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Smith normal forms of bivariate polynomial matrices

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ABSTRACT

In 1978, Frost and Storey asserted that any bivariate polynomial matrix is equivalent to its Smith normal form if and only if the reduced minors of each order of the matrix generate the unit ideal. In this paper, we first demonstrate by constructing an example that for any given positive integer s with $s \geq 2$, there exists a square bivariate polynomial matrix $\mathbf{M}$ over $K[x, y]$ with the degree of $\det(\mathbf{M})$ in y being equal to s, for which the condition that the reduced minors of each order of $\mathbf{M}$ generate the unit ideal in $K[x, y]$ is not a sufficient condition for $\mathbf{M}$ to be equivalent to its Smith normal form. Subsequently, we prove that for any square bivariate polynomial matrix $\mathbf{M}$ where the degree of $\det(\mathbf{M})$ in y is at most 1, Frost and Storey's assertion holds. Using the Quillen-Suslin

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Quillen-Suslin theorem

Theorem, we further extend our consideration of $\mathbf{M}$ to rank-deficient and non-square cases.

1. Introduction

The study of the equivalence between polynomial matrices and their Smith normal forms (see Definition 2.1) has long been a foundational topic in both algebra and applications, bridging algebraic theory with practical engineering problems (see [1,2] and the references therein). In recent decades, the Smith normal form has also found broad applications across algebra, combinatorics, and topology, particularly through its connections to symmetric functions, partition theory, and the study of 3-manifolds (see, e.g., [3–8]). For univariate polynomial matrices, the theory is well-established: since the ring of univariate polynomials over a field is a principal ideal domain (PID), every such matrix is equivalent to its Smith normal form [9,10]. However, for multivariate polynomial matrices (in two or more variables), the equivalence problem remains open due to the non-PID structure of multivariate polynomial rings, leading to rich and challenging research directions.

Early investigations concentrated on bivariate polynomial matrices. In 1978, Frost and Storey [11] asserted the following: a necessary and sufficient condition for any bivariate polynomial matrix to be equivalent to its Smith normal forms is that the reduced minors (see Definition 2.4) of each order of the matrix generate the unit ideal. But later, they constructed a counterexample in [12] to show that the above assertion does not hold. It has been proven that the condition that the reduced minors of each order generate the unit ideal is only a necessary condition for the equivalence of a multivariate polynomial matrix to its Smith normal form. Thus, researchers have begun to explore under what structural conditions of the matrix this condition serves as a necessary and sufficient condition.

To date, researchers have focused primarily on two major classes of multivariate polynomial matrices. Let $\mathbf{M} \in K[x_1, x_2, \ldots, x_n]^{l\times l}$ with $\det(\mathbf{M}) = (x_1 - f(x_2, \ldots, x_n))^t$, where K is a field, $x_1, x_2, \ldots, x_n$ are variables, $f \in K[x_2, \ldots, x_n]$ and t is a positive integer. When $t = 1$, Lin et al. [13] proved that $\mathbf{M}$ is equivalent to its Smith normal form. Subsequently, the main result obtained for $t = 1$ were extended in [14–17] to the case where $t \geq 2$. In particular, Liu et al. [17] proved that a necessary and sufficient condition for $\mathbf{M}$ to be equivalent to its Smith normal form is that the reduced minors of each order of $\mathbf{M}$ generate the unit ideal in $K[x_1, x_2, \ldots, x_n]$. In this case, the Smith normal form of $\mathbf{M}$ takes the form: $\mathrm{diag}\left((x_1 - f)^{t_1}, \ldots, (x_1 - f)^{t_l}\right)$, where $t_1, \ldots, t_l$ are nonnegative integers and satisfy $t_1 \leq \cdots \leq t_l$ and $\sum_{i=1}^{l} t_i = t$. Thus, they completely solved the equivalence problem between multivariate polynomial matrices of this type and their Smith normal forms. Let $\mathbf{M} \in K[x_1, x_2, \ldots, x_n]^{l\times l}$ with $\det(\mathbf{M}) = g(x_1)$, where

$g \in K[x_1] \setminus \{0\}$. When $n = 2$ and g is an irreducible polynomial in $K[x_1]$, Li et al. [18] proved that $\mathbf{M}$ is equivalent to its Smith normal form by constructing a homomorphism from $K[x_1, x_2]$ to $(K[x_1]/\langle g\rangle)[x_2]$. Subsequently, the cases where g is a power of an irreducible polynomial in $K[x_1]$ or $n \geq 2$ were investigated in [19–21]. These partially resolved the equivalence problem for such multivariate polynomial matrices and their Smith normal forms.

Although some progress has been made on the equivalence problem between the two aforementioned classes of multivariate polynomial matrices and their Smith normal forms, it remains as an open problem for many other classes.

This paper focuses on the equivalence problem between bivariate polynomial matrices and their Smith normal forms primarily. Let $\mathbf{M} \in K[x, y]^{l\times l}$ with $\det(\mathbf{M}) \neq 0$. Since $\det(\mathbf{M}) \in K[x, y] \setminus \{0\}$, we may without loss of generality write $\det(\mathbf{M})$ as:

$$\det(\mathbf{M}) = a_0(x) + a_1(x) \cdot y + \cdots + a_s(x) \cdot y^s,$$

where $a_0(x), a_1(x), \ldots, a_s(x) \in K[x]$ with $a_s(x) \neq 0$, and s is a nonnegative integer. Now, we construct an example to illustrate that for any positive integer s with $s \geq 2$, there exists a bivariate polynomial matrix $\mathbf{M}$ where the degree of $\det(\mathbf{M})$ in y is s, such that the reduced minors of each order of $\mathbf{M}$ generate the unit ideal in $K[x, y]$, while $\mathbf{M}$ is not equivalent to its Smith normal form.

Example 1.1. Let

$$\mathbf{M} = \begin{pmatrix} -1 + x + y^s & y \\ 0 & 1 + x \end{pmatrix}$$

be a bivariate polynomial matrix in $\mathbb{Q}[x, y]^{2\times 2}$, where s is a positive integer with $s \geq 2$, and $\mathbb{Q}$ is the field of rational numbers.

By computation, we have $\det(\mathbf{M}) = (-1+x+y^s)(1+x)$. Since $\langle -1+x+y^s, y, 1+x\rangle = \langle 1\rangle$, the 1×1 reduced minors of $\mathbf{M}$ generate the unit ideal in $\mathbb{Q}[x, y]$. As $\mathbf{M}$ is a square matrix, the only 2×2 reduced minor of $\mathbf{M}$ is 1, which clearly generates the unit ideal in $\mathbb{Q}[x, y]$. Next, we prove that $\mathbf{M}$ is not equivalent to its Smith normal form.

Let $\mathbf{A} = \begin{pmatrix} 1 - y^s & -y \\ 0 & -1 \end{pmatrix}$. Then $\mathbf{M} = x \cdot \mathbf{I}_2 - \mathbf{A}$. We construct the following two matrices:

$$\mathbf{M}_1 = \begin{pmatrix} -1 + x + y^s & 1 \\ 0 & 1 + x \end{pmatrix} \text{ and } \mathbf{A}_1 = \begin{pmatrix} 1 - y^s & -1 \\ 0 & -1 \end{pmatrix},$$

then $\mathbf{M}_1 = x \cdot \mathbf{I}_2 - \mathbf{A}_1$. Let

$$\mathbf{U} = \begin{pmatrix} 1 & 0 \\ -(1 + x) & 1 \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} 0 & -1 \\ 1 & -1 + x + y^s \end{pmatrix}.$$

Then $\det(\mathbf{U}) = \det(\mathbf{V}) = 1$, i.e., $\mathbf{U}, \mathbf{V}$ are unimodular matrices. Moreover, we get

$$\mathbf{U}\mathbf{M}_1\mathbf{V} = \begin{pmatrix} 1 & \\ & (-1+x+y^s)(1+x) \end{pmatrix}.$$

It follows that $\mathbf{M}_1$ is equivalent to its Smith normal form, which is the Smith normal form of $\mathbf{M}$. According to Corollary 16.4 in [2], $x \cdot \mathbf{I}_2 - \mathbf{A}$ and $x \cdot \mathbf{I}_2 - \mathbf{A}_1$ are equivalent over $\mathbb{Q}[x, y]$ if and only if $\mathbf{A}$ and $\mathbf{A}_1$ are similar over $\mathbb{Q}[y]$. We assume that there exists a unimodular matrix $\mathbf{Q} = (q_{ij})_{2\times 2} \in \mathbb{Q}[y]^{2\times 2}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{A}_1$. Then

$$\begin{cases} q_{11}q_{22} - q_{21}q_{12} = 1, \\ (1-y^s)q_{11} - yq_{21} = (1-y^s)q_{11}, \\ (1-y^s)q_{12} - yq_{22} = -q_{11} - q_{12}, \\ -q_{21} = (1-y^s)q_{21}, \\ -q_{22} = -q_{21} - q_{22}. \end{cases}$$

Solving the above equations, we obtain $q_{21} = 0$, $q_{11}q_{22} = 1$, and $yq_{22} - q_{11} = (2-y^s)q_{12}$. It follows from $q_{11}q_{22} = 1$ that the degree of $yq_{22} - q_{11}$ in y is 1. However, the degree of $(2-y^s)q_{12}$ in y is either 0 or greater than 1, which leads to a contradiction. Therefore, $\mathbf{M}$ is not equivalent to $\mathbf{M}_1$. This implies that $\mathbf{M}$ is not equivalent to its Smith normal form.

Example 1.1 reveals the fact that, in the general case, when the degree of $\det(\mathbf{M})$ in y is greater than 1, the property that the reduced minors of each order of $\mathbf{M}$ generate the unit ideal is not a sufficient condition for $\mathbf{M}$ to be equivalent to its Smith normal form. Based on the above fact, we shall consider the following problem.

Problem 1.2. Let $\mathbf{M} \in K[x, y]^{l\times l}$ be of full rank such that $\det(\mathbf{M}) = f(x)y + g(x)$. Prove that $\mathbf{M}$ is equivalent to its Smith normal form if and only if the reduced minors of each order of $\mathbf{M}$ generate the unit ideal in $K[x, y]$.

The rest of the paper is organized as follows. In Section 2, we recall some terminology and preliminary results needed for this paper. In Section 3, we solve Problem 1.2 and extend the square matrix case to rank-deficient and non-square matrix cases. Some concluding remarks are provided in Section 4.

2. Preliminaries

Let K be a field, $K[x, y]$ be the bivariate polynomial ring in the variables x, y with coefficients in K, and $K[x, y]^{l\times m}$ be the set of $l \times m$ matrices with entries in $K[x, y]$, where l, m are two positive integers. Throughout this paper, we assume without loss of generality that $l \le m$.

The arguments (x, y) and (x) are omitted whenever their omission does not cause confusion; for example, we denote $f(x, y)$ and $p(x)$ by f and p for simplicity, respectively.

Let $f_1, \ldots, f_l \in K[x, y]$; we use $\langle f_1, \ldots, f_l \rangle$ and $\mathrm{diag}(f_1, \ldots, f_l)$ to represent the ideal in $K[x, y]$ generated by $f_1, \ldots, f_l$ and the diagonal matrix whose diagonal elements are $f_1, \ldots, f_l$, respectively.

For convenience, we use bold letters to denote polynomial matrices. Let $\mathbf{M} \in K[x, y]^{l \times m}$. We use $\mathrm{rank}(\mathbf{M})$ to denote the rank of $\mathbf{M}$. For $i = 1, \ldots, l$, we use $I_i(\mathbf{M})$ and $d_i(\mathbf{M})$ to denote the ideal generated by all the $i \times i$ minors of $\mathbf{M}$ and the greatest common divisor of all the $i \times i$ minors of $\mathbf{M}$, respectively. Here, we make the convention that $d_0(\mathbf{M}) \equiv 1$.

2.1. Basic notions

We first introduce the concept of the Smith normal form of bivariate polynomial matrices.

Definition 2.1. Let $\mathbf{M} \in K[x, y]^{l \times m}$ with $\mathrm{rank}(\mathbf{M}) = r$, and f_i be a polynomial in $K[x, y]$ defined as follows:

$$f_i = \begin{cases} \frac{d_i(\mathbf{M})}{d_{i-1}(\mathbf{M})}, & 1 \le i \le r, \\ 0, & r < i \le l. \end{cases}$$

Then the Smith normal form of $\mathbf{M}$ is given by

$$\begin{pmatrix} \mathrm{diag}(f_1, \ldots, f_r) & \mathbf{0}_{r \times (m-r)} \\ \mathbf{0}_{(l-r) \times r} & \mathbf{0}_{(l-r) \times (m-r)} \end{pmatrix}.$$

In Definition 2.1, Li et al. [22] used localization and p-index techniques to prove that $f_1 \mid f_2 \mid \cdots \mid f_r$. This definition is a natural generalization of the concept of the Smith normal form of univariate polynomial matrices.

In the following, let R be an integral domain, i.e., a nonzero commutative ring with a multiplicative identity 1 and no zero divisors.

Definition 2.2. Let $\mathbf{U} \in R^{l \times l}$. Then $\mathbf{U}$ is said to be unimodular if $\det(\mathbf{U})$ is a unit in R. The set of all $l \times l$ unimodular matrices over R is denoted by $\mathrm{GL}_l(R)$.

If $R = K[x, y]$ in Definition 2.2, then $\mathbf{U}$ is unimodular if and only if $\det(\mathbf{U})$ is a nonzero constant in K. With the set of unimodular matrices defined, we can now specify what it means for two polynomial matrices to be equivalent.

Definition 2.3. Let $\mathbf{M}, \mathbf{N} \in R^{l \times m}$. We say that $\mathbf{M}$ is equivalent to $\mathbf{N}$ over R if there exist $\mathbf{U} \in \mathrm{GL}_l(R)$ and $\mathbf{V} \in \mathrm{GL}_m(R)$ such that $\mathbf{N} = \mathbf{UMV}$.

Denote by $\mathbf{M} \sim_R \mathbf{N}$ the equivalence of $\mathbf{M}$ and $\mathbf{N}$ over R. When $R = K[x, y]$, we write $\mathbf{M} \sim \mathbf{N}$ (omitting the subscript R) for the equivalence of $\mathbf{M}$ and $\mathbf{N}$ over $K[x, y]$.

Definition 2.4 *([23]).* Let $\mathbf{M} \in K[x,y]^{l\times m}$ with $\text{rank}(\mathbf{M}) = r$, where $1 \le r \le l$. For any given integer i with $1 \le i \le r$, let $a_1, \dots, a_{\beta_i}$ be all the $i \times i$ minors of $\mathbf{M}$, where $\beta_i = \binom{l}{i}\binom{m}{i}$. Extracting $d_i(\mathbf{M})$ from $a_1, \dots, a_{\beta_i}$ yields

$$a_j = d_i(\mathbf{M}) \cdot b_j, \quad j = 1, \dots, \beta_i.$$

Then, $b_1, \dots, b_{\beta_i}$ are called the $i\times i$ reduced minors of $\mathbf{M}$. For convenience, we use $J_i(\mathbf{M})$ to denote the ideal in $K[x,y]$ generated by $b_1, \dots, b_{\beta_i}$. In addition, set $J_0(\mathbf{M}) \equiv K[x,y]$.

In Definition 2.4, two special cases deserve attention: the first is when $l = m$, where $\beta_l = 1$ and the only $l \times l$ reduced minor of $\mathbf{M}$ is $b_1 = 1$; the second is when $d_i(\mathbf{M}) = 1$, in which case $I_i(\mathbf{M}) = J_i(\mathbf{M})$. In addition, if $\mathbf{M}$ is in Smith normal form, then $J_i(\mathbf{M}) = K[x,y]$ for $i = 1, \dots, r$.

Proposition 2.5 *(Cauchy-Binet Formula, [24]). Let $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$, where $\mathbf{M}_1 \in K[x,y]^{l\times t}$ and $\mathbf{M}_2 \in K[x,y]^{t\times m}$. Then an $r \times r$ minor of $\mathbf{M}$ is*

$$\det\left(\mathbf{M}\left(\begin{smallmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{smallmatrix}\right)\right) = \sum_{1\le s_1<\cdots<s_r\le t} \det\left(\mathbf{M}_1\left(\begin{smallmatrix} i_1 & \cdots & i_r \\ s_1 & \cdots & s_r \end{smallmatrix}\right)\right) \cdot \det\left(\mathbf{M}_2\left(\begin{smallmatrix} s_1 & \cdots & s_r \\ j_1 & \cdots & j_r \end{smallmatrix}\right)\right),$$

where $1 \le r \le \min\{t, l\}$.

A crucial observation is that $d_i(\mathbf{M})$, $I_i(\mathbf{M})$ and $J_i(\mathbf{M})$ are invariant under matrix equivalence, as shown in the following lemma, whose proof follows straightforwardly from the Cauchy-Binet Formula.

Lemma 2.6. *Let $\mathbf{M}, \mathbf{N} \in K[x,y]^{l\times m}$. If $\mathbf{M} \sim \mathbf{N}$, then $d_i(\mathbf{M}) = d_i(\mathbf{N})$, $I_i(\mathbf{M}) = I_i(\mathbf{N})$ and $J_i(\mathbf{M}) = J_i(\mathbf{N})$, where $i = 1, \dots, l$.*

2.2. Quillen-Suslin theorem

We begin by defining zero left prime and zero right prime matrices, which are closely related to unimodular matrices.

Definition 2.7 *([25]).* Let $\mathbf{M} \in K[x,y]^{l\times m}$ be of full row rank. Then $\mathbf{M}$ is said to be zero left prime (ZLP) if all the $l \times l$ minors of $\mathbf{M}$ generate the unit ideal in $K[x,y]$.

Let $\mathbf{W} \in K[x,y]^{m\times l}$ be of full column rank. Then $\mathbf{W}$ is said to be zero right prime (ZRP) if all the $l \times l$ minors of $\mathbf{W}$ generate the unit ideal in $K[x,y]$.

The Quillen-Suslin Theorem, which establishes the connection between ZLP matrices and unimodular matrices, comes from the independent work of Quillen [26] and Suslin [27] in 1976.

Theorem 2.8 *(Quillen-Suslin Theorem, [26,27]). Let* $\mathbf{M} \in K[x,y]^{l\times m}$ *be a ZLP matrix. Then there exists* $\mathbf{V} \in \mathrm{GL}_m(K[x,y])$ *such that* $\mathbf{MV} = (\mathbf{I}_l, \mathbf{0}_{l\times(m-l)})$*, where* $\mathbf{I}_l$ *is the* $l \times l$ *identity matrix.*

We now introduce the concept of localization, a crucial tool in the proof of the Quillen-Suslin Theorem, which also serves as an essential component in solving the matrix equivalence problem of the paper.

Definition 2.9 *(Definitions 29 and 30, [28]).* Let S be a subset of R.

(1) S is called a multiplicative subset of R if $1 \in S$, and $f, g \in S$ implies $fg \in S$.
(2) The localization of R with respect to the multiplicative subset S is defined as

$$R_S := \left\{ \frac{f}{g} \mid f \in R, g \in S \right\}.$$

(3) Let $S_1, \ldots, S_r$ be multiplicative subsets of R, where r is a positive integer. We say that $S_1, \ldots, S_r$ are comaximal if for any given $g_1 \in S_1, \ldots, g_r \in S_r$, there exist $h_1, \ldots, h_r \in R$ such that

$$1 = \sum_{i=1}^{r} h_i g_i.$$

Example 2.10. Let $f_1, \ldots, f_r \in K[x,y] \setminus \{0\}$ be pairwise distinct polynomials, where r is a positive integer. We define the subsets

$$S_i := \left\{ f_i^t \mid t \in \mathbb{N} \right\} \text{ for } i = 1, \ldots, r,$$

where $\mathbb{N}$ is the set of natural numbers including 0. Each S_i is a multiplicative subset of $K[x,y]$, where $i = 1, \ldots, r$. In addition, $S_1, \ldots, S_r$ are comaximal if and only if $\langle f_1, \ldots, f_r \rangle = \langle 1 \rangle$, i.e., $f_1, \ldots, f_r$ generate the unit ideal in $K[x,y]$.

By employing the technique of localization, we can establish the following matrix equivalence theorem.

Theorem 2.11 *([28,29]). Let* $\mathbf{M}(y)$ *be a matrix in* $R[y]^{l\times l}$*, and let* $S_1, \ldots, S_r$ *be comaximal multiplicative subsets of* R*, where* y *is a variable not in* R*, and* r *is a positive integer. Then* $\mathbf{M}(y)$ *and* $\mathbf{M}(0)$ *are equivalent over* $R[y]$ *if and only if they are equivalent over* $R_{S_i}[y]$ *for* $i = 1, \ldots, r$.

In the above theorem, $\mathbf{M}(0)$ denotes the matrix obtained by substituting y with 0 in $\mathbf{M}(y)$, which is clearly in $R^{l\times l}$. For a constructive proof of Theorem 2.11, we refer to Theorem 44 in [28] or Theorem 2.9 in [29] for more details. For $\mathbf{M}(y) \in R[y]^{l\times l}$ with

$\mathbf{M}(y)$ and $\mathbf{M}(0)$ being equivalent over $R_{S_i}[y]$ for $i = 1, \ldots, r$, the proof provides an explicit constriction of the matrices $\mathbf{U}, \mathbf{V} \in \mathrm{GL}_l(R[y])$ such that $\mathbf{U} \cdot \mathbf{M}(y) \cdot \mathbf{V} = \mathbf{M}(0)$.

3. Matrix equivalence theory

Let $\mathbf{M} \in K[x,y]^{l \times l}$ be of full rank such that $\det(\mathbf{M}) = fy + g$, where $f, g \in K[x]$. In Subsection 3.1, we consider the case $f = 0$. In Subsection 3.2, we address the case $f \neq 0$. In Subsection 3.3, we extend $\mathbf{M}$ to rank-deficient and non-square cases. In Subsection 3.4, we give an example to illustrate the effectiveness of Theorem 3.14.

3.1. The case $f = 0$

We first introduce two lemmas which play a crucial role in resolving the case $f = 0$.

Lemma 3.1 *(Primitive Factorization Theorem, [30]). Let $\mathbf{M} \in K[x,y]^{l \times m}$ be of full row rank, and $h \in K[x]$ be a divisor of $d_l(\mathbf{M})$. Then there exist $\mathbf{F} \in K[x,y]^{l \times l}$ and $\mathbf{G} \in K[x,y]^{l \times m}$ such that $\mathbf{M} = \mathbf{F}\mathbf{G}$ and $\det(\mathbf{F}) = h$.*

Lemma 3.2 *([19]). Let $\mathbf{M} \in K[x,y]^{l \times l}$ with $\det(\mathbf{M}) = \alpha p^r$, where $\alpha \in K \setminus \{0\}$ and $p \in K[x]$ is an irreducible polynomial. Then $\mathbf{M}$ is equivalent to its Smith normal form if and only if $J_i(\mathbf{M}) = K[x,y]$ for $i = 1, \ldots, l$.*

The main idea for solving the case $f = 0$ is as follows. Since $\det(\mathbf{M}) = g \in K[x] \setminus \{0\}$, let $g = \alpha p_1^{r_1} \cdots p_k^{r_k}$ be an irreducible factorization of g, where $\alpha \in K \setminus \{0\}$, $p_1, \ldots, p_k \in K[x]$ are pairwise coprime irreducible factors, and $r_1, \ldots, r_k, k$ are positive integers. We proceed by mathematical induction on k. If $k = 1$, the conclusion follows from Lemma 3.2. If $k \geq 2$, we first apply the Primitive Factorization Theorem to factor $\mathbf{M}$ as $\mathbf{M} = \mathbf{F}\mathbf{G}$, such that $\det(\mathbf{F}) = \alpha p_1^{r_1}$ and $\det(\mathbf{G}) = p_2^{r_2} \cdots p_k^{r_k}$. By the induction hypothesis, $\mathbf{F}$ and $\mathbf{G}$ are equivalent to their Smith normal forms $\mathbf{S}_{\mathbf{F}}$ and $\mathbf{S}_{\mathbf{G}}$, respectively. Then, using the technique of localization and Theorem 2.11, we combine $\mathbf{S}_{\mathbf{F}}$ and $\mathbf{S}_{\mathbf{G}}$ to conclude that $\mathbf{M}$ is equivalent to its Smith normal form $\mathbf{S}_{\mathbf{M}}$.

Lemma 3.3. *Let $\mathbf{M}, \mathbf{F}, \mathbf{G} \in K[x,y]^{l \times l}$ satisfy $\mathbf{M} = \mathbf{F}\mathbf{G}$ and $\gcd(\det(\mathbf{F}), \det(\mathbf{G})) = 1$. Then*

(1) *$d_i(\mathbf{M}) = d_i(\mathbf{F}) \cdot d_i(\mathbf{G})$ for $i = 1, \ldots, l$.*
(2) *If $J_i(\mathbf{M}) = K[x,y]$, then $J_i(\mathbf{F}) = J_i(\mathbf{G}) = K[x,y]$, where $i = 1, \ldots, l$.*

Proof. (1) For any given integer i with $1 \leq i \leq l$, let $h_1, \ldots, h_{\beta_i}, f_1, \ldots, f_{\beta_i}$ and $g_1, \ldots, g_{\beta_i}$ be all the $i \times i$ minors of $\mathbf{M}, \mathbf{F}$ and $\mathbf{G}$ respectively, where $\beta_i = \binom{l}{i}^2$. Let h_ξ be an arbitrary $i \times i$ minor of $\mathbf{M}$. It follows from the Cauchy-Binet Formula that there are $f_{\xi_1}, \ldots, f_{\xi_\eta}$ and $g_{\xi_1}, \ldots, g_{\xi_\eta}$ such that

$$h_\xi = \sum_{j=1}^{\eta} f_{\xi_j} \cdot g_{\xi_j}, \tag{1}$$

where $\eta = \binom{l}{i}$. For any given integer j with $1 \leq j \leq \eta$, it follows from $d_i(\mathbf{F}) \mid f_{\xi_j}$ and $d_i(\mathbf{G}) \mid g_{\xi_j}$ that $d_i(\mathbf{F}) \cdot d_i(\mathbf{G}) \mid f_{\xi_j} \cdot g_{\xi_j}$. Equation (1) gives $d_i(\mathbf{F}) \cdot d_i(\mathbf{G}) \mid h_\xi$. Since h_ξ is an arbitrary $i \times i$ minor of $\mathbf{M}$, we obtain

$$d_i(\mathbf{F}) \cdot d_i(\mathbf{G}) \mid d_i(\mathbf{M}).$$

Assume that $d_i(\mathbf{M}) = \alpha_i \cdot d_i(\mathbf{F}) \cdot d_i(\mathbf{G})$, where $\alpha_i \in K[x, y]$. In the following we only need to prove that $\alpha_i = 1$.

Let $\mathbf{G}^* \in K[x, y]$ be the adjoint matrix of $\mathbf{G}$. Then $\det(\mathbf{G}) \cdot \mathbf{F} = \mathbf{M} \cdot \mathbf{G}^*$. Let f_s be an arbitrary $i \times i$ minor of $\mathbf{F}$. It follows from the Cauchy-Binet Formula that there are $h_{s_1}, \ldots, h_{s_\eta}$ and $q_{s_1}, \ldots, q_{s_\eta}$ such that

$$(\det(\mathbf{G}))^i \cdot f_s = \sum_{j=1}^{\eta} h_{s_j} \cdot q_{s_j}, \tag{2}$$

where $h_{s_1}, \ldots, h_{s_\eta}$ and $q_{s_1}, \ldots, q_{s_\eta}$ are the $i \times i$ minors of $\mathbf{M}$ and $\mathbf{G}^*$, respectively. Since $d_i(\mathbf{M}) \mid h_{s_j}$ for any integer j with $1 \leq j \leq \eta$, Equation (2) shows that $d_i(\mathbf{M}) \mid (\det(\mathbf{G}))^i \cdot f_s$. Since f_s is an arbitrary $i \times i$ minor of $\mathbf{F}$, we have

$$d_i(\mathbf{M}) \mid \gcd((\det(\mathbf{G}))^i \cdot f_1, \ldots, (\det(\mathbf{G}))^i \cdot f_{\beta_i}) = (\det(\mathbf{G}))^i \cdot d_i(\mathbf{F}).$$

As $d_i(\mathbf{M}) = \alpha_i \cdot d_i(\mathbf{F}) \cdot d_i(\mathbf{G})$, we get $\alpha_i \cdot d_i(\mathbf{G}) \mid (\det(\mathbf{G}))^i$. This implies that

$$\alpha_i \mid (\det(\mathbf{G}))^i. \tag{3}$$

Using the same argument as above, we can get that

$$\alpha_i \mid (\det(\mathbf{F}))^i. \tag{4}$$

Combining Equations (3) and (4), we obtain

$$\alpha_i \mid \gcd((\det(\mathbf{F}))^i, (\det(\mathbf{G}))^i).$$

Since $\gcd(\det(\mathbf{F}), \det(\mathbf{G})) = 1$, we conclude that $\alpha_i = 1$.

(2) Dividing both sides of Equation (1) by $d_i(\mathbf{M})$, we get

$$\frac{h_\xi}{d_i(\mathbf{M})} = \sum_{j=1}^{\eta} \frac{f_{\xi_j}}{d_i(\mathbf{F})} \cdot \frac{g_{\xi_j}}{d_i(\mathbf{G})}, \tag{5}$$

where $\frac{h_1}{d_i(\mathbf{M})}, \ldots, \frac{h_{\beta_i}}{d_i(\mathbf{M})}$, $\frac{f_1}{d_i(\mathbf{F})}, \ldots, \frac{f_{\beta_i}}{d_i(\mathbf{F})}$ and $\frac{g_1}{d_i(\mathbf{G})}, \ldots, \frac{g_{\beta_i}}{d_i(\mathbf{G})}$ are all the $i \times i$ reduced minors of $\mathbf{M}, \mathbf{F}$ and $\mathbf{G}$, respectively. It follows from Equation (5) that

$$J_i(\mathbf{M}) \subseteq J_i(\mathbf{F}) \subseteq K[x,y] \text{ and } J_i(\mathbf{M}) \subseteq J_i(\mathbf{G}) \subseteq K[x,y].$$

Since $J_i(\mathbf{M}) = K[x,y]$, we have $J_i(\mathbf{F}) = J_i(\mathbf{G}) = K[x,y]$, where $i = 1, \dots, l$. $\square$

Lemma 3.4. *Let* $\mathbf{M} = \mathrm{diag}(f_1, \dots, f_l) \cdot \mathbf{U} \cdot \mathrm{diag}(g_1, \dots, g_l)$, *where* $\mathbf{U} \in \mathrm{GL}_l(K[x,y])$, $f_1, \dots, f_l \in K[x] \setminus \{0\}$ *satisfy* $f_1 \mid \cdots \mid f_l$, *and* $g_1, \dots, g_l \in K[x] \setminus \{0\}$ *satisfy* $g_1 \mid \cdots \mid g_l$. *If* $\gcd(f_l, g_l) = 1$, *then*

$$\mathbf{M} \sim \mathrm{diag}(f_1 g_1, \dots, f_l g_l).$$

Proof. We construct the following two sets

$$S_1 = \left\{ f_l^{t_1} \mid t_1 \in \mathbb{N} \right\} \text{ and } S_2 = \left\{ g_l^{t_2} \mid t_2 \in \mathbb{N} \right\}.$$

It is straightforward to show that $S_1, S_2 \subset K[x]$ are multiplicative sets. Moreover, it follows from $\gcd(f_l, g_l) = 1$ that S_1, S_2 are comaximal. Let

$$\mathbf{H}_1 = \mathrm{diag}(f_1, \dots, f_l) \cdot \mathbf{U} \cdot \mathrm{diag}(\frac{\tilde{f}_1}{f_l}, \dots, \frac{\tilde{f}_l}{f_l}),$$

where $\tilde{f}_i = \frac{f_l}{f_i} \in K[x]$ for $i = 1, \dots, l$. Then $\mathbf{H}_1 \in K[x]_{S_1}[y]^{l \times l}$. Since $\mathbf{U} \in \mathrm{GL}_l(K[x,y])$ and $\det(\mathbf{H}_1) = \det(\mathbf{U})$, $\mathbf{H}_1$ is a unimodular matrix in $K[x]_{S_1}[y]^{l \times l}$. Let $\mathbf{F} = \mathrm{diag}(f_1, \dots, f_l)$, $\mathbf{G} = \mathrm{diag}(g_1, \dots, g_l)$ and $\mathbf{D} = \mathbf{F}\mathbf{G}$. Given that $f_1 \mid \cdots \mid f_l$ and $g_1 \mid \cdots \mid g_l$, it follows that $\mathbf{D}$ is in Smith normal form. Since $\mathbf{M} = \mathbf{H}_1 \mathbf{D}$, we get

$$\mathbf{M} \sim_{K[x]_{S_1}[y]} \mathbf{D}. \tag{6}$$

We use $\mathbf{M}(y)$ and $\mathbf{U}(y)$ to denote $\mathbf{M}$ and $\mathbf{U}$ in $K[x][y]^{l \times l}$, respectively. Let $\mathbf{M}_0 = \mathbf{M}(0)$ and $\mathbf{U}_0 = \mathbf{U}(0)$. Then $\mathbf{M}_0 \in K[x]^{l \times l}$ and $\mathbf{U}_0 \in \mathrm{GL}_l(K[x])$. Since $\mathbf{M}_0 = \mathbf{F} \cdot \mathbf{U}_0 \cdot \mathbf{G}$ and $\mathbf{U}_0$ is unimodular, by Lemmas 2.6 and 3.3 we have

$$d_i(\mathbf{M}_0) = d_i(\mathbf{F}) \cdot d_i(\mathbf{G}) = d_i(\mathbf{D}) \text{ for } i = 1, \dots, l.$$

This implies that the Smith normal form of $\mathbf{M}_0$ is $\mathbf{D}$. By the fact that $K[x]$ is a Euclidean domain, $\mathbf{M}_0$ is equivalent to its Smith normal form over $K[x]$. That is,

$$\mathbf{M}_0 \sim_{K[x]} \mathbf{D}. \tag{7}$$

Since $K[x] \subset K[x]_{S_1}[y]$, it follows from Equations (6) and (7) that $\mathbf{M} \sim_{K[x]_{S_1}[y]} \mathbf{M}_0$. An analogous reasoning yields $\mathbf{M} \sim_{K[x]_{S_2}[y]} \mathbf{M}_0$. Since S_1 and S_2 are comaximal, by Theorem 2.11 we obtain

$$\mathbf{M} \sim \mathbf{M}_0. \tag{8}$$

Combining Equations (7) and (8), it follows from $K[x] \subset K[x,y]$ that $\mathbf{M} \sim \mathbf{D}$. $\square$

Theorem 3.5. *Let* $\mathbf{M} \in K[x,y]^{l\times l}$ *with* $\det(\mathbf{M}) = g$*, where* $g \in K[x] \setminus \{0\}$*. Then* $\mathbf{M}$ *is equivalent to its Smith normal form if and only if* $J_i(\mathbf{M}) = K[x,y]$ *for* $i = 1, \ldots, l$*.*

Proof. The necessity follows directly from Lemma 2.6. We now proceed to prove the sufficiency.

If $g \in K \setminus \{0\}$, then $\mathbf{M}$ is a unimodular matrix. As a consequence, $\mathbf{M} \sim \mathbf{I}_l$. Verification shows that $\mathbf{M}$ is equivalent to its Smith normal form. If $g \in K[x] \setminus K$, then let $g = \alpha p_1^{r_1} \cdots p_k^{r_k}$ be an irreducible factorization of g, where $\alpha \in K \setminus \{0\}$, $p_1, \ldots, p_k \in K[x]$ are pairwise coprime irreducible factors, and $r_1, \ldots, r_k, k$ are positive integers. We employ mathematical induction on k to prove the statement.

If $k = 1$, then the statement holds by Lemma 3.2. Suppose that the statement is true for $k-1$ with $k \geq 2$. We now show it also holds for k. According to the Primitive Factorization Theorem, there exist $\mathbf{F}, \mathbf{G} \in K[x,y]^{l\times l}$ such that $\mathbf{M} = \mathbf{FG}$, where $\det(\mathbf{F}) = \alpha p_1^{r_1}$ and $\det(\mathbf{G}) = p_2^{r_2} \cdots p_k^{r_k}$. It follows that $\gcd(\det(\mathbf{F}), \det(\mathbf{G})) = 1$. Based on Lemma 3.3, $J_i(\mathbf{F}) = J_i(\mathbf{G}) = K[x,y]$ for $i = 1, \ldots, l$. Assume that the Smith normal forms of $\mathbf{F}$ and $\mathbf{G}$ are $\mathbf{S_F} = \mathrm{diag}(f_1, \ldots, f_l)$ and $\mathbf{S_G} = \mathrm{diag}(g_1, \ldots, g_l)$ respectively, where $f_1, \ldots, f_l \in K[x] \setminus \{0\}$ satisfy $f_1 \mid \cdots \mid f_l$ and $f_1 \cdots f_l = \alpha p_1^{r_1}$, and $g_1, \ldots, g_l \in K[x] \setminus \{0\}$ satisfy $g_1 \mid \cdots \mid g_l$ and $g_1 \cdots g_l = p_2^{r_2} \cdots p_k^{r_k}$. By the induction hypothesis, we get $\mathbf{F} \sim \mathbf{S_F}$ and $\mathbf{G} \sim \mathbf{S_G}$. Then there exist $\mathbf{U}_1, \mathbf{V}_1, \mathbf{U}_2, \mathbf{V}_2 \in \mathrm{GL}_l(K[x,y])$ such that $\mathbf{F} = \mathbf{U}_1 \cdot \mathbf{S_F} \cdot \mathbf{V}_1$ and $\mathbf{G} = \mathbf{U}_2 \cdot \mathbf{S_G} \cdot \mathbf{V}_2$. The equation $\mathbf{M} = \mathbf{FG}$ gives

$$\mathbf{M} = \mathbf{U}_1 \cdot \mathbf{S_F} \cdot \mathbf{V}_1\mathbf{U}_2 \cdot \mathbf{S_G} \cdot \mathbf{V}_2.$$

Since $\mathbf{V}_1\mathbf{U}_2 \in \mathrm{GL}_l(K[x,y])$, it follows from Lemma 3.4 that

$$\mathbf{S_F} \cdot \mathbf{V}_1\mathbf{U}_2 \cdot \mathbf{S_G} \sim \mathrm{diag}(f_1g_1, \ldots, f_lg_l).$$

Therefore, $\mathbf{M}$ is equivalent to its Smith normal form. □

Theorem 3.5 resolves Problem 1.2 in the case where $f = 0$.

3.2. The case $f \neq 0$

In this subsection, we first consider the case where $\gcd(f,g) = 1$, and prove that $\mathbf{M}$ must be equivalent to its Smith normal form. Then, we deal with the case where $\gcd(f,g) \in K[x] \setminus K$, and solve the case $f \neq 0$ by means of mathematical induction.

Lemma 3.6. *Let* $h \in K[x,y] \setminus \{0\}$*, and* $p, q \in K[x]$ *be such that* $p \neq 0$ *and* $\gcd(p,q) = 1$*. Then* $(py+q) \mid h$ *if and only if* $h(x, -\frac{q}{p}) = 0$*.*

Proof. The necessity is straightforward to establish. It remains to prove the sufficiency. By the polynomial pseudo-division over $K[x,y]$ (see Page 425, Algorithm R in [31]), there exist $h_1 \in K[x,y]$, $\gamma \in K[x]$ and some nonnegative integer $s \in \mathbb{N}$ such that

$$p(x)^s \cdot h(x,y) = h_1(x,y) \cdot (p(x)y + q(x)) + \gamma(x). \tag{9}$$

Replacing y with $-\frac{q}{p}$ in Equation (9), we obtain $\gamma(x) = 0$. This implies that $(py+q) \mid p^s h$. It follows from $\gcd(p,q) = 1$ that $py + q$ is an irreducible polynomial in $K[x,y]$. Since $p \in K[x]$, we get $(py + q) \mid h$. $\square$

Lemma 3.7. *Let* $\mathbf{M} \in K[x,y]^{l\times l}$ *with* $\det(\mathbf{M}) = py + q$*, where* $p, q \in K[x]$ *satisfy* $p \neq 0$ *and* $\gcd(p,q) = 1$*. Then there exist* $\mathbf{V} \in \mathrm{GL}_l(K[x])$ *and* $\mathbf{G} \in \mathrm{GL}_l(K[x,y])$ *such that* $\mathbf{M} = \mathbf{V} \cdot \mathrm{diag}(1,\ldots,1,py+q) \cdot \mathbf{G}$*.*

Proof. Let $\mathbf{N} = \mathbf{M}(x, -\frac{q(x)}{p(x)})$. Then $\mathbf{N} \in K(x)^{l\times l}$ and $\det(\mathbf{N}) = 0$. Let $s \in \mathbb{N}$ be a sufficiently large positive integer such that $p^s \cdot \mathbf{N} \in K[x]^{l\times l}$. As $K[x]$ is a Euclidean domain, there exists $\mathbf{U} \in \mathrm{GL}_l(K[x])$ such that

$$\mathbf{U}(p^s \cdot \mathbf{N}) = \begin{pmatrix} \mathbf{N}' \\ \mathbf{0}_{1\times l} \end{pmatrix}, \tag{10}$$

where $\mathbf{N}' \in K[x]^{(l-1)\times l}$. Since $\mathbf{U}(p^s \cdot \mathbf{N}) = p^s \cdot (\mathbf{UN})$, it follows from Equation (10) that the last row of $\mathbf{UN}$ is a zero vector. Let $\mathbf{M}' = \mathbf{UM}$. Then

$$\mathbf{M}'(x, -\frac{q(x)}{p(x)}) = \mathbf{UN}.$$

Thus, the last row of $\mathbf{M}'(x, -\frac{q(x)}{p(x)})$ is a zero vector. By Lemma 3.6, all entries in the last row of $\mathbf{M}'$ are divisible by $py + q$. Hence, there exists $\mathbf{G} \in K[x,y]^{l\times l}$ such that

$$\mathbf{UM} = \mathrm{diag}(1,\ldots,1,py+q) \cdot \mathbf{G}. \tag{11}$$

Since $\det(\mathbf{M}) = py+q$ and $\mathbf{U}$ is a unimodular matrix, it follows from Equation (11) that $\mathbf{G} \in \mathrm{GL}_l(K[x,y])$. Let $\mathbf{V} = \mathbf{U}^{-1}$. Then

$$\mathbf{M} = \mathbf{V} \cdot \mathrm{diag}(1,\ldots,1,py+q) \cdot \mathbf{G}. \quad \square$$

Remark 3.8. If $q = 0$ in Lemmas 3.6 and 3.7, it follows from $\gcd(p,q) = 1$ that $p = 1$. In this case, s in both Equations (9) and (10) is 0, and $\mathbf{N}$ in Equation (10) is a polynomial matrix in $K[x]^{l\times l}$.

Since $\mathrm{diag}(1,\ldots,1,py+q)$ is the Smith normal form of $\mathbf{M}$ as given in Lemma 3.7, Lemma 3.7 provides a positive answer to Problem 1.2 when $\gcd(p,q) = 1$.

Lemma 3.9. *Let* $f, v_1, v_2 \in K[x]$ *with* $\gcd(f, v_1, v_2) = 1$*, where* $f \neq 0$*. Then there exists* $h \in K[x]$ *such that* $\gcd(f, v_1 + hv_2) = 1$*.*

Proof. There are two cases. If $f \in K \setminus \{0\}$, then the conclusion obviously holds for any polynomial $h \in K[x]$. If $f \in K[x] \setminus K$, then let $f = \alpha p_1^{r_1} \cdots p_k^{r_k}$ be an irreducible factorization of f, where $p_1, \ldots, p_k \in K[x]$ are pairwise coprime irreducible factors, $r_1, \ldots, r_k, k$ are positive integers and $\alpha \in K \setminus \{0\}$. For each integer i with $1 \leq i \leq k$, p_i cannot divide both v_1 and v_2; otherwise, it would contradict the fact that $\gcd(f, v_1, v_2) = 1$. This implies that there exists some integer $\delta_i \in \{0, 1\}$ such that

$$v_1 + \delta_i v_2 \neq 0 \pmod{p_i}. \tag{12}$$

By the Chinese Remainder Theorem (see Page 94, Theorem 2.1 in [32]), there exists $h \in K[x]$ such that

$$h \equiv \delta_i \pmod{p_i}. \tag{13}$$

Combining Equations (12) and (13), we have

$$v_1 + hv_2 \neq 0 \pmod{p_i} \text{ for each } i = 1, \ldots, k. \tag{14}$$

It follows from Equation (14) that

$$\gcd(f, v_1 + hv_2) = 1. \quad \square$$

Lemma 3.10. *Let*

$$\mathbf{W} = \begin{pmatrix} f_1 & 0 & \cdots & 0 & v_1(py+q) \\ 0 & f_2 & \ddots & \vdots & v_2(py+q) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & f_{l-1} & v_{l-1}(py+q) \\ 0 & \cdots & \cdots & 0 & v_l(py+q) \end{pmatrix} \in K[x,y]^{l \times l},$$

where $f_1, f_2, \ldots, f_{l-1} \in K[x] \setminus \{0\}$ satisfy $f_1 \mid f_2 \mid \cdots \mid f_{l-1}$, and $p, q, v_1, \ldots, v_l \in K[x]$. If $I_1(\mathbf{W}) = K[x,y]$, then $\langle f_1, py+q \rangle = K[x,y]$ and $\gcd(f_1, v_1, \ldots, v_l) = 1$.

Proof. As $I_1(\mathbf{W}) = \langle f_1, \ldots, f_{l-1}, v_1(py+q), \ldots, v_l(py+q) \rangle$, it follows from $f_1 \mid f_2 \mid \cdots \mid f_{l-1}$ that

$$I_1(\mathbf{W}) \subseteq \langle f_1, py+q \rangle \text{ and } I_1(\mathbf{W}) \subseteq \langle f_1, v_1, \ldots, v_l \rangle.$$

By the fact that $I_1(\mathbf{W}) = K[x,y]$, we have $\langle f_1, py+q \rangle = \langle f_1, v_1, \ldots, v_l \rangle = K[x,y]$. Then, $\gcd(f_1, v_1, \ldots, v_l) = 1$. $\square$

Lemma 3.11. *Let $\mathbf{M} \in K[x,y]^{l \times l}$ with $\det(\mathbf{M}) = fy + g$, where $f, g \in K[x]$ and $f \neq 0$. If $J_i(\mathbf{M}) = K[x,y]$ for $i = 1, \ldots, l$, then $\mathbf{M} \sim (\mathbf{B}_1, (py+q)\mathbf{b})$, where $\mathbf{B}_1 \in K[x]^{l \times (l-1)}$, $\mathbf{b} \in K[x]^{l \times 1}$ and $p, q \in K[x]$ satisfy $p \neq 0$ and $\gcd(p,q) = 1$.*

Proof. Let $h = \gcd(f, g)$. Then there exist $p, q \in K[x]$ such that $f = hp$ and $g = hq$, where $p \neq 0$ and $\gcd(p, q) = 1$. Since $\det(\mathbf{M}) = h(py+q)$ and $h \in K[x]$, by the Primitive Factorization Theorem (Lemma 3.1) there exist $\mathbf{F}, \mathbf{G} \in K[x, y]^{l\times l}$ such that

$$\mathbf{M} = \mathbf{F}\mathbf{G} \text{ and } \det(\mathbf{F}) = h. \tag{15}$$

Since $\det(\mathbf{M}) = \det(\mathbf{F}) \cdot \det(\mathbf{G})$, it follows from Equation (15) that $\det(\mathbf{G}) = py + q$. According to Lemma 3.3, we have $J_i(\mathbf{F}) = K[x, y]$ for all $i = 1, \dots, l$. Based on Theorem 3.5, $\mathbf{F}$ is equivalent to its Smith normal form $\mathbf{S}_{\mathbf{F}}$, where $\mathbf{S}_{\mathbf{F}} \in K[x]^{l\times l}$. That is, there exist $\mathbf{P}, \mathbf{Q} \in \mathrm{GL}_l(K[x, y])$ such that

$$\mathbf{F} = \mathbf{P} \cdot \mathbf{S}_{\mathbf{F}} \cdot \mathbf{Q}. \tag{16}$$

Combining Equations (15) and (16), we obtain

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{S}_{\mathbf{F}} \cdot \mathbf{Q} \cdot \mathbf{G}. \tag{17}$$

Let $\mathbf{G}' = \mathbf{Q}\cdot\mathbf{G}$. By the fact that $\mathbf{Q}$ is unimodular, we get $\det(\mathbf{G}') = \det(\mathbf{G}) = \alpha(py+q)$, where $\alpha \in K \setminus \{0\}$. Let $\mathbf{U} = \mathrm{diag}(1, \dots, 1, 1/\alpha)$ and $\mathbf{G}'' = \mathbf{U}\mathbf{G}'$. Then $\mathbf{U} \in \mathrm{GL}_l(K[x])$ and $\det(\mathbf{G}'') = py+q$. By Lemma 3.7, there exist $\mathbf{V}'' \in \mathrm{GL}_l(K[x])$ and $\mathbf{G}_1 \in \mathrm{GL}_l(K[x, y])$ such that $\mathbf{G}'' = \mathbf{V}'' \cdot \mathrm{diag}(1, \dots, 1, py + q) \cdot \mathbf{G}_1$. Let $\mathbf{V} = \mathbf{U}^{-1}\mathbf{V}''$. Then $\mathbf{V} \in \mathrm{GL}_l(K[x])$ and

$$\mathbf{G}' = \mathbf{V} \cdot \mathrm{diag}(1, \dots, 1, py + q) \cdot \mathbf{G}_1. \tag{18}$$

It follows from Equations (17) and (18) that

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{S}_{\mathbf{F}} \cdot \mathbf{V} \cdot \mathrm{diag}(1, \dots, 1, py + q) \cdot \mathbf{G}_1. \tag{19}$$

It follows from $\mathbf{S}_{\mathbf{F}} \in K[x]^{l\times l}$ and $\mathbf{V} \in \mathrm{GL}_l(K[x])$ that $\mathbf{S}_{\mathbf{F}} \cdot \mathbf{V} \in K[x]^{l\times l}$. Let $\mathbf{B}_1 \in K[x]^{l\times(l-1)}$ be the matrix formed by the first $l-1$ columns of $\mathbf{S}_{\mathbf{F}}\cdot\mathbf{V}$, and let $\mathbf{b} \in K[x]^{l\times 1}$ be the last column of $\mathbf{S}_{\mathbf{F}} \cdot \mathbf{V}$. Then

$$\mathbf{M} = \mathbf{P} \cdot (\mathbf{B}_1, (py + q)\mathbf{b}) \cdot \mathbf{G}_1.$$

Since $\mathbf{P}, \mathbf{G}_1$ are unimodular matrices, we have

$$\mathbf{M} \sim (\mathbf{B}_1, (py + q)\mathbf{b}). \quad \square$$

Lemma 3.12. *Let* $\mathbf{B} = (\mathbf{B}_1, (py + q)\mathbf{b}) \in K[x, y]^{l\times l}$ *with* $\det(\mathbf{B}) \neq 0$, *where* $\mathbf{B}_1 \in K[x]^{l\times(l-1)}$, $\mathbf{b} \in K[x]^{l\times 1}$, *and* $p, q \in K[x]$. *If* $I_1(\mathbf{B}) = K[x, y]$, *then*

$$\mathbf{B} \sim \begin{pmatrix} 1 & \mathbf{0}_{1\times(l-1)} \\ \mathbf{0}_{(l-1)\times 1} & \mathbf{M}_1 \end{pmatrix}.$$

Proof. Since $\det(\mathbf{B}) \neq 0$, $\mathbf{B}_1 \in K[x]^{l\times(l-1)}$ is a full column rank matrix. By the fact that $K[x]$ is a Euclidean domain, there exist $\mathbf{P} \in \mathrm{GL}_l(K[x])$ and $\mathbf{Q} \in \mathrm{GL}_{l-1}(K[x])$ such that

$$\mathbf{P}\mathbf{B}_1\mathbf{Q} = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & f_{l-1} \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $f_1, f_2, \dots, f_{l-1} \in K[x] \setminus \{0\}$ satisfy $f_1 \mid f_2 \mid \cdots \mid f_{l-1}$. Let $\mathbf{W} = \mathbf{P}\mathbf{B}\begin{pmatrix} \mathbf{Q} & \\ & 1 \end{pmatrix}$. Then

$$\mathbf{W} = \begin{pmatrix} f_1 & 0 & \cdots & 0 & v_1(py+q) \\ 0 & f_2 & \ddots & \vdots & v_2(py+q) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & f_{l-1} & v_{l-1}(py+q) \\ 0 & \cdots & \cdots & 0 & v_l(py+q) \end{pmatrix},$$

where $(v_1, \dots, v_l)^{\mathrm{T}} = \mathbf{P}\cdot\mathbf{b}$. By Lemma 2.6, we have $I_1(\mathbf{W}) = I_1(\mathbf{B}) = K[x,y]$. According to Lemma 3.10, we obtain

$$\langle f_1, py+q \rangle = K[x,y] \text{ and } \gcd(f_1, v_1, \dots, v_l) = 1. \tag{20}$$

Let $d = \gcd(v_2, \dots, v_l)$. Then there exist $c_2, \dots, c_n \in K[x]$ such that $d = c_2v_2 + \cdots + c_lv_l$. It follows from $\gcd(f_1, v_1, v_2, \dots, v_n) = 1$ that $\gcd(f_1, v_1, d) = 1$. By Lemma 3.9, there exists $h \in K[x]$ such that $\gcd(f_1, v_1 + hd) = 1$. As $f_1, v_1 + hd \in K[x] \subset K[x,y]$, we have

$$\langle f_1, v_1 + hd \rangle = K[x,y]. \tag{21}$$

Let

$$\mathbf{U} = \begin{pmatrix} 1 & hc_2 & \cdots & hc_{l-1} & hc_l \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \text{ and } \mathbf{W}' = \mathbf{U}\mathbf{W}.$$

Then

$$\mathbf{W}' = \begin{pmatrix} f_1 & hc_2f_2 & \cdots & hc_{l-1}f_{l-1} & (v_1+hd)(py+q) \\ 0 & f_2 & \cdots & 0 & v_2(py+q) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{l-1} & v_{l-1}(py+q) \\ 0 & 0 & \cdots & 0 & v_l(py+q) \end{pmatrix}.$$

Combining Equations (20) and (21), we deduce that

$$\langle f_1, (v_1+hd)(py+q)\rangle = K[x,y].$$

It follows that the first row of $\mathbf{W}'$ is a ZLP vector. By the Quillen-Suslin Theorem, there exists $\mathbf{V} \in \mathrm{GL}_l(K[x,y])$ such that

$$\mathbf{W}'\mathbf{V} = \begin{pmatrix} 1 & \mathbf{0}_{1\times(l-1)} \\ \mathbf{b}_1 & \mathbf{M}_1 \end{pmatrix},$$

where $\mathbf{b}_1 \in K[x,y]^{(l-1)\times 1}$ and $\mathbf{M}_1 \in K[x,y]^{(l-1)\times(l-1)}$. Furthermore, by performing elementary row operations on $\mathbf{W}'\mathbf{V}$, we get

$$\mathbf{W}'\mathbf{V} \sim \begin{pmatrix} 1 & \mathbf{0}_{1\times(l-1)} \\ \mathbf{0}_{(l-1)\times 1} & \mathbf{M}_1 \end{pmatrix}.$$

This implies that

$$\mathbf{B} \sim \begin{pmatrix} 1 & \\ & \mathbf{M}_1 \end{pmatrix}. \quad \square$$

Lemma 3.13. *Let* $\mathbf{M} = \begin{pmatrix} 1 & \mathbf{0}_{1\times(l-1)} \\ \mathbf{0}_{(l-1)\times 1} & \mathbf{M}_1 \end{pmatrix} \in K[x,y]^{l\times l}$. *Then* $J_i(\mathbf{M}) = J_{i-1}(\mathbf{M}_1)$ *for* $i = 2, \dots, l$.

Proof. For any integer i with $2 \le i \le l$, the structure of $\mathbf{M}$ implies the following: every $i \times i$ minor of $\mathbf{M}$ is either an $i \times i$ minor or an $(i-1)\times(i-1)$ minor of $\mathbf{M}_1$. This implies that

$$I_i(\mathbf{M}) = I_i(\mathbf{M}_1) + I_{i-1}(\mathbf{M}_1).$$

As $I_i(\mathbf{M}_1) \subseteq I_{i-1}(\mathbf{M}_1)$, we have

$$I_i(\mathbf{M}) = I_{i-1}(\mathbf{M}_1). \tag{22}$$

Since $I_i(\mathbf{M}) \subseteq \langle d_i(\mathbf{M})\rangle$, Equation (22) implies that every $(i-1)\times(i-1)$ minor of $\mathbf{M}_1$ is a multiple of $d_i(\mathbf{M})$, so $d_i(\mathbf{M}) \mid d_{i-1}(\mathbf{M})$. Conversely, we get $d_{i-1}(\mathbf{M}) \mid d_i(\mathbf{M})$. It follows that $d_i(\mathbf{M}) = d_{i-1}(\mathbf{M}_1)$. By the fact that

$$I_i(\mathbf{M}) = d_i(\mathbf{M}) \cdot J_i(\mathbf{M}) \text{ and } I_{i-1}(\mathbf{M}_1) = d_{i-1}(\mathbf{M}_1) \cdot J_{i-1}(\mathbf{M}_1),$$

we obtain $J_i(\mathbf{M}) = J_{i-1}(\mathbf{M}_1)$. $\square$

Theorem 3.14. *Let* $\mathbf{M} \in K[x,y]^{l\times l}$ *with* $\det(\mathbf{M}) = fy + g$, *where* $f, g \in K[x]$ *and* $f \neq 0$. *Then* $\mathbf{M}$ *is equivalent to its Smith normal form if and only if* $J_i(\mathbf{M}) = K[x,y]$ *for* $i = 1, \ldots, l$.

Proof. The necessity is obvious from Lemma 2.6. It suffices to prove the sufficiency. We proceed by induction on l. The statement is clearly true when $l = 1$. Assume that the statement holds for all $l < k$, where k is a positive integer with $k \geq 2$.

For $l = k$, we extract $d_1(\mathbf{M})$ from $\mathbf{M}$ and obtain

$$\mathbf{M} = d_1(\mathbf{M}) \cdot \mathbf{M}_0. \tag{23}$$

Given that $k \geq 2$ and $(d_1(\mathbf{M}))^k \mid \det(\mathbf{M})$, if $d_1(\mathbf{M})$ had a positive degree in y, it would follow that the degree of $\det(\mathbf{M})$ in y is greater than 1, thus contradicting the fact that $\det(\mathbf{M}) = fy + g$. Then $d_1(\mathbf{M}) \in K[x] \setminus \{0\}$. Let $f = \gcd(f,g) \cdot f_1$ and $g = \gcd(f,g) \cdot g_1$. Then $\det(\mathbf{M}) = \gcd(f,g) \cdot (f_1 y + g_1)$. Since $f_1 y + g_1$ is an irreducible polynomial in $K[x,y]$, it follows from $(d_1(\mathbf{M}))^k \mid \det(\mathbf{M})$ that $(d_1(\mathbf{M}))^k \mid \gcd(f,g)$. In addition, $d_1(\mathbf{M}_0) = 1$ and $\det(\mathbf{M}_0) \neq 0$. Equation (23) gives $d_i(\mathbf{M}) = (d_1(\mathbf{M}))^i \cdot d_i(\mathbf{M}_0)$ for $i = 1, \ldots, k$. This indicates that the $i \times i$ reduced minors of $\mathbf{M}$ are identical to those of $\mathbf{M}_0$ for $i = 1, \ldots, k$. It follows from $J_i(\mathbf{M}) = K[x,y]$ that $J_i(\mathbf{M}_0) = K[x,y]$ for $i = 1, \ldots, k$. By Lemma 3.11, we have

$$\mathbf{M}_0 \sim (\mathbf{B}_1, (py + q)\mathbf{b}) := \mathbf{B}, \tag{24}$$

where $\mathbf{B}_1 \in K[x,y]^{k\times(k-1)}$, $\mathbf{b} \in K[x,y]^{k\times 1}$, and $p, q \in K[x]$ satisfy $p \neq 0$ and $\gcd(p,q) = 1$. Lemma 2.6 shows that

$$\det(\mathbf{B}) \neq 0,\ d_1(\mathbf{B}) = d_1(\mathbf{M}_0) = 1, \text{ and } J_i(\mathbf{B}) = J_i(\mathbf{M}_0) = K[x,y] \text{ for } i = 1, \ldots, k.$$

This implies that $I_1(\mathbf{B}) = J_1(\mathbf{B}) = K[x,y]$. According to Lemma 3.12, we obtain

$$\mathbf{B} \sim \begin{pmatrix} 1 & \\ & \mathbf{M}_1 \end{pmatrix} := \mathbf{M}', \tag{25}$$

where $\mathbf{M}_1 \in K[x,y]^{(k-1)\times(k-1)}$. It follows from Lemma 3.13 that

$$J_i(\mathbf{M}_1) = J_{i+1}(\mathbf{M}') = J_{i+1}(\mathbf{B}) = K[x,y] \text{ for } i = 1, \ldots, k-1.$$

Since $\det(\mathbf{M}) = (d_1(\mathbf{M}))^k \cdot \det(\mathbf{M}_0)$ and $\mathbf{M}_0 \sim \mathbf{M}'$, we have

$$\det(\mathbf{M}_1) = \det(\mathbf{M}') = \alpha \cdot \det(\mathbf{M}_0) = \frac{\alpha}{(d_1(\mathbf{M}))^k} \cdot \det(\mathbf{M}) = \frac{\alpha}{(d_1(\mathbf{M}))^k} \cdot (fy + g),$$

where $\alpha \in K \setminus \{0\}$. Since $(d_1(\mathbf{M}))^k \mid \gcd(f, g)$, $\det(\mathbf{M}_1)$ is of the form $\tilde{f}y + \tilde{g}$, where $\tilde{f} = \frac{\alpha f}{(d_1(\mathbf{M}))^k}$ and $\tilde{g} = \frac{\alpha g}{(d_1(\mathbf{M}))^k}$ satisfy $\tilde{f}, \tilde{g} \in K[x]$. By the induction hypothesis, $\mathbf{M}_1$ is equivalent to its Smith normal form. That is,

$$\mathbf{M}_1 \sim \operatorname{diag}(\Psi_1, \ldots, \Psi_{k-1}), \tag{26}$$

where $\Psi_1, \ldots, \Psi_{k-1} \in K[x, y]$ satisfy $\Psi_1 \mid \Psi_2 \mid \cdots \mid \Psi_{k-1}$. Combining Equations (23)–(26), we obtain

$$\mathbf{M} \sim \operatorname{diag}(d_1(\mathbf{M}),\, d_1(\mathbf{M}) \cdot \Psi_1, \ldots, d_1(\mathbf{M}) \cdot \Psi_{k-1}).$$

Let $\Phi_1 = d_1(\mathbf{M})$ and $\Phi_i = d_1(\mathbf{M}) \cdot \Psi_{i-1}$ for $i = 2, \ldots, k$. Then

$$\mathbf{M} \sim \operatorname{diag}(\Phi_1, \Phi_2, \ldots, \Phi_k),$$

where $\Phi_i \mid \Phi_{i+1}$ for $i = 1, \ldots, k-1$. It follows that $\mathbf{M}$ is equivalent to its Smith normal form. □

Theorem 3.14 establishes an affirmative solution to Problem 1.2 for the case $f \neq 0$. Combining Theorems 3.5 and 3.14, we obtain the following main result, which provides a complete solution to Problem 1.2.

Theorem 3.15. *Let $\mathbf{M} \in K[x, y]^{l \times l}$ be of full rank such that $\det(\mathbf{M}) = fy + g$, where $f, g \in K[x]$. Then $\mathbf{M}$ is equivalent to its Smith normal form if and only if $J_i(\mathbf{M}) = K[x, y]$ for $i = 1, \ldots, l$.*

3.3. Rank-deficient and non-square cases

We begin by introducing a lemma that is related to the well-known Lin-Bose conjecture [33].

Lemma 3.16 *([34]). Let $\mathbf{M} \in K[x, y]^{l \times m}$ with $\operatorname{rank}(\mathbf{M}) = r$, where $1 \le r \le l$. If $J_r(\mathbf{M}) = K[x, y]$, then there exist $\mathbf{F} \in K[x, y]^{l \times r}$ and $\mathbf{G} \in K[x, y]^{r \times m}$ such that $\mathbf{M} = \mathbf{F}\mathbf{G}$, where $d_r(\mathbf{F}) = d_r(\mathbf{M})$ and $\mathbf{G}$ is a ZLP matrix.*

If $\mathbf{M} \in K[x, y]^{m \times l}$ in Lemma 3.16, then there exist $\mathbf{F} \in K[x, y]^{m \times r}$ and $\mathbf{G} \in K[x, y]^{r \times l}$ such that $\mathbf{M} = \mathbf{F}\mathbf{G}$, where $d_r(\mathbf{G}) = d_r(\mathbf{M})$ and $\mathbf{F}$ is a ZRP matrix. Leveraging Lemma 3.16 and the Quillen-Suslin Theorem, we establish an extension of Theorem 3.15 that accommodates both rank-deficient and non-square cases.

Corollary 3.17. *Let $\mathbf{M} \in K[x, y]^{l \times m}$ with $\operatorname{rank}(\mathbf{M}) = r$, and $d_r(\mathbf{M}) = fy + g$, where $1 \le r \le l$ and $f, g \in K[x]$. Then $\mathbf{M}$ is equivalent to its Smith normal form if and only if $J_i(\mathbf{M}) = K[x, y]$ for $i = 1, \ldots, r$.*

Proof. The necessity is obvious from Lemma 2.6. It suffices to prove the sufficiency. Since $J_r(\mathbf{M}) = K[x, y]$, by Lemma 3.16 there exist $\mathbf{F}_1 \in K[x, y]^{l\times r}$ and $\mathbf{G}_1 \in K[x, y]^{r\times m}$ such that

$$\mathbf{M} = \mathbf{F}_1\mathbf{G}_1, \tag{27}$$

where $\mathbf{G}_1$ is a ZLP matrix. By the Quillen-Suslin Theorem (Theorem 2.8), there exists $\mathbf{V} \in \mathrm{GL}_m(K[x, y])$ such that

$$\mathbf{G}_1\mathbf{V} = (\mathbf{I}_r, \mathbf{0}_{r\times(m-r)}). \tag{28}$$

Combining Equations (27) and (28), we have

$$\mathbf{M}\mathbf{V} = (\mathbf{F}_1, \mathbf{0}_{l\times(m-r)}). \tag{29}$$

By the fact that $\mathbf{V}$ is unimodular, it follows from Equation (29) and Lemma 2.6 that $J_r(\mathbf{F}_1) = J_r(\mathbf{M}) = K[x, y]$. Using Lemma 3.16 again, there exist $\mathbf{F}_2 \in K[x, y]^{l\times r}$ and $\mathbf{F}_3 \in K[x, y]^{r\times r}$ such that

$$\mathbf{F}_1 = \mathbf{F}_2\mathbf{F}_3, \tag{30}$$

where $\mathbf{F}_2$ is a ZRP matrix. According to the Quillen-Suslin Theorem, there exists $\mathbf{U} \in \mathrm{GL}_l(K[x, y])$ such that

$$\mathbf{U}\mathbf{F}_2 = \begin{pmatrix} \mathbf{I}_r \\ \mathbf{0}_{(l-r)\times r} \end{pmatrix}. \tag{31}$$

Combining Equations (29)–(31), we obtain

$$\mathbf{U}\mathbf{M}\mathbf{V} = \begin{pmatrix} \mathbf{F}_3 & \mathbf{0}_{r\times(m-r)} \\ \mathbf{0}_{(l-r)\times r} & \mathbf{0}_{(l-r)\times(m-r)} \end{pmatrix}.$$

Therefore, by Lemma 2.6 we get

$$d_i(\mathbf{F}_3) = d_i(\mathbf{M}) \text{ and } J_i(\mathbf{F}_3) = J_i(\mathbf{M}) \text{ for } i = 1, \ldots, r.$$

According to Theorem 3.15, there exist $\mathbf{U}_1, \mathbf{V}_1 \in \mathrm{GL}_r(K[x, y])$ such that

$$\mathbf{U}_1\mathbf{F}_3\mathbf{V}_1 = \mathrm{diag}(\omega_1, \ldots, \omega_r),$$

where $\omega_i = \frac{d_i(\mathbf{M})}{d_{i-1}(\mathbf{M})}$ for $i = 1, \ldots, r$. Let

$$\mathbf{U}_2 = \begin{pmatrix} \mathbf{U}_1 & \\ & \mathbf{I}_{l-r} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} \mathbf{V}_1 & \\ & \mathbf{I}_{m-r} \end{pmatrix}.$$

Then $\mathbf{U}_2, \mathbf{V}_2$ are unimodular matrices. Moreover, we have

$$\mathbf{U}_2\mathbf{UMVV}_2 = \begin{pmatrix} \mathrm{diag}(\omega_1, \dots, \omega_r) & \mathbf{0}_{r\times(m-r)} \\ \mathbf{0}_{(l-r)\times r} & \mathbf{0}_{(l-r)\times(m-r)} \end{pmatrix}. \tag{32}$$

It follows from Equation (32) that $\mathbf{M}$ is equivalent to its Smith normal form. □

Remark 3.18. All results in the above three subsections are constructive. In particular, whether or not the ideal $J_i(\mathbf{M})$ equals the unit ideal in $K[x, y]$ can be determined using Gröbner basis computations. Moreover, since all our arguments involve explicit transformations, the unimodular matrices $\mathbf{U}$ and $\mathbf{V}$ that establish the equivalence of $\mathbf{M}$ and its Smith normal form can be constructed by algorithmic procedures.

3.4. Illustrative example

Now we use an example to illustrate the effectiveness of Theorem 3.14.

Example 3.19. Let

$$\mathbf{M} = \begin{pmatrix} x^2y^2 - x^2y - xy^2 + 2x + y - 1 & x^3y^2 - x^3y - x^2y^2 - x^2y + x^2 + 2xy - 1 \\ xy^2 - xy - y^2 + y + 1 & x^2y^2 - x^2y - xy^2 + y \end{pmatrix}$$

be a bivariate polynomial matrix in $\mathbb{Q}[x, y]^{2\times 2}$, where $\mathbb{Q}$ is the field of rational numbers, $y > x$ and $\prec$ is the lexicographic order.

Upon calculation, we get $\det(\mathbf{M}) = (x - 1)(xy - x - 1)$. According to the definition of $I_1(\mathbf{M})$, we have

$$I_1(\mathbf{M}) = \langle \mathbf{M}[1, 1], \mathbf{M}[1, 2], \mathbf{M}[2, 1], \mathbf{M}[2, 2]\rangle,$$

where $\mathbf{M}[i, j]$ is the (i, j)-th entry of $\mathbf{M}$, for $1 \le i, j \le 2$. We compute a reduced Gröbner basis [35] of $I_1(\mathbf{M})$ with respect to $\prec$ and obtain $\{1\}$. This implies that $J_1(\mathbf{M}) = I_1(\mathbf{M}) = \mathbb{Q}[x, y]$ and $d_1(\mathbf{M}) = 1$. In addition, $J_2(\mathbf{M}) = \mathbb{Q}[x, y]$. Based on Theorem 3.14, $\mathbf{M}$ is equivalent to its Smith normal form. We proceed to construct $\mathbf{U}, \mathbf{V} \in \mathrm{GL}_2(\mathbb{Q}[x, y])$ satisfying

$$\mathbf{UMV} = \begin{pmatrix} 1 & 0 \\ 0 & (x - 1)(xy - x - 1) \end{pmatrix},$$

in a manner analogous to the proof of Theorem 3.14.

We first extract $d_1(\mathbf{M})$ from $\mathbf{M}$ to obtain $\mathbf{M}_0$. Since $d_1(\mathbf{M}) = 1$, we have $\mathbf{M}_0 = \mathbf{M}$. Then $J_1(\mathbf{M}_0) = J_2(\mathbf{M}_0) = \mathbb{Q}[x, y]$. It follows from Lemma 3.11 that there exist unimodular matrices

$$\mathbf{U}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ and}$$

$$\mathbf{V}_1 = \begin{pmatrix} -x^2y + 2x^2 + xy - 1 & x^2y^2 - 3x^2y - xy^2 + 2x^2 + 2x + y \\ xy - 2x - y + 1 & -xy^2 + 3xy + y^2 - 2x - y - 1 \end{pmatrix}$$

such that

$$\begin{aligned} \mathbf{U}_1\mathbf{M}_0\mathbf{V}_1 = (\mathbf{B}_1, (xy - x - 1)\mathbf{b}) &= \begin{pmatrix} 2x^3 - x^2 & (xy - x - 1)(-2x^2 + x - 1) \\ -2x^3 + 3x^2 - 1 & (xy - x - 1)(2x^2 - 3x + 1) \end{pmatrix} \\ &:= \mathbf{B}. \end{aligned}$$

Lemma 2.6 gives $I_1(\mathbf{B}) = I_1(\mathbf{M}_0) = \mathbb{Q}[x, y]$. According to Lemma 3.12, there exist unimodular matrices

$$\mathbf{U}_2 = \begin{pmatrix} -4x^2 + 4x + 1 & -4x^2 - 1 \\ 2x^3 - 3x^2 + 1 & 2x^3 - x^2 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 1 & 2xy - 2x - 2 \\ 0 & 1 \end{pmatrix}$$

such that

$$\mathbf{U}_2\mathbf{B}\mathbf{V}_2 = \begin{pmatrix} 1 & \\ & \mathbf{M}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (x - 1)(xy - x - 1) \end{pmatrix}.$$

Let

$$\mathbf{U} = \mathbf{U}_2\mathbf{U}_1 = \begin{pmatrix} 4x + 2 & -4x^2 - 1 \\ -2x^2 + 1 & 2x^3 - x^2 \end{pmatrix}$$

and

$$\mathbf{V} = \mathbf{V}_1\mathbf{V}_2 = \begin{pmatrix} -x^2y + 2x^2 + xy - 1 & -(x - 1)(2xy - 4x - y - 2)(xy - x - 1) \\ xy - 2x - y + 1 & 2x^2y^2 - 6x^2y + 4x^2 - 3xy^2 + 5xy + y^2 + y - 3 \end{pmatrix}.$$

Then,

$$\mathbf{UMV} = \begin{pmatrix} 1 & 0 \\ 0 & (x - 1)(xy - x - 1) \end{pmatrix}.$$

Remark 3.20. The construction of $\mathbf{U}_1, \mathbf{V}_1$ (resp. $\mathbf{U}_2, \mathbf{V}_2$) in Example 3.19 follows from Lemma 3.11 (resp. Lemma 3.12). More precisely, the proof of Lemma 3.11 relies on the Primitive Factorization Theorem, and a constructive proof of this result is provided in [30]. For Lemma 3.12, since any matrix over $K[x]$ can be reduced to its Smith normal form by finitely many unimodular row and column operations (see the proof of Theorem S1.1 on Page 317 of [36], where an inductive procedure is provided), the matrices $\mathbf{P}$ and $\mathbf{Q}$ therein can be constructed explicitly.

4. Concluding remarks

In this paper, we have investigated the equivalence problem between bivariate polynomial matrices and their Smith normal forms. Let $\mathbf{M} \in K[x,y]^{l\times m}$ with $\mathrm{rank}(\mathbf{M}) = r$, where $1 \leq r \leq l$. Corollary 3.17 shows that, for the case $d_r(\mathbf{M}) = fy+g$ with $f, g \in K[x]$, Frost and Storey's assertion holds. Consequently, we have completely solved the equivalence problem between this class of bivariate polynomial matrices and their Smith normal forms. We hope that the methods proposed herein will facilitate research on the equivalence problem between multivariate polynomial matrices (with more than 2 variables) and their Smith normal forms.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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