (shiftedLegendre n).eval x.2)
```

where `eval` denotes the evaluation of the shifted Legendre polynomial at a single point.

Since both $P_n(x)$ and $P_n(y)$ are polynomials with integer coefficients (4), we can express $P_n(x)$ and $P_n(y)$ as finite sums $\sum_{k=0}^{n} a_n x^k$ and $\sum_{k=0}^{n} a_n y^k$, respectively, where $a_i \in \mathbb{Z}$. By exchanging the order of the summation and the integral, we obtain

$$\mathfrak{J}_n = \sum_{k=0}^{n}\sum_{l=0}^{n} -a_k a_l \int_{(x,y)\in(0,1)^2} x^k y^l \frac{\log(xy)}{1-xy} d\mu = \sum_{k=0}^{n}\sum_{l=0}^{n} a_k a_l J_{kl}$$

By equalities (12) and $d_{\max\{k,l\}} | d_n$ for $k,l \le n$, one obtains

$$\mathfrak{J}_n := \sum_{k=0}^{n}\sum_{l=0}^{n} a_k a_l \left(a_{kl}\zeta(3) + \frac{b_{kl}}{d^3_{\max\{k,l\}}} \right) \in \mathbb{Z}\zeta(3) + \frac{\mathbb{Z}}{d_n^3}.$$

Hence, we obtain a sequence

$$\{\mathfrak{J}_n \cdot d_n^3 = a_n + b_n\zeta(3)\},$$

where $a_n, b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. This sequence has been used in [22, Theorem 2] for showing the irrationality of $\zeta(3)$.

The function $\mathfrak{J}_n \cdot d_n^3$ is formally defined in Lean 4 as follows:

```
noncomputable abbrev fun1 (n : ℕ) : ℝ := (d (Finset.Icc 1 n)) ^ 3 * JJ n
```

We have successfully constructed the sequence $\{a_n + b_n\zeta(3)\}$. The formal statement of the theorem in Lean 4 is:

```
theorem linear_int (n : ℕ) : ∃ a b : ℕ → ℤ, fun1 n = a n + b n *
    (d (Finset.Icc 1 n) : ℤ) ^ 3  * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3
```

5.3 From Double Integral to Triple Integral

Given the sequence $\{a_n + b_n\zeta(3)\}$, we need to prove two things: first, that the sequence is non-zero, and second, that it tends to 0 as $n \to \infty$.

First, we prove that the sequence is non-zero by demonstrating its positivity. Since d_n^3 is always positive, it suffices to show that $\mathfrak{J}_n$ is positive for all natural numbers n.

However, determining the sign of the shifted Legendre polynomial on the interval $(0,1)$ is not straightforward, so we prove the result by relating it to the triple integral.

$$\mathfrak{J}'_n := \int_{(x,y,z)\in(0,1)^3} \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-yz)x} \right)^n \frac{1}{1-(1-yz)x} d\mu.$$

It is formally defined in Lean 4 as follows:

```
noncomputable abbrev JJ' (n : ℕ) : ℝ :=
    ∫ (x : ℝ × ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
    (x.2.1 * (1 - x.2.1) * x.2.2 * (1 - x.2.2) * x.1 * (1 - x.1) /
    (1 - (1 - x.2.1 * x.2.2) * x.1)) ^ n / (1 - (1 - x.2.1 * x.2.2) * x.1)
```

Next, we demonstrate the following theorem:

Theorem 5.5 *For any $n \in \mathbb{N}$,*

$$\mathfrak{J}_n = \mathfrak{J}'_n.$$

The formal statement in Lean 4 is:

```
theorem JJ_eq_form (n : ℕ) : JJ n = JJ' n
```

We prove the equality by the following calculation

$$\begin{aligned}
\mathfrak{J}_n &= \int_{(x,y)\in(0,1)^2} P_n(x)P_n(y)\left(\int_0^1 \frac{1}{1-(1-xy)z}\,dz\right) d\mu && (18)\\
&= \int_0^1 \left(\int_{(x,y)\in(0,1)^2} P_n(x)P_n(y)\frac{1}{1-(1-xy)z} d\mu\right) dz && (19)\\
&= \int_0^1 \left(\int_{(x,y)\in(0,1)^2} \frac{P_n(x)(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}} d\mu\right) dz && (20)\\
&= \int_{(x,y)\in(0,1)^2} \left(\int_0^1 \frac{P_n(x)(1-z)^n(1-y)^n}{1-(1-xy)z}\,dz\right) d\mu && (21)\\
&= \int_0^1 (1-z)^n \left(\int_{(x,y)\in(0,1)^2} \frac{(xyz(1-x)(1-y))^n}{(1-(1-xy)z)^{n+1}} d\mu\right) dz && (22)\\
&= \mathfrak{J}'_n && (23)
\end{aligned}$$

In the course of the above proof, it is necessary to establish the integrability of the following three functions $f_1(x,y,z)$, $f_2(x,y,z)$ and $f_3(x,y,z)$, whose integrability is also required in the proofs of equality (19), equality (21) and equality (22), respectively:

$$\begin{aligned}
f_1(x,y,z) =& P_n(x)P_n(y)\frac{1}{1-(1-xy)z}, && (24)\\
f_2(x,y,z) =& P_n(x)(xyz)^n(1-y)^n\frac{1}{(1-(1-xy)z)^{n+1}},\\
=& P_n(x)\left(\frac{xyz(1-y)}{1-(1-xy)z}\right)^n \frac{1}{1-(1-xy)z}, && (25)\\
f_3(x,y,z) =& P_n(x)(1-z)^n(1-y)^n\frac{1}{1-(1-xy)z}. && (26)
\end{aligned}$$

By equation (4) and absolute value inequality, for any $n \in \mathbb{N}$ and $x \in (0,1)$,

$$P_n(x) \le |P_n(x)| \le \sum_{k=0}^{n}\left|(-1)^k\binom{n}{k}\binom{n+k}{n}x^k\right| \le \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}$$

Let $C_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{n}$, then for any $n \in \mathbb{N}$ and $(x,y,z) \in (0,1)^3$, we have:

$$P_n(x)P_n(y) \leqslant C_n^2,$$
$$P_n(x)\left(\frac{xyz(1-y)}{1-(1-xy)z}\right)^n \leqslant C_n\left(\frac{xyz(1-y)}{1-(1-xy)z}\right)^n \leqslant C_n,$$
$$P_n(x)(1-z)^n(1-y)^n \leqslant C_n.$$

It is necessary for the following inequalities to hold:

$$\frac{xyz(1-y)}{1-(1-xy)z} = \frac{xyz(1-y)}{1-z+xyz} \leqslant \frac{xyz(1-y)}{xyz} \leqslant 1-y \leqslant 1.$$

Since a bounded factor does not affect integrability, to prove that f_1, f_2 and f_3 have finite integrals over the unit cube $(0,1)^3$, it suffices to show that the function $\frac{1}{1-(1-xy)z}$ has a finite integral over the unit cube $(0,1)^3$. This will be proven in section 5.3.2.

5.3.1 Proof of the Equality (18)

Lemma 5.6 *For $0 < a < 1$, one has*

$$\int_0^1 \frac{1}{1-(1-a)z}dz = -\frac{\ln a}{1-a}, \tag{27}$$

which in Lean 4 is:

```
lemma integral1 {a : ℝ} (ha : 0 < a) (ha1 : a < 1) :
    ∫ (z : ℝ) in (0)..1, 1 / (1 - (1 - a) * z) = - a.log / (1 - a)
```

Proof By substituting $(1-a)z = u$ in the integral (27), with $du = (1-a)dz$, we obtain

$$\int_0^1 \frac{1}{1-(1-a)z}dz = \frac{1}{1-a}\int_0^{1-a} \frac{1}{1-u}du = \frac{1}{1-a}[-\ln(1-u)]_0^{1-a}$$
$$= -\frac{1}{1-a}[\ln a - \ln 1] = -\frac{\ln a}{1-a}.$$

□

Since $0 < x, y < 1$, it follows that $0 < xy < 1$. To prove equality (18), we set $a = xy$ in the integral (27).

5.3.2 Proof of the Equality (19)

To prove equality (19), we rely on the `MeasureTheory.integral_integral_swap` theorem from `Mathlib`, which enables the interchange of the order of integration.

```
theorem MeasureTheory.integral_integral_swap ⦃f : α → β → E⦄
    (hf : Integrable (uncurry f) (μ.prod ν)) :
    ∫ x, ∫ y, f x y ∂ν ∂μ = ∫ y, ∫ x, f x y ∂μ ∂ν
```

Here, `uncurry f` denotes the transformation of f into a function of type $\alpha \times \beta \to E$, and μ`.prod` ν refers to the product measure.

To apply this theorem, we need to demonstrate that $f_1(x, y, z)$ (24) is integrable on $(0, 1)^3$. It is equivalent to the previously mentioned proof of the integrability of $\frac{1}{1-(1-xy)z}$. A more general version of the result is as follows:

Lemma 5.7 *For* $r, s \in \mathbb{N}$,

$$\int^{-}_{(x,y)\in(0,1)^2} -\frac{\log(xy)}{1-xy} x^r y^s d\mu = \int^{-}_{(x,y,z)\in(0,1)^3} \frac{1}{1-(1-xy)z} x^r y^s d\mu. \tag{28}$$

The formal statement in Lean 4 is:

```
lemma JENN_eq_triple (r s : ℕ) : J_ENN r s =
    ∫⁻ (x : ℝ × ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
    ENNReal.ofReal (1 / (1 - (1 - x.2.1 * x.2.2) * x.1) * x.2.1 ^ r * x.2.2 ^ s)
```

Before proving Lemma 5.7, we present a result similar to Lemma 5.6 as follows:

Lemma 5.8 *For* $r, s \in \mathbb{N}$,

$$\int^{-}_{z\in(0,1)} \frac{1}{1-(1-xy)z} x^r y^s = \frac{-\log(xy)}{1-xy} x^r y^s. \tag{29}$$

The formal statement in Lean 4 is:

```
lemma JENN_eq_triple_aux (x : ℝ × ℝ) (hx : x ∈ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1) :
    ∫⁻ (w : ℝ) in Set.Ioo 0 1, ENNReal.ofReal (1 / (1 - (1 - x.1 * x.2) * w) * x.1 ^ r
    * x.2 ^ s) =
    ENNReal.ofReal (-Real.log (x.1 * x.2) / (1 - x.1 * x.2) * x.1 ^ r * x.2 ^ s)
```

Proof Since `ENNReal.ofReal (x^r*y^s)` is independent of z, we can factor it out of the integral on the left side of the equality (29). By comparing both sides of the equality (29), we now aim to prove the following equation:

$$\int^{-}_{z\in(0,1)} \frac{1}{1-(1-xy)z} = \frac{-\log(xy)}{1-xy}. \tag{30}$$

The formal statement in Lean 4 is:

```
lemma JENN_eq_triple_aux' (x : ℝ × ℝ)
    (hx : x ∈ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1) : ∫⁻ (w : ℝ) in Set.Ioo 0 1,
    ENNReal.ofReal (1 / (1 - (1 - x.1 * x.2) * w)) =
    ENNReal.ofReal (-Real.log (1 - (1 - x.1 * x.2)) / (1 - x.1 * x.2))
```

By applying the theorem (listing 3) in section 2 in reverse, we can take the `ENNReal.ofReal` out of "$\int^{-}$" symbolic. However, to apply the theorem (listing 3), we need to check the integrability condition first, i.e., we need to prove the function $f(z) = \frac{1}{1-(1-xy)z}$ is integrable on $(0, 1)$, where $(x, y) \in (0, 1)^2$. The antiderivative of $f(z)$ is

$$g(z) = \frac{-\log(1-(1-xy)z)}{1-xy}.$$

Since g is continuous and differentiable over the $[0,1]$, f must be integrable on $(0,1)$.

Furthermore, we also need to prove that f is greater than or equal to 0 almost everywhere which is the condition `f_nn` in theorem (listing 3). By theorem (listing 2), it suffices to verify that $f(z)$ is nonnegative at every point $(0,1)$, which is straightforward.

Then, by setting $a = xy$ in Lemma 5.6, the equality (30) can be proved. □

We now proceed to prove Lemma 5.7:

Proof We transform the triple integral on the right-hand side of the equation into a single integral with respect to z, followed by a double integral over x and y. We then compare both sides of the equation. For any $0 < x, y < 1$, Lemma 5.8 implies Lemma 5.7. □

By setting $r = s = 0$ in (29) and combining it with the previously proven equality (17), we can derive:

$$\int^{-}_{(x,y,z)\in(0,1)^3} \frac{1}{1-(1-yz)x} d\mu = 2 \cdot \sum_{n\in\mathbb{N}} \frac{1}{(n+1)^3}. \tag{31}$$

The formal statement in Lean 4 is:

```
∫⁻ (x : ℝ × ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, ENNReal.ofReal (1 /
    (1 - (1 - x.2.1 * x.2.2) * x.1)) =
ENNReal.ofReal (2 * Σ' (n : ℕ), 1 / ((n : ℝ) + 1) ^ 3)
```

At this point, since $2 \cdot \sum_{n\in\mathbb{N}} \frac{1}{(n+1)^3}$ is a real number, it naturally follows that $\frac{1}{1-(1-xy)z}$ has a finite integral on $(0,1)^3$. Therefore, we can obtain the integrability of f_1, f_2 and f_3 on $(0,1)^3$, where the integrability of f_1 proves the equality (19).

5.3.3 Proof of the Equality (20)

To prove equality 20, it suffices to show that the following two functions are equal for any $0 < z < 1$.

Lemma 5.9 *For $0 < z < 1$, one has*

$$\int_{(x,y)\in(0,1)^2} P_n(x)P_n(y)\frac{1}{1-(1-xy)z} d\mu = \int_{(x,y)\in(0,1)^2} \frac{P_n(x)(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}} d\mu \tag{32}$$

The formal statement in Lean 4 is:

```
lemma double_integral_eq1 (n : ℕ) (z : ℝ) (hz : z ∈ Set.Ioo 0 1) :
    ∫ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, eval x.1 (shiftedLegendre n) * eval
    x.2 (shiftedLegendre n) * (1 / (1 - (1 - x.1 * x.2) * z)) =
    ∫ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, eval x.1 (shiftedLegendre n) * (x.1
    * x.2 * z) ^ n * (1 - x.2) ^ n / (1 - (1 - x.1 * x.2) * z) ^ (n + 1)
```

Proof For $0 < z < 1$, let

$$f(x,y) = \frac{P_n(x)P_n(y)}{1-(1-xy)z},$$

and

$$g(x,y) = \frac{P_n(x)(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}}.$$

Since $0 < z < 1$, the functions f and g have no singular points within the unit square $[0,1]^2$, and both of them are continuously differentiable. Moreover, f and g are integrable on the unit square as they are continuous functions on the compact set $[0,1]^2$. Hence, to prove equality (32), one can convert the two integrals $\int_{(x,y)\in(0,1)^2} f(x,y)$ and $\int_{(x,y)\in(0,1)^2} g(x,y)$ into repeated integrals and compare the integrands for x. In other words, it suffices to prove the following equality:

$$\int_0^1 \frac{P_n(y)}{1-(1-xy)z}\,dy = \int_0^1 \frac{(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}}\,dy$$

where $x, z \in (0,1)$. This follows from Lemma 3.2. □

5.3.4 Proof of the Equality (21)

We begin by swapping the order of the double integral for x and y and the single integral for z. This step follows the pattern of equality (18) and requires the integrability of $f_2(x,y,z)$ (25), which has been proven in section 5.3.2.

To start, we prove the following lemma:

Lemma 5.10 *For $0 < x, y < 1$, one has*

$$\int_0^1 \frac{P_n(x)(xyz)^n(1-y)^n}{(1-(1-xy)z)^{n+1}}\,dz = \int_0^1 \frac{P_n(x)(1-z)^n(1-y)^n}{1-(1-xy)z}\,dz \tag{33}$$

The formal statement in Lean 4 is:

```
lemma double_integral_eq2 (n : ℕ) (x : ℝ × ℝ) (hx : x ∈ Set.Ioo 0 1 ×ˢ Set.Ioo 0 1)
    : ∫ (z : ℝ) in Set.Ioo 0 1, eval x.1 (shiftedLegendre n) * (x.1 * x.2 * z) ^ n *
    (1 - x.2) ^ n / (1 - (1 - x.1 * x.2) * z) ^ (n + 1) = ∫ (z : ℝ) in Set.Ioo 0 1,
    eval x.1 (shiftedLegendre n) * (1 - z) ^ n * (1 - x.2) ^ n / (1 - (1 - x.1 * x.2) *
     z)
```

Proof After removing the factors $P_n(x)(1-y)^n$ on both sides of equality (33) that are unrelated to the integral variable z, it is sufficient to show

$$\int_0^1 \frac{(xyz)^n}{(1-(1-xy)z)^{n+1}}\,dz = \int_0^1 \frac{(1-z)^n}{1-(1-xy)z}\,dz \tag{34}$$

If a function f has continuous derivative f' on $[a,b]$, and g is continuous, then by substituting $u = f(x)$, we have the equality:

$$\int_a^b (g\circ f)(x)f'(x)\,dx = \int_{f(a)}^{f(b)} g(u)\,du, \tag{35}$$

which follows from the theorem (listing 4) in section 2.

Let $f(z) = \frac{1-z}{1-(1-xy)z}$, with $f(0) = 1$ and $f(1) = 0$. Then for the right side of equality (34), it takes the following form:

$$\int_0^1 \frac{(1-z)^n}{1-(1-xy)z}\,dz = \int_0^1 \frac{(1-w)^n}{1-(1-xy)w}\,dw = \int_{f(1)}^{f(0)} \frac{(1-w)^n}{1-(1-xy)w}\,dw. \tag{36}$$

By substituting $w = f(z)$ into equality (36) and use equality (35), followed by straightforward calculations, we have the equality (34):

$$\begin{aligned}\int_{f(1)}^{f(0)} \frac{(1-w)^n}{1-(1-xy)w}\,dw &= -\int_{f(0)}^{f(1)} \frac{(1-w)^n}{1-(1-xy)w}\,dw \\ &= -\int_0^1 \frac{(1-f(z))^n}{1-(1-xy)f(z)} f'(z)\,dz \\ &= -\int_0^1 (\frac{xyz}{1-(1-xy)z})^n \frac{1-(1-xy)z}{xy} \frac{-xy}{(1-(1-xy)z)^2}\,dz \\ &= \int_0^1 \frac{(xyz)^n}{(1-(1-xy)z)^{n+1}}\,dz.\end{aligned}$$

□

5.3.5 Proof of the Equality (22)

To show the equality in (22), we first demonstrate the following:

$$\int_{(x,y)\in(0,1)^2} \left(\int_0^1 \frac{P_n(x)(1-z)^n(1-y)^n}{1-(1-xy)z}\,dz\right) d\mu$$
$$= \int_0^1 (1-z)^n \left(\int_{(x,y)\in(0,1)^2} \frac{P_n(x)(1-y)^n}{1-(1-xy)z} d\mu\right) dz.$$

It is sufficient to exchange the order of integration and use the integrability of $f_3(x,y,z)$ (as shown in (26)), which was established in section 5.3.2.

We then proceed to formalize the following lemma:

Lemma 5.11 *For* $0 < z < 1$, *one has*

$$\int_{(x,y)\in(0,1)^2} \frac{P_n(x)(1-y)^n}{1-(1-xy)z} d\mu = \int_{(x,y)\in(0,1)^2} \frac{(xyz(1-x)(1-y))^n}{(1-(1-xy)z)^{n+1}} d\mu.$$

The formal statement in Lean 4 is:

```
lemma double_integral_eq3 (n : ℕ) (z : ℝ) (hz : z ∈ Set.Ioo 0 1) : ∫ (x : ℝ × ℝ) in
    Set.Ioo 0 1 ×ˢ Set.Ioo 0 1,
    (x.1 * x.2 * z * (1 - x.1) * (1 - x.2)) ^ n / (1 - (1 - x.1 * x.2) * z) ^ (n + 1)
    = ∫ (x : ℝ × ℝ) in Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, eval x.1 (shiftedLegendre n) * (1
    - x.2) ^ n / (1 - (1 - x.1 * x.2) * z)
```

Proof The proof is similar to that of equality (20), achieved by interchanging the role of x and y. □

5.3.6 Proof of the Equality (23)

First, we move $(1-z)^2$ inside the integral "$\int_{(x,y)\in(0,1)^2}$". Next, we need to prove that a repeated integral is equal to a triple integral, as stated in Theorem (listing 5. This requires proving the following lemma regarding integrability:

Lemma 5.12 *For any $n \in \mathbb{N}$, the function*

$$\frac{(xyz(1-x)(1-y)(1-z))^n}{(1-(1-xy)z)^{n+1}} = \left(\frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z}\right)^n \frac{1}{1-(1-xy)z}$$

is integrable on $(0,1)^3$.

The formal statement in Lean 4 is:

```
lemma integrableOn_JJ' (n : ℕ) : MeasureTheory.Integrable
    (fun (x : ℝ × ℝ × ℝ) ↦ (x.2.1 * (1 - x.2.1) * x.2.2 * (1 - x.2.2) * x.1 * (1 -
    x.1) / (1 - (1 - x.2.1 * x.2.2) * x.1)) ^ n /
    (1 - (1 - x.2.1 * x.2.2) * x.1)) (MeasureTheory.volume.restrict (Set.Ioo 0 1 ×ˢ
    Set.Ioo 0 1 ×ˢ Set.Ioo 0 1))
```

We define a $[0,\infty]$ valued function $\mathcal{J}_n$:

$$\mathcal{J}_n := \int^{-}_{(x,y,z)\in(0,1)^3} \left(\frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z}\right)^n \frac{1}{1-(1-xy)z} d\mu.$$

The formal definition in Lean 4 is:

```
noncomputable abbrev JJENN (n : ℕ) : ENNReal := ∫⁻ (x : ℝ × ℝ × ℝ) in Set.Ioo 0 1 ×ˢ
     Set.Ioo 0 1 ×ˢ Set.Ioo 0 1, ENNReal.ofReal ((x.2.1 * (1 - x.2.1) * x.2.2 * (1 -
    x.2.2) * x.1 * (1 - x.1) / (1 - (1 - x.2.1 * x.2.2) * x.1)) ^ n / (1 - (1 - x.2.1 *
     x.2.2) * x.1))
```

Proof [Proof of Lemma 5.12] We need to prove that $\mathcal{J}_n$ has a finite integral. It suffices to show that

$$\mathcal{J}_n \leq 2\left(\frac{1}{24}\right)^n \sum_{n\in\mathbb{N}} \frac{1}{(n+1)^3} = 2\left(\frac{1}{24}\right)^n \zeta(3). \tag{37}$$

The formal statement in Lean 4 is:

```
lemma JJENN_upper (n : ℕ) : JJENN n ≤ ENNReal.ofReal
    (2 * (1 / 24) ^ n * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3)
```

To prove inequality (37), we first demonstrate that that for all $x, y, z \in (0,1)$, the following holds:

$$\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} < \frac{1}{24}. \tag{38}$$

The formal statement in Lean 4 is:

```
lemma bound' (x y z : ℝ) (x0 : 0 < x) (x1 : x < 1) (y0 : 0 < y) (y1 : y < 1) (z0 : 0
    < z) (z1 : z < 1) :
    x * (1 - x) * y * (1 - y) * z * (1 - z) / (1 - (1 - x * y) * z) < (1 / 24 : ℝ)
```

From the inequality

$$1-(1-xy)z = 1-z+xyz \geqslant 2\sqrt{1-z}\sqrt{xyz},$$

we can deduce that for $x, y, z \in (0,1)$, the following holds:

$$\frac{x(1-x)y(1-y)z(1-z)}{(1-(1-xy)z)} \leqslant \frac{x(1-x)y(1-y)z(1-z)}{2\sqrt{1-z}\sqrt{xyz}} = \frac{\sqrt{x}(1-x)\sqrt{y}(1-y)\sqrt{z}\sqrt{1-z}}{2}.$$

For $z \in (0,1)$, the maximum value of $\sqrt{z}\sqrt{1-z}$ is attained at $z = \frac{1}{2}$. For $y \in (0,1)$, we have

$$y(1-y)^2 - \frac{4}{27} = (y - \frac{4}{3})(y - \frac{1}{3})^2 \leqslant 0.$$

Hence, for $y \in (0,1)$, we have

$$\sqrt{y}(1-y) = \sqrt{y(1-y)^2} \leqslant \sqrt{\frac{4}{27}} \leqslant \sqrt{\frac{4}{25}} = \frac{2}{5}.$$

Finally, inequality (38) is proven

$$\frac{x(1-x)y(1-y)z(1-z)}{(1-(1-xy)z)} \leqslant \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{25} < \frac{1}{24}$$

Therefore, inequality (37) holds:

$$\begin{aligned}\mathcal{J}_n &= \int^{-}_{(x,y,z)\in(0,1)^3} \left(\frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z}\right)^n \frac{1}{1-(1-xy)z} d\mu \\ &\leq \int^{-}_{(x,y,z)\in(0,1)^3} \left(\frac{1}{24}\right)^n \frac{1}{1-(1-xy)z} d\mu \\ &= \left(\frac{1}{24}\right)^n \int^{-}_{(x,y,z)\in(0,1)^3} \frac{1}{1-(1-xy)z} d\mu \\ &= \left(\frac{1}{24}\right)^n 2\zeta(3).\end{aligned}$$

Consequently, we formalize `JJENN_upper` by using equation (28) and equation (31) with $r = 0$. □

5.4 Positive Sequences Converging to Zero

First, we establish that $\mathfrak{J}_n$ is always positive.

Theorem 5.13 *For any $n \in \mathbb{N}$,*

$$\mathfrak{J}_n > 0.$$

The formal statement in Lean 4 is:

```
theorem JJ_pos (n : ℕ) : 0 < JJ n
```

Proof By Theorem 5.5, it suffices to prove that $\mathfrak{J}'_n$ is always positive.

We apply the theorem (listing 6) from `Mathlib` in our proof. It suffices to show that the measure of the support of the function is larger than 0.

In this case, the measure μ is the measure on $\mathbb{R}$ restricted to $(0,1)^3$. Let

$$f(x,y,z) = \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}\right)^n \frac{1}{1-(1-xy)z}.$$

We can verify that for any $(x,y,z) \in (0,1)^3$, we have

$$f(x,y,z) > 0.$$

Therefore, $(0,1)^3 \subset \operatorname{supp} f$, and the measure of $\operatorname{supp} f$ must be greater than that of $(0,1)^3$, which is positive.

The integrability condition `hfi` is provided by Lemma 5.12. As for the condition `hf` in theorem (listing 6), we need to show $f(x,y,z)$ is almost everywhere greater than or equal to 0. Similar to the proof of Lemma 5.8, this can be established by using Theorem (listing 2), Lemma 5.12, and the fact that f is positive on $(0,1)^3$. □

By theorem (listing 3), we can move `ENNReal.ofReal` from inside to outside the integral. We obtain the following theorem:

Theorem 5.14 *For all $n \in \mathbb{N}$,*

$$\mathfrak{J}_n \leq 2 \cdot \left(\frac{1}{24}\right)^n \sum_{n\in\mathbb{N}} \frac{1}{(n+1)^3} = 2 \cdot \left(\frac{1}{24}\right)^n \zeta(3).$$

The formal statement in Lean 4 is:

```
Theorem JJ_upper (n : ℕ) :
    JJ n ≤ 2 * (1 / 24) ^ n * Σ' n : ℕ, 1 / ((n : ℝ) + 1) ^ 3
```

Next, we demonstrate that the sequence converges to zero.

Theorem 5.15 *The sequence $\{a_n + b_n\zeta(3)\}$ tends to 0 when $n \to \infty$.*

The formal statement in Lean 4 is:

```
theorem fun1_tendsto_zero : Filter.Tendsto (fun n ↦ ENNReal.ofReal (fun1 n))
    Filter.atTop (nhds 0)
```

Proof According to Theorem 5.14, we have

$$\mathfrak{J}_n \cdot d_n^3 \leq 2\left(\frac{1}{24}\right)^n d_n^3\zeta(3).$$

Since $2\zeta(3)$ is constant, we can analyze the asymptotic behavior of d_n^3 for sufficiently large n. Using Theorem 4.2 and equation (9), for sufficiently large n, we have $d_n^3 \leq 21^n$.

Therefore, we can conclude that for sufficiently large n, the following holds:

$$\mathfrak{J}_n \cdot d_n^3 \leq \left(\frac{21}{24}\right)^n 2\zeta(3).$$

When $n \to \infty$, $\left(\frac{21}{24}\right)^n \to 0$. Hence, the sequence $\mathfrak{J}_n \cdot d_n^3$ tends to 0, which implies that the sequence $\{a_n + b_n\zeta(3)\}$ converges to 0 as $n \to \infty$. □

5.5 Irrationality of $\zeta(3)$

Finally, we establish the irrationality of $\zeta(3)$:

Theorem 5.16 $\zeta(3)$ *is irrational.*

The formal statement in Lean 4 is:

```
theorem zeta_3_irratoinal : ¬ ∃ r : ℚ, r = riemannZeta 3
```

Proof Assume, for the sake of contradiction, that $\zeta(3) = \frac{p}{q}$, where $\gcd(p, q) = 1$ and $p, q > 0$. Then by Theorem 5.15, we have $qa_n + pb_n \to 0$ as $n \to \infty$, since q is a constant. Theorem 5.13 states that $a_n + b_n\zeta(3) > 0$ and $q > 0$, which implies that $qa_n + pb_n > 0$. Furthermore, since a_n, b_n are integers, $qa_n + pb_n \in \mathbb{Z}$. Therefore, we have $qa_n + pb_n \geqslant 1$ for all $n \in \mathbb{N}$. This leads to a contradiction, thereby implying that $\zeta(3)$ is irrational. □

6 Conclusion

Our work delivers the first complete formalization of the irrationality of $\zeta(3)$ in Lean 4. To transform Beukers' informal proof into a fully formal proof in Lean, we carefully adjusted and refined his arguments to meet the strict requirements of formalization. This process allowed us to bridge the gaps in the original proof and produce a complete, machine-verified demonstration within Lean's framework. We formally define the shifted Legendre polynomials and prove their fundamental properties. Additionally, we provide the first formal proof in Lean 4 of a version of the Prime Number Theorem with an error term which is stronger than what had previously been formalized. This achievement significantly advances Lean's analytical capabilities in number theory.

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